NEW SOLUTIONS FOR A FEW R.M.M. PROBLEMS

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Abstract: *In this paper are presented new solutions for a few R.M.M. problems.*

Proposition 1.(Cauchy-D'Alembert)

If $(a_n)_{n\geq 1}$ is a sequence of real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a > 0$, then:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = a$$

Proof. We have:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\frac{1}{n}\log n} \overset{Stolz-Cesaro}{=} e^{\lim_{n\to\infty} \frac{\log(a_{n+1})-\log a_n}{n+1-n}} = \tag{1}$$

$$= e^{\lim_{n\to\infty} \log(\frac{a_{n+1}}{a_n})} = e^{\log(\lim_{n\to\infty} \frac{a_{n+1}}{a_n})} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$$

Proposition 2.

If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} = a; t > 0$ then:

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^t} = \frac{a}{e^t} \tag{2}$$

Proof. We have:

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^t} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^{nt}}} \overset{C-D'A}{=} \lim_{n\to\inf} \frac{a_{n+1}}{(n+1)^{(n+1)t}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^t} \cdot \left(\frac{n}{n+1}\right)^{(n+1)t} = a \cdot \frac{1}{e^t} = \frac{a}{e^t}$$

APPLICATIONS

App. 1) If $(a_n)_{n\geq 1}$ is a sequence of real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} = a$ and $b, c \geq 1, b \neq c$, then find:

$$\Omega = \lim_{n\to\infty} (\sqrt[n]{b} - \sqrt[n]{c}) \cdot \sqrt[n]{a_n}$$

Solution 1. We have:

$$\Omega = \lim_{n\to\infty} (\sqrt[n]{b} - \sqrt[n]{c}) \cdot \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{c} \left(\sqrt[n]{\frac{b}{c}} - 1\right) \cdot n \cdot \frac{\sqrt[n]{a_n}}{n} =$$

$$= 1\cdot\lim_{n\to\infty}\frac{e^{\frac{1}{n}\log(\frac{b}{c})}-1}{\frac{1}{n}\log(\frac{b}{c})}\cdot\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\cdot\log(\frac{b}{c}) =$$

$$= 1\cdot 1\cdot\frac{a}{e}\cdot\log(\frac{b}{c}) = \frac{a}{e}(\log b - \log a)$$

Solution 2. We have:

$$(\sqrt[n]{b}-\sqrt[n]{c})\cdot\sqrt[n]{a_n} = (\sqrt[n]{b}-1+1-\sqrt[n]{c})\cdot n\cdot\frac{\sqrt[n]{a_n}}{n}; (\forall)n\in\mathbb{N}^*-1$$

$$\Omega = \lim_{n\to\infty}(\sqrt[n]{b}=\sqrt[n]{c})\cdot\sqrt[n]{a_n} = \frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{\sqrt[n]{b}-1}{\frac{1}{n}} - \frac{\sqrt[n]{c}-1}{\frac{1}{n}}\right) =$$

$$= \frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{e^{\frac{1}{n}\log b}-1}{\frac{1}{n}\log b}\cdot\log b - \frac{e^{\frac{1}{n}\log c}-1}{\frac{1}{n}\log c}\cdot\log c\right) =$$

$$= \frac{a}{e}(1\cdot\log b - 1\cdot\log c) = \frac{a}{e}(\log b - \log c)$$

If $c = 1$ and $a_n = (2n-1)!!$, then $a = 2$ and $\lim_{n\to\infty}(\sqrt[n]{b}-1)\cdot\sqrt[n]{(2n-1)!!} = \frac{2}{e}\log b$ i.e. we obtain the Proposed Problem UP.445 from R.M.M.

App. 2) Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty} e^{-2H_n}\cdot\sum_{k=1}^{n}\sqrt[k]{k!}} \tag{4}$$

Solution.

$$B_n = e^{-2H_n}\cdot\sum_{k=1}^{n}\sqrt[k]{k!} = n^2\cdot e^{-2H_n}\cdot\frac{\sum_{k=1}^{n}\sqrt[k]{k!}}{n^2} = e^{-2\gamma_n}\cdot\frac{\sum_{k=1}^{n}\sqrt[k]{k!}}{n^2}$$

$$\Omega = \lim_{n\to\infty} B_n \overset{Stolz-Cesaro}{=} \lim_{n\to\infty} e^{-2\gamma_n}\cdot\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^2-n^2} =$$

$$= e^{-2\gamma}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{2n+1} = e^{-2\gamma}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{2n+1} =$$

$$= \frac{e^{-2\gamma}}{2}\cdot\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{e^{-2\gamma}}{2}\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} =$$

$$\overset{C-D'A}{=} \frac{e^{-2\gamma}}{2}\cdot\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} =$$

$$= \frac{e^{-2\gamma}}{2}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \frac{e^{-2\gamma}}{2e} = \frac{1}{2e^{1+2\gamma}}$$

i.e. we obtain the Proposed Problem UP.442 from R.M.M.

