JI CHEN'S INEQUALITY REVISITED FORM

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Abstract: *In this paper are presented Ji Chen' s inequality revisited form.*

Proposition 1.

If $m, n, p, x, y, z > 0$, then:

$$\left(\frac{xy}{p}+\frac{yz}{m}+\frac{zx}{n}\right)\left(\frac{1}{(mx+ny)^2}+\frac{1}{(ny+pz)^2}+\frac{1}{(pz+mx)^2}\right)\geq\frac{9}{4mnp} \tag{1}$$

Proof. We have:

$$\left(\sum_{cyc}\frac{xy}{p}\right)\left(\sum_{cyc}\frac{1}{(mx+ny)^2}\right)=\frac{1}{mnp}\left(\sum_{cyc}mxy\right)\left(\frac{1}{(mx+ny)^2}\right)\overset{Ji\ Chen}{\geq}$$

$$\geq\frac{1}{mnp}\cdot\frac{9}{4}=\frac{9}{4mnp} \tag{2}$$

If $mnp = 1$, then inequality (2), becomes:

$$\left(\sum_{cyc}\frac{xy}{p}\right)\left(\sum_{cyc}\frac{1}{(mx+ny)^2}\right)\geq\frac{9}{4} \tag{3}$$

If $m = n = p = 1$, then we get:

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4} \tag{J.C}$$

i.e. Ji Chen's inequality.
In [1] Arkady M. Alt has proved the following inequality:

$$(x^2+t^2)(y^2+t^2)(z^2+t^2)\geq\frac{3}{4}(x+y+z)\cdot t^4 \tag{A.A.}$$

If $m = 1$, we get:

$$(x^2+1)(y^2+1)(z^2+1)\geq\frac{3}{4}(x+y+z)^2 \tag{4}$$

If $m, n, p, x, y, z > 0$, then:

$$(x^2y^2+p^2)(y^2z^2+m^2)(z^2x^2+n^2)\geq\frac{3}{4}(mnxy+npyz+pmzx)^2 \tag{5}$$

Proof We have:

$$(x^2y^2+p^2)(y^2z^2+m^2)(z^2x^2+n^2)=$$

$$=m^2n^2p^2\prod_{cyc}\left(\left(\frac{xy}{p}\right)^2+1\right)\overset{A.A.}{\geq}\frac{3}{4}m^2n^2p^2\left(\frac{xy}{p}+\frac{yz}{m}+\frac{zx}{n}\right)^2=$$

$$\frac{3}{4}m^2n^2p^2(mnxy+npyz+pmzx)^2\cdot\frac{1}{m^2n^2p^2}=$$

$$=\frac{3}{4}(mnxy+npyz+pmzx) \tag{6}$$

If $m,n,p,x,y,z>0$ then:

$$\prod_{cyc}\left(\frac{1}{(mx+ny)^4}+1\right)\overset{A.A}{\geq}\frac{3}{4}\left(\sum_{cyc}\frac{1}{(mx+ny)^2}\right)^2 \tag{7}$$

From (5) and (7), we get:

$$\prod_{cyc}(x^2y^2+p^2)\cdot\prod_{cyc}\left(\frac{1}{(mx+ny)^4}+1\right)\geq\frac{3}{4}\left(\sum_{cyc}nmxy\right)^2\cdot\frac{3}{4}\left(\sum_{cyc}\frac{1}{(mx+ny)^2}\right)^2=$$

$$=\frac{9}{16}\left[\left(\sum_{cyc}mnxy\right)\left(\sum_{cyc}\frac{1}{(mx+ny)^2}\right)\right]^2\overset{JiChen}{\geq}$$

$$\geq\frac{9}{16}\cdot\frac{9}{4mnp}=\frac{81}{64mnp} \tag{8}$$

If in (5), we take $mnp=1$, then:

$$\prod_{cyc}(x^2y^2+p^2)\geq\frac{3}{4}(mnxy+npyz+pmzx)^2$$

If $m=n=p=1$, it follows:

$$\prod_{cyc}(x^2y^2+1)\geq\frac{3}{4}(xy+yz+zx)^2 \tag{9}$$

If in (8) we take $mnp=1$, we get:

$$\prod_{cyc}(x^2y^2+p^2)\cdot\prod_{cyc}\left(\frac{1}{(x+y)^4}+1\right)\geq\frac{81}{64} \tag{10}$$

If in (8) we take $m=n=p=1$, we get:

$$\prod_{cyc}(x^2y^2+1)\cdot\prod_{cyc}\left(\frac{1}{(x+y)^4}+1\right)\geq\frac{81}{64} \tag{11}$$

References

[1] Alt M. Arkady, ABOUT ONE INEQUALITY FROM APMO,2004-NEW SOLUTION AND GENERALIZATIONS, OCTOGON MATHEMATICAL MAGAZINE, Vol.27,No.1, April 2019,pages 228-232

[2] ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro