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TECHNIQUES FOR INTEGRAL CALCULUS

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Abstract: *In this paper are presented few techniques for integral calculus and his applications.*

1. Introduction.

Proposition 1.1
Let $f : [a, b] \to \mathbb{R}, g : [a, b] \to \mathbb{R}^*$ and $u : [a, b] \to \mathbb{R}$ continuous function with properties $f(x) + f(s - x) = u(x)$, $g(x) = g(s - x), \forall x \in [a, b], s = a + b$ then:

$$\int_a^b \frac{f(x)}{g(x)}\, dx = \frac{1}{2}\int_a^b \frac{u(x)}{g(x)}\, dx$$

Proof. Using the substitution $x = s - t$, it follows:

$$\int_a^b \frac{f(x)}{g(x)}\, dx = \int_a^b \frac{f(s-t)}{g(s-t)}\,(-dt) = \int_a^b \frac{u(t) - f(t)}{g(t)}\, dt$$
$$= \int_a^b \frac{u(t)}{g(t)}\, dt - \int_a^b \frac{f(t)}{g(t)}\, dt$$

Hence,

$$\int_a^b \frac{f(x)}{g(x)}\, dx = \frac{1}{2}\int_a^b \frac{u(x)}{g(x)}\, dx$$

Application 1.1 Find:

$$I = \int_0^\pi \frac{(x+1)\sin x}{3+\cos^2 x}\, dx$$

Solution.

$$I = \int_0^\pi \frac{(x+1)\sin x}{3+\cos^2 x}\, dx = \int_0^\pi \frac{x\sin x}{3+\cos^2 x}\, dx + \int_0^\pi \frac{\sin x}{3+\cos^2 x}\, dx$$
$$= I_1 + I_2, \text{ where}$$

$$I_1 = \int_0^\pi \frac{x\sin x}{3+\cos^2 x}\, dx = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{3+\cos^2 x}\, dx =$$
$$= \pi\int_0^\pi \frac{\sin x}{3+\cos^2 x}\, dx - \int_0^\pi \frac{x\sin x}{3+\cos^2 x}\, dx, \text{ thus}$$

$$
\begin{aligned}
I_1 =& \pi I_2 - I_1 \text{ or } I_1 = \frac{\pi}{2} I_2, \text{ where} \\
I_2 =& \int_0^\pi \frac{\sin x}{3+\cos^2 x}\, dx = -\int_0^\pi \frac{d(\cos x)}{3+\cos^2 x}\, dx = \\
=& -\frac{1}{\sqrt{3}} \operatorname{arctg}(\frac{\cos x}{\sqrt{3}})\Big|_0^\pi = \frac{2\pi}{3\sqrt{3}}
\end{aligned}
$$

Application 1.2 Find:

$$
I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x \operatorname{arctg} x}{1+e^{\operatorname{tg} x}}\, dx
$$

Solution.

$$
\begin{aligned}
I =& \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x \operatorname{arctg} x}{1+e^{\operatorname{tg} x}}\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(-x) \operatorname{arctg}(-x)}{1+e^{\operatorname{tg}(-x)}}\, dx \\
=& \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x e^{\operatorname{tg} x} \operatorname{arctg} x}{1+e^{\operatorname{tg} x}}\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x(1+e^{\operatorname{tg} x}-1) \operatorname{arctg} x}{1+e^{\operatorname{tg} x}}\, dx \\
=& \int_{-\sqrt{3}}^{\sqrt{3}} x \operatorname{arctg} x\, dx - I \text{ or} \\
2I =& \int_{-\sqrt{3}}^{\sqrt{3}} x \operatorname{arctg} x\, dx = 2\int_0^{\sqrt{3}} x \operatorname{arctg} x\, dx, \text{ then} \\
I =& \int_0^{\sqrt{3}} x \operatorname{arctg} x\, dx = \left(\frac{x^2}{2} - \operatorname{arctg} x\right)\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\, dx \\
=& \frac{\pi}{3} - \frac{\sqrt{3}}{2}
\end{aligned}
$$

Application 1.3 Find:

$$
I = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x \cdot \ln(1+e^{x\sqrt{1-x^2}})\, dx
$$

Solution.

$$
\begin{aligned}
I =& \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x \cdot \ln(1+e^{x\sqrt{1-x^2}})\, dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (-x) \cdot \ln(1+e^{(-x)\sqrt{1-(-x)^2}})\, dx \\
=& -\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x \cdot \ln\left(\frac{1+e^{x\sqrt{1-x^2}}}{e^{x\sqrt{1-x^2}}}\right) dx = -I + \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x \cdot \ln\left(e^{x\sqrt{1-x^2}}\right)\, dx \\
=& -I + \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x^2\sqrt{1-x^2}\, dx \\
2I =& 2\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x^2\sqrt{1-x^2}\, dx \overset{x=\sin t}{=} 2\int_0^{\pi/4} \sin^2 t\sqrt{1-\sin^2 t} \cdot \cos t\, dt \text{ or}
\end{aligned}
$$

$$I = \int_0^{\pi/4} \sin^2 t \cos^2 t \, dt = \frac{1}{4} \int_0^{\pi/4} \sin^2 2t \, dt = \frac{1}{8} \left(x - \frac{1}{4} \sin 4t \right) \Big|_0^{\pi/4} = \frac{\pi}{32}$$

Application 1.4 Find:

$$I = \int_0^{\pi/4} \frac{\ln(1 + \operatorname{tg} x)}{2 + \sin 2x + \cos 2x} \, dx$$

Solution.

$$I = \int_0^{\pi/4} \frac{\ln(1 + \operatorname{tg} x)}{2 + \sin 2x + \cos 2x} \, dx \overset{x \to \pi/4 - x}{=} \int_0^{\pi/4} \frac{\ln\left(1 + \operatorname{tg}(\frac{\pi}{4} - x)\right)}{2 + \sin(\frac{\pi}{2} - 2x) + \cos(\frac{\pi}{2} - x)} \, dx$$

$$= \int_0^{\pi/4} \frac{\ln 2 - \ln(1 + \operatorname{tg} x)}{2 + \sin 2x + \cos 2x} \, dx$$

$$2I = \ln 2 \int_0^{\pi/4} \frac{1}{2 + \frac{2 \operatorname{tg} x}{1 + \operatorname{tg}^2 x} + \frac{1 - \operatorname{tg}^2 x}{1 + \operatorname{tg}^2 x}} \, dx \overset{t = \operatorname{tg} x}{=} \ln 2 \int_0^1 \frac{1}{t^2 + 2t + 3} \, dt, \text{ then}$$

$$I = \frac{\ln 2}{2\sqrt{2}} (\operatorname{arctg} \sqrt{2} - \frac{\pi}{4})$$

Application 1.5 Find:

$$I = \int_0^{2\pi} \frac{x + \operatorname{tg}(\sin x)}{2 + \cos x} \, dx$$

Solution 1.

$$I = \int_0^{2\pi} \frac{x + \operatorname{tg}(\sin x)}{2 + \cos x} \, dx \overset{x \to 2\pi - x}{=} \int_0^{2\pi} \frac{2\pi - x - \operatorname{tg}(\sin x)}{2 + \cos x} \, dx$$

$$= 2\pi \int_0^{2\pi} \frac{1}{2 + \cos x} \, dx - I$$

$$\frac{1}{\pi} I = \int_0^{2\pi} \frac{1}{2 + \cos x} \, dx = \int_0^{\pi} \frac{1}{2 + \cos x} \, dx + \int_{\pi}^{2\pi} \frac{1}{2 + \cos x} \, dx$$

$$= \int_0^{\pi} \frac{1}{2 + \cos x} \, dx + \int_0^{\pi} \frac{1}{2 - \cos x} \, dx$$

$$= \int_0^{\pi/2} \frac{1}{2 + \cos x} \, dx + \int_0^{\pi/2} \frac{1}{2 + \cos x} \, dx + \int_0^{\pi/2} \frac{1}{2 + \sin x} \, dx +$$

$$+ \int_0^{\pi/2} \frac{1}{2 - \sin x} \, dx$$

$$= 2 \int_0^1 \frac{1}{t^2 + 3} \, dt + \frac{2}{3} \int_0^1 \frac{1}{t^2 + \frac{1}{3}} \, dt + \int_0^1 \frac{1}{(t + 1)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dt +$$

$$+ \int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2\pi\sqrt{3}}{3}, \text{ then } I = \frac{2\pi^2\sqrt{3}}{3}$$

Solution 2.

$$I=\int_0^{2\pi}\frac{x+\operatorname{tg}(\sin x)}{2+\cos x}\,dx \overset{x\to 2\pi-x}{=} \int_{-\pi}^{\pi}\frac{\pi+x-\operatorname{tg}(\sin x)}{2-\cos x}\,dx$$

$$=\pi\int_{-\pi}^{\pi}\frac{1}{2-\cos x}\,dx+\int_{-\pi}^{\pi}\frac{x-\operatorname{tg}(\sin x)}{2-\cos x}\,dx=2\pi\int_0^{2\pi}\frac{1}{2-\cos x}\,dx$$

$$=2\pi\int_{-\pi/2}^{\pi/2}\frac{1}{2+\sin x}\,dx=2\pi\int_{-\pi/2}^{\pi/2}\frac{1}{2+\frac{2\operatorname{tg}(\frac{x}{2})}{1+\operatorname{tg}^2(\frac{x}{2})}}\,dx=2\pi\int_{-1}^{1}\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$=\frac{2\pi^2\sqrt{3}}{3}$$