App.3) Let $(a_n)_{n\geq 1}$ sequence of real numbers strictly positive such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0$ then find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}(e^{2H_{n+1}} - e^{2H_n})\cdot\frac{1}{\sqrt[n]{a_n}}} \tag{5}$$

Solution.

$$\text{Let: } G_n = (e^{2H_{n+1}} - e^{2H_n}) \cdot \frac{1}{\sqrt[n]{a_n}} = e^{2H_n}(e^{2(H_{n+1}-H_n)} - 1) \cdot \frac{1}{\sqrt[n]{a_n}} =$$

$$= e^{2H_n}\left(e^{\frac{2}{n+1}} - 1\right) \cdot \frac{1}{\sqrt[n]{a_n}} = \frac{e^{2H_n}}{n^2} \cdot \frac{n}{n+1} \cdot (n+1)\left(e^{\frac{2}{n+1}} - 1\right) \cdot \frac{n}{\sqrt[n]{a_n}} =$$

$$= e^{2\gamma_n} \cdot \frac{n}{n+1} \cdot 2 \cdot \frac{e^{\frac{2}{n+1}} - 1}{\frac{2}{n+1}} \cdot \frac{n}{\sqrt[n]{a_n}}; (\forall) n \in \mathbb{N}^*$$

$$\Omega = \lim_{n\to\infty} G_n = 2 \cdot e^{2\gamma} \cdot 1 \cdot \frac{e}{a} = \frac{2}{a} \cdot e^{1+2\gamma}.$$

If $a_n = n!$, then $\lim_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} = e$ and we get the Proposed Problem UP.444 from R.M.M.

App. 4) Let $u, v > 0$ and $(a_n)_{n\geq}, (b_n)_{n\geq 1}$ sequences of real numbers strictly positive such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^u \cdot a_n} = a > 0$, $\lim_{n\to\infty} \frac{b_{n+1}}{n^v \cdot b_n} = b > 0$, then find:

(6) $$\Omega = \lim_{n\to\infty} n^{u+v-2} \cdot \sqrt[n]{a_n} \cdot \sqrt[n]{b_n} \cdot \sin\frac{1}{n^u} \cdot \sin\frac{1}{n^v}$$

Solution. We have:

$$D_n = n^{u+v-2} \cdot \sqrt[n]{a_n} \cdot \sqrt[n]{b_n} \cdot \sin\frac{1}{n^u} \cdot \sin\frac{1}{n^v} = n^{u+v} \cdot \frac{\sqrt[n]{a_n}}{n} \cdot \frac{\sqrt[n]{b_n}}{n} \cdot \sin\frac{1}{n^u} \cdot \sin\frac{1}{n^v} =$$

$$= \frac{\sqrt[n]{a_n}}{n} \cdot \frac{\sqrt[n]{b_n}}{n} \cdot \frac{\sin\frac{1}{n^u}}{\frac{1}{n^u}} \cdot \frac{\sin\frac{1}{n^v}}{\frac{1}{n^v}}; (\forall) n \in \mathbb{N}^* - \{1\}$$

$$\Omega = \lim_{n\to\infty} D_n = \frac{a}{e} \cdot \frac{b}{e} \cdot 1 \cdot 1 = \frac{ab}{e^2}$$

If $a_n = n!, b_n = (2n-1)!!$, then $a = 1, b = 2$ and if $u = 2, v = 3$, we get:

$$D_n = n^3 \cdot \sqrt[n]{n!} \cdot \sqrt[n]{(2n-1)!!} \cdot \sin\frac{1}{n^2} \cdot \sin\frac{1}{n^3}, \text{ then } \lim_{n\to\infty} D_n = \frac{2}{e}$$

i.e. the Proposed Problem UP.446 from R.M.M.

App.5) If $u, v \in \mathbb{N}^*, n \geq 2, a \geq 1$ find $\lim_{n\to\infty} S_n$, where

$$S_n = e^{(u+v)H_n} \cdot (\sqrt[n^v]{a} - 1) \cdot \sin\frac{1}{n^u}$$

Solution. We have:

$$S_n = e^{(u+v)H_n} \cdot (\sqrt[n^v]{a} - 1) \cdot \sin\frac{1}{n^u} = \frac{e^{(u+v)H_n}}{n^{u+v}} \cdot \frac{n^{u+v}(\sqrt[n^v]{a} - 1)}{\frac{1}{n^v}} \cdot \frac{\sin\frac{1}{n^u}}{\frac{1}{n^u}} =$$

$$= e^{(u+v)\gamma_n} \cdot \frac{e^{\frac{1}{n^v}\log a} - 1}{\frac{1}{v}\log a} \cdot \log a \cdot \frac{\sin\frac{1}{n^u}}{\frac{1}{n^u}}; (\forall) n \in \mathbb{N}^* - \{1\}$$

$$\Omega = \lim_{n\to\infty} S_n = e^{(u+v)\gamma} \cdot 1 \cdot \log a \cdot 1 = e^{(u+v)\log a}$$

If $u = 3, v = 2$ then $S_n = e^{5H_n}(\sqrt[n^2]{a} - 1) \cdot \sin^2\frac{1}{n}$, then $\lim_{n\to\infty} S_n = e^{5\gamma} \cdot \log a$.

App.6) Let $u, v > 0$ and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real numbers such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0$, $\lim\limits_{n\to\infty} \frac{b_{n+1}}{n\cdot b_n} = b > 0$, then find:

$$\Omega = \lim_{n\to\infty} n^{t+u-2} \cdot \sqrt[n]{a_n} \cdot \sqrt[n]{b_n} \cdot \tan\frac{1}{n^t} \cdot \sin\frac{1}{n^u} \tag{7}$$

Solution. We have:

$$U_n = n^{t+u-2} \cdot \sqrt[n]{a_n} \cdot \sqrt[n]{b_n} \cdot \tan\frac{1}{n^t} \cdot \sin\frac{1}{n^u} =$$

$$= \frac{\sqrt[n]{a_n}}{n} \cdot \frac{\sqrt[n]{b_n}}{n} \cdot \frac{\tan\frac{1}{n^t}}{\frac{1}{n^t}} \cdot \frac{\sin\frac{1}{n^u}}{\frac{1}{n^u}}$$

$$\lim_{n\to\infty} U_n = \frac{a}{e} \cdot \frac{b}{e} \cdot 1 \cdot 1 = \frac{ab}{e^2}$$

References

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