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1901. For $k>0$ prove that: $\int_0^1 x^3\tan^{-1}\left(1+\frac{k}{x}\right)dx=$

$$=k^4\left(\frac{\pi}{64}-\frac{1}{16}\tan^{-1}\left(\frac{k+2}{k}\right)\right)+\frac{k^3}{16}-\frac{k^2}{16}+\frac{1}{4}\tan^{-1}(k+1)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Samar Das-India

$$\int_0^1 x^3\tan^{-1}\left(1+\frac{k}{x}\right)dx=\frac{x^4}{4}\tan^{-1}\left(1+\frac{k}{x}\right)\Big|_0^1+\frac{k}{4}\int_0^1\frac{x^2dx}{2+\frac{2k}{x}+\frac{k^2}{x^2}}=$$

$$=\frac{1}{4}\tan^{-1}(1+k)+\frac{k}{8}\int_0^1\left(x^2-kx+\frac{k^2}{2}-\frac{\left(\frac{k}{4}\right)^2}{k^2+kx+\frac{k^2}{2}}\right)dx=$$

$$=\frac{1}{4}\tan^{-1}(1+k)+\left[\frac{k}{8}\cdot\frac{x^3}{3}-\frac{k^2x^2}{16}+\frac{k^3}{16}x\right]_0^1-\frac{k^5}{32}\int_0^1\frac{dx}{x^2+kx+\frac{k^2}{2}}=$$

$$=\frac{1}{4}\tan^{-1}(1+k)+\frac{k}{24}-\frac{k^2}{16}+\frac{k^3}{16}-\frac{k^5}{32}\cdot\frac{1}{\frac{k}{2}}\tan^{-1}\left(1+\frac{2x}{k}\right)\Big|_0^2=$$

$$=\frac{1}{4}\tan^{-1}(1+k)+\frac{k}{24}-\frac{k^2}{16}+\frac{k^3}{16}-\frac{k^4}{16}\left(\tan^{-1}\left(1+\frac{2}{k}\right)-\frac{\pi}{4}\right)=$$

$$=k^4\left(\frac{\pi}{64}-\frac{1}{16}\tan^{-1}\left(1+\frac{2}{k}\right)\right)+\frac{k^3}{16}-\frac{k^2}{16}+\frac{k}{24}+\frac{1}{4}\tan^{-1}(k+1)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1 x^3\tan^{-1}\left(1+\frac{k}{x}\right)dx\overset{y=1+\frac{k}{x}}{=}k^4\int_{k+1}^{\infty}\frac{\tan^{-1}y}{(k-1)^5}dy\overset{IBP}{=}$$

$$=\left[-\frac{k^4\tan^{-1}y}{4(y-1)^4}\right]_{k+1}^{\infty}+\frac{k^4}{4}\int_{k+1}^{\infty}\frac{dy}{(1+y^2)(y-1)^4}=$$

$$=\frac{1}{4}\tan^{-1}(k+1)+\frac{k^4}{16}\int_{k+1}^{\infty}\left(\frac{2}{(y-1)^4}-\frac{2}{(y-1)^3}+\frac{1}{(y-1)^2}-\frac{1}{1+y^2}\right)dy=$$

$$=\frac{1}{4}\tan^{-1}(k+1)+\frac{k^4}{16}\left[-\frac{2}{3(y-1)^3}+\frac{1}{(y-1)^2}-\frac{1}{y-1}-\tan^{-1}y\right]_{k+1}^{\infty}=$$

$$=\frac{1}{4}\tan^{-1}(k+1)-\frac{\pi k^4}{32}+\frac{k}{24}-\frac{k^2}{16}+\frac{k^3}{16}+\frac{k^4}{16}\tan^{-1}(k+1)=$$

$$=\frac{k}{24}-\frac{k^2}{16}+\frac{k^3}{16}+\frac{1}{4}\tan^{-1}(k+1)-\frac{k^4}{16}\left(\frac{\pi}{2}-\tan^{-1}(k+1)\right)=$$

$$=\frac{k}{24}-\frac{k^2}{16}+\frac{k^3}{16}+\frac{1}{4}\tan^{-1}(k+1)-\frac{k^4}{16}\tan^{-1}\left(\frac{1}{k+1}\right)$$

$$\tan^{-1}\left(\frac{1}{k+1}\right)=\tan^{-1}\left(\frac{\frac{k+2}{k}-1}{1+\frac{k+2}{k}}\right)=\tan^{-1}\left(\frac{k+2}{k}\right)-\tan^{-1}(1)=$$

$$=\tan^{-1}\left(\frac{k+2}{k}\right)-\frac{\pi}{4}$$

Therefore,

$$\int_0^1 x^3\tan^{-1}\left(1+\frac{k}{x}\right)dx=k^4\left(\frac{\pi}{64}-\frac{1}{16}\tan^{-1}\left(\frac{k+2}{k}\right)\right)+\frac{k^3}{16}-\frac{k^2}{16}+\frac{1}{4}\tan^{-1}(k+1)$$

1902. Prove that:

$$\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\frac{H_{\frac{n}{2}}-H_{\frac{n-1}{2}}}{4^n(2n+1)}=8G+\pi\log(1+\sqrt{2})-4Li_2\left(\frac{1-i}{\sqrt{2}}\right)i+$$

$$+2i\left(2Li_2\left((-1)^{\frac{3}{4}}\right)+\pi\tan^{-1}(\sqrt[4]{-1})\right)$$

where $H_n=\int_0^1\frac{1-x^n}{1-x}dx$, $Li_2(x)$ is dilogarithm function, i is imaginary unit and G is Catalan's constant.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\frac{H_{\frac{n}{2}}-H_{\frac{n-1}{2}}}{4^n(2n+1)}$$

$$\text{Since: } H_{\frac{n}{2}}-H_{\frac{n-1}{2}}=2\int_0^1\frac{x^n}{1+x}dx$$

$$\frac{\binom{2n}{n}}{4^n}=\frac{1}{\pi}B\left(n+\frac{1}{2},\frac{1}{2}\right)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^{2n}y\,dy$$

$$\Omega=\frac{4}{\pi}\int_0^1\int_0^{\frac{\pi}{2}}\frac{1}{1+x}\left(\sum_{n=0}^{\infty}\frac{(-\sqrt{x}\cos y)^{2n}}{2n+1}\right)dydx=\frac{4}{\pi}\int_0^1\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\sqrt{x}\cos y)}{(1+x)(\sqrt{x}\cos y)}dydx$$

$$\Omega\overset{x\to x^2}{=}\frac{8}{\pi}\int_0^1\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(x\cos y)}{(1+x^2)\cos y}dy=\frac{8}{\pi}\int_0^1\frac{1}{1+x^2}\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(x\cos y)}{\cos y}dydx$$

$$\text{IBP: }\begin{cases}u=\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(x\cos y)}{\cos y}dy\\ dv=\frac{dx}{1+x^2}\end{cases}\Rightarrow\begin{cases}\frac{du}{dx}=\int_0^{\frac{\pi}{2}}\frac{dy}{1+x^2\cos^2 y}=\frac{\pi}{2\sqrt{1+x^2}}\\ v=\tan^{-1}x\end{cases}$$

$$\Omega=\frac{8}{\pi}\left[\tan^{-1}x\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(x\cos y)}{\cos y}dy\right]_0^1-4\int_0^1\frac{\tan^{-1}x}{\sqrt{1+x^2}}dx$$

$$\Omega=2\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\cos y)}{\cos y}dy-4\int_0^1\frac{\tan^{-1}x}{\sqrt{1+x^2}}dx$$

$$\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\cos y)}{\cos y}dy=\int_0^1\int_0^{\frac{\pi}{2}}\frac{1}{1+t^2\cos^2 y}dydt=\frac{\pi}{2}\int_0^1\frac{dt}{\sqrt{1+t^2}}=\frac{\pi}{2}\log(1+\sqrt{2})$$

$$I=\int\frac{\tan^{-1}x}{\sqrt{1+x^2}}dx\overset{x=\sinh y}{=}\int\tan^{-1}(\sinh y)\,dy$$

$$=\int gd(y)\,dy\ (\text{Gudermannian function})$$

$$I=-\frac{\pi}{2}y+i\left(Li_2(-ie^y)-Li_2(ie^y)\right)$$

$$I=\int\frac{\tan^{-1}x}{\sqrt{1+x^2}}dx=\left[i\left(Li_2(-ie^y)-Li_2(ie^y)\right)-\frac{\pi y}{2}\right]_0^{\log(1+\sqrt{2})}=$$

$$=i\left[Li_2\left(-i(1+\sqrt{2})\right)-Li_2\left(i(1+\sqrt{2})\right)\right]-\frac{\pi}{2}\log(1+\sqrt{2})-$$

$$-i\left(Li_2(-i)-Li_2(i)\right)$$

$$Li_2(-i)-Li_2(i)=-2iG$$

Therefore,

$$\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}\frac{H_{\frac{n}{2}}-H_{\frac{n-1}{2}}}{4^n(2n+1)}=8G+\pi\log(1+\sqrt{2})-4Li_2\left(\frac{1-i}{\sqrt{2}}\right)i+$$

$$+2i\left(2Li_2\left((-1)^{\frac{3}{4}}\right)+\pi\tan^{-1}(\sqrt[4]{-1})\right)$$

1903. $\Omega(x)=\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{2-x}{2}\right)\sin(\pi x);0<x<1$

Solve for real numbers:

$$x^2-\frac{4x}{\Omega(x)}+\frac{1}{\pi^4}=0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\Rightarrow\Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi x}{2}\right)}$$

$$\Gamma\left(\frac{x-1}{2}\right)\Gamma\left(1-\frac{x+1}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi(x+1)}{2}\right)}=\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{\pi x}{2}\right)}=\frac{\pi}{\cos\left(\frac{\pi x}{2}\right)}$$

So, we have:

$$\Omega(x)=\frac{\pi}{\sin\left(\frac{\pi x}{2}\right)}\cdot\frac{\pi}{\cos\left(\frac{\pi x}{2}\right)}\cdot 2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)=2\pi^2$$

$$x^2-\frac{4x}{\Omega(x)}+\frac{1}{\pi^4}=0\Leftrightarrow x^2-\frac{4x}{2\pi^2}+\frac{1}{\pi^4}=0\Leftrightarrow\left(x-\frac{1}{\pi^2}\right)^2=0\Rightarrow x=\frac{1}{\pi^2}$$

Solution 2 by Kader Tapsoba-Burkina Faso

$$\Omega(x)=\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{2-x}{2}\right)\sin(\pi x)=$$

$$=\Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)\Gamma\left(\frac{1}{2}+\frac{x}{2}\right)\Gamma\left(\frac{1}{2}-\frac{x}{2}\right)\sin(\pi x)=$$

$$=\frac{\pi}{\sin\left(\frac{\pi x}{2}\right)}\cdot\frac{\pi}{\cos\left(\frac{\pi x}{2}\right)}\cdot\sin(\pi x)=\frac{\pi^2\sin(\pi x)}{\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)}=\frac{\pi^2\sin(\pi x)}{\frac{1}{2}\sin(\pi x)}=2\pi^2$$

Therefore,

$$x^2-\frac{4x}{\Omega(x)}+\frac{1}{\pi^4}=0\Leftrightarrow x^2-\frac{4x}{2\pi^2}+\frac{1}{\pi^4}=0\Leftrightarrow\left(x-\frac{1}{\pi^2}\right)^2=0\Rightarrow x=\frac{1}{\pi^2}$$

Solution 3 by Ose Favour-Nigeria

$$\Omega(x)=\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{2-x}{2}\right)\sin(\pi x)=$$

$$=\Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\Gamma\left(1-\left(\frac{x}{2}+\frac{1}{2}\right)\right)\sin(\pi x)=$$

$$= \pi \csc\left(\frac{\pi x}{2}\right) \cdot \pi \csc\left(\frac{\pi(1+x)}{2}\right) \sin(\pi x) = 2\pi^2$$

Therefore,

$$x^2 - \frac{4x}{\Omega(x)} + \frac{1}{\pi^4} = 0 \Leftrightarrow x^2 - \frac{4x}{2\pi^2} + \frac{1}{\pi^4} = 0 \Leftrightarrow \left(x - \frac{1}{\pi^2}\right)^2 = 0 \Rightarrow x = \frac{1}{\pi^2}$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega(x) = \Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{2-x}{2}\right)\sin(\pi x) =$$

$$= \Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\Gamma\left(1-\left(\frac{x}{2}+\frac{1}{2}\right)\right)\sin(\pi x)$$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}$$

$$\Rightarrow \Gamma\left(\frac{1}{2}+p\right)\Gamma\left(\frac{1}{2}-p\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}+p\pi\right)} = \frac{\pi}{\cos(p\pi)}$$

$$\Omega(x) = \frac{\pi}{\sin\left(\frac{\pi x}{2}\right)} \cdot \frac{\pi}{\cos\left(\frac{\pi x}{2}\right)} \cdot \sin(\pi x) = \frac{\pi^2 \sin(\pi x)}{\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)} = \frac{\pi^2 \sin(\pi x)}{\frac{1}{2}\sin(\pi x)} = 2\pi^2$$

Therefore,

$$x^2 - \frac{4x}{\Omega(x)} + \frac{1}{\pi^4} = 0 \Leftrightarrow x^2 - \frac{4x}{2\pi^2} + \frac{1}{\pi^4} = 0 \Leftrightarrow \left(x - \frac{1}{\pi^2}\right)^2 = 0 \Rightarrow x = \frac{1}{\pi^2}$$

1904. Prove that:

$$\int_0^\infty \left(\sum_{n=0}^{\infty} x^{n-1} \sin^{(-1)^{\frac{1}{2}n(n+1)}}\left(\frac{\pi n}{4}\right)\right) dx = \pi$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^\infty \left(\sum_{n=0}^{\infty} x^{n-1} \sin^{(-1)^{\frac{1}{2}n(n+1)}}\left(\frac{\pi n}{4}\right)\right) dx$$

$$(-1)^{\frac{n(n+1)}{2}} = \begin{cases} 1, \text{if } n = 4k, 4k+3 \\ -1, \text{if } n = 4k+1, 4k+2 \end{cases}$$

$$\Omega=\int_0^\infty\sum_{k=0}^\infty x^{4k-1}\sin(k\pi)\,dx+\int_0^\infty x^{4k+2}\sin\left(k\pi+\frac{3\pi}{4}\right)dx+$$

$$+\int_0^\infty\sum_{k=0}^\infty\frac{x^{4k}}{\sin\left(k\pi+\frac{\pi}{4}\right)}dx+\int_0^\infty\sum_{k=0}^\infty\frac{x^{4k+1}}{\sin\left(k\pi+\frac{\pi}{2}\right)}dx=$$

$$=\int_0^\infty\sum_{k=0}^\infty(-x^4)^k\left(\frac{x^2}{\sqrt{2}}+\sqrt{2}+x\right)dx=$$

$$=\frac{1}{\sqrt{2}}\int_0^\infty\frac{x^2+x\sqrt{2}+2}{1+x^4}dx\overset{t=x^4}{=}\frac{1}{4\sqrt{2}}\int_0^\infty\frac{t^{\frac{3}{4}-1}+\sqrt{2}t^{\frac{1}{2}-1}+2t^{\frac{1}{4}-1}}{1+t}dt$$

$$\Omega=\frac{\pi}{4\sqrt{2}}\left(\frac{1}{\sin\left(\frac{3\pi}{4}\right)}+\frac{\sqrt{2}}{\sin\left(\frac{\pi}{2}\right)}+\frac{2}{\sin\left(\frac{\pi}{4}\right)}\right)=\pi$$

Therefore,

$$\int_0^\infty\left(\sum_{n=0}^\infty x^{n-1}\sin^{(-1)^{\frac{1}{2}n(n+1)}}\left(\frac{\pi n}{4}\right)\right)dx=\pi$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$\Omega=\int_0^\infty\left(\sum_{n=0}^\infty x^{n-1}\sin^{(-1)^{\frac{1}{2}n(n+1)}}\left(\frac{\pi n}{4}\right)\right)dx$$

$$(-1)^{\frac{n(n+1)}{2}}=\Re\left(e^{\frac{i\pi n(n+1)}{2}}\right)=\cos\left(\frac{\pi n^2}{2}\right)\cos\left(\frac{\pi n}{2}\right)-\sin\left(\frac{\pi n^2}{2}\right)\sin\left(\frac{\pi n}{2}\right)$$

Therefore, for $2n+1$ and $2n$ condition:

$$\Omega=\int_0^\infty\left(\sum_{n=0}^\infty x^{2n}\sin^{(-1)^{n+1}}\left(\frac{(2n+1)\pi}{4}\right)\right)dx+\int_0^\infty\left(\sum_{n=0}^\infty x^{2n-1}\sin^{(-1)^n}\left(\frac{n\pi}{2}\right)\right)dx=$$

$$=A+B$$

Similarly on applying the same condition:

$$A=\int_0^\infty\left(\sum_{n=0}^\infty\frac{x^{4n}}{\sin\left(\frac{(4n+1)\pi}{4}\right)}\right)dx+\int_0^\infty\left(\sum_{n=0}^\infty x^{4n+2}\sin\left(\frac{(4n+3)\pi}{4}\right)\right)dx$$

$$B=\int_0^\infty\left(\sum_{n=0}^\infty x^{4n-1}\sin(n\pi)\right)dx+\int_0^\infty\left(\sum_{n=0}^\infty\frac{x^{4n+1}}{\sin\left(\frac{(2n+1)\pi}{2}\right)}\right)dx\text{, where}$$

$$\sin\left(\frac{(4n+1)\pi}{4}\right)=\frac{(-1)^n}{\sqrt{2}};\sin\left(\frac{(4n+3)\pi}{4}\right)=\frac{(-1)^n}{\sqrt{2}}$$

$$\sin(n\pi)=0;\sin\left(\frac{(2n+1)\pi}{2}\right)=(-1)^n$$

$$\Omega=\int_0^\infty\left(\sum_{n=0}^\infty(-x^4)^n\left(\sqrt{2}+\frac{x^2}{\sqrt{2}}+x\right)\right)dx=\int_0^\infty\left(\frac{\sqrt{2}}{1+x^4}+\frac{x^2}{(1+x^4)\sqrt{2}}+\frac{x}{1+x^4}\right)dx$$

By Beta function it is known:

$$\int_0^\infty\frac{x^a}{1+x^4}dx=\frac{1}{4}\Gamma\left(\frac{a+1}{4}\right)\Gamma\left(1-\frac{a+1}{4}\right)$$

$$\int_0^\infty\frac{1}{1+x^4}dx=\int_0^\infty\frac{x^2}{1+x^4}dx=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{4}$$

$$\int_0^\infty\frac{x}{1+x^4}dx=\frac{1}{4}\Gamma^2\left(\frac{1}{2}\right)=\frac{\pi}{4},\qquad\Omega=\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{4}=\pi$$

Therefore,

$$\int_0^\infty\left(\sum_{n=0}^\infty x^{n-1}\sin^{(-1)^{\frac{1}{2}n(n+1)}}\left(\frac{\pi n}{4}\right)\right)dx=\pi$$

1905. Prove that:

$$\int_0^1\int_0^1 x^2y^2\sqrt{x^2+y^2}\,dxdy=\frac{3\sqrt{2}-\log(1+\sqrt{2})}{28}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Setif-Algerie

Polar coordinates substitution:

$$\begin{cases}x=r\cos\theta\left(0<\theta<\frac{\pi}{4},0<r<\frac{1}{\cos\theta}\right)\\ y=r\sin\theta\left(\frac{\pi}{4}<\theta<\frac{\pi}{2},0<r<\frac{1}{\sin\theta}\right)\end{cases};dxdy=rdrd\theta$$

$$\Omega=\int_0^1\int_0^1 x^2y^2\sqrt{x^2+y^2}\,dxdy=$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta \sin^2\theta \int_0^{\frac{1}{\cos\theta}} r^6\, dr d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta \sin^2\theta \int_0^{\frac{1}{\sin\theta}} r^6 dr d\theta =$$

$$= \frac{1}{7}\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^5\theta} d\theta + \frac{1}{7}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sin^5\theta} d\theta = \frac{2}{7}\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^5\theta} d\theta \text{ (by symmetry)}$$

$$\Omega = \frac{2}{7}\int_0^{\frac{\pi}{4}} \tan^2\theta \sec^3\theta\, d\theta = \frac{2}{7}\int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\sec^3\theta\, d\theta = \frac{2}{7}\int_0^{\frac{\pi}{4}} (\sec^5\theta - \sec^3\theta) d\theta$$

Using reduction formula:

$$\int_0^{\frac{\pi}{4}} \sec^5\theta\, d\theta = \left[\frac{1}{4}\sec^3\theta \tan\theta\right]_0^{\frac{\pi}{4}} + \frac{3}{4}\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \frac{\sqrt{2}}{2} + \frac{3}{4}\int_0^{\frac{\pi}{4}} \sec^3\theta\, t\theta$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \left[\frac{1}{2}\sec\theta\tan\theta\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec\theta\, d\theta =$$

$$= \frac{\sqrt{2}}{2} + \left[\frac{1}{2}\log(\sec\theta + \tan\theta)\right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{1}{2}\log(1+\sqrt{2})$$

$$\Omega = \frac{2}{7}\left(\frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{8} + \frac{3}{8}\log(1+\sqrt{2}) - \frac{\sqrt{2}}{2} - \frac{1}{2}\log(1+\sqrt{2})\right)$$

$$\int_0^1\int_0^1 x^2y^2\sqrt{x^2+y^2}\, dxdy = \frac{3\sqrt{2} - \log(1+\sqrt{2})}{28}$$

1906. **Prove that:**

$$\int_0^{\frac{\pi}{4}} \frac{x\sin x}{1+\sqrt{2}\cos x} dx = \frac{1}{8\sqrt{2}}(8G - 3\pi\log 2)$$

Proposed by Ose Favour-Nigeria

Solution 1 by Daniel Immarube-Nigeria

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{x\sin x}{1+\sqrt{2}\cos x} dx \overset{IBP}{=} -\frac{1}{\sqrt{2}}\left(x\log(1+\sqrt{cos\, x})\right)\Big|_0^{\frac{\pi}{4}} + \frac{1}{\sqrt{2}\int_0^{\frac{\pi}{4}}\log(1+\sqrt{2}\, cos\, x)\, dx}$$

$$= -\frac{1}{\sqrt{2}}\cdot\frac{\pi}{4}\log 2 + \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \log(1+\sqrt{2}\, cos\, x)\, dx$$

$$\Phi = \frac{1}{\sqrt{2}}\Theta, \text{where}$$

$$\Theta = \int_0^{\frac{\pi}{4}} \log(1+\sqrt{2}\cos x)\,dx =$$

$$= \frac{\pi}{4}\log 2 + \int_0^{\frac{\pi}{4}} \log\left(\sqrt{2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)dx + \int_0^{\frac{\pi}{4}} \log\left(\cos\frac{x}{2}\right)dx =$$

$$= \frac{\pi}{4}\log 2 + \frac{\pi}{8}\log 2 + \int_0^{\frac{\pi}{4}} \log\left(\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)dx + \int_0^{\frac{\pi}{4}} \log\left(\cos\frac{x}{2}\right)dx =$$

$$= \frac{3\pi}{8}\log 2 + \int_0^{\frac{\pi}{4}} \log\left(\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)dx + \int_0^{\frac{\pi}{4}} \log\left(\cos\frac{x}{2}\right)dx =$$

$$= \frac{3\pi}{8}\log 2 + \int_{\frac{3\pi}{4}}^{\pi} \log\left(\sin\frac{x}{2}\right)dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \log\left(\sin\frac{x}{2}\right)dx =$$

$$= \frac{3\pi}{8}\log 2 + \int_0^{\pi}\log\left(\sin\frac{x}{2}\right)dx - \int_0^{\frac{3\pi}{4}}\log\left(\sin\frac{x}{2}\right)dx + \int_0^{\frac{3\pi}{4}}\log\left(\sin\frac{x}{2}\right)dx - \int_0^{\frac{\pi}{2}}\log\left(\sin\frac{x}{2}\right)dx$$

$$= \frac{3\pi}{8}\log 2 - Cl_2(\pi) - \pi\log 2 + Cl_2\left(\frac{3\pi}{4}\right) + \frac{3\pi}{4}\log 2 - Cl_2\left(\frac{3\pi}{4}\right) - \frac{3\pi}{4}\log 2 + Cl_2\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\log 2 =$$

$$= \frac{3\pi}{8}\log 2 - \frac{\pi}{2}\log 2 + Cl_2\left(\frac{\pi}{2}\right) + 2\pi\log\left(\frac{G\left(\frac{2}{3}\right)}{G\left(\frac{1}{3}\right)}\right) + 2\pi\log\Gamma\left(\frac{3}{8}\right) + \frac{3\pi}{4}\log\left(\frac{2\pi}{\sqrt{2+\sqrt{2}}}\right) -$$

$$-2\pi\log\left(\frac{2\pi}{\sqrt{2+\sqrt{2}}}\right) - 2\pi\log\Gamma\left(\frac{3}{8}\right) - \frac{3\pi}{4}\log\left(\frac{2\pi}{\sqrt{2+\sqrt{2}}}\right)$$

$$\Theta = Cl_2\left(\frac{\pi}{2}\right) - \frac{\pi}{8}\log 2, \qquad \Phi = \frac{1}{\sqrt{2}}\left(Cl_2\left(\frac{\pi}{2}\right) - \frac{\pi}{8}\log 2\right)$$

$$\Omega = -\frac{\pi}{4\sqrt{2}}\log 2 - \frac{\pi}{8\sqrt{2}} + \frac{1}{\sqrt{2}}\left(Cl_2\left(\frac{\pi}{2}\right)\right) = -\frac{3\pi}{8\sqrt{2}}\log 2 + \frac{1}{\sqrt{2}}\left(Cl_2\left(\frac{\pi}{2}\right)\right) =$$

$$= \frac{1}{8\sqrt{2}}(8C - 3\pi\log 2)$$

Solution 2 by Probal Chakraborty-India

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{x\sin x}{1+\sqrt{2}\cos x}dx = \int_0^{\frac{\pi}{4}} \frac{\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-x\right)}{1+\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)}dx =$$

$$= \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{1+\cos x+\sin x}dx - \int_0^{\frac{\pi}{4}} \frac{x\sin\left(\frac{\pi}{4}-x\right)}{1+\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)}dx =$$

$$=\frac{\pi}{4\sqrt{2}}\log(1+\cos x+\sin x)\Big|_0^{\frac{\pi}{4}}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{x(\cos x-\sin x)}{1+\cos x+\sin x}dx=$$

$$=\frac{\pi}{4\sqrt{2}}\left[\log(1+\sqrt{2})-\log 2\right]-J,\text{ where}$$

$$J=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{x(\cos x-\sin x)}{1+\cos x+\sin x}dx=$$

$$=\frac{1}{\sqrt{2}}x\log(1+\cos x+\sin x)\Big|_0^{\frac{\pi}{4}}-\frac{1}{\sqrt{2}\int_0^{\frac{\pi}{4}}\log\left(\sin\frac{\pi}{4}+\sin\left(\frac{\pi}{4}+x\right)\right)dx}-\frac{\pi}{8\sqrt{2}}\log 2=$$

$$=\frac{\pi}{4\sqrt{2}}\left[\log(1+\sqrt{2})-\log 2\right]-\frac{\pi}{8\sqrt{2}}\log 2-\frac{1}{\sqrt{2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\log\left(\sin\frac{x}{2}\right)dx+\frac{1}{\sqrt{2}}\int_{\pi}^{\frac{3\pi}{4}}\log\left(\sin\frac{x}{2}\right)dx$$

$$\Omega=\frac{\pi}{8\sqrt{2}}\log 2-\frac{1}{\sqrt{2}}\int_{\frac{3\pi}{4}}^{\frac{\pi}{2}}\log\left(\sin\frac{x}{2}\right)dx-\frac{1}{\sqrt{2}}\int_{\pi}^{\frac{3\pi}{4}}\log\left(\sin\frac{x}{2}\right)dx=$$

$$=-\frac{3\pi}{8\sqrt{2}}\log 2+\frac{G}{\sqrt{2}}=\frac{1}{8\sqrt{2}}(8G-3\pi\log 2)$$

1907. Prove that:

$$\frac{1+\log(1-e^{-1})^{e-1}}{e-1}=\frac{1}{2}e^{-2}+\frac{2}{3}e^{-3}+\frac{3}{4}e^{-4}+\frac{4}{5}e^{-5}+\cdots$$

Proposed by Vincenzo Dima-Netro-Italy

Solution by Yen Tung Chung-Taichung-Taiwan

$$\log(1-x)=-\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\cdots\right),-1<x<1$$

$$\frac{\log(1-x)}{x}=-\left(1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+\cdots\right)$$

$$\frac{d}{dx}\left(\frac{\log(1-x)}{x}\right)=\frac{d}{dx}\left[-\left(1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+\cdots\right)\right]$$

$$\frac{-\frac{x}{1-x}-\log(1-x)}{x^2}=-\left(\frac{1}{2}+\frac{2}{3}x+\frac{3}{4}x^2+\frac{4}{5}x^3+\cdots\right),0<x<1$$

Put $x=e^{-1}$, we have:

$$\frac{-\frac{e^{-1}}{1-e^{-1}}-\log(1-e^{-1})}{e^{-2}}=-\left(\frac{1}{2}+\frac{2}{3}e^{-1}+\frac{3}{4}e^{-2}+\frac{4}{5}e^{-3}+\cdots\right)$$

$$\frac{e^{-1}}{1-e^{-1}}+\log(1+e^{-1})=\frac{1}{2}e^{-2}+\frac{2}{3}e^{-3}+\frac{3}{4}e^{-4}+\frac{4}{5}e^{-5}+\cdots$$

$$\frac{e^{-1}+(1-e^{-1})\log(1-e^{-1})}{1-e^{-1}}=\frac{1}{2}e^{-2}+\frac{2}{3}e^{-3}+\frac{3}{4}e^{-4}+\frac{4}{5}e^{-5}+\cdots$$

$$\frac{1+\log(1-e^{-1})^{e-1}}{e-1}=\frac{1}{2}e^{-2}+\frac{2}{3}e^{-3}+\frac{3}{4}e^{-4}+\frac{4}{5}e^{-5}+\cdots$$

1908. Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i+\frac{1}{4}\right)\right]^{-1}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Ahmed Yackoube Chach-Mauritania

$$\sum_{i=1}^{k}\left(i+\frac{1}{4}\right)=\sum_{i=1}^{k}i+\frac{k}{4}=\frac{2k^2+3k}{4}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i+\frac{1}{4}\right)\right]^{-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(2k+3)}=\frac{4}{3}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{2k+3}\right)=$$

$$=\frac{4}{3}\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{1}{k+1}-\frac{2}{2k+5}+\frac{2}{2k+4}-\frac{2}{2k+4}\right)=\frac{4}{3}\lim_{n\to\infty}A_n+\frac{8}{3}\lim_{n\to\infty}B_n\text{, where}$$

$$A_n=\sum_{k=0}^{n-1}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=1-\frac{1}{n+1}\xrightarrow{(n\to\infty)}1$$

$$B_n=\sum_{k=0}^{n-1}\left(\frac{1}{2k+4}-\frac{1}{2k+5}\right)=\frac{1}{4}\sum_{k=0}^{n-1}\frac{1}{(k+2)\left(k+\frac{5}{2}\right)}\xrightarrow{(n\to\infty)}\frac{1}{2}\left(\psi\left(\frac{5}{2}\right)-\psi(2)\right)$$

$$=\frac{5}{6}-\log 2,\qquad \Omega=\frac{4}{3}+\frac{8}{3}\left(\frac{5}{6}-\log 2\right)=\frac{32}{9}-\frac{8}{3}\log 2$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i+\frac{1}{4}\right)\right]^{-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k(k+1)}{2}+\frac{k}{4}\right)^{-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{4}{2k(k+1)+k}$$

$$\frac{\Omega}{4}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(2k+3)}\Rightarrow\frac{3}{4}\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{2k+3}\right)$$

$$\because\psi^{(0)}(1+z)-\psi^{(0)}(z)=\frac{1}{z},\ \frac{3}{4}\Omega=\frac{8}{3}+\gamma-2\log 2-\gamma=\frac{8}{3}-2\log 2$$

$$\because\psi^{(0)}(2z)=\frac{\psi^{(0)}(z)+\psi^{(0)}\left(z+\frac{1}{2}\right)}{2}+\log 2\,;\ \psi^{(0)}(1)=-\gamma$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i+\frac{1}{4}\right)\right]^{-1}=\frac{8}{9}(4-\log 8)$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i+\frac{1}{4}\right)\right]^{-1}=\frac{8}{3}\sum_{k=1}^{\infty}\left(\frac{1}{2k}-\frac{1}{2k+3}\right)=\frac{4}{3}H_{\frac{3}{2}}=\frac{32}{9}-\frac{8}{3}\log 2$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}\left(i+\frac{1}{4}\right)\right]^{-1}=\frac{8}{9}(4-\log 8)$$