2. General result.

Proposition 2.1
Let $f:\mathbb{R}\to\mathbb{R}$ be continuous function such that

$$\alpha f(x+b)+\beta f(c-x)=g(x+b),\forall x\in\mathbb{R};\ (1)$$

where $\alpha,\beta\in\mathbb{R}^*,\alpha+\beta\neq 0,b,c\in\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ continuous function with the primitives G. Then for all $m,n\in\mathbb{R}$ such that $m+n=b+c$,

$$I_{m,n}=\int_m^n f(x)\,dx=\frac{G(n)-G(m)}{\alpha+\beta};\ (2)$$

Proof. Replacing x with $x-b$, we get $\alpha f(x)+\beta f(b+c-x)=g(x)$, and hence

$$f(x)=\frac{1}{\alpha}g(x)-\frac{\beta}{\alpha}(b+c-x)$$

Integrating both sides on the interval $[m,n]$, it follows:

$$I_{m,n}=\int_m^n f(x)\,dx=\frac{1}{\alpha}\int_m^n g(x)\,dx-\frac{\beta}{\alpha}\int_m^n f(b+c-x)\,dx$$

$$=\frac{1}{\alpha}[G(n)-G(m)]-\frac{\beta}{\alpha}\int_m^n f(b+c-x)\,dx$$

$$\overset{(b+c-x=t)}{=}\frac{1}{\alpha}[G(n)-G(m)]+\frac{\beta}{\alpha}\int_n^m f(t)\,dt$$

$$I_{m,n}\left(1+\frac{\beta}{\alpha}\right)=\frac{1}{\alpha}[G(n)-G(m)],\text{ then}$$

$$I_{m,n}=\int_m^n f(x)\,dx=\frac{G(n)-G(m)}{\alpha+\beta}$$

Corolary 2.1
If $h:\mathbb{R}\to\mathbb{R}$ is odd function,$\theta-$periodic function with $f'-$continuous, then

$$I=\int_0^{2\theta}\frac{xh'(x)}{1+h^2(x)}\,dx=\theta[\operatorname{arctg}h(n)-\operatorname{arctg}h(m)],$$

where $m+n=2\theta$.
Proof. Let $f(x)=\frac{xh'(x)}{1+h^2(x)}$ then $f(x+2\theta)+f(-x)=2\theta\cdot\frac{h'(x)}{1+h^2(x)}$ and for

$$\alpha=\beta=1,\ b=2\theta, c=0, g(x)=2\theta\cdot\frac{h'(x)}{1+h^2(x)}$$

Let $G:\mathbb{R}\to\mathbb{R}, G(x)=2\theta\cdot\operatorname{arctg} x$ be an primitive of the function g. For $m+n=2\theta$ and using **Proposition 2.1**, we have:

$$I_{m,n}=\int_m^n f(x)\,dx=\theta[\operatorname{arctg} h(n)-\operatorname{arctg} h(m)]$$

Application 2.1 Find:

$$I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

Solution. Let $f(x)=\frac{x\sin x}{1+\cos^2 x}$, then $f(x)+f(\pi-x)=\frac{\pi\sin x}{1+\cos^2 x}$ and using **Proposition 2.1** for $m+n=\pi, m=0, n=\pi$, we get:

$$I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx=\frac{\pi^2}{4}$$

Application 2.2 Find:

$$I=\int_0^{\pi/4}\ln(1+\operatorname{tg} x)\,dx$$

Solution. Let $f(x)=\ln(1+\operatorname{tg} x)$, then

$$f(\frac{\pi}{4}-x)=\ln(1+\frac{1-\operatorname{tg} x}{1+\operatorname{tg} x})=\ln 2-\ln(1+\operatorname{tg} x)$$

$$f(x)+f(\frac{\pi}{4}-x)=\ln 2$$

Using **Proposition 2.1**, we get:

$$I_{m,n}=\int_m^n \ln(1+\operatorname{tg} x)\,dx=\frac{n-m}{2}\ln 2;\ m+n=\frac{\pi}{2}$$

For $m=0, n=\frac{\pi}{4}$, it follows $I=\frac{\pi}{8}\ln 2$.
Application 8. If $h:[0,1]\to\mathbb{R}$ is continuous function, then prove:

$$\int_0^\pi x\cdot h(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi h(\sin x)\,dx$$

Solution. Let $f(x)=xh(\sin x)-\frac{\pi}{2}h(\sin x)$, then $f(x)+f(\pi-x)=0$.
Using **Proposition 2.1**, we get

$$\int_m^n h(x)\,dx=0,\ m+n=\pi$$

For $m=0, n=\pi$, we get the problem.
Application 2.3 Prove that:

$$\int_0^1 \frac{dx}{\sqrt{x^4-4x^3+6x^2-4x+2}}=\int_0^1\frac{dx}{\sqrt{1+x^4}}$$

Solution. Let $f : [0,1] \to \mathbb{R}, f(x) = \frac{1}{\sqrt{x^4-4x^3+6x^2-4x+2}} - \frac{1}{\sqrt{1+x^4}}$, then $f(x) + f(1-x) = 0$.
Using **Proposition 2**, we have:

$$\int_m^n f(x)\,dx = 0;\ m + n = 1$$

For $m = 0, n = 1$ we get the problem.

3. Extension result.

Proposition 3.1
Let $f : [a-\theta, a+\theta] \to \mathbb{R}$ be continuous function such that $\alpha f(a+x) + \beta f(a-x) = \gamma, \forall x \in [-\theta, \theta], \alpha, \beta \in \mathbb{R}^*, \gamma \in \mathbb{R}$, then:

$$i) \int_{a-\theta}^{a+\theta} f(x)\,dx = \frac{2\gamma\theta}{\alpha+\beta};\ \alpha+\beta \neq 0$$

$$ii) \int_{a-\theta}^{a+\theta} f(x)\,dx = \frac{\gamma\theta}{\alpha} + \frac{\alpha-\beta}{\alpha}\int_a^{a+\theta} f(x)\,dx$$

Proof. If $f : [a-\theta, a+\theta] \to \mathbb{R}$ is continuous function and $u, v : [-\theta, \theta] \to [a-\theta, a+\theta], u(x) = a - t, v(x) = a + t$, then:

$$i) \int_{a-\theta}^{a+\theta} f(x)\,dx = \int_{-u(-\theta)}^{u(\theta)} f(x)\,dx = \int_{-\theta}^{\theta} f(u(t))u'(t)\,dt$$

$$= \int_{-\theta}^{\theta} f(a+t)\,dt = \int_{-\theta}^{\theta} \left(\frac{\gamma}{\alpha} - \frac{\beta}{\alpha} f(a-t)\right) dt =$$

$$= \frac{2\gamma\theta}{\alpha} - \frac{\beta}{\alpha}\int_{a-\theta}^{a+\theta} f(a-t)\,dx = \frac{2\gamma\theta}{\alpha} + \frac{\beta}{\alpha}\int_{-\theta}^{\theta} f(v(t))v'(t)\,dt =$$

$$= \frac{2\gamma\theta}{\alpha} + \int_{a-\theta}^{a+\theta} f(x)\,dx$$

Therefore,

$$\int_{a-\theta}^{a+\theta} f(x)\,dx + \frac{\beta}{\alpha}\int_{a-\theta}^{a+\theta} f(x)\,dx = \frac{2\gamma\theta}{\beta} \text{ or}$$

$$\int_{a-\theta}^{a+\theta} f(x)\,dx = \frac{2\gamma\theta}{\alpha+\beta};\ \alpha+\beta \neq 0$$

$ii) \int_{a-\theta}^{a+\theta} f(x)\,dx = \int_{a-\theta}^{a} f(x)\,dx + \int_a^{a+\theta} f(x)\,dx$, where

$$\int_{a-\theta}^{a} f(x)\,dx = \int u(-\theta)^{u(0)} f(x)\,dx = \int_{-\theta}^{0} f(u(t))u'(t)\,dt =$$

$$
\begin{aligned}
&= \int_{-\theta}^{0} f(a+t)\,dt = \int_{-\theta}^{0} \left(\frac{\gamma}{\alpha} - \frac{\beta}{\alpha} f(a-t)\right) dt = \\
&= \frac{\gamma\theta}{\alpha} - \frac{\beta}{\alpha}\int_{-\theta}^{0} f(a-t)\,dt = \frac{\gamma\theta}{\alpha} + \frac{\beta}{\alpha}\int_{v(-\theta)}^{v(0)} f(v(t))v'(t)\,dt = \\
&= \frac{\gamma\theta}{\alpha} - \frac{\beta}{\alpha}\int_{\theta}^{a+\theta} f(x)\,dx
\end{aligned}
$$

So, we have:

$$
\begin{aligned}
\int_{a-\theta}^{a+\theta} f(x)\,dx =& \frac{\gamma\theta}{\alpha} - \frac{\beta}{\alpha}\int_{a}^{a+\theta} f(x)\,dx + \int_{a}^{a+\theta} f(x)\,dx = \\
=& \frac{\gamma\theta}{\alpha} + \frac{\alpha-\beta}{\alpha}\int_{a}^{a+\theta} f(x)\,dx
\end{aligned}
$$

Observation:

3.1) The function $f : [a-\theta, a+\theta] \to \mathbb{R}$ is $a-even$ function iff

$$f(a-x) = f(a+x), \forall x \leq |\theta|.$$

3.2) The function $f : [a-\theta, a+\theta] \to \mathbb{R}$ is $a-odd$ function iff

$$f(a+x) = -f(a-x), \forall |x| \leq \theta.$$

Application 3.1 Find:

$$I_n = \int_0^1 \frac{4x^3-6x^2+8x-3}{(x^2-x+1)^n}\,dx,\; n \in \mathbb{N}$$

Solution. The function $g(x) = x^2-x+1$ is $\frac{1}{2}-even$ and $h(x) = 4x^3-6x^2+8x-3$ is $\frac{1}{2}-odd$, so $\frac{h(x)}{g^n(x)}$ is $\frac{1}{2}-odd$ and using **Proposition 3.1**, we get:

$$I_n = \int_0^1 \frac{4x^3-6x^2+8x-3}{(x^2-x+1)^n}\,dx = 0,\; n \in \mathbb{N}$$

Application 3.2 Find:

$$I_n = \int_0^1 (2x-1)^{2n+1} e^{x-x^2}\,dx,\; n \in \mathbb{N}$$

Solution. The function $g(x) = (2x-1)^{2n+1}$ is $\frac{1}{2}-odd$ and $h(x) = e^{x-x^2}$ is $\frac{1}{2}-even$, then $f(x) = g(x)\cdot h(x)$ is $\frac{1}{2}-odd$ function.
Using **Proposition 3.1**, we get

$$I_n = \int_0^1 (2x-1)^{2n+1} e^{x-x^2}\,dx = 0,\; \forall n \in \mathbb{N}$$

Corolary 3.1
For any $f : [a-\theta, a+\theta] \to \mathbb{R}$ exist a function $f_1, a-even$ and $f_2, b-odd$ such that:

$$\boldsymbol{f(x) = f_1(x) + f_2(x),\ \forall x \in [a-\theta, a+\theta].}$$

Corolary 3.2
Let $f, g : [a-\theta, a+\theta] \to \mathbb{R}$ integrable functions and f is $a - odd$, then:

$$\int_{a-\theta}^{a+\theta} f(x)g(x)\,dx = \int_{a}^{a+\theta} f(x)(g(x)+g(2a-x))dx$$

Application 3.3
Let $f : [-1,1] \to \mathbb{R}$ continuous function such that $f(x)+f(-x)=\pi$, $\forall x \in [-1,1]$. Find:

$$I_n = \int_0^{(2n+1)\pi} f(\cos x)\,dx,\ \forall n \in \mathbb{N}$$

Solution. We have:

$$I_n = I_{n-1} + \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x)\,dx$$

$$g(x) = f(\cos x) \text{ is } 2n\pi - odd, \text{ then}$$

$$I_n = \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x)\,dx = 2\int_{2n\pi}^{2n\pi+\pi} f(\cos x)\,dx =$$

$$=2\int_0^{\pi} f(\cos(t+2n\pi))\,dt = 2\int_0^{\pi} f(\cos t)\,dt = 2\int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}}\,du$$

$$g(u) = \frac{1}{\sqrt{1-u^2}} \text{ is } 0 - even, \text{ then}$$

$$I_n = 2\int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}}\,du = 2\int_0^1 \frac{f(u)+f(-u)}{\sqrt{1-u^2}}\,du = 2\pi\int_0^1 \frac{du}{\sqrt{1-u^2}} = \pi^2$$

Therefore, $I_n = I_{n-1} + \pi^2$ and hence $I_n = (n+1)\pi^2$.
Application 3.4 Find:

$$I_n = \int_0^{\pi/4} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx, n \in \mathbb{N}$$

Solution. Let $f(x) = \frac{x\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x}$ be $\frac{\pi}{2} - even$, then using **Proposition 3.1**, we have:

$$g(x) = 1 \text{ is } \frac{3\pi}{4} - even \text{ and } f(\frac{3\pi}{4} - x) = \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} = f(x)$$

$$I_n = \int_0^{\pi/4} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = \int_0^{\pi} xf(x)\,dx = \int_{\pi/2}^{\pi} f(x)(x+\pi-x)\,dx =$$

$$=\pi\int_{\pi/2}^{\pi} d(x)\,dx = \pi\int_{3\pi/4}^{\pi} \left(f(x)+f(\frac{3\pi}{2}-x)\right)dx = \pi\int_{3\pi/4}^{\pi} dx = \frac{\pi^2}{4}$$

Application 3.5 Find:

$$I = \int_0^{\pi/4} \frac{\ln(1+\operatorname{tg} x)}{\sin 2x + \cos 2x}\, dx$$

Solution.

