Another Flawed Proof

Introduction

In this article, we will study how an incorrect identity is derived while using RMT (Ramanujan's Master Theorem).

Prerequisites

Mellin Transform

The Mellin transform of a function f is given by

$$\{Mf\}(s) = \int_0^\infty x^{s-1} f(x) dx$$

where $s \in \mathbb{C}$ such that the above integral exists.

Ramanujan's Master Theorem

If the Taylor series expansion of f exists about $x = 0$ and is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{\phi(k)(-x)^k}{k!}$$

then Ramanujan's master theorem states that,

$$\int_0^\infty x^{n-1} f(x) dx = \Gamma(n)\phi(-n)$$

where ϕ satisfies the conditions mentioned in [1] and $n \in \mathbb{N}$.

Faulty Proof

Let us start from a well known result, which states that if $\Re(s) > 1$, then,

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s)\zeta(s)$$

substituting $x = mt$ in the above integral, where $m \in \mathbb{N}$, we have,

$$\int_0^\infty \frac{t^{s-1}}{e^{mt}-1}dt = \frac{\Gamma(s)\zeta(s)}{m^s}$$

summing up both the sides from $m = 1$ to $m = \infty$, we have,

$$\sum_{m=1}^\infty \int_0^\infty \frac{t^{s-1}}{e^{mt}-1}dt = \sum_{m=1}^\infty \frac{\Gamma(s)\zeta(s)}{m^s}$$

simplifying the above equation, we obtain,

$$\int_0^\infty t^{s-1} \sum_{m=1}^\infty \frac{1}{e^{mt}-1}dt = \Gamma(s)\zeta^2(s)$$

Therefore, if F is given as,

$$F(x) = \sum_{m=1}^\infty \frac{1}{e^{mx}-1} \tag{1}$$

where $x > 0$, then the Mellin transform of F exists and is given by

$$\{MF\}(s) = \Gamma(s)\zeta^2(s) \tag{2}$$

where $\Re(s) > 1$.
Let the series expansion of F about $x = 0$ takes the form,

$$F(x) = \sum_{k=0}^\infty \frac{\phi(k)(-x)^k}{k!} \tag{3}$$

thus from RMT, (2) and (3), we obtain that,

$$\phi(k) = \zeta^2(-k) = \frac{B_{k+1}^2}{(k+1)^2}$$

where $k \in \mathbb{N} \cup \{0\}$ and B_k is the k^{th} Bernoulli number.
Substituting ϕ in (3) and equating it with (1), we obtain,

$$\sum_{m=1}^\infty \frac{1}{e^{mx}-1} = \sum_{k=0}^\infty \frac{B_{k+1}^2(-x)^k}{(k+1)^2 k!} = \frac{1}{4} + \sum_{k=1}^\infty \frac{B_{k+1}^2(-x)^k}{(k+1)^2 k!} = \frac{1}{4} - \frac{1}{4}\sum_{k=1}^\infty \frac{B_{2k}^2 x^{2k-1}}{k^2(2k-1)!}$$

therefore, we finally have,

$$4\sum_{k=1}^\infty \frac{1}{e^{kx}-1} = 1 - \sum_{k=1}^\infty \frac{B_{2k}^2 x^{2k-1}}{k^2(2k-1)!}$$

which is incorrect.

Conclusion

It is not hard to prove that the equation obtained at the end is incorrect since the infinite series present in the R.H.S. is divergent for all $x \neq 0$. Since F diverges at $x = 0$, expanding F about $x = 0$ violates RMT conditions. Thus, we can't find the Mellin transform of F using RMT.
Using the formula

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}$$

where $k \in \mathbb{N}$, makes the proof even worse because the function obtained after the analytic continuation of the zeta function is not the same as the previous zeta function.

References

[1] B.C. Berndt, Ramanujan's Notebooks: Part I. New York: Springer-Verlag, 1985, 298-299.

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