1909. Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\frac{n}{(2n+1)\cdot 4^n}$$

Proposed by Vincenzo Dima-Netro-Italy

Solution 1 by Francesco Raso Stoia-Milan-Italy

$$f(x)=\sum_{n=1}^{\infty}\frac{n}{2n+1}\cdot x^{2n}\Rightarrow\Omega=f\left(\frac{1}{2}\right)$$

$$xf(x)=\sum_{n=1}^{\infty}\frac{n}{2n+1}\cdot x^{2n+1}$$

$$2\big(xf(x)\big)'=\sum_{n=2}^{\infty}2nx^{2n}=\sum_{n=1}^{\infty}(2n+1)x^{2n}-\sum_{n=1}^{\infty}x^{2n}$$

$$\sum_{n=1}^{\infty}(2n+1)x^{2n}=\frac{x^2}{1-x^2}+2\big(xf(x)\big)'$$

$$\sum_{n=1}^{\infty}x^{2n+1}=\frac{x^3}{1-x^2}=\int\frac{x^2}{1-x^2}dx+2xf(x)$$

$$f(x)=\frac{x^2}{2(1-x^2)}-\frac{1}{2x}\int\frac{x^2}{1-x^2}dx=\frac{x^2}{1-x^2}-\frac{1}{2x}\left(-x+\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|\right)=$$

$$=\frac{1}{2(1-x^2)}+\frac{1}{4x}\log\left|\frac{x-1}{x+1}\right|$$

Hence,

$$f\left(\frac{1}{2}\right)=\frac{1}{2\cdot\frac{3}{4}}+\frac{1}{2}\log\left|\frac{-\frac{1}{2}}{\frac{3}{4}}\right|=\frac{2}{3}-\frac{1}{2}\log 3$$

Solution 2 by Said Cerbach-Algiers-Algerie

$$S=\sum_{n=1}^{\infty}\frac{n}{2n+1}2^{-2n}\text{, let }f(x)=\sum_{n=1}^{\infty}\frac{n}{2n+1}x^n\text{ then }S=f\left(\frac{1}{4}\right)$$

$$f(x)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{2n}{2n+1}x^n=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2n+1}{2n+1}-\frac{1}{2n+1}\right)x^n$$

$$2f(x)=\sum_{n=1}^{\infty}x^n-\sum_{n=1}^{\infty}\frac{x^n}{2n+1}=\frac{x}{1-x}-\sum_{n=1}^{\infty}\frac{x^n}{2n+1}$$

$$2f(x)-\frac{x}{1-x}=-\sum_{n=1}^{\infty}\frac{x^n}{2n+1}\text{; let }x=t^2$$

$$2f(t^2)-\frac{t^2}{1-t^2}=-\sum_{n=1}^{\infty}\frac{t^{2n}}{2n+1}=-\sum_{n=1}^{\infty}t^{-1}\int_0^t u^{2n}\,du=$$

$$=-\frac{1}{t}\int_0^t\left(\sum_{n=1}^{\infty}u^{2n}\right)du=-\frac{1}{t}\int_0^t\frac{u^2}{1-u^2}du=\frac{1}{t}\int_0^t\frac{u^2}{1-u^2}du\text{ or}$$

$$\int_0^t\frac{u^2}{u^2-1}du=\log\left|\frac{u-1}{u+1}\right|\Bigg|_0^t+t=\log\left|\frac{t-1}{t+1}\right|+t$$

$$\frac{1}{t}\int_0^t\frac{u^2}{u^2-1}du=1+\frac{1}{2t}\log\left|\frac{t-1}{t+1}\right|$$

$$t=x^2, x=\frac{1}{4}\Rightarrow t=\frac{1}{2}\Rightarrow 2S-\frac{1}{3}=1-\log 3\,, S=\frac{2}{3}-\frac{1}{2}\log 3$$

Solution 3 by Santiago Alvarez –Quito-Ecuador

$$\sum_{n=1}^{\infty}\frac{n}{2n+1}2^{-2n}=\sum_{n=1}^{\infty}\frac{n}{4^n}\int_0^1 x^{2n}\,dx=\int_0^1\sum_{n=1}^{\infty}n\left(\frac{x^2}{4}\right)^n dx\overset{y=\frac{x^2}{4}}{=}$$

$$\int_0^1 y\frac{d}{dy}\left(\sum_{n=1}^{\infty}y^n\right)dx=\int_0^1 y\frac{d}{dy}\left(\sum_{n=0}^{\infty}y^n-1\right)dx=\int_0^1 y\frac{d}{dy}\left(\frac{1}{1-y}-1\right)dy=$$

$$=\int_0^1\frac{y}{(1-y)^2}dx=\frac{1}{4}\int_0^1\frac{x^2}{\left(1-\frac{x^2}{4}\right)^2}dx=-\frac{1}{2}\int_0^1\frac{1}{1-\frac{x^2}{4}}dx-\frac{1}{4}\int_0^1\frac{xdx}{\left(1-\frac{x^2}{4}\right)^2}=$$

$$=\frac{1}{2}\int_0^1\frac{d\left(1-\frac{x^2}{4}\right)}{\left(1-\frac{x^2}{4}\right)^2}-\int_0^1\frac{d\left(\frac{x}{2}\right)}{1-\left(\frac{x}{2}\right)^2}=2-\tanh\left(\frac{1}{2}\right)=\frac{2}{3}-\frac{1}{2}\log\left(\frac{2+1}{2-1}\right)=\frac{2}{3}-\frac{1}{2}\log 3$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\tanh^{-1}(x)=\int_0^x\frac{1}{1-t^2}dt;\ -1\le x\le 1$$

$$\tanh^{-1}(x)=\int_0^x\sum_{n=0}^{\infty}t^{2n}\,dt;\ -1\le x\le 1$$

Applying Fubini-Tonelli:

$$\tanh^{-1}(x)=\sum_{n=0}^{\infty}\int_0^x t^{2n}\,dt=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1};\ -1\le x\le 1$$

$$\frac{\tanh^{-1}(x)}{x}=\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1},\qquad \frac{1}{x(1-x^2)}-\frac{\tanh^{-1}(x)}{x^2}=2\sum_{n=1}^{\infty}\frac{nx^{2n-1}}{2n+1}$$

$$\frac{x}{2(1-x^2)}-\frac{\tanh^{-1}(x)}{2x}=\sum_{n=1}^{\infty}\frac{nx^{2n}}{2n+1}$$

$$\text{put } x=\frac{1}{2}\Rightarrow\frac{2}{3}-\tanh^{-1}\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{n2^{-2n}}{2n+1},\qquad \tanh^{-1}(z)=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

Hence,

$$\sum_{n=1}^{\infty}\frac{n2^{-2n}}{2n+1}=\frac{2}{3}-\frac{1}{2}\log 3$$

Solution 5 by Fayssal Abdelli-Bejaia-Algerie

$$\sum_{n=1}^{\infty}\frac{n2^{-2n}}{2n+1}=\sum_{n=1}^{\infty}\frac{2n+1-n-1}{2n+1}2^{-2n}=\sum_{n=1}^{\infty}2^{-2n}-\sum_{n=1}^{\infty}\frac{n+1}{2n+1}2^{-2n}=$$

$$=\sum_{n=1}^{\infty}2^{-2n}-\sum_{n=1}^{\infty}\frac{n2^{-2n}}{2n+1}-\sum_{n=1}^{\infty}\frac{2^{-2n}}{2n+1}$$

$$2\sum_{n=1}^{\infty}\frac{n}{2n+1}2^{-2n}=\sum_{n=1}^{\infty}2^{-2n}-\sum_{n=1}^{\infty}\frac{2^{-2n}}{2n+1}=A-B$$

$$A=\sum_{n=1}^{\infty}2^{-2n}=\lim_{n\to\infty}\left(-1+\frac{\left(\frac{1}{n}\right)^{n}-1}{\frac{1}{n}-1}\right)=\frac{4}{3}-1\Rightarrow\frac{A}{2}=\frac{2}{3}-\frac{1}{2}$$

$$-B=\sum_{n=1}^{\infty}\frac{2^{-2n}}{2n+1}=\frac{\frac{1}{4}}{3}+\frac{\left(\frac{1}{4}\right)^{2}}{5}+\frac{\left(\frac{1}{4}\right)^{3}}{7}+\cdots=\frac{\left(\frac{1}{2}\right)^{2}}{3}+\frac{\left(\frac{1}{2}\right)^{4}}{5}+\frac{\left(\frac{1}{2}\right)^{6}}{7}+\frac{\left(\frac{1}{2}\right)^{8}}{9}+\cdots$$

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots$$

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\cdots$$

$$\log(1+x)-\log(1-x)=2x+2\cdot\frac{x^3}{3}+2\cdot\frac{x^5}{5}+2\cdot\frac{x^7}{7}+\cdots$$

$$\log\left(\frac{1+x}{1-x}\right)=2x+2\cdot\frac{x^3}{3}+2\cdot\frac{x^5}{5}+2\cdot\frac{x^7}{7}+\cdots$$

$$\frac{1+x}{1-x}=3\Rightarrow x=\frac{1}{2}$$

$$\log 3=2\cdot\frac{1}{2}+2\cdot\frac{\left(\frac{1}{2}\right)^{3}}{3}+2\cdot\frac{\left(\frac{1}{2}\right)^{5}}{5}+2\cdot\frac{\left(\frac{1}{2}\right)^{7}}{7}+\cdots=$$

$$=1+\frac{\left(\frac{1}{2}\right)^{2}}{3}+\frac{\left(\frac{1}{2}\right)^{4}}{5}+\frac{\left(\frac{1}{2}\right)^{6}}{7}+\cdots$$

$$\log 3 - 1 = \sum_{n=1}^{\infty} \frac{2^{-2n}}{2n+1}$$

$$-B = 1 - \log 3 \Rightarrow -\frac{B}{2} = \frac{1}{2} - \frac{1}{2}\log 3$$

Finally,

$$2\sum_{n=1}^{\infty} \frac{n}{2n+1} 2^{-2n} = A - B = \frac{4}{3} - 1 - \log 3 + 1$$

Hence,

$$\sum_{n=1}^{\infty} \frac{n2^{-2n}}{2n+1} = \frac{2}{3} - \frac{1}{2}\log 3$$

1910. Prove that:

$$\frac{1}{3}\sum_{k=1}^{\infty}\left[\sum_{n=0}^{k}\frac{1}{n!}\right]\frac{k}{2^k} = \sqrt{e}$$

Proposed by Vincenzo Dima-Netro-Italy

Solution by Said Cerbach-Algiers-Algerie

We have:

$$\Omega = \frac{1}{3}\sum_{k=1}^{\infty}\left[\sum_{n=0}^{k}\frac{1}{n!}\right]\frac{k}{2^k} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{1}_{[0,k]}(n)\frac{k}{2^k}$$

$$\text{But: } \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{1}_{[0,k]}(n)\frac{k}{2^k} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=n}^{\infty}\frac{k}{2^k}\right)$$

$$\text{Let: } f(x) = \sum_{k=n}^{\infty} kx^k = x\sum_{k=n}^{\infty} kx^{k-1} = x\sum_{k=n}^{\infty}\left(x^k\right)', \text{then:}$$

$$f(x) = x\left(\sum_{k=n}^{\infty} x^k\right)' = x\left(\frac{x^n}{1-x}\right)'$$

For $x = \frac{1}{2}$ **we have:** $f\left(\frac{1}{2}\right) = \left[\frac{1}{2}n\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{2}\right)^n\right] \cdot 4$ **then:**

$$\Omega=4\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}n\left(\frac{1}{2}\right)^n+\frac{1}{2}\left(\frac{1}{2}\right)^n\right)=2\sum_{n=0}^{\infty}\frac{n}{n!}\left(\frac{1}{2}\right)^n+2\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\right)^n=$$

$$=2\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{(n-1)!}+2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n!}=\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^{n-1}}{(n-1)!}+2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n!}=$$

$$=\sqrt{e}+2\sqrt{e}=3\sqrt{e}$$

1911. If we define the function for $n\geq 1$

$$\phi(n)=\sum_{m=1}^{n}\sin^{(-1)^{\frac{1}{2}m(m+1)}}\left(\frac{\pi m}{4}\right)\sin^{(-1)^{\frac{1}{2}m(m-1)}}\left(\frac{\pi m}{2}\right)$$

then prove that:

$$\sum_{n=1}^{\infty}2^{-\frac{n}{2}}\phi(n)=\frac{3}{5}\left(2+\sqrt{2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Ahmed Yackoube Chach-Mauritania

$$\text{Let } U_m=\sin^{(-1)^{\frac{1}{2}m(m+1)}}\left(\frac{\pi m}{4}\right)\sin^{(-1)^{\frac{1}{2}m(m-1)}}\left(\frac{\pi m}{2}\right)$$

$$U_{2p}=0$$

$$a)\ U_{4p+1}=\sin^{-1}\left(\frac{(4p+1)\pi}{4}\right)\sin\left(\frac{(4p+1)\pi}{2}\right)=(-1)^p\sqrt{2}$$

$$b)\ U_{4p+3}=\sin\left(\frac{(4p+3)\pi}{4}\right)\sin^{(odd)}\left(\frac{(4p+3)\pi}{2}\right)=-\frac{1}{\sqrt{2}}(-1)^p$$

$$\phi(4k)=\sum_{k=1}^{4p}U_k=\phi(4p-1)+U_{4p}=\phi(4p-1)+0=$$

$$=\sum_{m=0}^{p-1}U_{4m+1}+\sum_{m=0}^{p-1}U_{4m+3}=\frac{1}{2\sqrt{2}}\left((-1)^{p+1}+1\right)$$

$$\phi(4p+1)=\sum_{k=1}^{4p+1}U_k=\phi(4p)+U_{4p+1}=\phi(4p+2)=$$

$$=\frac{1}{2\sqrt{2}}\left((-1)^{p+1}+1\right)+\sqrt{2}(-1)^p$$

$$\phi(4p)=\phi(4p-1)=\frac{1}{2\sqrt{2}}\left((-1)^{p+1}+1\right)$$

$$\phi(4p+3)=\frac{1}{2\sqrt{2}}\left((-1)^p+1\right)$$

$$\Omega=\sum_{n=1}^{\infty}\frac{\phi(n)}{2^{\frac{n}{2}}}=\sum_{k=0}^{\infty}\frac{\phi(4k+1)}{2^{\frac{4k+1}{2}}}+\sum_{k=0}^{\infty}\frac{\phi(4k+2)}{2^{\frac{4k+2}{2}}}+\sum_{k=0}^{\infty}\frac{\phi(4k+3)}{2^{\frac{4k+3}{2}}}+\sum_{k=1}^{\infty}\frac{\phi(4k)}{2^{\frac{4k}{2}}}$$

Hence, we have:

$$\sum_{k=0}^{\infty}\frac{\phi(4k+1)}{2^{\frac{4k+1}{2}}}=\frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{1}{2^{2k}}\left[\frac{1}{2\sqrt{2}}\left((-1)^{k+1}+1\right)+\sqrt{2}(-1)^k\right]=$$

$$=\frac{1}{\sqrt{2}}\left[\frac{1}{2\sqrt{2}}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{2^{2k}}+\frac{1}{2\sqrt{2}}\sum_{k=0}^{\infty}\frac{1}{2^{2k}}+\sqrt{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}}\right]=\frac{14}{15}$$

$$\sum_{k=0}^{\infty}\frac{\phi(4k+2)}{2^{\frac{4k+2}{2}}}=\sum_{k=0}^{\infty}\frac{\phi(4k+1)}{2^{\frac{4k+1+1}{2}}}=\frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{\phi(4k+1)}{2^{\frac{4k+1}{2}}}=\frac{14}{15\sqrt{2}}$$

$$\sum_{k=0}^{\infty}\frac{\phi(4k+3)}{2^{\frac{4k+3}{2}}}=\frac{1}{2\sqrt{2}}\sum_{k=0}^{\infty}\frac{\phi(4k+3)}{2^{2k}}=\frac{1}{2\sqrt{2}}\sum_{k=0}^{\infty}\frac{1}{2^{2k}}\left[\frac{1}{2\sqrt{2}}\left((-1)^k+1\right)\right]=$$

$$=\frac{1}{8}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}}+\sum_{k=0}^{\infty}\frac{1}{2^{2k}}\right)=\frac{4}{15}$$

$$\sum_{k=1}^{\infty}\frac{\phi(4k)}{2^{\frac{4k}{2}}}=\sum_{k=1}^{\infty}\frac{1}{2^{2k}}\left[\frac{1}{2\sqrt{2}}(-1)^{k+1}+1\right]=\frac{1}{2\sqrt{2}}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2^{2k}}+\sum_{k=1}^{\infty}\frac{1}{2^{2k}}\right]=\frac{4}{15\sqrt{2}}$$

Therefore,

$$\Omega=\frac{4}{15}\left(1+\frac{1}{\sqrt{2}}\right)+\frac{14}{15}\left(1+\frac{1}{\sqrt{2}}\right)=\frac{3}{5}\left(2+\sqrt{2}\right)$$

1912. Prove that:

$$\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{4}\right)}{(k!)\sqrt{2^k}}\pi^k=\sqrt{e^{\pi}}$$

Proposed by Vincenzo Dima-Netro-Italy

Solution 1 by Ankush Kumar Parcha-India

$$\Omega=\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{4}\right)}{(k!)\sqrt{2^k}}\pi^k=\Im\left\{\sum_{k=0}^{\infty}\frac{e^{ik\frac{\pi}{4}}}{(k!)\sqrt{2^k}}\pi^k\right\}=\Im\left\{\sum_{k=0}^{\infty}\frac{\left(e^{i\frac{\pi}{4}}\frac{\pi}{\sqrt{2}}\right)^k}{k!}\right\}=\Im\left\{e^{\frac{\pi}{\sqrt{2}}e^{i\frac{\pi}{4}}}\right\}=e^{\frac{\pi}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}$$

Therefore,

$$\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{4}\right)}{(k!)\sqrt{2^k}}\pi^k=\sqrt{e^{\pi}}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{4}\right)}{(k!)\sqrt{2^k}}\pi^k=\sum_{k=0}^{\infty}\frac{\Re\left\{e^{\frac{i\pi k}{4}}\right\}}{(k!)\sqrt{2^k}}\pi^k=\Re\left\{\sum_{k=0}^{\infty}\frac{e^{\frac{i\pi k}{4}}}{(k!)\sqrt{2^k}}\pi^k\right\}$$

$$\left|\sum_{k=0}^{\infty}\frac{e^{\frac{i\pi k}{4}}}{(k!)\sqrt{2^k}}\pi^k\right|\le\sum_{k=0}^{\infty}\left|\frac{e^{\frac{i\pi k}{4}}}{(k!)\sqrt{2^k}}\pi^k\right|=\sum_{k=0}^{\infty}\frac{1}{k!\sqrt{2^k}}\pi^k=e^{\frac{\pi}{\sqrt{2}}}<+\infty$$

Therefore,

$$\sum_{k=0}^{\infty}\frac{e^{\frac{i\pi k}{4}}}{(k!)\sqrt{2^k}}\pi^k<+\infty$$

Hence,

$$\Re\left\{\sum_{k=0}^{\infty}\frac{e^{\frac{i\pi k}{4}}}{(k!)\sqrt{2^k}}\pi^k\right\}=\Re\left\{e^{e^{\frac{i\pi}{4}}\frac{\pi}{\sqrt{2}}}\right\}$$

$$\because\ \Re(r^z)=e^{\Re\{z\}}\Rightarrow\Re\left\{e^{e^{\frac{i\pi}{4}}\frac{\pi}{\sqrt{2}}}\right\}=\frac{\pi}{\sqrt{2}}\Re\left(\cos\left(\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right)=\frac{\pi}{2}$$

$$\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{4}\right)}{(k!)\sqrt{2^k}}\pi^k=\sqrt{e^{\pi}}$$

Solution 3 by Toubal Fethi-Algerie

$$\Omega=\sum_{k=0}^{\infty}\frac{\sin\left(\frac{k\pi}{4}\right)}{(k!)\sqrt{2^k}}\pi^k$$

$$\text{Put } x=\frac{\pi}{\sqrt{2}}\Rightarrow \Omega=\sum_{k=0}^{\infty}\frac{x^k\sin\left(\frac{k\pi}{4}\right)}{k!}=\Im\left\{\sum_{k=0}^{\infty}\frac{x^k e^{\frac{ik\pi}{4}}}{k!}\right\}=\Im\left\{\sum_{k=0}^{\infty}\frac{\left(xe^{\frac{i\pi}{4}}\right)^k}{k!}\right\}$$

We have:

$$\Omega=\Im\left\{e^{xe^{\frac{i\pi}{4}}}\right\}=e^{x\sin\frac{\pi}{4}}=e^{\frac{\sqrt{2}x}{2}}=\sqrt{e^{\pi}}.$$

1913. Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\sin^{-1}\varepsilon}^{\sin^{-1}(1-\varepsilon)}\log\left((\cos x)^{\cot x}\cdot(\sin x)^{\frac{\cos x}{1+\sin x}}\right)dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ose Favour-Nigeria

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\sin^{-1}\varepsilon}^{\sin^{-1}(1-\varepsilon)}\log\left((\cos x)^{\cot x}\cdot(\sin x)^{\frac{\cos x}{1+\sin x}}\right)dx=$$

$$=\int_0^{\frac{\pi}{2}}\cot x\log(\cos x)\,dx+\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\sin x}\log(\sin x)\,dx=A+B$$

$$A\overset{u=\cos x}{=}\int_0^1\frac{u}{1-u^2}\log u\,du=\sum_{n=0}^{\infty}\int_0^1 u^{2n+1}\log u\,du\overset{IBP}{=}$$

$$=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 u^{2n+1}\,du=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}=-\frac{1}{4}\zeta(2)=-\frac{\pi^2}{24}$$

$$B\overset{u=\sin x}{=}\int_0^1\frac{\log u}{1+u}du=Li_2(-1)=-\frac{\pi^2}{12}\Rightarrow\Omega=A+B=-\frac{\pi^2}{8}$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\sin^{-1}\varepsilon}^{\sin^{-1}(1-\varepsilon)}\log\left((\cos x)^{\cot x}\cdot(\sin x)^{\frac{\cos x}{1+\sin x}}\right)dx=$$

$$=\int_0^{\frac{\pi}{2}}\cot x\log(\cos x)\,dx+\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\sin x}\log(\sin x)\,dx=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x}\log(1-\sin^2 x)\,dx+\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\sin x}\log(\sin x)\,dx\overset{y=\sin x}{=}$$

$$=\frac{1}{2}\int_0^1 \frac{\log(1-y^2)}{y}dy+\int_0^1 \frac{\log y}{1+y}dy=$$

$$=-\frac{1}{2}\int_0^1 \sum_{n=1}^{\infty}\frac{y^{2n}}{ny}dy+\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^n \log y\, dy=$$

$$\left(\because \int_0^1 x^m \log^n x\, dx=\frac{(-1)^n n!}{(m+1)^{n+1}}\right)$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 y^{2n-1}dy+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2}=-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}-\eta(2)=-\frac{\pi^2}{24}-\frac{\pi^2}{12}=-\frac{\pi^2}{8}$$

Solution 3 by Daniel Immarube-Nigeria

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\sin^{-1}\varepsilon}^{\sin^{-1}(1-\varepsilon)}\log\left((\cos x)^{\cot x}\cdot(\sin x)^{\frac{\cos x}{1+\sin x}}\right)dx=$$

$$=\int_0^{\frac{\pi}{2}}\cot x\log(\cos x)\,dx+\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\sin x}\log(\sin x)\,dx=\Psi+\Phi$$

$$\Psi=\int_0^{\frac{\pi}{2}}\cot x\log(\cos x)\,dx\overset{x\to\cos x}{=}\int_0^1\frac{x\log x}{1-x^2}dx\overset{x\to x^2}{=}\frac{1}{4}\int_0^1\frac{\log x}{1-x}dx=\frac{1}{4}\Delta$$

$$\Delta=\int_0^1\frac{\log^a x}{1-x}dx=\sum_{n=1}^{\infty}\int_0^1 x^{n-1}\log^a x\,dx=\Gamma(a+1)\sum_{n=1}^{\infty}\left(\frac{1}{n^{a+1}}\right)=$$

$$=(-1)^a\Gamma(a+1)\zeta(a+1)\Rightarrow\Psi=-\frac{1}{4}\Gamma(2)\zeta(2)=-\frac{\pi^2}{24}$$

$$\Phi=\int_0^{\frac{\pi}{2}}\frac{\cos x}{1+\sin x}\log(\sin x)\,dx\overset{x\to\sin x}{=}\int_0^1\frac{\cos x}{1+\sin x}dx=$$

$$=(-1)^n\sum_{n=0}^{\infty}\int_0^1 x^{a+n-2}dx=\frac{\delta}{\delta(a)}\left((-1)^{n-1}\sum_{n=0}^{\infty}\frac{1}{n+a-1}\right)=-(-1)^{n-1}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{\pi^2}{12}$$

Therefore,

$$\Omega=\Psi+\Phi=-\frac{\pi^2}{24}-\frac{\pi^2}{12}=-\frac{\pi^2}{8}$$

1914. Prove that:

$$\int_0^{\frac{\pi}{2}}e^{\cos(\tan x)}\cos(\sin(\tan x))\,dx=\frac{\pi}{2}\sqrt[e]{e}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Hamza Djahel-Algerie

$$\Omega=\int_0^{\frac{\pi}{2}} e^{\cos(\tan x)}\cos(\sin(\tan x))\,dx=\int_0^{\infty}\frac{e^{\cos x}\cos(\sin x)}{1+x^2}dx$$

$$e^{\cos x}\cos(\sin x)=\frac{e^{\cos x}}{2}\left[e^{i\sin x}+e^{-i\sin x}\right]=\frac{1}{2}\left[e^{\cos x+i\sin x}+e^{\cos x-i\sin x}\right]=$$

$$=\frac{1}{2}\left[e^{e^{ix}}+e^{e^{-ix}}\right]=\frac{1}{2}\sum_{n=0}^{\infty}\frac{e^{inx}+e^{-inx}}{n!}=\sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}$$

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\cos(nx)}{1+x^2}dx=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{\pi}{2}e^{-n}=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(e^{-1})^n}{n!}=\frac{\pi}{2}e^{e^{-1}}=\frac{\pi}{2}\sqrt[e]{e}$$

1915. Prove that:

$$\int_0^1\frac{\log x}{x^2-x-1}dx=\frac{1}{2\varphi-1}\left(Li_2\left(\frac{1}{\varphi}\right)-Li_2(-\varphi)\right)$$

where φ –Golden ration and $Li_2(x)$ –dilogarithm function.

Proposed by Ose Favour-Nigeria

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{\log x}{x^2-x-1}dx$$

$$x^2-x-1=\left(x-\frac{1}{2}\right)^2-\frac{5}{4}=\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{\sqrt{5}-1}{2}\right)=(x-\varphi)\left(x+\frac{1}{\varphi}\right)$$

$$\Omega=\int_0^1\frac{\log x}{(x-\varphi)\left(x+\frac{1}{\varphi}\right)}dx=\frac{1}{\varphi+\frac{1}{\varphi}}\int_0^1\frac{\log x}{x-\varphi}dx-\frac{1}{\varphi+\frac{1}{\varphi}}\int_0^1\frac{\log x}{x+\frac{1}{\varphi}}dx$$

$$\text{Since: }\varphi+\frac{1}{\varphi}=2\varphi-1;\ Li_2(z)=-\int_0^1\frac{z\log x}{1-zx}dx=\int_0^1\frac{\log x}{x-\frac{1}{z}}dx$$

$$\Omega=\frac{1}{2\varphi-1}\underbrace{\int_0^1\frac{\log x}{x-\varphi}dx}_{Li_2\left(\frac{1}{\varphi}\right)}-\frac{1}{2\varphi-1}\underbrace{\int_0^1\frac{\log x}{x+\frac{1}{\varphi}}dx}_{Li_2(-\varphi)}$$

Therefore,

$$\int_0^1 \frac{\log x}{x^2-x-1}dx = \frac{1}{2\varphi-1}\left(Li_2\left(\frac{1}{\varphi}\right)-Li_2(-\varphi)\right)$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega = \int_0^1 \frac{\log x}{x^2-x-1}dx = \int_0^1 \frac{\log x}{(x-\varphi)\left(x+\frac{1}{\varphi}\right)}dx =$$

$$= \frac{1}{\sqrt{5}}\int_0^1 \left[\frac{1}{x-\varphi}-\frac{1}{x+\frac{1}{\varphi}}\right]\log x\,dx$$

$$\sqrt{5}\Omega = -\int_0^1 \frac{\log x}{\varphi-x}dx - \varphi\int_0^1 \frac{\log x}{x\varphi+1}dx =$$

$$= -\frac{1}{\varphi}\int_0^1 \sum_{n=0}^{\infty}\left(\frac{x}{\varphi}\right)^n \log x\,dx - \varphi\int_0^1 \sum_{n=0}^{\infty}(-x\varphi)^n \log x\,dx$$

$$\left(\because \int_0^1 x^m \log^n x\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}};m\neq -1;n>-1\right)$$

$$\sqrt{5}\Omega = -\frac{1}{\varphi}\sum_{n=0}^{\infty}\frac{1}{\varphi^n}\cdot\frac{-1}{(n+1)^2} - \varphi\sum_{n=0}^{\infty}\frac{(-\varphi)^n(-1)}{(n+1)^2} \Rightarrow$$

$$\left[2\left(\frac{1+\sqrt{5}}{2}\right)-1\right]\Omega = \sum_{n=1}^{\infty}\frac{\frac{1}{\varphi^n}}{n^2} - \sum_{n=0}^{\infty}\frac{(-\varphi)^n}{n^2}$$

$$\left(\because Li_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}\right)$$

Therefore,

$$\int_0^1 \frac{\log x}{x^2-x-1}dx = \frac{1}{2\varphi-1}\left(Li_2\left(\frac{1}{\varphi}\right)-Li_2(-\varphi)\right)$$

1916. **If** $\Omega = \underbrace{\int_0^1 \dots \int_0^1}_{2023} \frac{(-1)}{\sum_{i=1}^{2023}\log(x_i)}\prod_{i=1}^{2023} dx_i$ **, then find** Ω^{-1}.

Proposed by Syed Shahabudeen-Kerala-India

Solution by Asmat Qatea-Afghanistan

$$\Omega=\underbrace{\int_0^1\cdots\int_0^1}_{2023}\frac{(-1)}{\sum_{i=1}^{2023}\log(x_i)}\prod_{i=1}^{2023}dx_i=\int_0^\infty\underbrace{\int_0^1\cdots\int_0^1}_{2023}e^{-t\log\left(\prod_{i=1}^{2023}x_k\right)}\prod_{i=1}^{2023}dx_i\,dt=$$

$$=\int_0^\infty\left(\int_0^1 x^{-t}\,dx\right)^{2023}dt=\int_0^\infty\left(\frac{1}{1-t}\right)^{2023}dt=$$

$$=\int_0^\infty(1-t)^{-2023}\,dt=\left[\frac{(1-t)^{-2022}}{2022}\right]_0^\infty=-\frac{1}{2022}$$

1917. $\Omega(\alpha,\beta)=\int_{-1}^{1}\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx,\ \alpha,\beta>0$

Find a closed form and prove that:

$$\Omega(3,5)>\sqrt{\Omega(4,5)\cdot\Omega(3,6)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ose Favour-Nigeria

$$\Omega(\alpha,\beta)=\int_{-1}^{1}\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx=2\int_0^1\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx=$$

$$\overset{x=\frac{1-x}{1+x}}{=}4\int_0^1\frac{\left(\frac{2}{1+x}\right)^{2\alpha-1}\left(\frac{2x}{1+x}\right)^{2\beta-1}}{(1+x)^2\left(\frac{2(1+x^2)}{(1+x)^2}\right)^{\alpha+\beta}}dx=2^{\alpha+\beta}\int_0^1\frac{x^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx$$

$$\Phi=\int_0^1\frac{x^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx=\int_0^\infty\frac{x^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx-\underbrace{\int_1^\infty\frac{x^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx}_{x=\frac{1}{x}}=$$

$$=\int_1^\infty\frac{x^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx-\int_1^\infty\frac{x^{2\alpha-1}}{(1+x^2)^{\alpha+\beta}}dx$$

$$2\Phi=\int_0^1\frac{x^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx+\int_0^1\frac{x^{2\alpha-1}+x^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx\overset{x=\sqrt{x}}{=}$$

$$=\frac{1}{2}\underbrace{\int_0^\infty\frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}dx}_{B(\beta,\alpha)}+\frac{1}{2}\underbrace{\int_0^1\frac{\left(x^{\alpha-\frac{1}{2}}+x^{\beta-\frac{1}{2}}\right)x^{\frac{1}{2}-1}}{(1+x)^{\alpha+\beta}}dx}_{B(\alpha,\beta)}$$

$$\Phi=\frac{1}{2}B(\alpha,\beta)$$

$$\Omega(\alpha,\beta)=2^{\alpha+\beta-1}B(\alpha,\beta)$$

$$\Omega(3,5)=2^7B(3,5)=\frac{128}{105},\Omega(4,5)=2^8B(4,5)=\frac{32}{35},\Omega(3,6)=2^8B(3,6)=\frac{32}{21}$$

Hence,

$$\frac{128}{105}>\sqrt{\frac{32}{35}\cdot\frac{32}{21}}\Leftrightarrow\Omega(3,5)>\sqrt{\Omega(4,5)\cdot\Omega(3,6)}$$

Solution 2 by Adrian Popa-Romania

$$\Omega(\alpha,\beta)=\int_{-1}^{1}\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx\overset{\frac{1-x}{1+x}=t}{=}$$

$$=\int_0^\infty\frac{2^{2\alpha+2\beta-2}t^{2\alpha-1}}{(1+t)^{2\alpha+2\beta-2}}\cdot\frac{(1+t)^{2\alpha+2\beta}}{2^{\alpha+\beta}(1+t^2)^{\alpha+\beta}}\cdot\left(\frac{2}{1+t^2}\right)dt\overset{u=t^2}{=}$$

$$=2^{\alpha+\beta-2}\int_0^\infty\frac{(\sqrt{u})^{2\alpha-1}}{(1+u)^{\alpha+\beta}}\cdot\frac{du}{2\sqrt{u}}=2^{\alpha+\beta-2}\int_0^\infty\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du=2^{\alpha+\beta-2}B(\alpha,\beta)$$

$$\Omega(\alpha,\beta)=2^{\alpha+\beta-2}B(\alpha,\beta)$$

$$\Omega(3,5)=2^6\cdot B(3,5)=2^6\cdot\frac{\Gamma(3)\Gamma(5)}{\Gamma(3+5)}=2^6\cdot\frac{2!\cdot 4!}{7!}$$

$$\Omega(4,5)=2^7\cdot B(4,5)=2^7\cdot\frac{\Gamma(4)\Gamma(5)}{\Gamma(4+5)}=2^7\cdot\frac{3!\cdot 4!}{8!}$$

$$\Omega(3,6)=2^7\cdot B(3,6)=2^7\cdot\frac{\Gamma(3)\Gamma(6)}{\Gamma(3+6)}=2^7\cdot\frac{2!\cdot 5!}{8!}$$

$$\sqrt{\Omega(4,5)\cdot\Omega(3,6)}=\frac{2^7}{8!}\sqrt{2!\cdot 3!\cdot 4!\cdot 5!}\overset{(?)}{\le}\frac{2^6}{7!}\cdot 2!\cdot 4!\Leftrightarrow$$

$$\frac{1}{4}\sqrt{(2!)^2\cdot 3\cdot(4!)^2\cdot 5}\le 2!\cdot 4!\Leftrightarrow\sqrt{15}\le\sqrt{16}.$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\Omega(\alpha,\beta)=\int_{-1}^{1}\frac{(1+x)^{2\alpha-1}(1-x)^{2\beta-1}}{(1+x^2)^{\alpha+\beta}}dx=$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\left(\sqrt{2}\right)^{2\alpha+2\beta-2}\cos^{2\alpha-1}\left(x-\frac{\pi}{4}\right)\cos^{2\beta-1}\left(x+\frac{\pi}{4}\right)}{\cos^2 x\cos^{2\alpha+2\beta-2}x}\cos^{2(\alpha+\beta)}x\,dx=$$

$$=2^{\alpha+\beta-1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^{2\alpha-1}\left(x-\frac{\pi}{4}\right)\cos^{2\beta-1}\left(x+\frac{\pi}{4}\right)dx\overset{y=x+\frac{\pi}{4}}{=}$$

$$=2^{\alpha+\beta-1}\int_0^{\frac{\pi}{2}}\sin^{2\alpha-1}y\cos^{2\beta-1}y\,dy=2^{\alpha+\beta-2}\cdot 2\int_0^{\frac{\pi}{2}}\sin^{2\alpha-1}y\cos^{2\beta-1}y\,dy$$

$$\Omega(3,5)=2^6\cdot B(3,5)=2^6\cdot\frac{\Gamma(3)\Gamma(5)}{\Gamma(3+5)}=2^6\cdot\frac{2!\cdot 4!}{7!}$$

$$\Omega(4,5)=2^7\cdot B(4,5)=2^7\cdot\frac{\Gamma(4)\Gamma(5)}{\Gamma(4+5)}=2^7\cdot\frac{3!\cdot 4!}{8!}$$

$$\Omega(3,6)=2^7\cdot B(3,6)=2^7\cdot\frac{\Gamma(3)\Gamma(6)}{\Gamma(3+6)}=2^7\cdot\frac{2!\cdot 5!}{8!}$$

$$\sqrt{\Omega(4,5)\cdot\Omega(3,6)}=\frac{2^7}{8!}\sqrt{2!\cdot 3!\cdot 4!\cdot 5!}\overset{(?)}{\leq}\frac{2^6}{7!}\cdot 2!\cdot 4!\Leftrightarrow$$

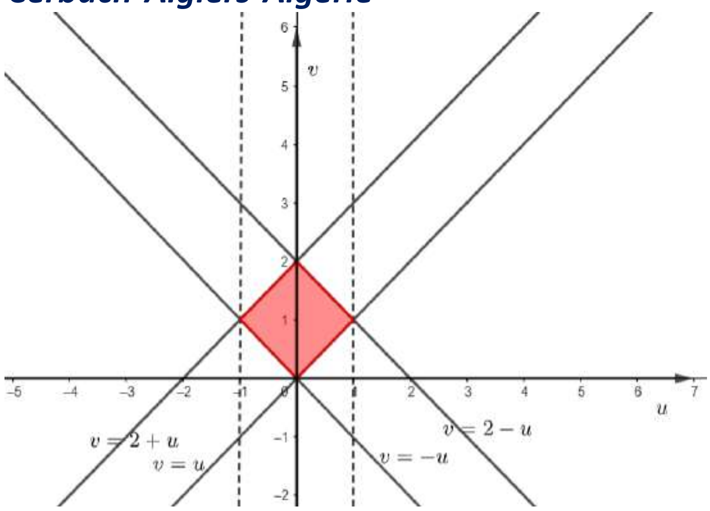
$$\frac{1}{4}\sqrt{(2!)^2\cdot 3\cdot(4!)^2\cdot 5}\leq 2!\cdot 4!\Leftrightarrow 8>\sqrt{60}.$$

1918. Prove without any software:

$$4e\left|1-\int_0^1 e^{x^2}dx\int_0^1 e^{-x^2}dx\right|<(e-1)^2$$

Proposed by Daniel Sitaru-Romania

Solution by Said Cerbach-Algiers-Algerie



Using Cauchy-Schwarz's inequality, we have:

$$\int_0^1 e^{x^2}dx\int_0^1 e^{-x^2}dx \geq 1 \text{ then}$$

$$I=\int_0^1 e^{x^2}dx\int_0^1 e^{-x^2}dx-1=\int_0^1\int_0^1\left(e^{x^2-y^2}-1\right)dxdy.\ \text{We have:}$$

$$\frac{(e-1)^2}{4e}=\frac{\cosh 1-1}{2}$$

We use a change of variables: $u=x-y$ and $v=x+y$ then we have:

$$I=\iint_D\frac{e^{uv}-1}{2}dudv,\qquad \text{where } D \text{ is red sector.}$$

$$I=\int_0^1\int_{-u}^{2+u}\frac{e^{uv}-1}{2}dv\,du+\int_0^1\int_u^{2-u}\frac{e^{uv}-1}{2}dvdu=I_-+I_+,\text{where}$$

$$I_-=\int_0^1\int_{-u}^{2+u}\frac{e^{uv}-1}{2}dv\,du,\qquad I_+=\int_0^1\int_u^{2-u}\frac{e^{uv}-1}{2}dvdu$$

With change u in I_-, we have:

$$I_-=\int_0^1\int_u^{2-u}\frac{e^{-uv}-1}{2}dudv,\text{then:}$$

$$I=\int_0^1\int_u^{2-u}\frac{e^{uv}+e^{-uv}-2}{2}dvdu=\int_0^1\int_u^{2-u}\frac{\left(e^{\frac{uv}{2}}-e^{-\frac{uv}{2}}\right)^2}{2}dvdu=$$

$$=\int_0^1\int_u^{2-u}\sinh^2\left(\frac{uv}{2}\right)dvdu=\int_0^1\int_u^{2-u}\frac{\cosh v-1}{2}dv\,du;(0\leq v\leq 2)$$

$$I\leq\int_0^{\frac{1}{2}}\frac{\cosh x-1}{2}dx\leq\int_0^1\frac{\cosh 1-1}{2}dx=\frac{\cosh 1-1}{2}$$

1919. Find:

$$\Omega=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{3^{k+1}-1}{2^{k+1}-1}\frac{\eta(k+2)}{2^{k+1}}$$

$$\text{where } \eta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s}\text{ is the Dirichlet's eta function.}$$

Proposed by Ankush Kumar Parcha-India

Solution by Ahmed Yackoube Chach-Mauritania

$$\eta(k+2)=\left(1-2^{1-(k+2)}\right)\zeta(k+2)=\left(1-\frac{1}{2^{k+1}}\right)\zeta(k+2)=\frac{2^{k+1}-1}{2^{k+1}}\zeta(k+2)$$

Thus, we have:

$$\Omega=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{3^{k+1}-1}{2^{k+1}-1}\frac{\eta(k+2)}{2^{k+1}}=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{3^{k+1}-1}{2^{k+1}-1}\frac{2^{k+1}-1}{2^{k+1}}\frac{\zeta(k+2)}{2^{k+1}}=$$

$$=\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{3^{k+1}-1}{4^{k+1}}\zeta(k+2)=\frac{1}{\pi}\left[\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^{k+1}\zeta(k+2)-\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k+1}\zeta(k+2)\right]=$$

$$=\frac{1}{\pi}\left[\sum_{k=2}^{\infty}\left(\frac{3}{4}\right)^{k-1}\zeta(k)-\sum_{k=2}^{\infty}\left(\frac{1}{4}\right)^{k-1}\zeta(k)\right]=\frac{1}{\pi}\left[-\psi_0\left(1-\frac{3}{4}\right)-\gamma+\psi_0\left(1-\frac{1}{4}\right)+\gamma\right]=$$

$$=\frac{1}{\pi}\left[\frac{\pi}{2}+2\log 2+\frac{\pi}{2}-2\log 2\right]=1$$

1920. Prove the summation:

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}n(n+1)}+(-1)^n}{8n^3+4n^2+2n+1}=\frac{\pi}{4}\left(1+\frac{e^{\frac{\pi}{4}}\operatorname{sech}\left(\frac{\pi}{4}\right)}{e^{\frac{\pi}{4}}-1}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$(-1)^{\frac{1}{2}n(n+1)}+(-1)^n=\begin{cases}2 & ,\text{for } n=4k\\ -2 & ,\text{for } n=4k+1\\ 0, & \text{otherwise}\end{cases}$$

$$\Omega=\sum_{n=-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}n(n+1)}+(-1)^n}{8n^3+4n^2+2n+1}=$$

$$=\sum_{n=-\infty}^{\infty}\frac{2}{512n^3+64n^2+8n+1}-\sum_{n=-\infty}^{\infty}\frac{2}{512n^3+448n^2+136\,n+15}=$$

$$=\Omega_1-\Omega_2$$

Using Residue Theorem: $\sum_{n=-\infty}^{\infty}f(n)=-\pi\sum\{\text{Rezidues of } \cot(\pi z) \text{ at } f\text{'s poles}\}$

$$\Omega_1=-2\pi\sum\left\{\text{Rezidues of }\frac{\cot(\pi z)}{(8z+1)(8z+i)(8z-i)}\text{ at } z=-\frac{1}{8};-\frac{i}{8};\frac{i}{8}\right\}$$

$$Res\left[\frac{\cot(\pi z)}{(8z+1)(8z+i)(8z-i)};-\frac{1}{8}\right]=-\frac{1}{16}\cot\left(\frac{\pi}{8}\right)$$

$$Res\left[\frac{\cot(\pi z)}{(8z+1)(8z+i)(8z-i)};-\frac{i}{8}\right]=\frac{1-i}{32}\cot\left(\frac{i\pi}{8}\right)$$

$$Res\left[\frac{\cot(\pi z)}{(8z+1)(8z+i)(8z-i)};\frac{i}{8}\right]=-\frac{1+i}{32}\cot\left(\frac{i\pi}{8}\right)$$

$$\Omega_1=\frac{\pi}{8}\cot\left(\frac{\pi}{8}\right)+\frac{\pi}{8}\coth\left(\frac{\pi}{8}\right)$$

$$\Omega_2=-2\pi\sum\left\{\text{Rezidues}\frac{\text{of }\cot(\pi z)}{(8z+3)(8z+2+i)(8z+2-i)}\text{ at }z\right.$$

$$\left.=-\frac{3}{8};-\frac{1}{4}-\frac{i}{8};-\frac{1}{4}+\frac{i}{8}\right\}$$

$$Res\left[\frac{\cot(\pi z)}{(8z+3)(8z+2+i)(8z+2-i)};-\frac{1}{8}\right]=-\frac{1}{16}\tan\left(\frac{\pi}{8}\right)$$

$$Res\left[\frac{\cot(\pi z)}{(8z+3)(8z+2+i)(8z+2-i)};-\frac{i}{8}\right]=\frac{1-i}{32}\cot\left(\frac{\pi}{4}+\frac{i\pi}{8}\right)$$

$$Res\left[\frac{\cot(\pi z)}{(8z+3)(8z+2+i)(8z+2-i)};\frac{i}{8}\right]=-\frac{1+i}{32}\cot\left(\frac{\pi}{4}-\frac{i\pi}{8}\right)$$

$$\Omega_2=\frac{\pi}{8}\tan\left(\frac{\pi}{8}\right)-\frac{\pi}{16}\cot\left(\frac{\pi}{4}+\frac{i\pi}{8}\right)-\frac{\pi}{16}+\frac{i\pi}{16}\cot\left(\frac{\pi}{4}+\frac{i\pi}{8}\right)-\frac{i\pi}{16}\cot\left(\frac{\pi}{4}-\frac{i\pi}{8}\right)=$$

$$\Omega_2=\frac{\pi}{8}\tan\left(\frac{\pi}{8}\right)+\frac{\pi}{8}\tan\left(\frac{\pi}{4}\right)-\frac{\pi}{8}\text{sech}\left(\frac{\pi}{4}\right)$$

$$\Omega=\frac{\pi}{8}\left[\cot\left(\frac{\pi}{8}\right)-\tan\left(\frac{\pi}{8}\right)\right]+\frac{\pi}{8}\left[\coth\left(\frac{\pi}{8}\right)-\tanh\left(\frac{\pi}{4}\right)+\text{sech}\left(\frac{\pi}{4}\right)\right]$$

$$\because\ \coth\left(\frac{\pi}{8}\right)=\frac{1+\cosh\left(\frac{\pi}{4}\right)}{\sinh\left(\frac{\pi}{4}\right)}$$

$$\Omega=\frac{\pi}{4}+\frac{\pi}{8}\left[\frac{1+\cosh\left(\frac{\pi}{4}\right)}{\sinh\left(\frac{\pi}{4}\right)}-\frac{1-\sinh\left(\frac{\pi}{4}\right)}{\cosh\left(\frac{\pi}{4}\right)}\right]=\frac{\pi}{4}+\frac{\pi}{4}\left[\frac{\left(1+e^{\frac{\pi}{4}}\right)\text{sech}\left(\frac{\pi}{4}\right)}{2\sinh\left(\frac{\pi}{4}\right)}\right]=$$

$$=\frac{\pi}{4}+\frac{\pi}{4}\left[\frac{\left(1+e^{\frac{\pi}{4}}\right)\text{sech}\left(\frac{\pi}{4}\right)}{e^{\frac{\pi}{2}}-e^{-\frac{\pi}{4}}}\right]=\frac{\pi}{4}+\frac{\pi}{4}\left[\frac{e^{\frac{\pi}{8}}\left(e^{\frac{\pi}{8}}+e^{-\frac{\pi}{8}}\right)\text{sech}\left(\frac{\pi}{4}\right)}{\left(e^{\frac{\pi}{8}}+e^{-\frac{\pi}{8}}\right)\left(e^{\frac{\pi}{8}}-e^{-\frac{\pi}{8}}\right)}\right]$$

$$=\frac{\pi}{4}+\frac{\pi}{4}\left[\frac{e^{\frac{\pi}{8}}\text{sech}\left(\frac{\pi}{4}\right)}{e^{\frac{\pi}{8}}-e^{-\frac{\pi}{8}}}\right]=\frac{\pi}{4}+\frac{\pi}{4}\left[\frac{e^{\frac{\pi}{4}}\text{sech}\left(\frac{\pi}{4}\right)}{e^{\frac{\pi}{4}}-1}\right]$$

Therefore,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^{\frac{1}{2}n(n+1)}+(-1)^n}{8n^3+4n^2+2n+1}=\frac{\pi}{4}\left(1+\frac{e^{\frac{\pi}{4}}\operatorname{sech}\left(\frac{\pi}{4}\right)}{e^{\frac{\pi}{4}}-1}\right)$$

1921. Find:

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\cos x\cdot\sinh x}{5+e^x\sin\left(x+\frac{\pi}{4}\right)+e^{-x}\cos\left(x+\frac{\pi}{4}\right)}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$\text{Let } P=5+e^x\sin\left(x+\frac{\pi}{4}\right)+e^{-x}\cos\left(x+\frac{\pi}{4}\right)$$

$$dP=(e^x-e^{-x})\left[\sin\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{\pi}{4}\right)\right]=dx$$

$$=(e^x-e^{-x})\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right]dx=$$

$$=(e^x-e^{-x})\sqrt{2}\cos x\,dx=2\sinh x\sqrt{2}\cos x\,dx=2\sqrt{2}\sinh x\cos x$$

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\cos x\cdot\sinh x}{5+e^x\sin\left(x+\frac{\pi}{4}\right)+e^{-x}\cos\left(x+\frac{\pi}{4}\right)}dx=$$

$$=\frac{1}{2\sqrt{2}}\int_{5+\sqrt{2}}^{5+e^{\frac{\pi}{4}}}\frac{dP}{P}=\frac{1}{2\sqrt{2}}\log P\Big|_{5+\sqrt{2}}^{5+e^{\frac{\pi}{4}}}=\frac{1}{2\sqrt{2}}\log\left(\frac{5+e^{\frac{\pi}{4}}}{5+\sqrt{2}}\right)$$

Solution 2 by Sakthi Vel-India

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\cos x\cdot\sinh x}{5+e^x\sin\left(x+\frac{\pi}{4}\right)+e^{-x}\cos\left(x+\frac{\pi}{4}\right)}dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{\cos x\sinh x\,dx}{5+e^x\left(\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}\right)+e^{-x}\left(\frac{\cos x}{\sqrt{2}}-\frac{\sin x}{\sqrt{2}}\right)}=$$

$$=\sqrt{2}\int_0^{\frac{\pi}{4}}\frac{\cos x\sinh x}{5\sqrt{2}+\sin x\,2\sinh x+\cos x\,2\cosh x}dx=$$

$$\text{let } y=5\sqrt{2}+\sin x\,2\sinh x+\cos x\,2\cosh x,\qquad dy=4\cos x\sinh x\,dx$$

$$=\sqrt{2}\int_{5\sqrt{2}+2}^{5\sqrt{2}+\sqrt{2}\sinh\frac{\pi}{4}+\sqrt{2}\cosh\frac{\pi}{4}}\frac{dy}{y}=\frac{1}{2\sqrt{2}}\log\left(\frac{5+e^{\frac{\pi}{4}}}{5+\sqrt{2}}\right)$$

1922. If we have the integral relation

$$\left(\int_{-\infty}^{\infty}\frac{\sin\left(x-\frac{1}{x}\right)\cos\left(x+\frac{1}{x}\right)}{\sqrt[3]{x}}dx\right)^2=\frac{3\pi\beta}{4}+\left(\int_{-\infty}^{\infty}\frac{\sin\left(x+\frac{1}{x}\right)\cos\left(x-\frac{1}{x}\right)}{\sqrt[3]{x}}dx\right)^2$$

then prove that $\beta^4+3\beta^2+9=0$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\left(\int_{-\infty}^{\infty}\frac{\sin\left(x-\frac{1}{x}\right)\cos\left(x+\frac{1}{x}\right)}{\sqrt[3]{x}}dx\right)^2-\left(\int_{-\infty}^{\infty}\frac{\sin\left(x+\frac{1}{x}\right)\cos\left(x-\frac{1}{x}\right)}{\sqrt[3]{x}}dx\right)^2=\frac{3\pi\beta}{4}$$

$$\Omega=\frac{1}{4}\left(\int_{-\infty}^{\infty}\frac{\sin(2x)}{\sqrt[3]{x}}dx-\int_{-\infty}^{\infty}\frac{\sin\left(\frac{2}{x}\right)}{\sqrt[3]{x}}dx\right)^2-\left(\int_{-\infty}^{\infty}\frac{\sin(2x)}{\sqrt[3]{x}}dx+\int_{-\infty}^{\infty}\frac{\sin\left(\frac{2}{x}\right)}{\sqrt[3]{x}}dx\right)^2$$

$$\Omega=\frac{(A-B)^2-(A+B)^2}{4}=-AB$$

$$A=\underbrace{\int_0^{\infty}\frac{\sin(2x)}{\sqrt[3]{x}}dx}_{2x\to x}+\underbrace{\int_{-\infty}^{0}\frac{\sin(2x)}{\sqrt[3]{x}}dx}_{2x\to -x}=\frac{1}{\sqrt[3]{4}}\int_0^{\infty}\frac{\sin x}{\sqrt[3]{x}}dx-\frac{e^{-\frac{i\pi}{3}}}{\sqrt[3]{4}}\int_0^{\infty}\frac{\sin x}{\sqrt[3]{x}}dx=$$

$$=\frac{e^{\frac{i\pi}{3}}}{\sqrt[3]{4}}\int_0^{\infty}\frac{\sin x}{\sqrt[3]{x}}dx=\frac{e^{\frac{i\pi}{3}}}{\sqrt[3]{4}}\int_0^{\infty}x^{\frac{2}{3}-1}\sin x\,dx$$

$$\text{Mellin Transform: }\int_0^{\infty}x^{s-1}\sin x\,dx=\sin\left(\frac{\pi s}{2}\right)\Gamma(s)$$

$$A=\frac{e^{\frac{i\pi}{3}}}{\sqrt[3]{4}}\sin\left(\frac{\pi}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{e^{\frac{i\pi}{3}}}{\sqrt[3]{4}\Gamma\left(\frac{1}{3}\right)}\sin\left(\frac{\pi}{3}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\pi e^{\frac{i\pi}{3}}}{\sqrt[3]{4}\Gamma\left(\frac{1}{3}\right)}$$

$$B=\underbrace{\int_0^{\infty}\frac{\sin\left(\frac{2}{x}\right)}{\sqrt[3]{x}}dx}_{\frac{2}{x}\to x}+\underbrace{\int_{-\infty}^{0}\frac{\sin\left(\frac{2}{x}\right)}{\sqrt[3]{x}}dx}_{\frac{2}{x}\to -x}=\sqrt[3]{4}\int_0^{\infty}\frac{\sin x}{\sqrt[3]{x^5}}dx-\sqrt[3]{4}e^{-\frac{i\pi}{3}}\int_0^{\infty}\frac{\sin x}{\sqrt[3]{x^5}}dx=$$

$$=\sqrt[3]{4}e^{\frac{i\pi}{3}}\int_0^{\infty}x^{-\frac{2}{3}-1}\sin x\,dx=\sqrt[3]{4}e^{\frac{i\pi}{3}}\sin\left(-\frac{\pi}{3}\right)\Gamma\left(-\frac{2}{3}\right)$$

$$\Gamma\left(\frac{1}{3}\right)=\Gamma\left(1-\frac{2}{3}\right)=-\frac{2}{3}\Gamma\left(-\frac{2}{3}\right),\qquad B=\frac{3\sqrt{3}\sqrt[3]{4}e^{\frac{i\pi}{3}}}{4}\Gamma\left(\frac{1}{3}\right)$$

$$\Omega=-AB=-\frac{3\sqrt{3}\pi}{4}e^{\frac{2i\pi}{3}}=\frac{3\sqrt{3}\pi}{4}e^{-\frac{i\pi}{3}}=\frac{3\pi\beta}{4}$$

$$\beta=\sqrt{3}e^{-\frac{i\pi}{3}}\Rightarrow\beta^4+3\beta^2+9=9\left(e^{-\frac{4i\pi}{3}}+e^{-\frac{2i\pi}{3}}+1\right)=9\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}-\frac{1}{2}-\frac{i\sqrt{3}}{2}+1\right)=0$$

1923. Find:

$$\Omega=\int_0^1\frac{\log^2 y\left(Li_2(y)-\zeta(2)\right)}{(1-y)^2}dy$$

where $Li_n(1)=\sum_{k=1}^{\infty}\frac{1}{k^n}$ and $\zeta(2)=Li_2(1)$.

Proposed by Narendra Bhandari-Bajura-Nepal

Solution 1 by Togrul Ehmedov-Azerbaijan

We know that: $Li_2(x)-\zeta(2)=Li_2(x)-Li_2(1)=$

$$=-\int_1^x\frac{\log(1-y)}{y}dy$$

$$I=-\int_0^1\frac{\log^2 x}{(1-x)^2}\int_1^x\frac{\log(1-y)}{y}dxdy$$

$$u=\int_1^x\frac{\log(1-y)}{y}dy\,;du=\frac{\log(1-x)}{x}dx$$

$$dv=\frac{\log^2 x}{(1-x)^2}dx;v=\frac{\log^2 x}{1-x}-\log^2 x-2Li_2(1-x)$$

$$I=-\left(\frac{x\log^2 x}{1-x}-2Li_2(1-x)\right)\int_1^x\frac{\log(1-y)}{y}dy\Bigg|_0^1+\int_0^1\frac{\log^2 x\log(1-x)}{1-x}dx-$$

$$-2\int_0^1\frac{\log(1-x)\,Li_2(1-x)}{x}dx=$$

$$=2\zeta(2)\underbrace{\int_0^1\frac{\log(1-y)}{y}dy}_{I_1}+\underbrace{\int_0^1\frac{\log^2 x\log(1-x)}{1-x}dx}_{I_2}-2\underbrace{\int_0^1\frac{\log(1-x)\,Li_2(1-x)}{x}dx}_{I_3}$$

$$I_1=\int_0^1\frac{\log(1-y)}{y}dy=-\zeta(2)$$

$$I_2=\int_0^1 \frac{\log^2 x\log(1-x)}{1-x}dx=-\sum_{k=1}^{\infty} H_k\int_0^1 x^k\log^2 x\,dx=$$

$$=-2\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3}=-2\left[\sum_{k=1}^{\infty}\frac{H_k}{k^3}-\sum_{k=1}^{\infty}\frac{1}{k^4}\right]=-\frac{1}{2}\zeta(4)$$

$$I_3=\int_0^1 \frac{\log(1-x)\,Li_2(1-x)}{x}dx=\int_0^1\frac{\log x\,Li_2(x)}{1-x}dx=-\frac{3}{4}\zeta(4)$$

$$I=2\zeta(2)I_1+I_2-2I_3=-2\zeta^2(2)-\frac{1}{2}\zeta(4)+\frac{3}{2}\zeta(3)=-4\zeta(4)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{\log^2 y\left(Li_2(y)-\zeta(2)\right)}{(1-y)^2}dy$$

Integration by parts gives: $\int\frac{\log^2 y}{(1-y)^2}dy=\frac{y\log^2 y}{1-y}-2Li_2(1-y)$

$$\Omega\overset{IBP}{=}\left[\left(\frac{y\log^2 y}{1-y}-2Li_2(1-y)\right)\left(Li_2(y)-\zeta(2)\right)\right]\Big|_0^1+$$

$$+\int_0^1\frac{\log^2 y\log(1-y)}{1-y}dy-2\int_0^1\frac{\log(1-y)\,Li_2(1-y)}{y}dy=$$

$$=-5\zeta(4)+\underbrace{\int_0^1\frac{\log^2(1-y)\log y}{y}dy}_{A}-2\underbrace{\int_0^1\frac{\log y\,Li_2(y)}{1-y}dy}_{B}$$

$$\log^2(1-y)=2\sum_{n=1}^{\infty}\frac{H_n}{n}y^n-2Li_2(y)$$

$$A=2\sum_{n=1}^{\infty}\frac{H_n}{n}\int_0^1 y^{n-1}\log y\,dy-2\int_0^1\frac{Li_2(y)\log y}{y}dy=$$

$$=-2\sum_{n=1}^{\infty}\frac{H_n}{n^3}-2[Li_3(y)\log y-Li_4(y)]_0^1$$

$$A=-\frac{5}{2}\zeta(4)+2Li_4(1)=-\frac{1}{2}\zeta(4)$$

$$B=\sum_{n=1}^{\infty}H_n^{(2)}\int_0^1 y^n\log y\,dy=-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{(n+1)^2}$$

$$B=\sum_{n=1}^{\infty}\frac{1}{(n+1)^4}-\sum_{n=1}^{\infty}\frac{H_{n+1}^{(2)}}{(n+1)^2}=\sum_{n=2}^{\infty}\frac{1}{n^4}-\sum_{n=2}^{\infty}\frac{H_n^{(2)}}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^4}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}$$

$$B=\zeta(4)-\frac{7}{4}\zeta(4)=-\frac{3}{4}\zeta(4)$$

$$\Omega=-5\zeta(4)-\frac{1}{2}\zeta(4)+\frac{3}{2}\zeta(4)=-4\zeta(4)$$

$$\Omega=\int_0^1\frac{\log^2 y\left(Li_2(y)-\zeta(2)\right)}{(1-y)^2}dy=-4\zeta(4)$$

1924. Prove that:

$$\int_0^1\frac{(1+x)\log^2 x}{(1-x)^3}\left(Li_2(x)-\frac{\pi^2}{6}\right)dx=2\zeta(2)-10\zeta(3)$$

where $Li_n(x)$ is polylogarithm function, defined as $Li_n(x)=\sum_{k=1}^{\infty}\frac{x^k}{k^n}$.

Proposed by Narendra Bhandari-Bajura-Nepal

Solution by Togrul Ehmedov-Baku-Azerbaijan

$$I=\int_0^1\frac{(1+x)\log^2 x}{(1-x)^3}\left(Li_2(x)-\frac{\pi^2}{6}\right)dx=$$

$$=\underbrace{\left(\frac{x\log^2 x}{(1-x)^2}-\frac{2x\log x}{1-x}-2\log(1-x)\right)\left(Li_2(x)-\zeta(2)\right)\Big|_0^1}_{=0}+$$

$$+\underbrace{\int_0^1\frac{\log^2 x\log(1-x)}{(1-x)^2}dx}_{I_1}-2\underbrace{\int_0^1\frac{\log x\log(1-x)}{1-x}dx}_{I_2}-2\underbrace{\int_0^1\frac{\log^2(1-x)}{x}dx}_{I_3}=$$

$$=I_1-2I_2-2I_3$$

$$I_1=\int_0^1\frac{\log^2 x\log(1-x)}{(1-x)^2}dx\overset{IBP}{=}-2\int_0^1\frac{\log x\log(1-x)}{x(1-x)}dx+\int_0^1\frac{\log^2 x}{(1-x)^2}dx=$$

$$=-4\underbrace{\int_0^1\frac{\log x\log(1-x)}{1-x}dx}_{I_2}+\underbrace{\int_0^1\frac{\log^2(1-x)}{x^2}dx}_{I_4}$$

$$I_1=-4I_2+I_4$$

$$I_2=\int_0^1\frac{\log x\log(1-x)}{1-x}dx=\zeta(3),\qquad I_3=\int_0^1\frac{\log^2(1-x)}{x}dx=2\zeta(3)$$

$$I_4=-4I_2+I_4=2\zeta(2)-4\zeta(3)$$

$$I=I_1-2I_2-2I_3=2\zeta(2)-20\zeta(3)$$

1925. **Prove that:**

$$\int_0^1\dots\int_0^1\frac{1}{\log x_1+\log x_2+\dots+\log x_n}dx_n\dots dx_2dx_1=\frac{(-1)^{n-1}}{n-1}$$

Proposed by Fao Ler-Iraq

Solution by proposer

$$\int_0^1\dots\int_0^1\frac{1}{\log x_1+\log x_2+\dots+\log x_n}dx_n\dots dx_2dx_1=$$

$$\overset{x_i\to e^{-x_i}}{=}\int_{e^{-x_1}\to 0}^{e^{-x_1}\to 1}\dots\int_{e^{-x_n}\to 0}^{e^{-x_n}\to 1}\frac{1}{-x_1-x_2-\dots-x_n}d(e^{-x_n})\dots d(e^{-x_2})d(e^{-x_1})=$$

$$=(-1)^{n+1}\int_0^\infty\dots\int_0^\infty\frac{e^{-(x_1+x_2+\dots+x_n)}}{x_n+\dots+x_2+x_1}dx_n\dots dx_2dx_1=$$

$$=(-1)^{n+1}\int_1^\infty\int_0^\infty\dots\int_0^\infty\frac{1}{x_n+\dots+x_2+x_1}\left(\frac{-d}{dz}e^{-(x_n+\dots+x_1)}\right)dx_n\dots dx_1dz=$$

$$=(-1)^{n+1}\int_1^\infty\left(\int_0^\infty e^{-zx}dx\right)^n dz==(-1)^{n+1}\int_1^\infty\left(\int_0^\infty e^{-zx}\,dx\right)^n dz=$$

$$=(-1)^{n+1}\int_1^\infty\left(\frac{e^{-zx}}{-z}\Bigg|_0^\infty\right)^n dz==(-1)^{n+1}\int_1^\infty z^{-n}\,dz=(-1)^{n+1}\frac{z^{-n+1}}{-n+1}\Bigg|_1^\infty=\frac{(-1)^{n-1}}{n-1}$$

1926. **Find:**

$$\Omega=\int_0^1\frac{\frac{\pi}{4}-\tan^{-1}x}{1-x}dx$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{\frac{\pi}{4}-\tan^{-1}x}{1-x}dx=\int_0^1\frac{\tan^{-1}\left(\frac{1-x}{1+x}\right)}{1-x}dx\overset{x\to\frac{1-x}{1+x}}{=}$$

$$= \int_0^1 \frac{\tan^{-1} x}{x(1+x)} dx = \underbrace{\int_0^1 \frac{\tan^{-1} x}{x} dx}_{G} - \underbrace{\int_0^1 \frac{\tan^{-1} x}{1+x} dx}_{I}$$

$$I \overset{x \to \frac{1-x}{1+x}}{=} \int_0^1 \frac{\tan^{-1}\left(\frac{1-x}{1+x}\right)}{1+x} dx = \int_0^1 \frac{\frac{\pi}{4} - \tan^{-1} x}{1+x} dx = \frac{\pi}{4} \log 2 - I$$

$$\text{Therefore, } \Omega = \int_0^1 \frac{\frac{\pi}{4} - \tan^{-1} x}{1-x} dx = G - \frac{\pi}{8} \log 2$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega = \int_0^1 \frac{\frac{\pi}{4} - \tan^{-1} x}{1-x} dx = \int_0^1 \frac{\tan^{-1}\left(\frac{1-x}{1+x}\right)}{1-x} dx ;\left(t = \frac{1-x}{1+x} ; dt = \frac{-2}{(1+t)^2} dt\right)$$

$$\Omega = 2 \int_0^1 \frac{\tan^{-1} t}{2t(1+t)} dt = 2 \int_0^1 \left(\frac{\tan^{-1} t}{t} - \frac{\tan^{-1} t}{1+t}\right) dt$$

$$A = \int_0^1 \frac{\tan^{-1} t}{t} dt = G = \beta(2)$$

$$B = \int_0^1 \frac{\tan^{-1} t}{1+t} dt = \log(1+t) \tan^{-1} t\Big|_0^1 - \int_0^1 \frac{\log(1+t)}{1+t^2} dt = \frac{\pi}{8} \log 2$$

$$\text{Hence, } \Omega = 2G - \frac{\pi}{4} \log 2$$

Solution 3 by Daniel Immarube-Nigeria

$$\Omega = \int_0^1 \frac{\frac{\pi}{4} - \tan^{-1} x}{1-x} dx = \frac{\pi}{4} \int_0^1 \frac{dx}{1-x} - \int_0^1 \frac{\tan^{-1} x}{1-x} dx = -\int_0^1 \frac{\tan^{-1} x}{1-x} dx = -\mathrm{A}$$

$$\mathrm{A} = \int_0^1 \frac{\tan^{-1} x}{1-x} dx \overset{\mathrm{IBP}}{=} \int_0^1 \frac{\log(1-x)}{1+x^2} dx = \int_0^{\frac{\pi}{4}} \log\left(\sqrt{2}\, sin\left(\frac{\pi}{4} - x\right) sec\, x\right) dx =$$

$$= \log\sqrt{2} \int_0^{\frac{\pi}{4}} dx + \int_0^{\frac{\pi}{4}} \log\left(sin\left(\frac{\pi}{4} - x\right)\right) dx + \int_0^{\frac{\pi}{4}} \log(\sec x)\, \mathrm{dx} = \frac{\pi}{8} \log 2 + \mathrm{B} + \mathrm{C}$$

$$\mathrm{B} = \int_0^{\frac{\pi}{4}} \log\left(sin\left(\frac{\pi}{4} - x\right)\right) dx \overset{x \to \frac{\pi}{4} - x}{=} \int_0^{\frac{\pi}{4}} \log(sin\, x)\, \mathrm{dx}$$

$$\int_0^{\mathrm{x}} \log(\sin x)\, dx = -\frac{1}{2} Cl_2(2x) - x \log 2 ,\, Cl_2(x) \sim \text{Clausen function}$$

$$A=-\frac{1}{2}Cl_2\left(2\cdot\frac{\pi}{4}\right)-\frac{\pi}{4}\log 2=-\frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{2}\right)-\frac{\pi}{4}\log 2$$

$$C=-\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx=\frac{1}{2}Cl_2(\pi-2x)-x\log 2$$

$$Cl_2\left(\frac{\pi}{2}\right)=G=\beta(2), G-\text{Catalan constant}, \beta-\text{diriclet Beta function.}$$

$$C=-\int_0^{\frac{\pi}{4}}\log(\cos x)\,dx=-\left(\frac{1}{2}Cl_2\left(\pi-2\cdot\frac{\pi}{4}\right)-\frac{\pi}{4}\log 2\right)=-\frac{1}{2}Cl_2\left(\frac{\pi}{2}\right)+\frac{\pi}{4}\log 2$$

$$\Psi=B+C=\frac{\pi}{8}\log 2-\frac{1}{2}Cl_2\left(\frac{\pi}{2}\right)-\frac{\pi}{4}\log 2-\frac{1}{2}Cl_2\left(\frac{\pi}{2}\right)+\frac{\pi}{4}\log 2=$$

$$\frac{\pi}{8}\log 2-\beta(2)=-\Psi$$

$$\text{Hence, }\Omega=\beta(2)-\frac{\pi}{8}\log 2$$

1927. If $\frac{dy}{dx}+\frac{1}{x+(1-y)^{n-1}e^{-y}}=0; y(0)=1, y=f_n(x)$ then find:

$$\Omega=\lim_{n\to\infty}\left(n^2\int_0^{\frac{1}{n}}f_n(x)\,dx\right)^n$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Fao Ler-Iraq

$$\frac{dy}{dx}+\frac{1}{x+(1-y)^{n-1}e^{-y}}=0$$

$$\frac{dy}{dx}=\frac{-1}{x+(1-y)^{n-1}e^{-y}},\quad x'=-x-(1-y)^{n-1}e^{-y}$$

$$x'+x=-(1-y)^{n-1}e^{-y},\qquad xe^y=\int(-e^y(1-y)^{n-1}e^{-y})\,dx$$

$$x=-e^y\int(1-y)^{n-1}dy,\quad x=e^{-y}\frac{(1-y)^n}{n}+ce^{-y}$$

$$0=e^{-y(0)}\frac{\left(1-y(0)\right)^n}{n}+ce^{-y(0)},\quad ce^{-1}=0\Rightarrow c=0\Rightarrow x=e^{-y}\frac{(1-y)^n}{n}$$

$$e^{-y}(1-y)^n=nx,\quad e^{1-y}(1-y)^n=nxe$$

$$e^{\frac{(1-y)}{n}}\left(\frac{1-y}{n}\right)=\frac{1}{n}(nxe)^{\frac{1}{n}},\quad \frac{1-y}{n}=W\left(\frac{1}{n}(nxe)^{\frac{1}{n}}\right)$$

$$y=1-nW\left(\frac{1}{n}(nxe)^{\frac{1}{n}}\right)$$

$$\Omega=\lim_{n\to\infty}\left(n^2\int_0^{\frac{1}{n}}\left(1-nW\left(\frac{1}{n}(nxe)^{\frac{1}{n}}\right)\right)dx\right)^n=\lim_{n\to\infty}\left(n-n^3\int_0^{\frac{1}{n}}W\left(\frac{1}{n}(nxe)^{\frac{1}{n}}\right)dx\right)^n=$$

$$=\lim_{n\to\infty}\left(n-n^3\int_0^{\frac{1}{n}}W\left(n^{\frac{1}{n}-1}e^{\frac{1}{n}}x^{\frac{1}{n}}\right)dx\right)^n=$$

$$=\lim_{n\to\infty}\left(n-n^3\left(n^{\frac{1}{n}-1}e^{\frac{1}{n}}\right)^{-n}\int_0^{\left(n^{\frac{1}{n}-1}e^{\frac{1}{n}}\right)^n\frac{1}{n}}W\left(x^{\frac{1}{n}}\right)dx\right)^n=$$

$$=\lim_{n\to\infty}\left(n-n^{n+2}e^{-1}\int_0^{n^{-n}e}W(x)\,d(x^n)\right)^n=$$

$$=\lim_{n\to\infty}\left(n-n^{n+3}e^{-1}\int_0^{\frac{1}{n}e^{\frac{1}{n}}}(xe^x)^{n-1}W(xe^x)\,d(xe^x)\right)^n=$$

$$=\lim_{n\to\infty}\left(n-n^{n+3}e^{-1}\int_0^{\frac{1}{n}}x^ne^{nx}\left(\frac{x}{n}+1\right)d\left(\frac{x}{n}\right)\right)^n=$$

$$=\lim_{n\to\infty}\left(n-ne^{-1}\int_0^1x^ne^x(x+n)\,dx\right)^n=\lim_{n\to\infty}n^n\left(1-e^{-1}\int_0^1x^n(x+n)\sum_{k=0}^{\infty}\frac{x^k}{k!}dx\right)^n=$$

$$=\lim_{n\to\infty}n^n\left(1-\frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\left(x^{n+k+1}+nx^{n+k}\right)dx\right)^n=$$

$$=\lim_{n\to\infty}n^n\left(1-\frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{n+k+2}-\frac{k+1}{n+k+1}+1\right)\right)^n=$$

$$=\lim_{n\to\infty}n^n\left(\frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{k+1}{n+k+1}-\frac{1}{n+k+2}\right)\right)^n=$$

$$=\lim_{n\to\infty}\left(\frac{n}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\left((k+1)x^{n+k}-x^{n+k+1}\right)dx\right)^n=$$

$$=\lim_{n\to\infty}\left(\frac{n}{e}\int_0^1x^n\left(\sum_{k=0}^{\infty}\frac{k+1}{k!}x^k-x\sum_{k=0}^{\infty}\frac{1}{k!}x^k\right)dx\right)^n=$$

$$=\lim_{n\to\infty}\left(\frac{n}{e}\int_0^1x^n\left(e^x(x+1)-xe^x\right)dx\right)^n=$$

$$=\lim_{n\to\infty}\left(\frac{n}{e}\int_0^1x^ne^x\,dx\right)^n=\lim_{n\to\infty}\left(\frac{n}{e}(-1)^{n+1}(n!-e!\,n)\right)^n=$$

$$=\lim_{n\to\infty}\left(\frac{n}{e}\left(n!-e\sum_{k=0}^{n}\frac{n!\,(-1)^k}{k!}\right)\right)^n=\lim_{n\to\infty}\left(n\cdot n!\left(e-\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)\right)^n=$$

$$=\lim_{n\to\infty}\left(n\cdot n!\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}-\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)\right)^n=$$

$$=\lim_{n\to\infty}\left(n\cdot n!\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k!}\right)^n=\lim_{n\to\infty}\left(n\cdot n!\sum_{k=0}^{\infty}\frac{(-1)^{k+n+1}}{(k+n+1)!}\right)^n=$$

$$=\lim_{n\to\infty}\left(n\sum_{k=0}^{\infty}(-1)^k\frac{1}{(n+1)^{k+1}}\right)^n=\lim_{n\to\infty}\left(\frac{n}{n+1}\sum_{k=0}^{\infty}\left(-\frac{1}{n+1}\right)^k\right)^n=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{1}{1+\frac{1}{n+1}}\right)^n=\lim_{n\to\infty}\left(\frac{n}{n+2}\right)^n=\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^{-n}=\frac{1}{e^2}$$

1928. Find:

$$\Omega=\sum_{k=1}^{\infty}\frac{(4k)^3+3}{(4k)!}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Fao Ler-Iraq

$$\Omega=\sum_{k=1}^{\infty}\frac{(4k)^3+3}{(4k)!}=\frac{1}{4}\sum_{k=1}^{\infty}\frac{k^3+3}{k!}\left(1+(-1)^k+i^k+(-i)^k\right)=$$

$$=\frac{1}{4}\sum_{k=0}^{\infty}\frac{k^3}{k!}\left(1+(-1)^k+i^k+(-i)^k\right)+\frac{3}{4}\sum_{k=0}^{\infty}\frac{1}{k!}\left(1+(-1)^k+i^k+(-i)^k\right)-3=$$

$$=\frac{1}{4}\left(f(1)+f(-1)+f(i)+f(-i)\right)+\frac{3}{4}\left(e+e^{-1}+e^{i}+e^{-i}\right)-3$$

$$f(x)=\sum_{i=1}^{\infty}\frac{k^3}{k!}x^k=xe^e(x^2+3x+1)\Rightarrow f(1)=5e,f(-1)=\frac{1}{e},$$

$$f(i)=-3e^i,f(-i)=-3e^{-i}$$

$$\Omega=\frac{1}{4}\left(5e+\frac{1}{e}-3e^i-3e^{-i}\right)+\frac{3}{4}\left(e+\frac{1}{e}\right)+\frac{3}{2}Re(e^i)-3=$$

$$=\frac{1}{4}\left(5e+\frac{1}{e}-3(2\cos 1)\right)+\frac{3}{4}\left(e+\frac{1}{e}\right)+\frac{3}{2}\cos 2-3=2e+\frac{1}{e}-3$$

Solution 2 by Said Cerbach-Algerie

$$\Omega=\sum_{k=1}^{\infty}\left(\frac{(4k)^3}{(4k)!}+\frac{3}{(4k)!}\right)=\sum_{k=1}^{\infty}\left(\frac{(4k)^2}{(4k-1)!}+\frac{3}{(4k)!}\right)=$$

$$=\sum_{k=1}^{\infty}\left(\frac{(4k)^2-1+1}{(4k-1)!}+\frac{3}{(4k)!}\right)=\sum_{k=1}^{\infty}\left(\frac{4k+1}{(4k-2)!}+\frac{1}{(4k-1)!}+\frac{3}{(4k)!}\right)=$$

$$=\sum_{k=1}^{\infty}\left(\frac{4k-2+3}{(4k-2)!}+\frac{1}{(4k-1)!}+\frac{3}{(4k)!}\right)=$$

$$=\sum_{k=1}^{\infty}\left(\frac{1}{(4k-3)!}+\frac{3}{(4k-2)!}+\frac{1}{(4k-1)!}+\frac{3}{(4k)!}\right)=$$

$$=\sum_{k=1}^{\infty}\left(\frac{1}{(4k)!}+\frac{1}{(4k-1)!}+\frac{1}{(4k-2)!}+\frac{1}{(4k-3)!}\right)+2\underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{(4k)!}+\frac{1}{(4k-2)!}\right)}_{\text{inverse of even numbers}}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k!}+2\sum_{k=1}^{\infty}\frac{1}{(2k)!}=(e-1)+2(\cosh 1-1)$$

$$\text{Therefore, }\Omega=e+2\cosh 1-3$$

1929. Lambert series type representation for $\sqrt{-1}$ factorial

$$(i^i)!\,(-i^i)!=1-2\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Lucas Paes Barreto-Brazil

$$\text{By }\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2-x^2}-\frac{\pi\csc(\pi x)}{2x}-\frac{1}{2x^2}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-x^2}-\frac{\pi\csc(\pi x)}{2x}+\frac{1}{2x^2}$$

$$\text{Taking }x=e^{-\frac{\pi}{2}}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}e^{\pi}=-\frac{\pi e^{\frac{\pi}{2}}\csc\left(\pi e^{-\frac{\pi}{2}}\right)}{2}+\frac{e^{\pi}}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}=-\frac{\pi\csc\left(\pi e^{-\frac{\pi}{2}}\right)}{2e^{\frac{\pi}{2}}}+\frac{1}{2}\Rightarrow 1-2\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}=\frac{1}{e^{\frac{\pi}{2}}}\cdot\frac{\pi}{\sin\left(\pi e^{-\frac{\pi}{2}}\right)}$$

$$\Rightarrow 1-2\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}=\frac{\Gamma\left(1+e^{-\frac{\pi}{2}}\right)\Gamma\left(1-e^{-\frac{\pi}{2}}\right)}{\Gamma\left(1+e^{-\frac{\pi}{2}}-e^{-\frac{\pi}{2}}\right)}$$

$$\Rightarrow 1-2\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}=\left(e^{\frac{\pi}{2}}\right)!\left(e^{-\frac{\pi}{2}}\right)!\Rightarrow 1-2\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}=\left(e^{-i\cdot i\cdot\frac{\pi}{2}}\right)!\left(e^{i\cdot i\cdot\frac{\pi}{2}}\right)!$$

$$(i^i)!(-i^i)!=1-2\sum_{n=1}^{\infty}\frac{(-1)^n}{e^{\pi}n^2-1}$$

1930. Find:

$$\Omega=\int_0^1\frac{\log^2 x}{(1-x)^2}\left(Li_2(x)-\zeta(2)\right)dx$$

where $Li_n(x)=\sum_{k=1}^{\infty}\frac{x^k}{k^n}$ and $Li_2(1)=\zeta(2)=\frac{\pi^2}{6}$.

Proposed by Narendra Bhandari-Bajura-Nepal

Solution by Said Attaoui-Oran-Algerie

$$H_n=\sum_{k=1}^{n}\frac{1}{k}$$

First of all, with a simple integration, we can verify that:

$$\int\frac{\log^2 x}{(1-x)^2}dx=-2Li_2(1-x)+\frac{x\log^2 x}{1-x}$$

Applying the property $Li_2(x)+Li_2(1-x)=\zeta(2)-\log x\log(1-x)$, we obtain, with integration by parts:

$$\Omega=\underbrace{\left[\left(-2Li_2(1-x)+\frac{x\log^2 x}{1-x}\right)\left(-Li_2(1-x)-\log x\log(1-x)\right)\right]_0^1}_{=-2(Li_2(2))^2=-2\zeta^2(2)}+$$

$$+\int_0^1\left(-2Li_2(1-x)+\frac{x\log^2 x}{1-x}\right)\frac{\log(1-x)}{x}dx=$$

$$=-2\zeta^2(2)+\int_0^1 \frac{-2Li_2(1-x)}{x}\log(1-x)\,dx+\int_0^1 \frac{\log^2 x\log(1-x)}{1-x}dx=$$

$$=-2\zeta^2(2)+\int_0^1 \frac{-2Li_2(1-x)}{x}\log(1-x)\,dx+\int_0^1 \frac{\log^2(1-x)\log x}{x}dx=$$

$$=-2\zeta^2(2)+\underbrace{\int_0^1 \frac{\log(1-x)}{x}(-2Li_2(1-x)+\log(1-x)\log x)dx}_{=K}$$

Again integrating by parts and using the result $\int_0^1 \frac{\log x}{1-x}Li_2(x)dx=-\frac{3}{4}\zeta(4)$, we obtain

$$K=-Li_2(x)(-2Li_2(1-x)+\log(1-x)\log x)|_0^1$$
$$+\int_0^1 Li_2(x)\left(\frac{\log(1-x)}{x}-\frac{3\log x}{1-x}\right)dx$$
$$=-\frac{1}{2}\left(Li_2(x)\right)^2\Big|_0^1-3\int_0^1 \frac{\log x}{1-x}Li_2(x)\,dx=-\frac{1}{2}\zeta^2(2)+\frac{9}{4}\zeta(4)$$

Finally,

$$\Omega=-\frac{5}{2}\zeta^2(2)+\frac{9}{4}\zeta(4)=-\frac{25}{4}\zeta(4)+\frac{9}{4}\zeta(4)=-4\zeta(4)$$

1931. Find:

$$I=\int_0^1\int_0^\infty \frac{x}{(1+x)(1+x^2)(1+a^2x^2)}dx\,da$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^\infty \frac{x}{(1+x)(1+x^2)(1+a^2x^2)}dx\,da=\int_0^\infty \frac{x}{(1+x)(1+x^2)}\int_0^1 \frac{1}{(1+a^2x^2)}da\,dx$$

$$=\int_0^\infty \frac{\arctan x}{(1+x)(1+x^2)}dx$$

Let $x=\tan y$

$$I=\int_0^\infty \frac{\arctan x}{(1+x)(1+x^2)}dx=\int_0^{\frac{\pi}{2}} \frac{y\cos y}{\sin y+\cos y}dy$$

$$M = \int_0^{\frac{\pi}{2}} \frac{y\sin y}{\sin y + \cos y} dy$$

$$\begin{cases} I + M = \int_0^{\frac{\pi}{2}} \frac{y\cos y}{\sin y + \cos y} dy + \int_0^{\frac{\pi}{2}} \frac{y\sin y}{\sin y + \cos y} dy \\ I - M = \int_0^{\frac{\pi}{2}} \frac{y\cos y}{\sin y + \cos y} dy - \int_0^{\frac{\pi}{2}} \frac{y\sin y}{\sin y + \cos y} dy = \int_0^{\frac{\pi}{2}} \frac{y(\cos y - \sin y)}{\sin y + \cos y} dy \end{cases}$$

$$I + M = \frac{\pi^2}{8}$$

$$I - M = \int_0^{\frac{\pi}{2}} \frac{y(\cos y - \sin y)}{\sin y + \cos y} dy \overset{IBP}{=} -\int_0^{\frac{\pi}{2}} \log(\sin y + \cos y)\, dy$$

$$= -\int_0^{\frac{\pi}{2}} \log\left(\sqrt{2}\sin\left(y + \frac{\pi}{4}\right)\right) dy = -\frac{\pi}{4}\log(2) - \int_0^{\frac{\pi}{2}} \log\left(\sin\left(y + \frac{\pi}{4}\right)\right) dy$$

$$= -\frac{\pi}{4}\log(2) - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \log(\sin y)\, dy = -\frac{\pi}{4}\log(2) - \left(-\frac{\pi}{2}\log(2) + G\right)$$

$$= \frac{\pi}{4}\log(2) - G$$

$$\begin{cases} I + M = \frac{\pi^2}{8} \\ I - M = \frac{\pi}{4}\log(2) - G \end{cases} \Rightarrow I = \frac{\pi^2}{16} + \frac{\pi}{8}\log(2) - \frac{G}{2}$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^{\infty}\int_0^1 \frac{x}{(1-x)(1+x^2)(1+a^2x^2)} da\, dx =$$

$$\overset{x=\tan t}{=} \int_0^{\frac{\pi}{2}} \frac{t}{1+\tan t} dt = \int_0^{\frac{\pi}{4}} \frac{t}{1+\tan t} + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{t}{1+\tan t} dt$$

Change t by $\frac{\pi}{4} - t$ in the 1^{sr} integral and by $\frac{\pi}{4} = t$ in the 2nd

$$\Omega=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\left(\frac{\pi}{4}-t\right)(1+\tan t)+\left(\frac{\pi}{4}+t\right)(1-\tan t)\right]dt=$$

$$=\frac{\pi}{4}\int_0^{\frac{\pi}{4}}dt-\int_0^{\frac{\pi}{4}}t\cdot\tan t\,dt=\frac{\pi^2}{16}+[t\cdot\log(\cos t)]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\log(\cos t)\,dt=$$

$$=\frac{\pi^2}{16}-\frac{\pi}{8}\log 2-\frac{G}{2}+\frac{\pi}{4}\log 2$$

Therefore, $\Omega=\int_0^{\infty}\int_0^1\frac{x}{(1-x)(1+x^2)(1+a^2x^2)}da\,dx=\frac{\pi^2-8G+\log 4}{16}$

1932. Find:

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Samar Das-India

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx=\int\frac{\sin x+\sqrt{3}\cos x}{3\sin x-4\sin^3 x}dx=$$

$$=\int\frac{\sin x+\sqrt{3}\cos x}{\sin x(3-4\sin^2 x)}dx=\int\frac{(\sin x+\sqrt{3}\cos x)dx}{\sin x(3\sin^2 x+3\cos^2 x-4\sin^2 x)}=$$

$$=\int\frac{(\sin x+\sqrt{3}\cos x)dx}{\sin x\cos^2 x(3-\tan^2 x)}=\int\frac{(1+\sqrt{3}\cos x)\sec^2 x\,dx}{3-\tan^2 x}\overset{y=\tan x}{=}$$

$$=\int\frac{\left(1+\frac{\sqrt{3}}{y}\right)dy}{3-y^2}=\int\frac{y+\sqrt{3}}{y(3-y^2)}dy=\frac{1}{\sqrt{3}}\int\left(\frac{1}{y}+\frac{1}{\sqrt{3}-y}\right)dy=$$

$$=\frac{1}{\sqrt{3}}\left(\log|y|-\log|\sqrt{3}-y|\right)+C=\frac{1}{\sqrt{3}}\log\left|\frac{y}{\sqrt{3}-y}\right|+C=$$

$$=\frac{1}{\sqrt{3}}\log\left|\frac{\sin x}{\sqrt{3}\cos x-\sin x}\right|+C$$

Solution 2 by Fayssal Abdelli-Algerie

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx=\int\frac{\sin x}{\sin(3x)}dx+\sqrt{3}\int\frac{\cos x}{\sin(3x)}dx=\Omega_1+\Omega_2$$

$$\Omega_1=\int\frac{\sin x}{\sin(3x)}dx=\int\frac{\sin x}{3\sin x-4\sin^3 x}dx=\int\frac{dx}{3-4\sin^2 x}=$$

$$=\int\frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 x}-4\tan^2 x}dx=\int\frac{\sec^2 x}{3\sec^2 x-4\tan^2 x}dx=\int\frac{\sec^2 x}{3-\tan^2 x}dx\overset{y=\tan x}{=}$$

$$=\int\frac{dy}{3-y^2}=\int\frac{dy}{(\sqrt{3}-y)(\sqrt{3}+y)}=\frac{1}{2\sqrt{3}}\int\left(\frac{1}{\sqrt{3}-y}+\frac{1}{\sqrt{3}+y}\right)dy=$$

$$=\frac{1}{2\sqrt{3}}\log|3-tan^2 x|$$

$$\Omega_2=\int\frac{\sqrt{3}\cos x}{\sin(3x)}dx=\int\frac{\sqrt{3}\cos x\,dx}{\sin 2x\cos x+\cos 2x\sin x}=$$

$$=\sqrt{3}\int\frac{\cos x\,dx}{\sin x\,(2\cos^2 x+2\cos^2 x-1)}=\sqrt{3}\int\frac{\cos x\,dx}{\sin x\,(3-4\sin^2 x)}\overset{y=\cos x}{=}$$

$$=\sqrt{3}\int\frac{dy}{y(3-4y^2)}=\sqrt{3}\int\frac{dy}{y(\sqrt{3}-2y)(\sqrt{3}+2y)}=$$

$$=\frac{\sqrt{3}}{3}\int\left(\frac{1}{y}-\frac{1}{\sqrt{3}-2y}+\frac{1}{\sqrt{3}+2y}\right)dy=\frac{\sqrt{3}}{3}\log\left|\frac{y(\sqrt{3}+2y)}{\sqrt{3}-2y}\right|+C=$$

$$=\frac{\sqrt{3}}{3}\log\left|\frac{\sin x\,(\sqrt{3}+2\sin x)}{\sqrt{3}-2\sin x}\right|+C$$

$$\Omega=\frac{\sqrt{3}}{3}\log\left|\frac{\sin x\,(\sqrt{3}+2\sin x)}{\sqrt{3}-2\sin x}\right|+\frac{1}{2\sqrt{3}}\log|3-tan^2 x|+C$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx=2\int\frac{\sin\left(x+\frac{\pi}{3}\right)}{\sin(3x)}dx\overset{x+\frac{\pi}{3}=y}{=}$$

$$=2\int\frac{\sin y}{\sin(3y-\pi)}dy=-2\int\frac{\sin y}{\sin(3y)}dy$$

$$\because\ \sin(3x)=\sin x\,(2\cos(2x)+1)$$

$$\Omega=-2\int\frac{\sin y}{\sin y\,(2\cos 2y+1)}dy\overset{t=\tan y}{=}-2\int\frac{1+t^2}{2-2t^2+1+t^2}\frac{dt}{1+t^2}=$$

$$=-2\int\frac{dt}{(\sqrt{3})^2-t^2}=\frac{2}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+C=$$

$$=\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x-\sqrt{3}+3\tan x}{\sqrt{3}+\tan x+\sqrt{3}-3\tan x}\right|$$

$$\Omega=\frac{1}{\sqrt{3}}\log\left|\frac{2\tan x}{\sqrt{3}-\tan x}\right|+C$$

Solution 4 by Abner Chinga Bazo-Lima-Peru

$$\Omega=\int\frac{\tan x+\sqrt{3}}{\frac{\sin x\,(2\cos 2x+1)}{\cos x}}dx=\int\frac{\tan x+\sqrt{3}}{\tan x\left(\frac{2(1-\tan^2 x)}{1+\tan^2 x}+1\right)}dx=$$

$$=\int\frac{(\tan x+\sqrt{3})\sec^2 x}{\tan x\,(3-\tan^2 x)}dx\overset{u=\tan x}{=}\int\frac{du}{u(\sqrt{3}-u)}=\frac{1}{\sqrt{3}}\int\frac{\sqrt{3}}{u(\sqrt{3}-u)}du=$$

$$=\frac{1}{\sqrt{3}}\int\left(\frac{1}{u}+\frac{1}{\sqrt{3}-u}\right)du=\frac{1}{\sqrt{3}}\left(\log|u|-\log|u-\sqrt{3}|\right)+C=$$

$$=\frac{1}{\sqrt{3}}\log\left|\frac{2\tan x}{\sqrt{3}-\tan x}\right|+C$$

Solution 5 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\int\frac{\sin x+\sqrt{3}\cos x}{\sin(3x)}dx=2\int\frac{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}{\sin 3x}dx=$$

$$=2\int\frac{\sin\left(x+\frac{\pi}{3}\right)}{\sin 3x}dx\overset{\theta=x+\frac{\pi}{3}}{=}2\int\frac{\sin\theta}{\sin(3\theta-\pi)}d\theta=-2\int\frac{\sin\theta}{\sin 3\theta}d\theta=$$

$$=-2\int\frac{\sin\theta}{\sin\theta-4\sin\theta\cos^2\theta}d\theta=2\int\frac{d\theta}{4\cos^2\theta-1}=$$

$$=2\int\frac{1}{3\cos^2\theta-\sin^2\theta}d\theta=2\int\frac{1}{3-\tan^2\theta}d(\tan\theta)=$$

$$=\frac{2}{\sqrt{3}}\tanh^{-1}\left(\frac{\tan\theta}{\sqrt{3}}\right)+C=\frac{2}{\sqrt{3}}\tanh^{-1}\left(\frac{\tan\left(x+\frac{\pi}{3}\right)}{\sqrt{3}}\right)+C$$

1933. Prove the integral relation:

$$\int_0^{\pi}\min\left(\sin(2x),\cos(2x),\sin\left(\frac{x}{2}\right),\cos\left(\frac{x}{2}\right)\right)dx=$$

$$=\frac{1}{8}\left(12-\sqrt{2\left(25\sqrt{5}+40\sqrt{\sqrt{5}+5}+141\right)}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Adrian Popa-Romania

$$\sin(2x)=\cos(2x)\Rightarrow\sin(2x)=\sin\left(\frac{\pi}{2}-2x\right)\Rightarrow 2x=(-1)^k\left(\frac{\pi}{2}-2x\right)+k\pi$$

$$k-\text{even}\Rightarrow 2x=\frac{\pi}{2}-2x+k\pi\Rightarrow 4x=\frac{\pi}{2}+k\pi\Rightarrow x=\frac{\pi}{8}+\frac{k\pi}{2}$$

$$k=0\Rightarrow x=\frac{\pi}{8};k=1\Rightarrow x=\frac{5\pi}{8}$$

$$k-\text{odd}\Rightarrow 2x=-\frac{\pi}{2}+2x+k\pi(\text{no solution})$$

$$\sin\left(\frac{x}{2}\right)=\cos(2x)\Rightarrow\sin\left(\frac{x}{2}\right)=\sin\left(\frac{\pi}{2}-x\right)\Rightarrow\frac{x}{2}=(-1)^k\left(\frac{\pi}{2}-2x\right)+k\pi$$

$$k-\text{even}\Rightarrow\frac{x}{2}=\frac{\pi}{2}-2x+2\pi\Rightarrow\frac{5x}{2}+\frac{\pi}{2}+k\pi$$

$$k=0\Rightarrow\frac{5x}{2}=\frac{\pi}{2}\Rightarrow x=\frac{\pi}{5}$$

$$\text{So},\min\left(\sin(2x),\cos(2x),\sin\left(\frac{x}{2}\right),\cos\left(\frac{x}{2}\right)\right)=\begin{cases}\sin\left(\frac{x}{2}\right),\text{if } x\in\left(0,\frac{\pi}{5}\right)\\ \cos(2x),\text{if } x\in\left(\frac{\pi}{5},\frac{5\pi}{8}\right)\\ \sin(2x),\text{if } x\in\left(\frac{5\pi}{8},\pi\right)\end{cases}$$

$$\Omega=\int_0^{\pi}\min\left(\sin(2x),\cos(2x),\sin\left(\frac{x}{2}\right),\cos\left(\frac{x}{2}\right)\right)dx=$$

$$=\int_0^{\frac{\pi}{5}}\sin\left(\frac{x}{2}\right)dx+\int_{\frac{\pi}{5}}^{\frac{5\pi}{8}}\cos(2x)\,dx+\int_{\frac{5\pi}{8}}^{\pi}\sin(2x)\,dx=$$

$$=-2\cos\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{5}}+\frac{1}{2}\sin(2x)\Big|_{\frac{\pi}{5}}^{\frac{5\pi}{8}}-\frac{1}{2}\cos(2x)\Big|_{\frac{5\pi}{8}}^{\pi}=$$

$$=-\cos\left(\frac{\pi}{10}\right)+2+\frac{1}{2}\sin\left(\frac{5\pi}{4}\right)-\frac{1}{2}\sin\left(\frac{2\pi}{5}\right)-\frac{1}{2}+\frac{1}{2}\cos\left(\frac{5\pi}{4}\right)$$

$$\text{Now},\cos\left(\frac{\pi}{10}\right)=\frac{\sqrt{10+2\sqrt{5}}}{4}=\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}$$

$$\sin\left(\frac{5\pi}{4}\right)=-\frac{\sqrt{2}}{2},\qquad\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{2}}{4}\sqrt{5+\sqrt{5}}$$

$$\cos\left(\frac{5\pi}{4}\right)=-\frac{\sqrt{2}}{2}$$

$$\Omega=-\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{2}+\frac{3}{2}-\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{8}\sqrt{5+\sqrt{5}}-\frac{\sqrt{2}}{4}=$$

$$=\frac{3}{2}-\frac{5\sqrt{2}\sqrt{5+\sqrt{5}}}{8}-\frac{\sqrt{2}}{2}=\frac{1}{8}\left(12-\sqrt{2}\left(5\sqrt{5+\sqrt{5}}+4\right)\right)=$$

$$=\frac{1}{8}(12-\sqrt{2\left(5\sqrt{5+\sqrt{5}}+4\right)^{2}}=\frac{1}{8}\left(12-\sqrt{2\left(25(5+\sqrt{5})+40\sqrt{5+\sqrt{5}}+16\right)}\right)=$$

$$=\frac{1}{8}\left(12-\sqrt{2\left(25\sqrt{5}+40\sqrt{\sqrt{5}+5}+141\right)}\right)$$

1934. If we have the integrals:

$$\alpha=\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(x^2+x)}(-1)^{\frac{x}{2}(x+1)}\sqrt{1+\cosh\left(\frac{\pi x}{2}\right)}\,dx$$

$$\beta=\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(x^2+x)}(-1)^{x}\sqrt{1+\cosh\left(\frac{\pi x}{2}\right)}\,dx$$

then show that

$$\left|\frac{\alpha}{\beta}\right|=\frac{e^{\frac{31\pi}{64}}\left(1+e^{\frac{\pi}{4}}\right)}{\sqrt[4]{2}\sqrt{1+e^{\frac{\pi}{2}}}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\alpha=\frac{1}{\sqrt{2}}e^{\frac{\pi}{64}(1-7i)}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(1-i)\left(x+\frac{3-i}{8}\right)^2}dx+\frac{1}{\sqrt{2}}e^{\frac{\pi}{64}(17-7i)}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(1-i)\left(x+\frac{5+i}{8}\right)^2}dx=$$

$$=\frac{\sqrt{1+i}}{\sqrt{2}}e^{\frac{\pi}{64}(1-7i)}+\frac{\sqrt{1+i}}{\sqrt{2}}e^{\frac{\pi}{64}(17-7i)}=\frac{1}{\sqrt[4]{2}}e^{\frac{\pi}{64}(1+i)}+\frac{1}{\sqrt[4]{2}}e^{\frac{\pi}{64}(17+i)}=$$

$$=\frac{1}{\sqrt[4]{2}}e^{\frac{\pi}{64}(1+i)}\left(1+e^{\frac{\pi}{4}}\right)$$

$$\beta=\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(x^2+x)}(-1)^{x}\sqrt{1+\cosh\left(\frac{\pi x}{2}\right)}\,dx=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(x^2+x)}e^{i\pi x}\left(e^{\frac{\pi x}{4}}+e^{-\frac{\pi x}{4}}\right)dx=$$

$$=\frac{e^{-\frac{\pi}{32}(15+8i)}}{\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}\left(x+\frac{1-4i}{4}\right)^2}dx+\frac{e^{-\frac{\pi}{32}(7+4i)}}{\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}\left(x+\frac{3-4i}{4}\right)^2}dx=$$

$$=e^{-\frac{\pi}{32}(15+8i)}+e^{-\frac{\pi}{32}(7+24i)}=e^{-\frac{\pi}{32}(7+8i)}\left(e^{-\frac{\pi}{4}}+e^{-\frac{\pi}{2}}\right)=$$

$$=e^{-\frac{\pi}{32}(7+8i)}\left(e^{-\frac{pi}{4}}-i\right)=e^{-\frac{\pi}{32}(15+8i)}\left(1-ie^{\frac{\pi}{4}}\right)$$

$$\left|\frac{\alpha}{\beta}\right|=\left|\frac{e^{\frac{\pi(31+17i)}{64}}}{\sqrt[4]{2}}\left(\frac{1+e^{\frac{\pi}{4}}}{1-ie^{\frac{\pi}{4}}}\right)\right|=\frac{e^{\frac{31\pi}{64}}\left(1+e^{\frac{\pi}{4}}\right)}{\sqrt[4]{2}}\left|\frac{e^{\frac{17i\pi}{64}}}{1-ie^{\frac{\pi}{4}}}\right|$$

Therefore,

$$\left|\frac{\alpha}{\beta}\right|=\frac{e^{\frac{31\pi}{64}}\left(1+e^{\frac{\pi}{4}}\right)}{\sqrt[4]{2}\sqrt{1+e^{\frac{\pi}{2}}}}$$

1935. Solve for $n\in\mathbb{C}$:

$$\int_{-\infty}^{\infty}(-1)^x e^{\left(-\frac{\pi x}{2}\right)(nx+1)}dx=\frac{1}{\sqrt{n}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_{-\infty}^{\infty}(-1)^x e^{\left(-\frac{\pi x}{2}\right)(nx+1)}dx=\int_{-\infty}^{\infty}e^{i\pi x-\frac{\pi x}{2}(1+nx)}dx=\int_{-\infty}^{\infty}e^{-\frac{\pi n}{2}\left(x^2+\frac{1-2i}{n}x\right)}dx=$$

$$=e^{-\frac{\pi(3+4i)}{8n}}\int_{-\infty}^{\infty}e^{\frac{\pi n}{2}\left(x+\frac{1-2i}{2n}\right)^2}dx=e^{-\frac{\pi(3+4i)}{8n}}\int_{-\infty}^{\infty}e^{-\frac{\pi n}{2}t^2}dt=\sqrt{\frac{2}{n}}e^{-\frac{\pi(3+4i)}{8n}}$$

$$\Omega=\frac{1}{\sqrt{n}}\Rightarrow e^{-\frac{\pi(3+4i)}{8n}}=\frac{1}{\sqrt{2}}\Rightarrow-\frac{\pi(3+4i)}{8n}=-\frac{1}{2}\log 2$$

$$n=\frac{\pi(3+4i)}{4\log 2}$$

$$\int_{-\infty}^{\infty}(-1)^x e^{\left(-\frac{\pi x}{2}\right)(nx+1)}dx=\frac{1}{\sqrt{n}}\Rightarrow n=\frac{\pi(3+4i)}{4\log 2}$$

1936. Prove that:

$$\int_0^{\frac{\pi}{2}}\int_0^1 \tan^{-1}\left(\frac{\sin x}{u+\cos x}\right)du\,dx=\frac{\pi^2}{16}+\frac{3}{2}\log 2-\frac{\pi}{4}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$\int_0^{\frac{\pi}{2}}\int_0^1 \tan^{-1}\left(\frac{\sin x}{u+\cos x}\right)du\,dx=\int_0^{\frac{\pi}{2}} I(x)\,dx$$

$$I(x)=\int_0^1 \tan^{-1}\left(\frac{\sin x}{u+\cos x}\right)du=\int_0^1\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{u+\cos x}{\sin x}\right)\right)du$$

$$\because \tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2}$$

$$\Rightarrow I(x)=\frac{\pi}{2}-\sin x\int_0^1\left(\frac{u}{\sin x}+\cot x\right)'\tan^{-1}\left(\frac{u}{\sin x}+\cot x\right)du$$

$$z=\frac{u}{\sin x}+\cot x\Rightarrow dz=\left(\frac{u}{\sin x}+\cot x\right)'dx$$

$$u=0\Rightarrow z=\cot x,\ u=1\Rightarrow z=\frac{1}{\sin x}+\cot x=\cot\frac{x}{2}$$

$$\Rightarrow I(x)=\frac{\pi}{2}-\sin x\int_{\cot x}^{\cot\frac{x}{2}}\tan^{-1}z\,dz=$$

$$=\frac{\pi}{2}-\sin x\,(x\cdot\tan^{-1}z\Big|_{\cot x}^{\cot\frac{x}{2}}-\frac{1}{2}\int_{\cot x}^{\cot\frac{x}{2}}\frac{2z}{z^2+1}dz=$$

$$=\frac{\pi}{2}-\sin x\left(\cot\frac{x}{2}\cdot\tan^{-1}\left(\cot\frac{x}{2}\right)-\cot x\cdot\tan^{-1}(\cot x)-\frac{1}{2}\log(z^2+1)\Big|_{\cot x}^{\cot\frac{x}{2}}\right)=$$

$$=\frac{\pi}{2}-\sin x\left(\frac{\pi}{2}\cot\frac{x}{2}-\frac{x}{2}\cot\frac{x}{2}-\frac{\pi}{2}\cot x+x\cot x-\frac{1}{2}\log\left(\frac{cot^2\frac{x}{2}+1}{cot^2 x+1}\right)\right)=$$

$$=\frac{\pi}{2}-\frac{\pi}{2}\cdot\frac{2\sin\frac{x}{2}\cos^2\frac{x}{2}}{\sin\frac{x}{2}}+\frac{x}{2}\cdot\frac{2\sin\frac{x}{2}\cos^2\frac{x}{2}}{\sin\frac{x}{2}}+\frac{\pi}{2}\cos x-x\cos x+$$

$$+\frac{\sin x}{2}\log\left(\frac{cot^2\frac{x}{2}+1}{cot^2 x+1}\right)=$$

$$=\frac{\pi}{2}-\pi\cos^2\frac{x}{2}+x\cos^2\frac{x}{2}+\frac{\pi}{2}\cos x-x\cos x+\frac{\sin x}{2}\log\left(\frac{\cot^2\frac{x}{2}+1}{\cot^2 x+1}\right)$$

Therefore,

$$\int_0^{\frac{\pi}{2}}\int_0^1\tan^{-1}\left(\frac{\sin x}{u+\cos x}\right)du\,dx=$$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\pi\cos^2\frac{x}{2}+x\cos^2\frac{x}{2}+\frac{\pi}{2}\cos x-x\cos x+\frac{\sin x}{2}\log\left(\frac{\cot^2\frac{x}{2}+1}{\cot^2 x+1}\right)\right)dx=$$

$$=\frac{\pi^2}{4}+\frac{\pi}{2}\sin x\Big|_0^{\frac{\pi}{2}}-\pi\int_0^{\frac{\pi}{2}}\cos^2\frac{x}{2}dx+\int_0^{\frac{\pi}{2}}x\left(\cos^2\frac{x}{2}-\cos x\right)dx+$$

$$+\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin x\cdot\log\left(\frac{cot^2\frac{x}{2}+1}{cot^2 x+1}\right)dx=\frac{\pi^2}{4}+\frac{\pi}{2}-\pi A+B+\frac{C}{2};(1)$$

$$A=\int_0^{\frac{\pi}{2}}\cos^2\frac{x}{2}dx=\int_0^{\frac{\pi}{2}}\frac{1+\cos x}{2}dx=\frac{1}{2}(x+\sin x)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{4}+\frac{1}{2};(2)$$

$$B=\int_0^{\frac{\pi}{2}}x\left(\cos^2\frac{x}{2}-\cos x\right)dx=\int_0^{\frac{\pi}{2}}x\left(\frac{1+\cos x}{2}-\cos x\right)dx=$$

$$=\int_0^{\frac{\pi}{2}}x\cdot\frac{1-\cos x}{2}dx=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}x\,dx-\int_0^{\frac{\pi}{2}}x\cos x\,dx\right)=$$

$$=\frac{1}{2}\left(\frac{\pi^2}{8}-x\sin x|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(1-\sin x)\,dx\right)=\frac{1}{2}\left(\frac{\pi^2}{8}-\frac{\pi}{2}\cos x|_0^{\frac{\pi}{2}}\right)=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2};(3)$$

$$C=\int_0^{\frac{\pi}{2}}\sin x\cdot\log\left(\frac{cot^2\frac{x}{2}+1}{cot^2 x+1}\right)dx=\int_0^{\frac{\pi}{2}}\sin x\cdot\log\left(\frac{\frac{cos^2\frac{x}{2}}{sin^2\frac{x}{2}}+1}{\frac{cos^2 x}{sin^2 x}+1}\right)dx=$$

$$=\int_0^{\frac{\pi}{2}} \sin x\cdot\log\left(\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}}\cdot\frac{\sin^2 x}{\cos^2 x+\sin^2 x}\right)dx=$$

$$=\int_0^{\frac{\pi}{2}}\sin x\cdot\log\left(\frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}}\right)dx=\int_0^{\frac{\pi}{2}}\sin x\cdot\log\left(4\cos^2\frac{x}{2}\right)dx=$$

$$=2\int_0^{\frac{\pi}{2}}\sin x\cdot\log\left(2\cos\frac{x}{2}\right)dx=2\left(\log 2\int_0^{\frac{\pi}{2}}\sin x\,dx+\int_0^{\frac{\pi}{2}}\sin x\cdot\log\left(\cos\frac{x}{2}\right)dx\right)=$$

$$=2\left(\log 2+2\int_0^{\frac{\pi}{2}}\sin\frac{x}{2}\cos\frac{x}{2}\log\left(\cos\frac{x}{2}\right)dx\right)=2(\log 2-4\int_0^{\frac{\pi}{2}}\left(\cos\frac{x}{2}\right)'\cos\frac{x}{2}\log\left(\cos\frac{x}{2}\right)dx$$

$$t=\cos\frac{x}{2}\Rightarrow dt=\left(\cos\frac{x}{2}\right)'dx,\qquad x=0\Rightarrow t=1,x=\frac{\pi}{2}\Rightarrow t=\frac{\sqrt{2}}{2}$$

$$C=2\left(\log 2-4\int_1^{\frac{\sqrt{2}}{2}}t\log t\,dt\right)=2\left(\log 2-\log\left(\frac{\sqrt{2}}{2}\right)+t^2\Big|_1^{\frac{\sqrt{2}}{2}}\right)=2\left(\frac{3}{2}\log 2-\frac{1}{2}\right);(4)$$

From (1),(2),(3) and (4), it follows that:

$$\int_0^{\frac{\pi}{2}}\int_0^1\tan^{-1}\left(\frac{\sin x}{u+\cos x}\right)du\,dx=\frac{\pi^2}{16}+\frac{3}{2}\log 2-\frac{\pi}{4}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$J=\int_0^{\frac{\pi}{2}}\int_0^1\tan^{-1}\left(\frac{\sin x}{y+\cos x}\right)dydx\overset{u=\tan\frac{x}{2}}{=}2\int_0^1\int_0^1\frac{\tan^{-1}\left(\frac{2u}{(1+y)-(1-y)u^2}\right)dydu}{1+u^2}=$$

$$\overset{v=\frac{1-y}{1+y}}{=}4\int_0^1\int_0^1\frac{\tan^{-1}\left(\frac{u(1+v)}{1-vu^2}\right)}{(1+u^2)(1+v)^2}dudv=4\int_0^1\int_0^1\frac{\tan u+\tan(uv)}{(1+u^2)(1+v)^2}dudv=$$

$$=4\left(1-\frac{1}{2}\right)\cdot\frac{\pi^2}{32}-2\int_0^1\frac{\tan^{-1}u}{1+u^2}du+\int_0^1\int_0^1\frac{ududv}{(1+u^2)(1+v)(1+u^2v^2)}\overset{t=u^2}{=}$$

$$=2\int_0^1\int_0^1\frac{dtdv}{(1+t)(1+v)(1+tv^2)}\overset{t=u^2}{=}-2\int_0^1\frac{\log\left(\frac{1+v^2}{2}\right)}{(1-v^2)(1+v)}dv=$$

$$\overset{v=\frac{1-y}{1+y}}{=}-\frac{1}{2}\int_0^1\frac{\log\left(\frac{1+y^2}{(1+y)^2}\right)(1+y)dy}{y}=-\frac{1}{2}\int_0^1\int_0^1\frac{\log\left(\frac{1+y^2}{(1+y)^2}\right)(1+y)dy}{y}=$$

$$=-\frac{1}{2}\int_0^1\int_0^1\frac{\log(1+y^2)-2\log(1+y)}{y}dy-\frac{1}{2}\int_0^1\log(1+y^2)\,dy+\int_0^1(1+y)dy=$$

$$=-\frac{1}{2}\left(-\frac{1}{2}Li_2(-1)+2Li_2(-1)\right)-\frac{1}{2}\left(\log 2-2\int_0^1\frac{y^2+1-1}{1+y^2}dy\right)+2\log 2-1=$$

$$=\frac{\pi^2}{16}+\frac{3}{2}\log 2-\frac{\pi}{4}$$

Therefore,

$$\int_0^{\frac{\pi}{2}}\int_0^1\tan^{-1}\left(\frac{\sin x}{u+\cos x}\right)du\,dx=\frac{\pi^2}{16}+\frac{3}{2}\log 2-\frac{\pi}{4}$$

Solution 3 by Togrul-Ehmedov-Azerbaijan

$$I=\int_0^{\frac{\pi}{2}}\int_0^1\arctan\left(\frac{\sin x}{a+\cos x}\right)da\,dx=\int_0^1\int_0^{\frac{\pi}{2}}\arctan\left(\frac{\sin x}{a+\cos x}\right)dx\,da=$$

$$=\int_0^1\left[\left[x\arctan\left(\frac{\sin x}{a+\cos x}\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}x\frac{1+a\cos x}{a^2+2a\cos x+1}dx\right]da=$$

$$=\frac{\pi}{2}\int_0^1\arctan\left(\frac{1}{a}\right)da-\int_0^1\int_0^{\frac{\pi}{2}}x\frac{1+a\cos x}{a^2+2a\cos x+1}dx\,da=$$

$$=\frac{\pi}{2}\left[\frac{\log(2)}{2}+\frac{\pi}{4}\right]-\int_0^{\frac{\pi}{2}}x\int_0^1\frac{1+a\cos x}{a^2+2a\cos x+1}da\,dx=\frac{\pi}{2}\left[\frac{\log(2)}{2}+\frac{\pi}{4}\right]-\int_0^{\frac{\pi}{2}}x\Omega(x)\,dx$$

$$\Omega(x)=\int_0^1\frac{1+a\cos x}{a^2+2a\cos x+1}da$$

$$=\left[\cos x\frac{\log(a(a+2\cos x)+1)}{2}+\sin x\arctan\left(\frac{a+\cos x}{\sin x}\right)\right]_0^1=$$

$$=\cos x\frac{\log(2)+\log(1+\cos x)}{2}+\sin x\arctan\left(\frac{1+\cos x}{\sin x}\right)-\sin x\arctan(\mathrm{ctg}x)=$$

$$=\frac{\log(2)}{2}\cos x+\frac{1}{2}\cos x\log(1+\cos x)+\sin x\left(\frac{\pi}{2}-\frac{x}{2}\right)-\sin x\left(\frac{\pi}{2}-x\right)=$$

$$=\frac{\log(2)}{2}\cos x+\frac{1}{2}\cos x\log(1+\cos x)+\frac{x}{2}\sin x$$

$$I=\frac{\pi}{2}\left[\frac{\log(2)}{2}+\frac{\pi}{4}\right]-\int_0^{\frac{\pi}{2}} x\Omega(x)\,dx=$$

$$=\frac{\pi}{2}\left[\frac{\log(2)}{2}+\frac{\pi}{4}\right]-\int_0^{\frac{\pi}{2}} x\left[\frac{\log(2)}{2}\cos x+\frac{1}{2}\cos x\log(1+\cos x)+\frac{x}{2}\sin x\right]dx=$$

$$=\frac{\pi}{2}\left[\frac{\log(2)}{2}+\frac{\pi}{4}\right]-\frac{\log(2)}{2}\int_0^{\frac{\pi}{2}} x\cos x\,dx-\frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos x\log(1+\cos x)\,dx-\frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\sin x\,dx=$$

$$=\frac{\pi}{2}\left[\frac{\log(2)}{2}+\frac{\pi}{4}\right]-\frac{\log(2)}{2}\left[\frac{\pi}{2}-1\right]-\frac{1}{2}M-\frac{1}{2}[\pi-2]=\frac{\pi^2}{8}+\frac{\log(2)}{2}-\frac{\pi}{2}+1-\frac{1}{2}M$$

$$M=\int_0^{\frac{\pi}{2}} x\cos x\log(1+\cos x)\,dx=\log(2)\int_0^{\frac{\pi}{2}} x\cos x\,dx+2\int_0^{\frac{\pi}{2}} x\cos x\log\left(\cos\left(\frac{x}{2}\right)\right)dx=$$

$$=\log(2)\left[\frac{\pi}{2}-1\right]+2\int_0^{\frac{\pi}{2}} x\cos x\log\left(\cos\left(\frac{x}{2}\right)\right)dx=\log(2)\left[\frac{\pi}{2}-1\right]+2N$$

$$N=\int_0^{\frac{\pi}{2}} x\cos x\log\left(\cos\left(\frac{x}{2}\right)\right)dx=4\int_0^{\frac{\pi}{4}} x\cos(2x)\log(\cos x)\,dx=$$

$$=8\int_0^{\frac{\pi}{4}} x\cos^2 x\log(\cos x)\,dx-4\int_0^{\frac{\pi}{4}} x\log(\cos x)\,dx=8K-4Q$$

$$Q=\int_0^{\frac{\pi}{4}} x\log(\cos x)\,dx=\frac{\pi G}{8}-\frac{21}{128}\zeta(3)-\frac{\pi^2}{32}\log(2)$$

$$K=\int_0^{\frac{\pi}{4}} x\cos^2 x\log(\cos x)\,dx=$$

$$=\left[\left(\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x+\frac{x^2}{4}\right)\log(\cos x)\right]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\left(\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x+\frac{x^2}{4}\right)\tan x\,dx=$$

$$=-\frac{1}{2}\log(2)\left[\frac{\pi}{16}+\frac{\pi^2}{64}\right]+\frac{1}{2}\int_0^{\frac{\pi}{4}} x\sin^2 x\,dx+\frac{1}{8}\int_0^{\frac{\pi}{4}}\sin 2x\,dx-\frac{1}{8}\int_0^{\frac{\pi}{4}}\tan x\,dx+\frac{1}{4}\int_0^{\frac{\pi}{4}} x^2\tan x\,dx$$

$$=$$

$$= -\frac{1}{2}\log(2)\left[\frac{\pi}{16}+\frac{\pi^2}{64}\right]+\frac{1}{2}\left[\frac{\pi^2}{64}-\frac{\pi}{16}+\frac{1}{8}\right]+\frac{1}{16}-\frac{1}{16}\log(2)+\frac{1}{4}\int_0^{\frac{\pi}{4}} x^2\tan x\,dx$$

$$L = \int_0^{\frac{\pi}{4}} x^2\tan x\,dx = \frac{\pi^2}{32}\log(2)+2\int_0^{\frac{\pi}{4}} x\log(\cos x)\,dx =$$

$$= \frac{\pi^2}{32}\log(2)+\frac{1}{64}(16\pi G-21\zeta(3)-4\pi^2\log(2)) = \frac{\pi G}{4}-\frac{21}{64}\zeta(3)-\frac{\pi^2}{32}\log(2)$$

$$K = -\frac{1}{2}\log(2)\left[\frac{\pi}{16}+\frac{\pi^2}{64}\right]+\frac{1}{2}\left[\frac{\pi^2}{64}-\frac{\pi}{16}+\frac{1}{8}\right]+\frac{1}{16}-\frac{1}{16}\log(2)+\frac{1}{4}L =$$

$$= \frac{\pi G}{16}-\frac{21}{256}\zeta(3)+\frac{1}{8}-\frac{\pi}{32}\log(2)+\frac{\pi^2}{128}-\frac{\pi}{32}-\frac{1}{16}\log(2)-\frac{\pi^2}{64}\log(2)$$

$$N = 8K-4Q = 1-\frac{\pi}{4}\log(2)+\frac{\pi^2}{16}-\frac{1}{2}\log(2)-\frac{\pi}{4}$$

$$M = \log(2)\left[\frac{\pi}{2}-1\right]+2N = 2+\frac{\pi^2}{8}-2\log(2)-\frac{\pi}{2}$$

$$I = \frac{\pi^2}{8}+\frac{\log(2)}{2}-\frac{\pi}{2}+1-\frac{1}{2}M = \frac{\pi^2}{16}+\frac{3}{2}\log(2)-\frac{\pi}{4}$$

1937. Prove:

$$I = \int_\infty^1 \frac{1+x}{x(1+x^2)}\log\left(\frac{x-1}{x+1}\right)dx = \beta(2)+\frac{3}{8}\zeta(2)$$

Where $\zeta(s)$ is the Euler – Riemann zeta function.
And $\beta(x)$ is the Dirichlet – beta function.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Togrul Ehmedov-Azerbaijan

$$I = \int_\infty^1 \frac{1+x}{x(1+x^2)}\log\left(\frac{x-1}{x+1}\right)dx = -\int_0^1 \frac{1+x}{1+x^2}\log\left(\frac{1-x}{1+x}\right)dx$$

$$Let\ \frac{1-x}{1+x} = y$$

$$I = -2\int_0^1 \frac{\log(y)}{(1+y)(1+y^2)}dy$$

$$I_1 = \int_0^1 \frac{\log(y)}{(1+y)(1+y^2)}dy = \int_0^1 \frac{\log(y)}{1+y^2}dy - \int_0^1 \frac{y\log(y)}{(1+y)(1+y^2)}dy = -G-\left(\frac{\pi^2}{32}-\frac{G}{2}\right)$$

$$= -\frac{G}{2}-\frac{\pi^2}{32}$$

$$I=-2I_1=-2\left(-\frac{G}{2}-\frac{\pi^2}{32}\right)=G+\frac{\pi^2}{16}=\beta(2)+\frac{3}{8}\zeta(2)$$

$$Note:\Omega=\int_0^1\frac{y\log(y)}{(1+y)(1+y^2)}dy=\frac{\pi^2}{32}-\frac{G}{2}$$

Solution 2 by Daniel Immarube-Nigeria

$$\Omega=\int_0^1\frac{1+y}{y(1+y^2)}\log\left(\frac{y-1}{y+1}\right)dy\overset{y\to\frac{y-1}{y+1}}{=}-2\int_0^1\frac{\log y}{(1+y)(1+y^2)}dy=2I$$

$$I=\int_0^1\frac{\log y}{(1+y)(1+y^2)}dy=\frac{1}{2}\int_0^1\frac{\log y}{y^2+1}dy-\frac{1}{2}\int_0^1\frac{y\log y}{y^2+1}dy+\frac{1}{2}\int_0^1\frac{\log y}{y+1}dy$$

$$\because\int_0^1\frac{\log^a y}{1+y}dy=(-1)^a\Gamma(a+1)(1-2^{-a})\zeta(a+1)$$

$$\because\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\beta(2)$$

$$I=\frac{(-1)^n}{2}\sum_{n=0}^{\infty}\frac{d}{da}\int_0^1 y^{2n+a}dx-\frac{1}{2^3}\int_0^1\frac{\log y}{y+1}dy+\frac{1}{2}\int_0^1\frac{\log y}{y+1}dy=$$

$$=-\frac{1}{2}\frac{(-1)^n}{(2n+1)^2}-\frac{1}{2^3}\Gamma(1-2^{-1})\zeta(2)+\frac{1}{2}(-1)\Gamma(1-2^{-1})\zeta(2)=$$

$$=-\frac{1}{2}\beta(2)+\frac{1}{16}\zeta(2)-\frac{1}{4}\zeta(2)=$$

$$=-\frac{\beta(2)}{2}-\frac{3}{16}\zeta(2)$$

$$\Omega=-2I=-2\left(-\frac{\beta(2)}{2}-\frac{3}{16}\zeta(2)\right)=\beta(2)+\frac{3}{8}\zeta(2)$$

1938. Evaluate

$$I=4\int_0^{\infty}\frac{\log(x)\log(1+x^2)}{x^5+x}dx$$

Proposed by Daniel Immarube-Nigeria

Solution by Togrul Ehmedov

$$I=4\int_0^{\infty}\frac{\log(x)\log(1+x^2)}{x^5+x}dx=4\int_0^{\infty}\frac{\log(x)\log(1+x^2)}{x(x^4+1)}dx$$

$$\text{Let }x^2=y$$

$$I=\int_0^\infty \frac{\log(y)\log(1+y)}{y(y^2+1)}dy=\int_0^1 \frac{\log(y)\log(1+y)}{y(y^2+1)}dy+\int_1^\infty \frac{\log(y)\log(1+y)}{y(y^2+1)}dy$$

$$=\int_0^1 \frac{\log(y)\log(1+y)}{y(y^2+1)}dy-\int_0^1 \frac{y\log(y)\left(\log(1+y)-\log(y)\right)}{y^2+1}dy$$

$$=\int_0^1 \frac{\log(y)\log(1+y)}{y(y^2+1)}dy-\int_0^1 \frac{y\log(y)\log(1+y)}{y^2+1}dy+\int_0^1 \frac{y\log^2(y)}{y^2+1}dy$$

$$=\int_0^1 \frac{\log(y)\log(1+y)}{y}dy-2\int_0^1 \frac{y\log(y)\log(1+y)}{y^2+1}dy+\int_0^1 \frac{y\log^2(y)}{y^2+1}dy$$

$$=I_1-2I_2+I_3$$

$$I_1=\int_0^1 \frac{\log(y)\log(1+y)}{y}dy=-\frac{3}{4}\zeta(3)$$

$$I_2=\int_0^1 \frac{y\log(y)\log(1+y)}{y^2+1}dy=\frac{1}{32}\left[\pi^2\log(2)-\frac{15}{2}\zeta(3)\right]$$

$$I_3=\int_0^1 \frac{y\log^2(y)}{y^2+1}dy=\frac{3}{16}\zeta(3)$$

$$I=I_1-2I_2+I_3=-\frac{3}{32}\zeta(3)-\frac{\pi^2}{16}\log(2)$$

1939. If $0<\alpha\le\beta<\frac{\pi}{2}$ then:

$$\int_\alpha^\beta\int_\alpha^\beta \sqrt{4-(\sin x+\sin y)^2}\,dxdy\ge 2(\beta-\alpha)(\sin\beta-\sin\alpha)$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

Let be $a,b\in(0,1)$.

$$(a-b)^2\ge 0\Rightarrow -a^2-b^2\le -2ab$$

$$1-a^2-b^2+a^2b^2\le 1-2ab+a^2b^2$$

$$(1-a^2)(1-b^2)\le(1-ab)^2$$

$$2\sqrt{(1-a^2)(1-b^2)}\le 2-2ab$$

$$2-a^2-b^2+2\sqrt{(1-a^2)(1-b^2)}\le 4-a^2-b^2-2ab$$

$$1-a^2+1-b^2+2\sqrt{(1-a^2)(1-b^2)}\le 4-(a+b)^2$$

$$\sqrt{1-a^2}+\sqrt{1-b^2}\le\sqrt{4-(a+b)^2}$$

Let be $a=sinx, b=siny, x,y\in\left(0,\frac{\pi}{2}\right)\Rightarrow a,b\in(0,1)$

$$\sqrt{1-sin^2x}+\sqrt{1-sin^2y}\le\sqrt{4-(sinx+siny)^2}$$

$$\int_\alpha^\beta\int_\alpha^\beta\sqrt{4-(sinx+siny)^2}\,dxdy\ge\int_\alpha^\beta\int_\alpha^\beta(cosx+cosy)\,dxdy$$

$$\int_\alpha^\beta\int_\alpha^\beta\sqrt{4-(sinx+siny)^2}\,dxdy\ge 2(\beta-\alpha)(sin\beta-sin\alpha)$$

Equality holds for $\alpha=\beta$.

1940. If $n\in(-1,1]$ then prove:

$$\int_0^\pi\cot\frac{x}{2}\tan^{-1}\left(\frac{n\sin x}{1+n\cos x}\right)dx=\pi\log(1+n)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\tan^{-1}\left(\frac{n\sin x}{1+n\cos x}\right)=\tan^{-1}\left(\frac{y\sin x}{1+y\cos x}\right)\Bigg|_0^n=\int_0^n\frac{\frac{\sin x}{(1+y\cos x)^2}}{1+\frac{y^2\sin^2 x}{(1+y\cos x)^2}}dy=$$

$$=\int_0^n\frac{\sin x}{y^2+2y\cos x+1}dy$$

$$\Omega(n)=\int_0^n\int_0^\pi\frac{\sin x\cot\left(\frac{x}{2}\right)}{y^2+2y\cos x+1}dxdy=\int_0^n\int_0^\pi\frac{1+\cos x}{y^2+2y\cos x+1}dxdy=$$

$$\overset{t=\tan\frac{x}{2}}{=}4\int_0^n\int_0^\infty\frac{1}{(1+t^2)((1+y)^2+(1-y)^2t^2)}dtdy=$$

$$=\int_0^n\frac{1}{y}\int_0^\infty\left(\frac{1}{1+t^2}-\frac{(1-y)^2}{(1+y)^2+(1-y)^2t^2}\right)dtdy=$$

$$=\frac{\pi}{2}\int_0^n\frac{1}{y}\left(1-\frac{1-y}{1+y}\right)dy=\pi\int_0^n\frac{1}{1+y}dy=\pi\log(1+n)$$

Therefore,

$$\int_0^\pi\cot\frac{x}{2}\tan^{-1}\left(\frac{n\sin x}{1+n\cos x}\right)dx=\pi\log(1+n)$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega(n)=\int_0^{\pi}\cot\frac{x}{2}\tan^{-1}\left(\frac{n\sin x}{1+n\cos x}\right)dx\overset{m=\frac{n-1}{n+1}}{=}2\int_0^{\infty}\frac{1}{t(1+t^2)}\tan^{-1}\left(\frac{(1+m)t^2}{1-mt^2}\right)dt=$$

$$=2\int_0^{\infty}\frac{1}{t(1+t^2)[\tan^{-1}(t)+\tan^{-1}(mt)]}dt=$$

$$=2\int_{-1}^{m}\int_0^{\infty}\frac{dtda}{(1+t^2)(1+a^2t^2)}=2\int_{-1}^{m}\frac{1}{1-a^2}\int_0^{\infty}\left(\frac{1}{1+t^2}-\frac{a^2}{1+a^2t^2}\right)dtda=$$

$$=2\int_{-1}^{m}\frac{1}{1-a^2}\left(\frac{\pi}{2}+\frac{a\pi}{2}\right)da=\pi\log\left(\frac{2}{1-m}\right)$$

Therefore,

$$\int_0^{\pi}\cot\frac{x}{2}\tan^{-1}\left(\frac{n\sin x}{1+n\cos x}\right)dx=\pi\log(1+n)$$

1941. Find a closed form:

$$\Omega=\int_0^1\int_0^1\int_0^1\tan^{-1}(xyz)\,dx\,dy\,dz$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by proposer

$$xyz=u$$

$$I=\int_0^1\int_0^1\frac{1}{xy}\int_0^{xy}\tan^{-1}(u)\,du\,dy\,dx$$

$$=\int_0^1\left[\left[\frac{\log(y)}{x}\int_0^{xy}\tan^{-1}(u)\,du\right]_0^1-\int_0^1\log(y)\tan^{-1}(xy)\,dy\right]dx$$

$$=-\int_0^1\int_0^1\log(y)\tan^{-1}(xy)\,dy\,dx=-\int_0^1\log(y)\int_0^1\tan^{-1}(xy)\,dx\,dy$$

$$=-\int_0^1\log(y)\left[\tan^{-1}(y)-\frac{\log(y^2+1)}{2y}\right]dy$$

$$=-\int_0^1\log(y)\tan^{-1}(y)\,dy+\frac{1}{2}\int_0^1\frac{\log(y)\log(y^2+1)}{y}dy=-I_1+\frac{1}{2}I_2$$

$$I_1=\int_0^1 \log(y)\tan^{-1}(y)\,dy=\frac{\pi^2}{48}+\frac{1}{2}\log(2)-\frac{\pi}{4}$$

$$I_2=\int_0^1 \frac{\log(y)\log(y^2+1)}{y}dy=-\frac{3}{16}\zeta(3)$$

$$I=-I_1+\frac{1}{2}I_2=\frac{\pi}{4}-\frac{1}{2}\log(2)-\frac{\pi^2}{48}-\frac{3}{32}\zeta(3)$$

1942. Find:

$$\Omega=\int_0^1 \frac{x\sqrt{x}\log x}{x^2+x+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Ankush Kumar Parcha-India

$$\Omega=\int_0^1 \frac{x\sqrt{x}\log x}{x^2+x+1}dx=\int_0^1 \frac{(1-x)x\sqrt{x}\log x}{(1-x)(x^2+x+1)}dx=$$

$$=\int_0^1 \frac{x\sqrt{x}\log x}{1-x^3}dx-\int_0^1 \frac{x^2\sqrt{x}\log x}{1-x^3}dx=$$

$$=\sum_{n=0}^{\infty}\int_0^1 x^{3n+\frac{3}{2}}\log x\,dx-\sum_{n=0}^{\infty}\int_0^1 x^{3n+\frac{5}{2}}\log x\,dx=$$

$$\left(\because \int_0^1 x^m \log^n x\,dx=\frac{(-1)^n n!}{(m+1)^{n+1}}, n>-1, m\neq -1\right)$$

$$\left(\because \psi^{(m)}(z)=(-1)^{m+1}m!\,\zeta(m+1,z)\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)}{\left(3n+\frac{5}{2}\right)^2}-\sum_{n=0}^{\infty}\frac{(-1)}{\left(3n+\frac{7}{2}\right)^2}=\frac{1}{9}\zeta\left(2;\frac{7}{6}\right)-\frac{1}{9}\zeta\left(2;\frac{5}{6}\right)=\frac{1}{9}\left(\psi'\left(\frac{7}{6}\right)-\psi'\left(\frac{5}{6}\right)\right)$$

Solution 2 by Sakthi Vel-India

$$\Omega=\int_0^1 \frac{x\sqrt{x}\log x}{x^2+x+1}dx=\int_0^1 \frac{(1-x)x\sqrt{x}\log x}{1-x^3}dx=$$

$$=\sum_{n=0}^{\infty}\left(x^{3n}\int_0^1 x\sqrt{x}\log x\,dx-\int_0^1 x^{3n}x^2\sqrt{x}\log x\,dx\right)=$$

$$=\sum_{n=0}^{\infty}\left(\int_0^1 x^{3n}(x\sqrt{x})\log x\,dx-\int_0^1 x^{3n}(x^2\sqrt{x})\log x\,dx\right)\overset{\sqrt{x}=y}{=}$$

$$=\sum_{n=0}^{\infty}\left(\int_0^1 y^{6n}y^3\log y^2\cdot 2y\,dy-\int_0^1 y^{6n}y^5\log y^2\cdot 2y\,dy\right)=$$

$$=\sum_{n=0}^{\infty}\left(4\int_0^1 y^{6n+4}\log y\,dy-4\int_0^1 y^{6n+6}\log y\,dy\right)\overset{(*)}{=}$$

$$(*):\log y=z\Rightarrow ye^{-z}dy=-dz\cdot e^{-z}$$

$$\overset{(*)}{=}4\sum_{n=0}^{\infty}\left(\int_0^{\infty}e^{-z(6n+7)}z\,dz-\int_0^{\infty}e^{-z(6n+6)}\,zdz\right)\overset{(**)}{=}$$

$$(**):z(6n+7)=a,z(6n+5)=b$$

$$\overset{(**)}{=}4\sum_{n=0}^{\infty}\left(\int_0^{\infty}e^{-a}\cdot\frac{a}{6n+7}\frac{da}{6n+7}-\int_0^{\infty}e^{-a}\cdot\frac{a}{6n+5}\frac{da}{6n+5}\right)=$$

$$=\frac{4}{36}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{7}{6}\right)^2}-\frac{4}{36}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{5}{6}\right)^2}=\frac{1}{9}\left(\psi'\left(\frac{7}{6}\right)-\psi'\left(\frac{5}{6}\right)\right)$$

Solution 3 by Togrul Ehmedov-Azerbaijan

$$\textit{Let } x=t^2$$

$$I=\int_0^1\frac{x\sqrt{x}\log(x)}{x^2+x+1}dx=4\int_0^1\frac{t^4\log(t)}{t^4+t^2+1}dt=4\int_0^1\frac{t^4(1-t^2)\log(t)}{(t^4+t^2+1)(1-t^2)}dt$$

$$=4\int_0^1\frac{(t^4-t^6)\log(t)}{1-t^6}dt$$

$$\textit{Let } t^6=y$$

$$I=\frac{1}{9}\int_0^1\frac{\left(y^{\frac{2}{3}}-y\right)\log(y)}{1-y}y^{-\frac{5}{6}}dy=\frac{1}{9}\int_0^1\frac{\left(y^{-\frac{1}{6}}-y^{\frac{1}{6}}\right)\log(y)}{1-y}dy$$

$$=\frac{1}{9}\int_0^1\frac{\left(y^{\frac{5}{6}-1}-y^{\frac{7}{6}-1}\right)\log(y)}{1-y}dy$$

$$\textit{We know that}\int_0^1\frac{t^{x-1}\log(t)}{1-t}dt=-\varphi^{(1)}(x)$$

$$I=\frac{1}{9}\int_0^1\frac{\left(y^{\frac{5}{6}-1}-y^{\frac{7}{6}-1}\right)\log(y)}{1-y}dy=\frac{1}{9}\left\{\varphi^{(1)}\left(\frac{7}{6}\right)-\varphi^{(1)}\left(\frac{5}{6}\right)\right\}$$

Solution 4 by Ose Favour-Nigeria

$$\Omega=\int_0^1\frac{x\sqrt{x}\log x}{x^2+x+1}dx=\int_0^1\frac{x\sqrt{x}(1-x)\log x}{1-x^3}dx=$$

$$=\int_0^1\frac{x^{\frac{3}{2}}\log x}{1-x^3}dx-\int_0^1\frac{x^{\frac{5}{2}}\log x}{1-x^3}dx\overset{x\to x^{\frac{1}{3}}}{=}$$

$$=\frac{1}{9}\left(\int_0^1\frac{x^{\frac{1}{2}+\frac{1}{3}-1}\log x}{1-x}dx-\int_0^1\frac{x^{\frac{5}{6}+\frac{1}{3}-1}\log x}{1-x}dx\right)=$$

$$\because\psi^{(1)}(x)=-\int_0^1\frac{t^{x-1}\log x}{1-t}dx\text{ and }\psi^{(1)}(1+z)=\psi^{(1)}(z)+\frac{1}{z^2}$$

$$\because\psi^{(1)}(1-z)+\psi^{(1)}(z)=\pi^2\csc^2(\pi z)$$

$$=\frac{1}{9}\left(-\psi^{(1)}\left(\frac{5}{6}\right)+\psi^{(1)}\left(\frac{7}{6}\right)\right)=\frac{2}{9}\left(\psi^{(1)}\left(\frac{1}{6}\right)-2\pi^2+18\right)$$

Solution 5 by Daniel Immarube -Nigeria

$$\Omega=\int_0^1\frac{x\sqrt{x}\log x}{x^2+x+1}dx=\int_0^1\frac{x^{\frac{3}{2}}\log x}{1-x^3}dx-\int_0^1\frac{x^{\frac{5}{2}}\log x}{1-x^3}dx=\mathrm{I}_1-I_2$$

$$I_1=\int_0^1\frac{x^{\frac{3}{2}}\log x}{1-x^3}dx=\frac{1}{9}\int_0^1\frac{x^{-\frac{1}{6}}\log x}{1-x}dx=\frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{d}{da}\int_0^1x^{n+\left(a-\frac{1}{6}\right)}dx\right)\Big|_{a=0}=$$

$$=-\frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{5}{6}\right)^2}=-\frac{1}{9}\psi^{(1)}\left(\frac{5}{6}\right)$$

$$\because\psi^{(1)}(x)=-\int_0^1\frac{t^{x-1}\log x}{1-t}dx$$

$$I_2=\int_0^1\frac{x^{\frac{5}{2}}\log x}{1-x^3}dx=\frac{1}{9}\int_0^1\frac{x^{\frac{1}{6}}\log x}{1-x}dx=\frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{d}{da}\int_0^1x^{n+\left(a+\frac{1}{6}\right)}dx\right)\Big|_{a=0}=$$

$$=-\frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{7}{6}\right)^2}=-\frac{1}{9}\psi^{(1)}\left(\frac{7}{6}\right)$$

Therefore,

$$\Omega=\frac{1}{9}\left\{\varphi^{(1)}\left(\frac{7}{6}\right)-\varphi^{(1)}\left(\frac{5}{6}\right)\right\}$$

Solution 6 by Fethi Toubal-Algerie

$$\Omega=\int_0^1\frac{x\sqrt{x}\log x}{x^2+x+1}dx\overset{t=\sqrt{x}}{=}4\int_0^1\frac{t^4\log t}{t^4+t^2+t}dt=4\int_0^1\frac{t^4(1-t^2)\log t}{1-t^6}dt$$

$$=4\int_0^1\frac{t^4\log t}{1-t^6}dt-4\int_0^1\frac{t^6\log t}{1-t^6}dt=$$

$$=4\int_0^1 t^4\log t\sum_{n=1}^{\infty}t^{6(n-1)}\,dt-4\int_0^1 t^6\log t\sum_{n=1}^{\infty}t^{6(n-1)}\,dt=$$

$$=4\sum_{n=1}^{\infty}\int_0^1 t^{6n-2}\log t\,dt-4\sum_{n=1}^{\infty}\int_0^1 t^{6n}\log t\,dt=$$

$$=-4\sum_{n=1}^{\infty}\frac{1}{(6n-1)^2}+4\sum_{n=1}^{\infty}\frac{1}{(6n+1)^2}=-4\sum_{n=0}^{\infty}\frac{1}{(6n+5)^2}+4\sum_{n=0}^{\infty}\frac{1}{(6n+1)^2}=$$

$$=-\frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{5}{6}\right)^2}+\frac{1}{9}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{7}{6}\right)^2}=$$

$$=-\frac{1}{9}\psi^{(1)}\left(\frac{5}{6}\right)+\frac{1}{9}\psi^{(1)}\left(\frac{7}{6}\right)=\frac{1}{9}\left\{\varphi^{(1)}\left(\frac{7}{6}\right)-\varphi^{(1)}\left(\frac{5}{6}\right)\right\}$$

1943. Find:

$$\Omega=\int_0^1\int_0^1\int_0^1\tan^{-1}(xyz)\,dxdydz$$

Proposed by Togrul Ehmedov-Azerbaijan

Solutions 1 by Asmat Qatea-Afghanistan

$$\Omega=\int_0^1\int_0^1\int_0^1\tan^{-1}(xyz)\,dxdydz\overset{\substack{xyz=t\\ dx=\frac{dt}{yz}}}{=}\int_0^1\int_0^1\int_0^{yz}\tan^{-1}t\frac{dt}{yz}dydz=$$

$$=\int_0^1\int_0^1\left[t\tan^{-1}t-\frac{1}{2}\log(1+t^2)\right]_0^{yz}\frac{1}{yz}dydz=$$

$$=\int_0^1\underbrace{\int_0^1\tan^{-1}(yz)\,dydz}_{A}-\frac{1}{2}\int_0^1\int_0^1\underbrace{\frac{\log(1+y^2z^2)}{yz}}_{B}dtdz$$

$$B=\int_0^1\int_0^1\frac{\log(1+y^2z^2)}{yz}dtdz=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1\int_0^1 x^{2n-1}\cdot y^{2n-1}\,dxdy=$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=\frac{1}{4}\eta(3)$$

$$A=\int_0^1\int_0^1\tan^{-1}(yz)\,dydz\overset{yz=t,dy=\frac{dz}{t}}{=}\int_0^1\int_0^z\tan^{-1}t\frac{dt}{z}dz=$$

$$=\int_0^1\left[t\tan^{-1}t-\frac{1}{2}\log(1+t^2)\right]_0^z\frac{dz}{z}=\int_0^1\tan^{-1}z\,dz-\frac{1}{2}\int_0^1\frac{\log(1+z^2)}{z}dz=$$

$$=\left[z\tan^{-1}z-\frac{1}{2}\log(1+z^2)\right]_0^1-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 z^{2n-1}\,dz=$$

$$=\frac{\pi}{4}-\frac{1}{2}\log 2-\frac{1}{4}\cdot\frac{\pi^2}{12}$$

$$\Omega=A-\frac{1}{2}B=\frac{\pi}{4}-\frac{1}{2}\log 2-\frac{1}{4}\cdot\frac{\pi^2}{12}-\frac{1}{8}\eta(3)$$

Solution 2 by Asmat Qatea-Afghanistan

$$\Omega=\int_0^1\int_0^1\int_0^1\tan^{-1}(xyz)\,dxdydz=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1\int_0^1\int_0^1(xyz)^{2n-1}\,dxdydz=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\int_0^1 x^{2n-1}\,dx\right)^3=\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)n^3}=$$

$$=\frac{1}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)n^3}=\frac{1}{8}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{8}{2n-1}-\frac{4}{n}-\frac{2}{n^2}-\frac{1}{n^3}\right)=$$

$$=\frac{\pi}{4}-\frac{1}{2}\log 2-\frac{1}{4}\cdot\frac{\pi^2}{12}-\frac{1}{8}\eta(3)$$

$$\because\ \eta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z}$$

$$\because\ \tan^{-1}z=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}z^{2n-1}$$

Solution 3 by Sakthi Vel-India

$$\Omega = \int_0^1 \int_0^1 \int_0^1 \tan^{-1}(xyz)\, dxdydz = \int_0^1 \int_0^1 \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (xyz)^{2n+1}\, dxdydz =$$

$$= \int_0^1 \int_0^1 \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} y^{2n+1} z^{2n+1}\, dxdydz =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 x^{2n+1}\, dx \int_0^1 y^{2n+1}\, dy \int_0^1 z^{2n+1}\, dz =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{x^{2n+2}}{2n+2}\Bigg|_0^1 \frac{y^{2n+2}}{2n+2}\Bigg|_0^1 \frac{z^{2n+2}}{2n+2}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)^3} =$$

$$= \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n+1)^3} = \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \left[\frac{8}{2n+1} - \frac{4}{n+1} - \frac{2}{(n+1)^2} - \frac{1}{(n+1)^3}\right] =$$

$$= \frac{1}{8} \left[\sum_{n=0}^{\infty} \frac{(-1)^n 8}{2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n 4}{n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n 2}{(n+1)^2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3}\right] =$$

$$= \frac{1}{8} \tan^{-1} 1 - \frac{\eta(1)}{2} - \frac{\eta(2)}{4} - \frac{\eta(3)}{8} = \frac{\pi}{4} - \frac{1}{2} \log 2 - \frac{\pi^2}{48} - \frac{1}{8} \eta(3)$$

1944. Find:

$$\Omega(m) = \int_0^{\infty} \int_0^{\infty} \frac{\log^m(xy)}{(1+x^2)(1+y^2)}\, dx\, dy,\ m \in \mathbb{R}, m > 0$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by proposer

$$\text{Let } xy = z$$

$$\Omega(m) = \int_0^{\infty} \int_0^{\infty} \frac{x \log^m(z)}{(1+x^2)(z^2+x^2)}\, dz\, dx = \int_0^{\infty} \log^m(z) \int_0^{\infty} \frac{x}{(1+x^2)(z^2+x^2)}\, dx\, dz$$

$$= \int_0^{\infty} \frac{\log^{m+1}(z)}{z^2-1}\, dz = \int_0^1 \frac{\log^{m+1}(z)}{z^2-1}\, dz + \int_1^{\infty} \frac{\log^{m+1}(z)}{z^2-1}\, dz$$

$$= \int_0^1 \frac{\log^{m+1}(z)}{z^2-1}\, dz + (-1)^m \int_0^1 \frac{\log^{m+1}(z)}{z^2-1}\, dz$$

$$= (1+(-1)^m) \int_0^1 \frac{\log^{m+1}(z)}{z^2-1}\, dz$$

$$\text{Note: } \int_0^1 \frac{\log^n(z)}{z^2-1}dz = \left(1-\frac{1}{2^{n+1}}\right)(-1)^{n+1}n!\,\zeta(n+1)$$

Then we can write

$$\int_0^1 \frac{\log^{m+1}(z)}{z^2-1}dz = \left(1-\frac{1}{2^{m+2}}\right)(-1)^{m+2}(m+1)!\,\zeta(m+2)$$

$$\Omega(m) = (1+(-1)^m)\left(1-\frac{1}{2^{m+2}}\right)(-1)^{m+2}(m+1)!\,\zeta(m+2)$$

$$= ((-1)^{m+2}+(-1)^{2m+2})\left(1-\frac{1}{2^{m+2}}\right)(m+1)!\,\zeta(m+2)$$

$$= (1+(-1)^m)\left(1-\frac{1}{2^{m+2}}\right)(m+1)!\,\zeta(m+2)$$

1945. Find:

$$\Omega = \int_0^1 \frac{x\log^2(1+x)}{(x^2+1)^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Artan Ajredini-Presheva-Serbie

$$\Omega = \int_0^1 \frac{x\log^2(1+x)}{(x^2+1)^2} \overset{IBP}{=} -\frac{\log^2(1+x)}{2(x^2+1)}\Bigg|_0^1 + \int_0^1 \frac{\log(1+x)}{(1+x)(1+x^2)}dx =$$

$$= -\frac{1}{4}\log^2 2 + \frac{1}{2}\int_0^1 \frac{\log(1+x)}{1+x}dx + \frac{1}{2}\int_0^1 \frac{(1-x)\log(1+x)}{x^2+1}dx =$$

$$= -\frac{1}{4}\log^2 2 + \frac{1}{4}\log^2(1+x)\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{\log(1+x)}{x^2+1}dx - \frac{1}{2}\int_0^1 \frac{x\log(1+x)}{x^2+1}dx =$$

$$= -\frac{1}{4}\log^2 2 + \frac{1}{4}\log^2 2 + \frac{\pi}{16}\log 2 - \frac{\pi^2}{192} - \frac{12}{192}\log^2 2 =$$

$$= \frac{\pi}{16}\log 2 - \frac{\pi^2}{192} - \frac{12}{192}\log^2 2$$

Solution 2 by Hamza Djahel-Algerie

$$\begin{cases} \dfrac{1-x}{1+x} = y \Rightarrow x = \dfrac{1-y}{1+y} \\ dx = \dfrac{2dy}{(1+y)^2} \end{cases}$$

$$\Omega=\int_0^1\frac{x\log^2(1+x)}{1+x^2}dx\overset{(*)}{=}\int_0^1\frac{\frac{1-y}{1+y}}{1+y^2}\log^2\left(\frac{2}{1+y}\right)dy=$$

$$=\int_0^1\left(\frac{1}{1+y}-\frac{y}{1+y^2}\right)[\log 2-\log(1+y)]^2\,dy=$$

$$=\int_0^1\frac{[\log 2-\log(1+y)]^2dy}{1+y}$$

$$-\int_0^1\frac{y}{1+y^2}[\log^2(1+y)-2\log 2\log(1+y)+\log^2 2]\,dy$$

$$=\int_0^1\frac{\log^2(1+y)}{1+y}dy-\Omega+2\log 2\int_0^1\frac{y\log(1+y)}{1+y^2}dy-\log^2 2\int_0^1\frac{y}{1+y^2}dy$$

$$2\Omega=\frac{1}{3}\log^3 2+2\log 2\left(\frac{\pi^2}{96}+\frac{\log^2 2}{8}\right)-\frac{1}{2}\log^3 2$$

$$\Omega=\frac{\log^3 2}{24}+\frac{\pi^2}{96}\log 2$$

Solution 3 by proposer

$$I=\int_0^1\frac{x\log^2(1+x)}{(x^2+1)^2}dx=-\frac{1}{2}\frac{\log^2(1+x)}{1+x^2}\bigg|_0^1+\int_0^1\frac{\log(1+x)}{(1+x)(1+x^2)}dx$$

$$=-\frac{\log^2(2)}{4}+\int_0^1\frac{\log(1+x)}{(1+x)(1+x^2)}dx$$

$$I_1=\int_0^1\frac{\log(1+x)}{(1+x)(1+x^2)}dx=\frac{1}{2}\left[\int_0^1\frac{\log(1+x)}{1+x}dx-\int_0^1\frac{x\log(1+x)}{1+x^2}dx+\int_0^1\frac{\log(1+x)}{1+x^2}dx\right]$$

$$=\frac{1}{2}\left[\left\{\frac{1}{2}\log^2(2)\right\}-\left\{\frac{1}{8}\log^2(2)+\frac{1}{16}\zeta(2)\right\}+\left\{\frac{\pi}{8}\log(2)\right\}\right]$$

$$=\frac{1}{16}\left[3\log^2(2)-\frac{1}{2}\zeta(2)+\pi\log(2)\right]$$

$$I=-\frac{\log^2(2)}{4}+\frac{1}{16}\left[3\log^2(2)-\frac{1}{2}\zeta(2)+\pi\log(2)\right]$$

$$=\frac{1}{16}\left[-\log^2(2)-\frac{1}{2}\zeta(2)+\pi\log(2)\right]$$

Note:

$$\int_0^1 \frac{\log(1+x)}{1+x}dx = \frac{1}{2}\log^2(2)$$

$$\int_0^1 \frac{x\log(1+x)}{1+x^2}dx = \frac{1}{8}\log^2(2) + \frac{1}{16}\zeta(2)$$

$$\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log(2)$$

1946. Find:

$$\Omega = \int_0^1 \int_0^1 \log(x^5 + x^2y^3 + y^2x^3 + y^5)\,dxdy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Ankush Kumar Parcha-India

$$\Omega = \int_0^1 \int_0^1 \log(x^5 + x^2y^3 + y^2x^3 + y^5)\,dxdy = \int_0^1 \int_0^1 \log[(x^2+y^2)(x^3+y^3)]\,dxdy =$$

$$= \int_0^1 \int_0^1 [\log(x^2+y^2) + \log(x^3+y^3)]dxdy =$$

$$= \int_0^1 [x\log(x^2+y^2)]_0^1 dy - \int_0^1 \int_0^1 \frac{2x^2}{x^2+y^2}dxdy + \int_0^1 [x\log(x^3+y^3)]_0^1 dy -$$

$$- \int_0^1 \int_0^1 \frac{3x^3}{x^3+y^3}dxdy =$$

$$= \int_0^1 \log(1+y^2)\,dy - 2\int_0^1 \int_0^1 \frac{x^2}{x^2+y^2}dxdy + \int_0^1 \log(1+y^3)\,dy$$

$$- 3\int_0 \int_0^1 \frac{x^3}{x^3+y^3}dxdy\,;(1)$$

$$\Omega = \int_0^1 \log(1+y^2)\,dy - 2\int_0^1 \int_0^1 \frac{y^2}{x^2+y^2}dxdy + \int_0^1 \log(1+y^3)\,dy$$

$$- 3\int_0^1 \int_0^1 \frac{y^3}{x^3+y^3}dxdy\,;\ \ (2)$$

By adding (1) and (2), we get:

$$\Omega=\int_0^1 \log(1+y^2)\,dy-1+\int_0^1 \log(1+y^3)\,dy-\frac{3}{2}=$$

$$=\left[y\log(1+y^2)\right]_0^1-2\int_0^1 \frac{y^2+1-1}{1+y^2}dy-1+\left[y\log(1+y^3)\right]_0^1-2\int_0^1 \frac{y^3+1-1}{1+y^3}dy-\frac{3}{2}$$

$$\Omega=\log 2+\frac{\pi}{2}-3-\frac{9}{2}+2\log 2-\frac{1}{2}\int_0^1 \frac{2y-1}{y^2-y+1}dy+\frac{3}{2}\int_0^1 \frac{dy}{y^2-y+1}$$

$$\because \int \frac{dx}{ax^2+bx+c}=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+C$$

Therefore,

$$\Omega=\int_0^1\int_0^1 \log\left(x^5+x^2y^3+y^2x^3+y^5\right)dxdy=\frac{\pi\left(3+2\sqrt{2}\right)+6\log 8-45}{6}$$

1947. Find:

$$\Omega=\int_0^1 x\log(x^2+1)\tan^{-1}\left(\frac{1}{x}\right)dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1 x\log(x^2+1)\tan^{-1}\left(\frac{1}{x}\right)dx \overset{IBP}{=}$$

$$==\frac{1}{2}\left[\left((1+x^2)\log(1+x^2)-(1+x^2)\right)\tan^{-1}\left(\frac{1}{x}\right)\right]_0^1+\frac{1}{2}\int_0^1 (\log(1+x^2)-1)dx=$$

$$=\frac{\pi}{4}\log 2-\frac{\pi}{8}+\frac{1}{2}\int_0^1 \log(1+x^2)\,dx-\frac{1}{2}\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$$

$$\int_0^1 \log(1+x^2)\,dx=\left[x\log(1+x^2)\right]_0^1-2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx=\log 2+\frac{\pi}{2}-2$$

Therefore,

$$\Omega=\int_0^1 x\log(x^2+1)\tan^{-1}\left(\frac{1}{x}\right)dx=\frac{\pi}{4}\log 2+\frac{1}{2}\log 2+\frac{\pi}{4}-\frac{3}{2}$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega=\int_0^1 x\log(x^2+1)\tan^{-1}\left(\frac{1}{x}\right)dx$$

$$=\frac{\pi}{2}\int_0^1 x\log(1+x^2)\,dx-\int_0^1 x\log(1+x^2)\tan^{-1}x\,dx$$

$$=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{2n+1}\,dx-\left[\frac{\tan^{-1}x}{2}\Big((x^2+1)\log(1+x^2)-x^2\Big)\right]_0^1+$$

$$+\frac{1}{2}\int_0^1\frac{1}{1+x^2}[(x^2+1)\log(1+x^2)-x^2]dx=$$

$$=\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}-\frac{\pi}{8}(2\log 2-1)+\frac{1}{2}\int_0^1\left(\log(1+x^2)-\frac{x^2}{1+x^2}\right)dx=$$

$$=\frac{\pi}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{\pi\log 2}{4}+\frac{\pi}{8}+$$

$$+\frac{1}{2}\left[x\log(1+x^2)\Big|_0^1-2\int_0^1\frac{x^2}{1+x^2}dx-\int_0^1\frac{x^2}{1+x^2}dx\right]=$$

$$=\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n+1}-\frac{\pi\log 2}{4}+\frac{\pi}{8}+\frac{1}{2}\left[\log 2-3\int_0^1\frac{1+x^2-1}{1+x^2}dx\right]=$$

$$=\frac{\pi\log 2}{4}-\frac{\pi}{4}+\frac{\pi}{8}+\frac{\log 2}{2}-\frac{3}{2}+\frac{3\pi}{8}$$

Therefore,

$$\Omega=\int_0^1 x\log(x^2+1)\tan^{-1}\left(\frac{1}{x}\right)dx=\frac{\pi\log(2e)+\log 4-6}{4}$$

Solution 3 by Ose Favour-Nigeria

$$\Omega=\int_0^1 x\log(x^2+1)\tan^{-1}\left(\frac{1}{x}\right)dx\overset{IBP}{=}$$

$$=\frac{1}{2}\left[\Big((1+x^2)\log(1+x^2)-(1+x^2)\Big)\tan^{-1}\left(\frac{1}{x}\right)\right]_0^1+\frac{1}{2}\int_0^1(\log(1+x^2)-1)dx=$$

$$=\frac{\pi}{4}(\log 2+1)+\frac{1}{2}\int_0^1\log(1+x^2)\,dx-\frac{1}{2}\int_0^1 dx=$$

$$=\frac{\pi}{4}(\log 2+1)+\frac{1}{2}x\log(1+x^2)\Big|_0^1-\int_0^1\frac{x^2}{1+x^2}dx-\frac{1}{2}==\frac{\pi}{4}(\log 2+1)+\frac{1}{2}\log 2+\frac{\pi}{4}-\frac{3}{2}$$

1948. Prove that:

$$\int_0^{\frac{1}{2}}\frac{\tan^{-1}\left(\sqrt{1-2x^2}\right)}{1+2x^2}dx=\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^2+\frac{\pi}{4\sqrt{2}}\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

Proposed by Hamza Djahel-Algerie

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{1}{2}}\frac{\tan^{-1}\left(\sqrt{1-2x^2}\right)}{1+2x^2}dx=\frac{1}{\sqrt{2}}\int_0^{\frac{1}{\sqrt{2}}}\frac{\tan^{-1}\left(\sqrt{1-x^2}\right)}{1+x^2}dx$$

$$\overset{x=\sin t}{=}\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{\cos t\tan^{-1}(\cos t)}{1+\sin^2 t}dt=\frac{1}{\sqrt{2}}\int_0^1\int_0^{\frac{\pi}{4}}\frac{\cos^2 t}{(1+\sin^2 t)(1+y^2\cos^2 t)}dtdy$$

$$\overset{x=\tan t}{=}\frac{1}{\sqrt{2}}\int_0^1\int_0^1\frac{dxdy}{(1+2x^2)(1+x^2+y^2)}$$

$$=\frac{1}{2\sqrt{2}}\int_0^1\int_0^1\frac{dxdy}{(1+2x^2)(1+x^2+y^2)}+\frac{1}{2\sqrt{2}}\int_0^1\int_0^1\frac{dxdy}{(1+2y^2)(1+x^2+y^2)}$$

$$\int_0^1\int_0^1\frac{dxdy}{(1+2y^2)(1+x^2+y^2)}=$$

$$=2\int_0^1\int_0^1\frac{dxdy}{(2x^2+1)(2y^2+1)}-\int_0^1\int_0^1\frac{dxdy}{(1+2x^2)(1+x^2+y^2)}$$

$$\Omega=\frac{1}{\sqrt{2}}\int_0^1\int_0^1\frac{dxdy}{(2x^2+1)(2y^2+1)}=$$

$$=\frac{1}{\sqrt{2}}\int_0^1\frac{dx}{2x^2+1}\int_0^1\frac{dy}{2y^2+1}=\frac{1}{2\sqrt{2}}\left(\tan^{-1}(\sqrt{2})\right)^2$$

$$\Omega=\frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^2=$$

$$=\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^2+\frac{\pi}{4\sqrt{2}}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$=\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)+\frac{\pi}{4\sqrt{2}}\left(\tan^{-1}(\sqrt{2})-\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$=\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^2+\frac{\pi}{4\sqrt{2}}\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

Therefore,

$$\int_0^{\frac{1}{2}} \frac{\tan^{-1}(\sqrt{1-2x^2})}{1+2x^2}dx = \frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)^2 + \frac{\pi}{4\sqrt{2}}\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

1949. If $0 \le a \le b$ then :

$$\int_a^b \int_a^b \frac{1}{x^2+y^2+4xy}dxdy \ge \frac{(a-b)^2}{2(a^2+ab+b^2)}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\int_a^b \int_a^b \frac{1}{x^2+y^2+4xy}dxdy = \int_a^b \int_a^b \frac{2}{3(x+y)^2-(x-y)^2}dxdy \ge \int_a^b \int_a^b \frac{2}{3(x+y)^2}dxdy =$$

$$= \frac{2}{3}\int_a^b \left[-\frac{1}{x+y}\right]_a^b dy = \frac{2}{3}\int_a^b \left(\frac{1}{a+y} - \frac{1}{b+y}\right)dy = \frac{8(b-a)}{3}\int_a^b \frac{1}{4(a+y)(b+y)}dy \ge$$

$$\overset{AM-GM}{\ge} \frac{8(b-a)}{3}\int_a^b \frac{1}{[(a+y)+(b+y)]^2}dy = \frac{8(b-a)}{3}\left[-\frac{1}{2(2y+a+b)}\right]_a^b =$$

$$= \frac{4(b-a)}{3}\left(\frac{1}{3a+b} - \frac{1}{a+3b}\right) = \frac{8(b-a)^2}{3(3a+b)(a+3b)} = \frac{8(b-a)^2}{16(a^2+ab+b^2)-7(b-a)^2}$$
$$\ge \frac{8(b-a)^2}{16(a^2+ab+b^2)}$$

$$\textit{Therefore,} \int_a^b \int_a^b \frac{1}{x^2+y^2+4xy}dxdy \ge \frac{(a-b)^2}{2(a^2+ab+b^2)}.$$

1950. If $0 < a \le b$ then:

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx \le \tan^{-1}\left(\frac{a+b}{2}\right) - \tan^{-1}(\sqrt{ab})$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

Let $f(x) = e^{-x^2(1+x^2)} - 1$, then

$f'(x) = -2x^3e^{-x^2} \le 0 \Rightarrow f$ –decreasing, then $f(x) < f(0) \Rightarrow$

$$e^{-x^2}(1+x^2)-1\le 0\Rightarrow e^{-x^2}(1+x^2)\le 1\Rightarrow e^{-x^2}\le\frac{1}{x^2+1}$$

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx\le\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{1}{1+x^2}dx=\tan^{-1}x\Big|_{\sqrt{ab}}^{\frac{a+b}{2}}=\tan^{-1}\left(\frac{a+b}{2}\right)-\tan^{-1}(\sqrt{ab})$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$e^{x^2}\ge 1+x^2,\forall x\in\mathbb{R}\Rightarrow e^{-x^2}\le\frac{1}{x^2+1}$$

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx\le\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{1}{1+x^2}dx=\tan^{-1}x\Big|_{\sqrt{ab}}^{\frac{a+b}{2}}=\tan^{-1}\left(\frac{a+b}{2}\right)-\tan^{-1}(\sqrt{ab})$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$e^{x^2}\ge 1+x^2,\forall x\in\mathbb{R}\Rightarrow e^{-x^2}\le\frac{1}{x^2+1}$$

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx\le\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{1}{1+x^2}dx=\tan^{-1}x\Big|_{\sqrt{ab}}^{\frac{a+b}{2}}=\tan^{-1}\left(\frac{a+b}{2}\right)-\tan^{-1}(\sqrt{ab})$$

Solution 4 by Christos Tsifakis-Greece

$$e^{t}\ge 1+t,\forall t\in\mathbb{R}\Rightarrow e^{x^2}\ge 1+x^2,\forall x\in\mathbb{R}\Rightarrow e^{-x^2}\le\frac{1}{x^2+1}$$

$$\int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx\le\int_{\sqrt{ab}}^{\frac{a+b}{2}}\frac{1}{1+x^2}dx=\tan^{-1}x\Big|_{\sqrt{ab}}^{\frac{a+b}{2}}=\tan^{-1}\left(\frac{a+b}{2}\right)-\tan^{-1}(\sqrt{ab})$$

1951. If $a,b>0$ then:

$$\frac{4}{a+b}\le\int_0^1\frac{dx}{ax+(1-x)b}+\int_0^\infty\frac{dx}{(x+a)(x+b)}\le\frac{2}{\sqrt{ab}}$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

Let $b>a$:

$$\int_0^1\frac{dx}{xa+(1-x)b}=\int_0^1\frac{dx}{x(a-b)+b}=\frac{1}{a-b}\log[(x(a-b)+b]\Big|_0^1=$$

$$=\frac{1}{a-b}(\log b-\log a)$$

$$\int_0^\infty \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\int_0^\infty \left(\frac{1}{x+b} - \frac{1}{x+a}\right)dx = \frac{1}{a-b}\log\left(\frac{x+b}{x+a}\right)\Big|_0^\infty =$$

$$= \frac{1}{a-b}\left(\log 1 - \log\left(\frac{b}{a}\right)\right) = \frac{\log a - \log b}{a-b}$$

Thus, we must to show that:

$$\frac{4}{a+b} \le 2\cdot\frac{\log a - \log b}{a-b} \le \frac{2}{\sqrt{ab}} \Leftrightarrow$$

$$\frac{2}{a+b} \le \frac{\log a - \log b}{a-b} \le \frac{1}{\sqrt{ab}}; (1)$$

For $t \ge 0$, we have:

$$t^2 + 2t\sqrt{ab} + ab \le t^2 + (a+b)t + ab \le t^2 + (a+b)t + ab + \left(\frac{a-b}{2}\right)^2 =$$

$$= t^2 + (a+b)t + \left(\frac{a+b}{2}\right)^2 \le \left(t + \frac{a+b}{2}\right)^2$$

Hence,

$$\frac{1}{\left(t+\frac{a+b}{2}\right)^2} \le \frac{1}{(t+a)(t+b)} \le \frac{1}{(t+\sqrt{ab})^2}$$

$$\int_0^\infty \frac{dt}{\left(t+\frac{a+b}{2}\right)^2} \le \int_0^\infty \frac{dt}{(t+a)(t+b)} \le \int_0^\infty \frac{dt}{(t+\sqrt{ab})^2}$$

$$-\frac{1}{t+\frac{a+b}{2}}\Bigg|_0^\infty \le \frac{\log a - \log b}{a-b} \le -\frac{1}{t+\sqrt{ab}}\Bigg|_0^\infty$$

$$\frac{2}{a+b} \le \frac{\log a - \log b}{a-b} \le \frac{1}{\sqrt{ab}}$$

For $a = b$, all the three expressions are equal to $\frac{2}{a}$.

1952. If $a, b > 0$ then prove that :

$$\int_a^b\int_a^b \frac{1}{x^3+y^3}dxdy \ge \frac{2(a-b)^2}{a^3+a^2b+ab^2+b^3}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM\ inequality\ we\ have:$$

$$\frac{1}{x^3+y^3}+\frac{4(x^3+y^3)}{(a^3+a^2b+ab^2+b^3)^2}\ge 2\sqrt{\frac{1}{x^3+y^3}\cdot\frac{4(x^3+y^3)}{(a^3+a^2b+ab^2+b^3)^2}}$$
$$=\frac{4}{a^3+a^2b+ab^2+b^3}.$$

$$Then:$$

$$\frac{1}{x^3+y^3}\ge\frac{4}{a^3+a^2b+ab^2+b^3}-\frac{4(x^3+y^3)}{(a^3+a^2b+ab^2+b^3)^2}.$$

$$\int_a^b\int_a^b\frac{1}{x^3+y^3}dxdy\ge\int_a^b\int_a^b\left(\frac{4}{a^3+a^2b+ab^2+b^3}-\frac{4(x^3+y^3)}{(a^3+a^2b+ab^2+b^3)^2}\right)dxdy=$$

$$=\frac{4(b-a)^2}{a^3+a^2b+ab^2+b^3}-\frac{2(b-a)(b^4-a^4)}{(a^3+a^2b+ab^2+b^3)^2}$$
$$=\frac{4(b-a)^2}{a^3+a^2b+ab^2+b^3}-\frac{2(b-a)^2}{a^3+a^2b+ab^2+b^3}$$

$$\int_a^b\int_a^b\frac{1}{x^3+y^3}dxdy\ge\frac{2(a-b)^2}{a^3+a^2b+ab^2+b^3}.$$

1953. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{(n^2-2)(n+1)!}\left(\sum_{k=2}^{n}(k^3-1)k!\right)\right)^{(n^2-2)(n+1)!}$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$(k^3-1)k!=k^3k!-k!=k^2\cdot k\cdot k!-k!=k^2\big((k+1)!-k!\big)-k!=$$
$$=k^2(k+1)!-k^2\cdot k!-k!=(k+1-1)^2(k+1)!-k^2\cdot k!-k!=$$
$$=(k+1)^2(k+1)!-2(k+1)(k+1)!+(k+1)!-k^2\cdot k!-k!=$$
$$=(k+1)^2(k+1)!-k^2\cdot k!+(k+1)!-k!-2\big((k+2)!-(k+1)!\big)$$

$$S=\sum_{k=2}^{n}(k^3-1)k!=\sum_{k=2}^{n}\left((k+1)^2(k+1)!-k^2\cdot k!\right)+\sum_{k=2}^{n}\left((k+1)!-k!\right)-$$

$$-2\sum_{k=2}^{n}\left((k+2)!-(k+1)!\right)=(n+1)^2(n+1)!-8+(n+1)!-2-2\left((n+2)!-6\right)$$

$$=(n+1)!\,(n^2+2n+1+1-2n-4)+2=(n+1)!\,(n^2-2)+2$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{(n^2-2)(n+1)!}\left(\sum_{k=2}^{n}(k^3-1)k!\right)\right)^{(n^2-2)(n+1)!}=$$

$$=\lim_{n\to\infty}\left(1+\frac{2}{(n^2-2)(n+1)!}\right)^{(n^2-2)(n+1)!}=e^2$$

1954. Prove that:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\beta(n+2,1)\beta(1,m+2)H_{\left[\frac{n+2}{2}\right]}H_{\left[\frac{m+2}{2}\right]}=\log^4 2$$

where $\beta(m,n)$ –is the Euler integral of the first kind, H_n is the n^{th} harmonic number and $[\cdot]$ is the floor function.

Proposed by Ankush Kumar Parcha-India

Solution by Syed Shahabudeen-India

$$\Omega=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\beta(n+2,1)\beta(1,m+2)H_{\left[\frac{n+2}{2}\right]}H_{\left[\frac{m+2}{2}\right]}=$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\frac{H_{\left[\frac{n+2}{2}\right]}H_{\left[\frac{m+2}{2}\right]}}{(n+2)(m+2)}=\sum_{m=1}^{\infty}\frac{(-1)^m H_{\left[\frac{m+1}{2}\right]}}{m+1}\sum_{n=1}^{\infty}\frac{(-1)^n H_{\left[\frac{n+1}{2}\right]}}{n+1}=$$

$$=\left(\sum_{k=1}^{\infty}\frac{(-1)^k H_{\left[\frac{k+1}{2}\right]}}{k+1}\right)^2,\text{ where}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k H_{\left[\frac{k+1}{2}\right]}}{k+1}=\frac{(-1)}{2}H_1+\frac{1}{3}H_1+\frac{(-1)}{4}H_2+\frac{1}{5}H_2-\cdots+\cdots=$$

$$=\sum_{k=1}^{\infty}\left(\frac{1}{2k+1}-\frac{1}{2k}\right)H_k=-\sum_{k=1}^{\infty}\frac{H_k}{(2k+1)(2k)}$$

$$\sum_{k=1}^{\infty}\frac{H_k}{(2k+1)(2k)}=\frac{1}{2}\int_0^1\sum_{k=1}^{\infty}\frac{H_k}{k}x^{2k}dx=\frac{1}{2}\int_0^1\left(Li_2(x^2)+\frac{1}{2}\log^2(1-x^2)\right)$$

$$\int_0^1 Li_2(x^2)dx=\sum_{k=1}^{\infty}\frac{1}{k^2(2k+1)}=\sum_{k=1}^{\infty}\left(\frac{1}{k^2}-\frac{2}{k(2k+1)}\right)=\zeta(2)+2\log 4-4$$

$$\int_0^1\log^2(1-x^2)\,dx=\frac{1}{2}\frac{d^2}{da^2}\int_0^1\frac{(1-t)^a}{\sqrt{t}}dt=\frac{1}{2}\frac{d^2}{da^2}\left(\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(a+1)}{\Gamma\left(a+\frac{3}{2}\right)}\right),$$

$$\text{when } a=0,\int_0^1\log^2(1-x^2)\,dx=\log^2 4-4\log 4+8-2\zeta(2)$$

$$\sum_{k=1}^{\infty}\frac{H_k}{(2k+2)2k}=\frac{1}{2}\left(\zeta(2)+2\log 4-4+\frac{\log^2 4}{2}-2\log 4+4-\zeta(2)\right)=$$

$$=\frac{\log^2(4)}{4}$$

$$\Omega=\left(\frac{\log^2(4)}{4}\right)^2=\frac{\log^4(2^2)}{16}=\log^2(2)$$

1955. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}\left|\left(\frac{\sin x}{x}\right)^{(k)}\right|,x\in\left(0,\frac{\pi}{2}\right),\qquad (*)^{(k)}-k^{th}\text{derivative.}$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right)=\frac{1}{x^{k+1}}\int_0^x t^k\cos\left(t+\frac{k\pi}{2}\right)dt\overset{t=ux}{\Longrightarrow}$$

$$\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right)=\frac{1}{x^{k+1}}\int_0^1 u^k x^k\cos\left(ux+\frac{k\pi}{2}\right)x\,du$$

$$=\frac{1}{x^{k+1}}\int_0^1 u^k x^{k+1}\cos\left(ux+\frac{k\pi}{2}\right)du=\int_0^1 u^k\cdot\cos\left(ux+\frac{k\pi}{2}\right)du$$

$$\because -1\le\cos z\le 1,\forall z\in\mathbb{R}\text{ and }\int_a^b f(x)dx\le\int_a^b g(x)dx,x\in[a,b]$$

$$-\int_0^1 u^k\,du\le\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\le\int_0^1 u^k\,du,0\le u\le 1$$

$$\left|\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\right|\le\int_0^1 u^k\,du$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right)\right|=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{1}{x^{k+1}}\int_0^x t^k\cos\left(t+\frac{k\pi}{2}\right)dt\right|$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\right|$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\int_0^1 u^k\cos\left(ux+\frac{k\pi}{2}\right)du\right|\le\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1 u^k\,du=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k+1}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k+1}\overset{C-S}{=}\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}\frac{1}{k+1}-\sum_{k=1}^{n}\frac{1}{k+1}}{(n+1)-n}=\lim_{n\to\infty}\frac{1}{n+2}=0$$

$$0\le\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d^k}{dx^k}\left(\frac{\sin x}{x}\right)\right|\le\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k+1}$$

$$\text{Therefore,}\qquad\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}\left|\left(\frac{\sin x}{x}\right)^{(k)}\right|=0$$

1956. Prove that:

$$\int_0^\infty\frac{\log(1+y)}{y(1+y^2)}dy=\frac{5}{8}\zeta(2)$$

where $\zeta(s),\Re(s)>1$ is the Euler-Riemann zeta function.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Daniel Immarube-Nigeria

$$\Omega=\int_0^\infty\frac{\log(1+y)}{y(1+y^2)}dy=\int_0^1\frac{\log(1+y)}{y(1+y^2)}dy+\int_1^\infty\frac{\log(1+y)}{y(1+y^2)}dy=I_1+I_2$$

$$I_1=\int_0^1\frac{\log(1+y)}{y(1+y^2)}dy=\int_0^1\frac{\log(1+y)}{y}dy-\int_0^1\frac{y\log(1+y)}{1+y^2}dy=$$

$$=-\int_0^1\frac{\log y}{1+y}dy-\int_0^1\frac{y\log(1+y)}{1+y^2}dy=\frac{1}{2}\Gamma(2)\zeta(2)-I_3=\frac{\zeta(2)}{2}-I_3$$

$$I_3=\int_0^1\frac{y\log(1+y)}{1+y^2}dy=$$

$$=\frac{1}{2}\left(-\int_0^1\frac{x}{1+x^2}\log\left(\frac{1-x}{1+x}\right)dx+\int_0^1\frac{x}{1+x^2}\log(1-x^2)\,dx\right)=\frac{1}{2}(B-A)$$

$$A=\int_0^1\frac{x}{1+x^2}\log\left(\frac{1-x}{1+x}\right)dx=$$

$$=-\left(\int_0^1\frac{\log x}{1+x}dx+\int_0^1\frac{x\log x}{1+x^2}dx\right)=-\frac{3}{4}\int_0^1\frac{\log x}{1+x}dx=\frac{3}{8}\zeta(2)$$

$$B=\int_0^1\frac{x}{1+x^2}\log(1-x^2)\,dx\overset{x\to x^2}{=}\frac{1}{2}\int_0^1\frac{\log x}{2-x}dx=$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{t^{n-1}}{2^n}\right)\int_0^1 t^{n-1}\log^a t\,dt=\frac{1}{4}\log^2 2-\frac{\pi^2}{24}=\frac{1}{4}\log^2 2-\frac{\pi^2}{24}$$

$$I_3=\frac{1}{2}(A+B)=\frac{1}{2}\left(\frac{3}{8}\cdot\frac{\pi^2}{6}-\frac{\pi^2}{24}+\frac{1}{4}\log^2 2\right)=\frac{1}{2}\left(\frac{\pi^2}{48}+\frac{1}{4}\log^2 2\right)=\frac{\pi^2}{96}+\frac{1}{8}\log^2 2$$

$$I_1=\frac{\zeta(2)}{2}-I_3=\frac{\pi^2}{12}-\frac{\pi^2}{96}-\frac{1}{8}\log^2 2$$

$$I_2=\int_1^\infty\frac{\log(1+y)}{y(1+y^2)}dy\overset{y\to\frac{1}{y}}{=}\int_0^1\frac{y\log\left(1+\frac{1}{y}\right)}{y^2+1}dy=$$

$$=\int_0^1\frac{y\log\left(\frac{y+1}{y}\right)}{y(1+y^2)}dy=\int_0^1\frac{y\log(y+1)}{y^2+1}dy-\int_0^1\frac{y\log y}{y^2+1}dy=$$

$$=I_3-\frac{1}{4}\int_0^1\frac{\log y}{y+1}dy=I_3+\frac{1}{8}\zeta(2)=\frac{\pi^2}{96}+\frac{1}{8}\log^2 2$$

$$\Omega=I_1+I_2=\frac{5}{48}\pi^2=\frac{5}{8}\zeta(2)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\mathrm{I}=\int_0^\infty\frac{\log(1+x)}{x(x^2+1)}dx=\int_0^\infty\log(1+x)\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$$

$$=\int_0^\infty\frac{\log(1+x)}{x}dx-\int_0^\infty\frac{x\log(1+x)}{x^2+1}dx$$

$$\mathrm{I}_1=\int_0^\infty\frac{\log(1+x)}{x}dx=\int_0^1\frac{\log(1+x)}{x}dx+\int_1^\infty\frac{\log(1+x)}{x}dx$$

$$=\int_0^1\frac{\log(1+x)}{x}dx+\int_0^1\frac{\log(1+x)-\log(x)}{x}dx$$

$$=2\int_0^1\frac{\log(1+x)}{x}dx-\int_0^1\frac{\log(x)}{x}dx$$

$$I_2=\int_0^{\infty}\frac{x\log(1+x)}{x^2+1}dx=\int_0^1\frac{x\log(1+x)}{x^2+1}dx+\int_1^{\infty}\frac{x\log(1+x)}{x^2+1}dx$$

$$=\int_0^1\frac{x\log(1+x)}{x^2+1}dx+\int_0^1\frac{\log(1+x)-\log(x)}{x(x^2+1)}dx$$

$$=\int_0^1\frac{\log(1+x)}{x}dx-\int_0^1\frac{\log(x)}{x(x^2+1)}dx$$

$$=\int_0^1\frac{\log(1+x)}{x}dx-\int_0^1\frac{\log(x)}{x}dx+\int_0^1\frac{x\log(x)}{x^2+1}dx$$

$$I=I_1-I_2=\left\{2\int_0^1\frac{\log(1+x)}{x}dx-\int_0^1\frac{\log(x)}{x}dx\right\}$$

$$-\left\{\int_0^1\frac{\log(1+x)}{x}dx-\int_0^1\frac{\log(x)}{x}dx+\int_0^1\frac{x\log(x)}{x^2+1}dx\right\}$$

$$=\int_0^1\frac{\log(1+x)}{x}dx-\int_0^1\frac{x\log(x)}{x^2+1}dx=\frac{\zeta(2)}{2}-\left(-\frac{\zeta(2)}{8}\right)=\frac{5}{8}\zeta(2)$$

Solution 3 by Kartick Chandra Betal-India

$$\Omega=\int_0^{\infty}\frac{\log(1+y)}{y(1+y^2)}dy=\int_0^1\frac{\log(1+y)}{y(1+y^2)}dy+\int_1^{\infty}\frac{\log(1+y)}{y(1+y^2)}dy=$$

$$=\int_0^1\frac{\log(1+y)}{y(1+y^2)}dy+\int_0^1\frac{y[\log(1+y)-\log y]}{1+y^2}dy=$$

$$=\int_0^1\frac{(1+y^2)\log(1+y)}{y(1+y^2)}dy-\int_0^1\frac{y\log y}{1+y^2}dy=\int_0^1\frac{\log(1+y)}{y}dy-\frac{1}{4}\int_0^1\frac{\log y}{1+y}dy=$$

$$=\int_0^1\frac{\log(1+y)}{y}dy-\frac{1}{4}[\log y\log(1+y)]_0^1+\frac{1}{4}\int_0^1\frac{\log(1+y)}{y}dy=$$

$$=\frac{5}{4}\int_0^1\frac{\log(1+y)}{y}dy=\frac{5}{4}\eta(2)+\frac{5}{8}\zeta(2)$$

1957. For $n>0$ prove that:

$$\int_0^{\frac{\pi}{2}}\frac{\log(\sin^n x+\cos^n x)}{\sin x\cos x}dx=\frac{\pi^2}{24}\left(\frac{4}{n}-n\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\log(\sin^n x+\cos^n x)}{\sin x\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{\log(\tan^n x+1)+\log(\cos^n x)}{\tan x\cos^2 x}dx=$$

$$=\int_0^{\frac{\pi}{2}}\frac{\log(\tan^n x+1)-\frac{n}{2}\log\left(\frac{1}{\cos^2 x}\right)}{\tan x\cos^2 x}dx\overset{t=\tan x}{=}\int_0^{\infty}\frac{\log(t^n+1)-\frac{n}{2}\log(t^2+1)}{t}dt=$$

$$=\int_0^1\frac{\log(t^n+1)-\frac{n}{2}\log(t^2+1)}{t}dt+\int_1^{\infty}\frac{\log(t^n+1)-\frac{n}{2}\log(t^2+1)}{t}dt$$

$$\int_1^{\infty}\frac{\log(t^n+1)-\frac{n}{2}\log(t^2+1)}{t}dt\overset{t=\frac{1}{u}}{=}\int_0^1\frac{\log\left(\frac{u^n+1}{u^n}\right)-\frac{n}{2}\log\left(\frac{u^2+1}{u^2}\right)}{\frac{1}{u}\cdot u^2}du=$$

$$=\int_0^1\frac{\log\left(\frac{u^n+1}{u^n}\right)-\frac{n}{2}\log\left(\frac{u^2+1}{u^2}\right)}{u}du=$$

$$=\int_0^1\frac{\log(t^n+1)-\log t^n-\frac{n}{2}\log(t^2+1)+\frac{n}{2}\log t^2}{t}dt=$$

$$=\int_0^1\frac{\log(t^n+1)-\frac{n}{2}\log(t^2+1)}{t}dt$$

$$\Omega=\int_0^1\frac{2\log(t^n+1)-n\log(t^2+1)}{t}dt=2\int_0^1\frac{\log(t^n+1)}{t}dt-n\int_0^1\frac{\log(t^2+1)}{t}dt=$$

$$=2J-nK.$$

$$J=\int_0^1\frac{\log(t^n+1)}{t}dt\overset{t^n=v}{=}\int_0^1\frac{\log(v+1)}{v^{\frac{1}{n}}}\cdot\frac{1}{n}v^{\frac{1}{n}-1}dv=\frac{1}{n}\int_0^1\frac{\log(v+1)}{v}dv$$

$$K=\int_0^1\frac{\log(t^2+1)}{t}dt\overset{v=t^2}{=}\int_0^1\frac{\log(v+1)}{\sqrt{v}}\cdot\frac{1}{2\sqrt{v}}dv=\frac{1}{2}\int_0^1\frac{\log(v+1)}{v}dv$$

$$I=2J-nK=\left(\frac{2}{n}-\frac{n}{2}\right)\int_0^1\frac{\log(v+1)}{v}dv=\frac{\pi^2}{24}\left(\frac{4}{n}-n\right)$$

1958. Prove that:

$$\int_0^{\infty}\int_0^{\frac{\pi}{2}}\frac{x}{y}\frac{\sin y}{\sin x}dxdy=\pi\beta(2)$$

where $\beta(s)$ is the Dirichlet's beta function.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{x}{y}\frac{\sin y}{\sin x} dxdy = \underbrace{\int_0^\infty \frac{\sin y}{y} dy}_{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx \overset{IBP}{=} =$$

$$= \frac{\pi}{2}\left[x\log\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \log\left(tan\left(\frac{x}{2}\right)\right) dx$$

$$\because \log\left(tan\left(\frac{x}{2}\right)\right) = -2\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{2n+1}$$

$$\Omega = \pi\sum_{n=0}^{\infty} \frac{1}{2n+1}\int_0^{\frac{\pi}{2}} \cos(2n+1)x\, dx = \pi\sum_{n=0}^{\infty} \frac{\sin\left(n\pi + \frac{\pi}{2}\right)}{(2n+1)^2} = \pi\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