$f(x) = \dfrac{1}{\sin 2x + \cos 2x} = \dfrac{1}{\sqrt{2}\cos\left(2x - \frac{\pi}{4}\right)}$ then f is $\dfrac{\pi}{8} - even$.

$$I = \int_0^{\pi/4} \frac{\ln(1+\operatorname{tg} x)}{\sin 2x + \cos 2x}\, dx = \int_0^{\pi/4} f(x)\ln(1+\operatorname{tg} x)\, dx =$$

$$= \int_{\pi/8}^{\pi/4} f(x)\left(\ln(1+\operatorname{tg} x) + \ln(\frac{2}{1+\operatorname{tg} x})\right) dx = \frac{\ln 2}{\sqrt{2}}\int_{\pi/8}^{\pi/4} \frac{dx}{\cos(2x - \frac{\pi}{4})}$$

Proposition 3.2
Let $a, b \in \mathbb{R}, a < b$ and $f : [0, b-a] \to \mathbb{R}$ continuous function, then:

$$\int_a^b \frac{f(x-a)}{f(x-a) + f(b-x)}\, dx = \frac{b-a}{2}$$

Proof. We have:

$$I = \int_a^b \frac{f(x-a)}{f(x-a)+f(b-x)}\, dx =$$

$$= \int_a^{\frac{a+b}{2}} \frac{f(x-a)}{f(x-a)+f(b-x)}\, dx + \int_{\frac{a+b}{2}}^b \frac{f(x-a)}{f(x-a)+f(b-x)}\, dx$$

$$= \int_a^{\frac{a+b}{2}} \frac{f(x-a)}{f(x-a)+f(b-x)}\, dx - \int_{\frac{a+b}{2}}^a \frac{f(b-t)}{f(b-t)+f(t-a)}\, dt$$

$$= \int_a^{\frac{a+b}{2}} \frac{f(x-a)+f(b-x)}{f(x-a)+f(b-x)} dx$$

$$= \frac{b-a}{2}$$

Application 3.6 Find:

$$I = \int_n^{n+1} \frac{\operatorname{arctg}(x-n)}{\operatorname{arctg}(x-n) + \operatorname{arctg}(n+1-x)} dx,\ n \in \mathbb{N}$$

Solution. Using **Proposition 3.2**, we have:

$$I = \int_n^{n+1} \frac{\operatorname{arctg}(x-n)}{\operatorname{arctg}(x-n) + \operatorname{arctg}(n+1-x)} dx = \frac{1}{2}$$

Application 3.7 For $n \in \mathbb{N}, n > 1$ find:

$$I = \int_{n-1}^{n+1} \frac{\operatorname{arctg}(x-n+1)}{\operatorname{arctg}(\frac{2}{(x-n)^2})}\, dx$$

Solution. Using the relation $\operatorname{arctg}\alpha + \operatorname{arctg}\beta = \operatorname{arctg}(\frac{\alpha+\beta}{1-\alpha\beta})$ we have:

$$\operatorname{arctg}(x-(n-1)) + \operatorname{arctg}((n+1)-x) = \operatorname{arctg}\left(\frac{x-(n-1)+(n+1)-x}{1-(x-(n-1))((n+1)-x)}\right) = \\ = \operatorname{arctg}\left(\frac{2}{(x-n)^2}\right)$$

Now, making the substitution $t = 2n - x$, we have:

$$I = \int_{n-1}^{n+1} \frac{\operatorname{arctg}(n+1-t)}{\operatorname{arctg}\left(\frac{2}{(t-n)^2}\right)} dt$$

hence,

$$2I = \int_{n-1}^{n+1} \frac{\operatorname{arctg}(x-n+1) + \operatorname{arctg}(n+1-x)}{\operatorname{arctg}\left(\frac{2}{(x-n)^2}\right)} dx =$$

$$= \int_{n-1}^{n+1} \frac{\operatorname{arctg}\left(\frac{2}{(x-n)^2}\right)}{\operatorname{arctg}\left(\frac{2}{(x-n)^2}\right)} dx = 2$$

$$I = 1$$

4. Improper integrals with parameter.

Application 4.1 Find:

$$I = \int_0^{\pi/2} \ln\left(\frac{1-t\cos x}{1+t\cos x}\right)\frac{dx}{\cos x},\ |t| < 1$$

Solution. Let us denote:

$F(t) = \int_0^{\pi/2} \ln\left(\frac{1-t\cos x}{1+t\cos x}\right)\frac{dx}{\cos x}$ then $F(0) = 0$ and

$F'(t) = \int_0^{\pi/2} \frac{2}{t^2\cos^2 x - 1} dx = -2\int_0^{+\infty} \frac{1}{1-t^2-u^2} du = -\frac{2}{\sqrt{1-t^2}} \cdot \frac{\pi}{2}$ or

$F(t) = -\pi \arcsin t + C,$ but from $F(0) = 0$, we get $C = 0$.

Application 4.2 Find:

$$I = \int_0^{\pi/2} \ln\left(\frac{a+b\sin x}{a-b\sin x}\right)\frac{dx}{\sin x},\ 0 \le b < a.$$

Solution.

$$I = \int_0^{\pi/2} \ln\left(\frac{a+b\sin x}{a-b\sin x}\right)\frac{dx}{\sin x} = -\int_0^{\pi/2} \ln\left(\frac{a-b\sin x}{a+b\sin x}\right)\frac{dx}{\sin x}$$

$$= -\int_0^{\pi/2} \ln\left(\frac{1-\frac{b}{a}\cos(\frac{\pi}{2}-x)}{1+\frac{b}{a}\cos(\frac{\pi}{2}-x)}\right)\frac{dx}{\cos(\frac{\pi}{2}-x)} \overset{\frac{\pi}{2}-x\to x;\frac{a}{b}=t}{=}$$

$$= -\int_0^{\pi/2} \ln\left(\frac{1-t\cos x}{1+t\cos x}\right)\frac{dx}{\cos x} = \pi\arcsin t = \pi\arcsin(\frac{b}{a})$$

Application 4.3 Find:

$$I = \int_0^{\pi} \ln(1+t\cos x)\frac{dx}{\cos x},\ |t|<1$$

Solution. We have:

$$I = \int_0^{\pi} \ln(1+t\cos x)\frac{dx}{\cos x} = \int_0^{\pi/2} \ln(1+t\cos x)\frac{dx}{\cos x} + \int_{\pi/2}^{\pi} \ln(1+t\cos x)\frac{dx}{\cos x} =$$

$$= \int_0^{\pi/2} \ln(1+t\cos x)\frac{dx}{\cos x} - \int_{\pi/2}^{\pi} \ln(1-t\cos(\pi-x))\frac{dx}{\cos(\pi-x)} =$$

$$= \int_0^{\pi/2} \ln(1+t\cos x)\frac{dx}{\cos x} - \int_0^{\pi/2} \ln(1-t\cos x)\frac{dx}{\cos x} =$$

$$= \int_0^{\pi/2} \ln\left(\frac{1+t\cos x}{1-t\cos x}\right)\frac{dx}{\cos x} = \pi\arcsin t$$

Application 4.4 Find:

$$I = \int_0^{\pi/2} \frac{\operatorname{arctg}(t\operatorname{tg} x)}{\operatorname{tg} x}\,dx,\ |t|<1$$

Solution.

$$I = \int_0^{\pi/2} \frac{\operatorname{arctg}(t\operatorname{tg} x)}{\operatorname{tg} x}\,dx = \int_0^{\pi/2}\int_0^t \frac{du}{1+u^2\operatorname{tg}^2 x}dx = \int_0^t\int_0^{\pi/2} \frac{dx}{1+u^2\operatorname{tg}^2 x}du =$$

$$= \int_0^t\int_0^{+\infty} \frac{dv}{(1+v^2)(1+u^2v^2)}du = \int_0^t \frac{1}{u^2-1}\int_0^{+\infty}\left(\frac{u^2}{1+u^2v^2} - \frac{1}{1+v^2}dv\right)du =$$

$$= \int_0^t \frac{1}{u^2-1}\cdot\frac{\pi}{2}(u-1)\,du = \frac{\pi}{2}\ln(1+t)$$

Application 4.5 Find:

$$I = \int_0^{\pi} \ln(1-2t\cos x+t^2)\,dx,\ |t|<1$$

Solution. Let be the function:

$$F(t) = \int_0^{\pi} \ln(1-2t\cos x+t^2)\,dx,\ F(0)=0$$

$$F'(t) = \int_0^{\pi} \frac{2(t-\cos x)}{1-2t\cos x+t^2}\,dx = 4\int_0^{+\infty} \frac{t-1+(t+1)u^2}{(1+u^2)[(1-t)^2+(1+t)^2u^2]}du =$$

$$=2\int_0^{+\infty}\left(\frac{1}{t(1+u^2)} + \frac{t^2-1}{t}\cdot\frac{1}{(t-1)^2+(t+1)^2u^2}\right)du =$$

$$=\frac{2}{t}\left[\frac{\pi}{2} - \lim_{v\to\infty}\operatorname{arctg}(\frac{1+t}{1-t}u)\Big|_0^v\right] = 0.\ \text{Therefore, } F(t)=0$$

Application 4.6 Let $f\left[\frac{1}{a},a\right]\to\mathbb{R}, a>1$ continuous function. Find:

$$I=\int_{1/a}^{a} f\left(\frac{x^{2\lambda}+1}{x^{\lambda}}\right)\frac{\ln x}{x}\,dx,\ \lambda\in\mathbb{R}$$

Solution. We have:

$$\begin{aligned}
I=&\int_{1/a}^{a} f\left(\frac{x^{2\lambda}+1}{x^{\lambda}}\right)\frac{\ln x}{x}\,dx\\
=&\int_{1/a}^{1} f\left(\frac{x^{2\lambda}+1}{x^{\lambda}}\right)\frac{\ln x}{x}\,dx+\int_{1}^{a} f\left(\frac{x^{2\lambda}+1}{x^{\lambda}}\right)\frac{\ln x}{x}\,dx\\
=&I_1+I_2,\text{ where}\\
I_1=&\int_{1/a}^{1} f\left(\frac{x^{2\lambda}+1}{x^{\lambda}}\right)\frac{\ln x}{x}\,dx\overset{x\to 1/x}{=}-I_2,\text{ thus}\\
I=&0
\end{aligned}$$

Application 4.7 Find:

$$I=\int_{0}^{\pi/2}\ln(\sin x)\,dx$$

Solution.

$$\begin{aligned}
I=&\int_{0}^{\pi/2}\ln(\sin x)\,dx=\int_{0}^{\pi/2}\ln(2\sin\frac{x}{2}\cos\frac{x}{2})\,dx\\
=&\frac{\pi}{2}\ln 2+\int_{0}^{\pi/2}\ln(\sin\frac{x}{2})\,dx+\int_{0}^{\pi/2}\ln(\cos\frac{x}{2})\,dx\\
=&\frac{\pi}{2}\ln 2+2\int_{0}^{\pi/4}\ln(\sin y)\,dy+2\int_{0}^{\pi/4}\ln(\cos y)\,dy\\
=&\frac{\pi}{2}\ln 2+2\int_{0}^{\pi/4}\ln(\sin y)\,dy+2\int_{0}^{\pi/4}\ln(\sin(\frac{\pi}{2}-y))\,dy\\
=&\frac{\pi}{2}\ln 2+2\int_{0}^{\pi/4}\ln(\sin y)\,dy+2\int_{\pi/4}^{\pi/2}\ln(\sin y)\,dy\\
=&\frac{\pi}{2}\ln 2+2I,\text{ therefore}\\
I=&-\frac{\pi}{2}\ln 2
\end{aligned}$$

Application 4.8 Find:

$$I=\int_{0}^{1}\frac{\ln(1-t^2x^2)}{\sqrt{1-x^2}}\,dx,\ |t|<1$$

Solution. Let us denote:

$$F(t)=\int_{0}^{1}\frac{\ln(1-t^2x^2)}{\sqrt{1-x^2}}\,dx,\ |t|<1, F(0)=0$$

$$F'(t) = \int_0^1 \frac{-2tx^2}{(1-t^2x^2)\sqrt{1-x^2}}\,dx = -2t\int_0^{\pi/2} \frac{\sin^2 u}{1-t^2\sin^2 u}\,du$$
$$= -2t\int_0^{+\infty} \frac{v^2}{(1+(1-t^2)v^2)(1+v^2)}dv$$
$$= -\frac{2}{t}\int_0^{+\infty} \left(\frac{1}{1+v^2} - \frac{1}{1+(1-t^2)v^2}\right) dv = \pi\left(\frac{1}{t} - \frac{1}{t\sqrt{1-t^2}}\right)$$
$$F(t) = \pi\ln\left(\frac{1+\sqrt{1-t^2}}{2}\right)$$

Application 4.9 Find:

$$I = \int_0^{2\pi} \ln(1-t\cos x)\,dx,\ |t|<1$$

Solution. Let be the function:

$$F(t) = \int_0^{2\pi} \ln(1-t\cos x)\,dx = 2\int_0^{\pi} \ln(1-t\cos x)\,dx, |t|<1,\ F(0)=0$$
$$F'(t) = -2\int_0^{\pi} \frac{\cos x}{1-t\cos x}\,dx = -4\int_0^{+\infty} \frac{1-u^2}{(1+u^2)(1-t+(1+t)u^2)}du$$
$$= \frac{4}{t}\int_0^{+\infty} \left(\frac{1}{1+u^2} - \frac{1}{1-t+(1+t)u^2}\right) du = 2\pi\left(\frac{1}{t} - \frac{1}{t\sqrt{1-t^2}}\right)$$
$$F(t) = 2\pi\ln\left(\frac{1+\sqrt{1-t^2}}{2}\right)$$

Application 4.10 Find:

$$I = \int_0^{\pi/2} \ln(t^2-\sin^2 x)\,dx,\ |t|>1$$

Solution. Let be the function:

$$F(t) = \int_0^{\pi/2} \ln(t^2-\sin^2 x)\,dx = \frac{\pi}{2}\ln t + \int_0^{\pi/2} \ln(1-\frac{\sin^2 x}{t^2}dx$$
$$= \frac{\pi}{2}\ln t + \int_0^1 \frac{\ln(1-\frac{y^2}{t^2})}{\sqrt{1-y^2}}dy = \pi\ln\frac{t+\sqrt{t^2-1}}{2t}$$

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