Therefore,

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \frac{x}{y}\frac{\sin y}{\sin x} dxdy = \pi\beta(2)$$

Solution 2 by Probal Chakraborty-Kokata-India

$$\Omega = \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{x}{y}\frac{\sin y}{\sin x} dxdy = \int_0^\infty \frac{\sin y}{y} dy \int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx =$$

$$= \frac{\pi}{2}\int_0^1 \frac{2x(\sin^{-1} x)}{\sqrt{1-x^2}}\frac{dx}{2x^2} = \frac{\pi}{2}\int_0^1 \frac{dx}{x^2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}} = \pi G = \pi\beta(2)$$

$$\text{As } 2G = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(\frac{1+\sin x}{1-\sin x}\right) dx = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}}$$

1959. Find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt{H_1} + \sqrt{\frac{1}{2}H_2} + \sqrt{\frac{1}{3}H_3} + \cdots + \sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1 + H_2 + \cdots + H_n)}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\left(\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}\right)^2 \overset{CBS}{\leq}$$

$$\leq\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)(H_1+H_2+\cdots+H_n)=H_n(H_1+H_2+\cdots+H_n)$$

$$\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}\leq\sqrt{H_n(H_1+H_2+\cdots+H_n)}$$

$$0\leq\frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}}\leq\frac{1}{n}\to 0$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}}$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $a_k=\sqrt{\frac{1}{k}}$ and $b_k=\sqrt{H_k}$ for $1\leq k\leq n$. By CBS inequality:

$$\left(\sum_{cyc}a_kb_k\right)^2\leq\left(\sum_{k=1}^{n}a_k\right)^2\left(\sum_{k=1}^{n}b_k\right)^2 \text{ we get:}$$

$$\sum_{k=1}^{n}\sqrt{\frac{1}{k}H_k}\leq\sqrt{\left(\sum_{k=1}^{n}\frac{1}{k}\right)\left(\sum_{k=1}^{n}H_k\right)}=\sqrt{H_n\sum_{k=1}^{n}H_k}$$

$$0\leq\frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}}\leq\frac{1}{n}\to 0$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}}$$

Solution 3 by Syed Shahabudeen-Kerala-India

By CBS inequality:

$$\left(\sum_{cyc} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k\right)^2 \left(\sum_{k=1}^{n} b_k\right)^2 \text{ we get:}$$

$$\frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{\sqrt{H_1+H_2+\cdots+H_n}} \le \sqrt{H_n},$$

$$0 \le \frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}} \le \frac{1}{n} \to 0$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{\sqrt{H_1}+\sqrt{\frac{1}{2}H_2}+\sqrt{\frac{1}{3}H_3}+\cdots+\sqrt{\frac{1}{n}H_n}}{n\sqrt{H_n(H_1+H_2+\cdots+H_n)}}$$

1960. Prove that:

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)}{1+x^2}dx = \frac{13\pi^2}{288}$$

Proposed by Hamza Djahel-Algerie

Solution 1 by Pham Duc Nam-Vietnam

$$\Omega = \int_0^{\frac{1}{\sqrt{3}}} \frac{\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)}{1+x^2}dx \overset{x=\frac{t}{\sqrt{2}}}{=} \int_0^1 \frac{\tan^{-1}\left(\sqrt{t^2+2}\right)}{(t^2+1)\sqrt{t^2+2}}dt =$$

$$= \frac{\pi}{2}\int_0^1 \frac{dt}{(t^2+1)\sqrt{t^2+2}} - \int_0^1 \frac{\tan^{-1}\left(\frac{1}{\sqrt{t^2+2}}\right)}{(t^2+1)\sqrt{t^2+1}}dt = J + K$$

$$J = \int_0^1 \frac{dt}{(t^2+1)\sqrt{t^2+2}} \overset{t=\sqrt{2}\tan u}{=} \int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)} \frac{du}{(2\tan^2 u+1)\sqrt{2}} \cdot \frac{\sqrt{2}}{\cos^2 u} =$$

$$=\int_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}\frac{\frac{1}{\sqrt{2}}du\cos u}{\sin^2 u+1}=\tan^{-1}(\sin u)\Big|_0^{\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}=\frac{\pi}{6}$$

$$\because \tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2}.$$

$$K=\int_0^1\frac{\tan^{-1}\left(\frac{1}{\sqrt{t^2+2}}\right)}{(t^2+1)\sqrt{t^2+2}}dt=\int_0^1\frac{\tan^{-1}\left(\frac{1}{\sqrt{x^2+2}}\right)}{(x^2+1)\sqrt{x^2+2}}dx=$$

$$=\int_0^1\int_0^1\frac{dxdy}{(x^2+1)(x^2+y^2+2)}=\int_0^1\int_0^1\frac{1}{y^2+1}\left(\frac{1}{x^2+1}-\frac{1}{x^2+y^2+2}\right)dxdy=$$

$$=\int_0^1\int_0^1\frac{dxdy}{(1+x^2)(1+y^2)}-\int_0^1\int_0^1\frac{dxdy}{(y^2+1)(x^2+y^2+2)}=\left(\int_0^1\frac{dx}{x^2+1}\right)^2-K$$

$$2K=\left(\tan^{-1}x\Big|_0^1\right)^2=\frac{\pi^2}{16}\Rightarrow K^2=\frac{\pi^2}{32}\Rightarrow \Omega=\frac{\pi}{2}\cdot\frac{\pi}{6}-\frac{\pi^2}{32}=\frac{5\pi^2}{96}$$

$$\Omega=\int_0^{\frac{1}{\sqrt{3}}}\frac{\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)}{1+x^2}dx=\frac{\pi}{2}\int_0^{\frac{1}{\sqrt{3}}}\frac{dx}{x^2+1}-\int_0^{\frac{1}{\sqrt{3}}}\frac{\tan^{-1}\left(\sqrt{1-2x^2}\right)}{x^2+1}dx=$$

$$=\frac{\pi}{2}\tan^{-1}x\Big|_0^{\frac{1}{\sqrt{3}}}-\int_0^{\frac{1}{\sqrt{3}}}\frac{\tan^{-1}\left(\sqrt{1-2x^2}\right)}{x^2+1}dx=\frac{\pi^2}{12}-F$$

$$F=\int_0^{\frac{1}{\sqrt{3}}}\frac{\tan^{-1}\left(\sqrt{1-2x^2}\right)}{x^2+1}dx\overset{x=\frac{1}{\sqrt{2}}\sin u}{=}\frac{1}{\sqrt{2}}\int_0^{\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)}\frac{\tan^{-1}(\cos u)\cdot\cos u\,du}{1+\frac{1}{2}\sin^2 u}=$$

$$=\sqrt{2}\int_0^{\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)}\frac{\tan^{-1}(\cos u)\,d(\sin u)}{2+\sin^2 u}=$$

$$=\tan^{-1}\left(\frac{\sin u}{\sqrt{2}}\right)\Big|_0^{\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)}+\int_0^{\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)}\frac{\sin u\cdot\tan^{-1}\left(\frac{\sin u}{\sqrt{2}}\right)}{1+\cos^2 u}du=\frac{\pi^2}{36}+G$$

$$G=\int_0^{\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)}\frac{\sin u\cdot\tan^{-1}\left(\frac{\sin u}{\sqrt{2}}\right)}{1+\cos^2 u}du=\int_0^{\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)}\int_0^1\frac{\sin^2 u}{(1+\cos^2 u)(1+k^2\sin^2 u)}dk\,du$$

$$=\int_0^{\frac{1}{\sqrt{2}}}\left(\frac{\sqrt{2}\tan^{-1}\left(\frac{\tan u}{\sqrt{2}}\right)}{2k^2+1}-\frac{\tan^{-1}\left(\tan u\sqrt{k^2+1}\right)}{(2k^2+1)\sqrt{k^2+1}}\right)_0^{\sin^{-1}\sqrt{\frac{2}{3}}}dk=$$

$$=\frac{\pi}{4}\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{dk}{2k^2+1}-\int_0^{\frac{1}{\sqrt{2}}}\frac{\tan^{-1}\left(\tan u\sqrt{2k^2+1}\right)}{(2k^2+1)\sqrt{k^2+1}}dk=\frac{\pi^2}{16}-\frac{5\pi^2}{96}$$

$$\Omega=\int_0^{\frac{1}{\sqrt{3}}}\frac{\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)}{1+x^2}dx=\frac{\pi^2}{12}-\frac{\pi^2}{36}-\frac{\pi^2}{16}+\frac{5\pi^2}{96}=\frac{13\pi^2}{288}$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{1}{\sqrt{3}}}\frac{\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)}{1+x^2}dx\overset{IBP}{=}\left[\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)\tan^{-1}x\right]_0^{\frac{1}{\sqrt{3}}}$$

$$-\int_0^{\frac{1}{\sqrt{3}}}\frac{x\tan^{-1}x}{(1-x^2)\sqrt{1-2x^2}}dx=\frac{\pi^2}{18}-I$$

$$\text{Let: }\sqrt{1-2x^2}=\frac{1}{1+t^2}$$

$$I=\int_0^{\sqrt{2}}\frac{t\tan^{-1}\left(\frac{t}{\sqrt{2+2t^2}}\right)}{(2+t^2)\sqrt{1+t^2}}dt\overset{t=x\sqrt{2}}{=}\int_0^1\frac{x\tan^{-1}\left(\frac{x}{\sqrt{1+2x^2}}\right)}{(1+x^2)\sqrt{1+2x^2}}dx=$$

$$=\int_0^1\int_0^1\frac{x^2}{(1+x^2)(1+2x^2+y^2x^2)}dxdy=$$

$$=\int_0^1\frac{1}{1+y^2}\int_0^1\left(\frac{1}{1+x^2}-\frac{1}{1+(2+y^2)x^2}\right)dxdy=\frac{\pi^2}{16}-\int_0^1\frac{\tan^{-1}\left(\sqrt{2+y^2}\right)}{(1+y^2)\sqrt{2+y^2}}dy$$

$$\int_0^1\frac{\tan^{-1}\left(\sqrt{2+y^2}\right)}{(1+y^2)\sqrt{2+y^2}}dy=\frac{5\pi^2}{96}\text{ (Ahmad's Integral)}$$

$$\Omega=\int_0^{\frac{1}{\sqrt{3}}}\frac{\tan^{-1}\left(\frac{1}{\sqrt{1-2x^2}}\right)}{1+x^2}dx=\frac{\pi^2}{12}-\frac{\pi^2}{36}-\frac{\pi^2}{16}+\frac{5\pi^2}{96}=\frac{13\pi^2}{288}$$

1961. Prove that:

$$\int_1^\infty\int_1^\infty\frac{\sqrt{x}+\sqrt{y}}{\left(1+\sqrt{xy}\right)\left(1+xy(1+xy(1+xy))\right)}dydx=\pi\left(\sqrt{2}-\frac{3}{2}\right)+\log\left(\frac{1}{8}(2\sqrt{2}+3)^{\sqrt{2}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_1^\infty\int_1^\infty\frac{\sqrt{x}+\sqrt{y}}{(1+\sqrt{xy})\left(1+xy(1+xy(1+xy))\right)}dydx=$$

$$=2\int_1^\infty\int_1^\infty\frac{\sqrt{x}}{(1+\sqrt{xy})(1+xy)(1+x^2y^2)}dxdy$$

$$\Omega=2\int_1^\infty\int_1^\infty\frac{\sqrt{x}(1-\sqrt{xy})}{1-x^4y^4}dxdy\overset{xy=u^2;y=v^2}{=}8\int_1^\infty\frac{1}{v^2}\int_v^\infty\frac{u^2(1-u)}{1-u^8}dudv=$$

$$=8\left[-\frac{1}{v}\int_v^\infty\frac{u^2(1-u)}{1-u^8}du\right]_1^\infty-8\int_1^\infty\frac{v(1-v)}{1-v^8}dv=$$

$$=8\int_1^\infty\frac{u^2(1-u)}{1-u^8}du-\int_1^\infty\frac{v(1-v)}{1-v^8}dv$$

$$\Omega=-8\int_1^\infty\frac{u(1-u)^2}{1-u^8}du=8\int_1^\infty\frac{u^2-u}{(1+u)(1+u^2)(1+u^4)}du=$$

$$=4\int_1^\infty\left(\frac{1}{1+u}-\frac{1}{1+u^2}-\frac{u^3}{1+u^4}-\frac{u}{1+u^4}+\frac{2u^2}{1+u^4}\right)du=$$

$$=4\left[\log(1+u)-\tan^{-1}u-\frac{1}{4}\log(1+x^4)-\frac{1}{2}\tan^{-1}(u^2)\right]_1^\infty$$

$$+4\int_1^\infty\frac{u^2+1+u^2-1}{1+u^4}du=$$

$$=-\frac{3\pi}{4}-3\log 2+4\underbrace{\int_1^\infty\frac{1+\frac{1}{u^2}}{u^2+\frac{1}{u^2}}du}_{y=u-\frac{1}{u}}+4\underbrace{\int_1^\infty\frac{1-\frac{1}{u^2}}{u^2+\frac{1}{u^2}}du}_{y=u+\frac{1}{u}}=$$

$$=-\frac{3\pi}{2}-\log 8+4\int_1^\infty\frac{dx}{x^2+2}+4\int_2^\infty\frac{dy}{y^2-2}=$$

$$=-\frac{3\pi}{2}-\log 8+\pi\sqrt{2}+\sqrt{2}\log(3+2\sqrt{2})=\pi\left(\sqrt{2}-\frac{3}{2}\right)+\log\left(\frac{1}{8}(3+2\sqrt{2})^{\sqrt{2}}\right)$$

Therefore,

$$\int_1^\infty\int_1^\infty\frac{\sqrt{x}+\sqrt{y}}{(1+\sqrt{xy})\left(1+xy(1+xy(1+xy))\right)}dydx=\pi\left(\sqrt{2}-\frac{3}{2}\right)+\log\left(\frac{1}{8}(2\sqrt{2}+3)^{\sqrt{2}}\right)$$

1962. Prove that:

$$I=\int_0^1\int_0^1\frac{xy}{\sqrt{x^2+y^2}}\arctan\sqrt{x^2+y^2}\arctan\left(\frac{x^2}{y^2}\right)dx\,dy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\frac{xy}{\sqrt{x^2+y^2}}\arctan\sqrt{x^2+y^2}\arctan\left(\frac{x^2}{y^2}\right)dx\,dy=$$

$$=\int_0^1\int_0^1\frac{xy}{\sqrt{x^2+y^2}}\arctan\sqrt{x^2+y^2}\arctan\left(\frac{y^2}{x^2}\right)dx\,dy$$

$$2I=\int_0^1\int_0^1\frac{xy}{\sqrt{x^2+y^2}}\arctan\sqrt{x^2+y^2}\left\{\arctan\left(\frac{x^2}{y^2}\right)+\arctan\left(\frac{y^2}{x^2}\right)\right\}dx\,dy$$

$$=\frac{\pi}{2}\int_0^1\int_0^1\frac{xy}{\sqrt{x^2+y^2}}\arctan\sqrt{x^2+y^2}\,dx\,dy$$

$$I=\frac{\pi}{4}\int_0^1\int_0^1\frac{xy}{\sqrt{x^2+y^2}}\arctan\sqrt{x^2+y^2}\,dx\,dy$$

$$I_1=\int_0^1\int_0^1\frac{xy}{\sqrt{x^2+y^2}}\arctan\sqrt{x^2+y^2}\,dx\,dy\Bigg]_{x^2+y^2=z^2}=\int_0^1 x\int_x^{\sqrt{x^2+1}}\arctan(z)\,dz\,dx$$

$$=\left[\frac{x^2}{2}\int_x^{\sqrt{x^2+1}}\arctan(z)\,dz\right]_{x=0}^{x=1}-\frac{1}{2}\int_0^1\frac{x^3}{\sqrt{x^2+1}}\arctan\sqrt{x^2+1}\,dx+\frac{1}{2}\int_0^1 x^2\arctan(x)\,dx$$

$$=\frac{1}{2}\int_1^{\sqrt{2}}\arctan(z)\,dz-\frac{1}{2}\int_0^1\frac{x^3}{\sqrt{x^2+1}}\arctan\sqrt{x^2+1}\,dx+\frac{1}{2}\int_0^1 x^2\arctan(x)\,dx$$

$$=\frac{1}{2}[I_{1a}-I_{1b}+I_{1c}]$$

$$I_{1a}=\int_1^{\sqrt{2}}\arctan(z)\,dz=\sqrt{2}\arctan\sqrt{2}-\frac{\log(3)}{2}+\frac{\log(2)}{2}-\frac{\pi}{4}$$

$$I_{1b}=\int_0^1\frac{x^3}{\sqrt{x^2+1}}\arctan\sqrt{x^2+1}\,dx\Bigg]_{x^2+1=k^2}=\int_1^{\sqrt{2}}k^2\arctan(k)\,dk-\int_1^{\sqrt{2}}\arctan(k)\,dk$$

$$=\left[\frac{2\sqrt{2}}{3}\arctan\sqrt{2}+\frac{\log(3)}{6}-\frac{\log(2)}{6}-\frac{\pi}{12}-\frac{1}{6}\right]$$

$$-\left[\sqrt{2}\arctan\sqrt{2}-\frac{\log(3)}{2}+\frac{\log(2)}{2}-\frac{\pi}{4}\right]$$

$$=-\frac{\sqrt{2}}{3}\arctan\sqrt{2}+\frac{2\log(3)}{3}-\frac{2\log(2)}{3}+\frac{\pi}{6}-\frac{1}{6}$$

$$I_{1c}=\int_0^1 x^2\arctan(x)\,dx=\frac{\log(2)}{6}+\frac{\pi}{12}-\frac{1}{6}$$

$$I_1=\frac{1}{2}[I_{1a}-I_{1b}+I_{1c}]=\frac{2\sqrt{2}}{3}\arctan\sqrt{2}-\frac{7\log(3)}{12}+\frac{2\log(2)}{3}-\frac{\pi}{6}$$

$$I=\frac{\pi}{4}I_1=\frac{\sqrt{2}}{6}\pi\arctan\sqrt{2}-\frac{7\pi\log(3)}{48}+\frac{\pi\log(2)}{6}-\frac{\pi^2}{24}$$

1963. Find:

$$\Omega=\lim_{n\to\infty}\left(\int_0^\infty\frac{x^{2n}}{e^{x^2}}dx\right)^{\frac{1}{n}}\left(\int_0^\infty\frac{x^{2n+1}}{e^{x^2}}dx\right)^{\frac{-1}{n}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$I_1(n)=\int_0^\infty\frac{x^{2n}}{e^{x^2}}dx\overset{x^2=t}{\triangleq}\int_0^\infty\frac{t^n}{e^t}\cdot\frac{dt}{2\sqrt{t}}=\frac{1}{2}\int_0^\infty t^{\left(n+\frac{1}{2}\right)-1}e^{-t}dt=\frac{1}{2}\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2\cdot 4^n\cdot n!}\sqrt{\pi}$$

$$I_2(n)=\int_0^\infty\frac{x^{2n+1}}{e^{x^2}}dx\overset{x^2=t}{\triangleq}\frac{1}{2}\int_0^\infty\frac{t^n}{e^t}dt=\frac{1}{2}\int_0^\infty t^{(n+1)-1}e^{-t}dt=\frac{1}{2}\Gamma(n+1)=\frac{n!}{2}$$

$$\Omega=\lim_{n\to\infty}\left(\int_0^\infty\frac{x^{2n}}{e^{x^2}}dx\right)^{\frac{1}{n}}\left(\int_0^\infty\frac{x^{2n+1}}{e^{x^2}}dx\right)^{\frac{-1}{n}}=\lim_{n\to\infty}\sqrt[n]{\frac{I_1(n)}{I_2(n)}}=$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{\frac{(2n)!}{2\cdot 4^n\cdot n!}\sqrt{\pi}}{\frac{n!}{2}}} = \lim_{n\to\infty} \frac{(2n+2)!}{2\cdot 4^{n+1}\cdot (n+1)!}\sqrt{\pi}\cdot \frac{2\cdot n!}{\frac{(2n)!}{2\cdot 4^n\cdot n!}\sqrt{\pi}} =$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{4(n+1)^2} = 1$$

Solution 2 by Ravi Prakash-New Delhi-India

$$a_n = \int_0^\infty \frac{x^{2n}}{e^{x^2}}dx, \qquad b_n = \int_0^\infty \frac{x^{2n+1}}{e^{x^2}}, \qquad x^2 = t$$

$$a_n = \frac{1}{2}\Gamma\left(n+\frac{1}{2}\right), b_n = \frac{1}{2}\Gamma(n+1) = \frac{1}{2}n!$$

$$\Omega = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{a_{n+1}}{b_{n+1}}\cdot\frac{b_n}{a_n} = \lim_{n\to\infty}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{n+\frac{1}{2}}{n+1} = 1$$

1964. Find:

$$\Omega = \lim_{n\to\infty}\frac{(n-1)!\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)}{n\Gamma\left(n+\frac{1}{2}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Asmat Qatea-Afghanistan

$$\Omega = \lim_{n\to\infty}\frac{(n-1)!\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)}{n\Gamma\left(n+\frac{1}{2}\right)} =$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\cdot\lim_{n\to\infty}\sqrt{n}\cdot\frac{(n-1)!}{\Gamma\left(n+\frac{1}{2}\right)} = \int_0^1\sqrt{x}\,dx\cdot\lim_{n\to\infty}\frac{\sqrt{n}\cdot\Gamma(n)}{n\cdot\Gamma\left(n+\frac{1}{2}\right)} =$$

$$= \frac{2}{3}\cdot\lim_{n\to\infty} n^{-\frac{1}{2}}\cdot n^{\frac{1}{2}} = \frac{2}{3}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

Note: $(n-1)! = \Gamma(n)$

$$\sqrt{\frac{2\pi}{x}}x^x e^{-x} \le \Gamma(x) \le \sqrt{\frac{2\pi}{x}}x^x e^{-x}e^{\frac{1}{12x}}$$

We also need bound on $\sum_{k=1}^{n}\sqrt{k}$

$$\sum_{k=1}^{n}\sqrt{k}>\int_{0}^{n}\sqrt{u}\,du=\frac{2}{3}n^{\frac{3}{2}}$$

and

$$\sum_{k=1}^{n}\sqrt{k}<\int_{0}^{n+1}\sqrt{u}\,du=\frac{2}{3}(n+1)^{\frac{3}{2}}$$

$$\Rightarrow\frac{2}{3}n^{\frac{3}{2}}<\sum_{k=1}^{n}\sqrt{k}<\frac{2}{3}(n+1)^{\frac{3}{2}}$$

$$\frac{\frac{2}{3}n^{\frac{3}{2}}\sqrt{2\pi}n^{n-\frac{1}{2}}e^{-n}}{n\sqrt{2\pi}\left(n+\frac{1}{2}\right)^{n}e^{-n-\frac{1}{2}}e^{\frac{1}{12\left(n+\frac{1}{2}\right)}}}\leq\Omega\leq\frac{\frac{2}{3}(n+1)^{\frac{3}{2}}\sqrt{2\pi}n^{n-\frac{1}{2}}e^{-n}e^{\frac{1}{12n}}}{n\sqrt{2\pi}\left(n+\frac{1}{2}\right)^{n}e^{-n-\frac{1}{2}}}$$

$$\lim_{n\to\infty}\frac{\frac{2}{3}(n+1)^{\frac{3}{2}}n^{n-\frac{1}{2}}\sqrt{e}}{n\left(n+\frac{1}{2}\right)^{n}}=\lim_{n\to\infty}\frac{2}{3}\sqrt{e}\frac{\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}{\left(1+\frac{\frac{1}{2}}{n}\right)^{n}}=\lim_{n\to\infty}\frac{2}{3}\sqrt{e}\frac{1}{\sqrt{e}}=\frac{2}{3}$$

$$\lim_{n\to\infty}\frac{\frac{2}{3}n^{\frac{3}{2}}n^{n-\frac{1}{2}}\sqrt{e}}{n\left(n+\frac{1}{2}\right)^{n}}=\lim_{n\to\infty}\frac{2}{3}\sqrt{e}\frac{1}{\left(1+\frac{\frac{1}{2}}{n}\right)^{n}}=\lim_{n\to\infty}\frac{2}{3}\sqrt{e}\frac{1}{\sqrt{e}}=\frac{2}{3}$$

Therefore, $\Omega=\frac{2}{3}$

1965. If we define the integral-function

$$\psi(n)=\int_{-\infty}^{\infty}\sqrt{1+\cosh(\pi nx)}\,e^{-\frac{\pi x(x+1)}{2}}dx$$

then show that

$$\int_{-\infty}^{\infty}\psi(n+1)\,\psi(n-1)e^{-\frac{\pi n(n+1)}{2}}dn=2e^{\frac{\pi}{4}}(1+e^{\pi})\left(1+e^{\frac{\pi}{4}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\psi(n)=\int_{-\infty}^{\infty}\sqrt{1+\cosh(n\pi x)}\,e^{-\frac{\pi x(x+1)}{2}}dx$$

$$\psi(n)=\sqrt{2}\int_{-\infty}^{\infty}\cosh\left(\frac{\pi n x}{2}\right)e^{-\frac{\pi(x^2+x)}{2}}dx=$$

$$=\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(x^2-(n-1)x)}\,dx+\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}(x^2+(1+n)x)}\,dx$$

$$\psi(n)=\frac{e^{\frac{\pi(n-1)^2}{8}}}{\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}\left(x+\frac{1-n}{2}\right)^2}\,dx+\frac{e^{\frac{\pi(n+1)^2}{8}}}{\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}\left(x+\frac{1+n}{2}\right)^2}\,dx$$

$$\psi(n)=e^{\frac{\pi(n-1)^2}{8}}+e^{\frac{\pi(n+1)^2}{8}}$$

$$\Omega=\int_{-\infty}^{\infty}\psi(n-1)\,\psi(n+1)e^{-\frac{\pi n(n+1)}{2}}dn$$

$$\Omega=\int_{-\infty}^{\infty}\left(e^{\frac{\pi(n-2)^2}{8}}+e^{\frac{\pi n^2}{8}}\right)\left(e^{\frac{\pi n^2}{8}}+e^{\frac{\pi(n+2)^2}{8}}\right)e^{-\frac{\pi n(n+1)}{2}}dn$$

$$\Omega=\int_{-\infty}^{\infty}\left(1+e^{\frac{\pi(1-n)}{2}}\right)\left(1+e^{\frac{\pi(1+n)}{2}}\right)e^{-\frac{\pi n(n+2)}{4}}dn=$$

$$=\int_{-\infty}^{\infty}\left(1+e^{\pi}+e^{\frac{\pi(1-n)}{2}}+e^{\frac{\pi(1+n)}{2}}\right)\,e^{-\frac{\pi n(n+2)}{4}}dn$$

$$\Omega=\int_{-\infty}^{\infty}e^{-\frac{\pi n(n+2)}{4}}dn+\int_{-\infty}^{\infty}e^{-\frac{\pi(n^2+2n-4)}{4}}dn+$$

$$+\int_{-\infty}^{\infty}-\frac{\pi(n^2+4n-2)}{4}dn+e^{\frac{\pi}{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi n^2}{4}}dn$$

$$\Omega=e^{\frac{\pi}{4}}\int_{-\infty}^{\infty}e^{-\frac{\pi(n+1)^2}{4}}dn+e^{\frac{5\pi}{4}}\int_{-\infty}^{\infty}e^{-\frac{\pi(n+1)^2}{4}}dn+e^{\frac{3\pi}{2}}\int_{-\infty}^{\infty}e^{-\frac{\pi(n+2)^2}{4}}dn+2e^{\frac{\pi}{2}}$$

$$\Omega=2e^{\frac{\pi}{4}}+2e^{\frac{5\pi}{4}}+2e^{\frac{3\pi}{2}}+2e^{\frac{\pi}{2}}=$$

$$=2e^{\frac{\pi}{4}}\left(1+e^{\pi}+e^{\frac{5\pi}{4}}+e^{\frac{\pi}{4}}\right)=2e^{\frac{\pi}{4}}\left(1+e^{\pi}+(1+e^{\pi})e^{\frac{\pi}{4}}\right)$$

$$\int_{-\infty}^{\infty}\pi(n-1)\pi(n+1)\,e^{-\frac{\pi n(n+1)}{2}}dn=2e^{\frac{\pi}{4}}(1+e^{\pi})\left(1+e^{\frac{\pi}{4}}\right)$$

1966.

$$\Omega=\left(\int_0^\infty \frac{x^{2n}}{e^{x^2}}dx\right)^{\frac{1}{n}}\left(\int_0^\infty \frac{x^{2n+1}}{e^{x^2}}\right)^{-\frac{1}{n}}$$

Prove that:

$$\lim_{n\to\infty}\frac{\Omega}{\beta^{\frac{1}{n}}(n,n)}=4=\lim_{n\to\infty}\frac{\beta^{\frac{1}{n}}(n,n)}{\beta^{\frac{1}{n}}(2n,2n)}$$

Where $\beta(x,y),\mathcal{R}(x)>0,\mathcal{R}(y)>0$ is the Euler integral of the first kind.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Bui Hong Suc-Vietnam

$$\Omega=\left(\int_0^\infty \frac{x^{2n}}{e^{x^2}}dx\right)^{\frac{1}{n}}\left(\int_0^\infty \frac{x^{2n+1}}{e^{x^2}}\right)^{-\frac{1}{n}}=$$

$$=\left(\int_0^\infty \frac{1}{2}(x^2)^{n-\frac{1}{2}}e^{-x^2}d(x^2)\right)^{\frac{1}{n}}\cdot\left(\frac{1}{2}\int_0^\infty (x^2)^{n+1-1}e^{-x^2}d(x^2)\right)^{-\frac{1}{n}}$$

$$=\left(\frac{1}{2}\cdot\sqrt{\left(n+\frac{1}{2}\right)}\right)^{\frac{1}{n}}\cdot\left(\frac{1}{2}\sqrt{(n+1)}\right)^{-\frac{1}{n}}=\left(\frac{\sqrt{\left(n+\frac{1}{2}\right)}}{\sqrt{(n+1)}}\right)^{\frac{1}{n}}$$

$$\beta(n,n)=\frac{\sqrt{(n)}\cdot\sqrt{(n)}}{\sqrt{(2n)}}\to\beta^{\frac{1}{n}}(n,n)=\frac{\sqrt{(n)}^{\frac{2}{n}}}{\sqrt{(2n)}^{\frac{1}{n}}};\ \beta^{\frac{1}{n}}(2n,2n)=\frac{\sqrt{(2n)}^{\frac{2}{n}}}{\sqrt{(4n)}^{\frac{1}{n}}}$$

We use Stirling's formula:

$$\frac{\sqrt{\left(n+\frac{1}{2}\right)}}{\sqrt{(n+1)}}\sim\frac{\sqrt{2n}\cdot e^{-n-\frac{1}{2}}\cdot\left(n+\frac{1}{2}\right)^n}{\sqrt{2n}\cdot e^{-n-1}\cdot(n+1)^{n+\frac{1}{2}}}=e^{\frac{1}{2}}\cdot\left(n+\frac{1}{2}\right)^n\cdot(n+1)^{-n-\frac{1}{2}}$$

$$\frac{\sqrt{(n)}^2}{\sqrt{(2n)}}\sim\frac{\left(\sqrt{2n}\right)^2}{\sqrt{2n}}\cdot\frac{e^{-2n}n^{2\left(n-\frac{1}{2}\right)}}{e^{-2n}(2n)^{2n-\frac{1}{2}}}=\sqrt{2n}\cdot 2^{-2n+\frac{1}{2}}\cdot n^{-\frac{1}{2}}$$

$$\to \frac{\Omega}{\beta^{\frac{1}{n}}(n,n)} \sim \frac{\left(\frac{\sqrt{\left(n+\frac{1}{2}\right)}}{\sqrt{(n+1)}}\right)^{\frac{1}{n}}}{\left(\frac{\sqrt{(n)}^2}{\sqrt{(2n)}}\right)^{\frac{1}{n}}} \sim \left(\frac{e^{\frac{1}{2}}}{\sqrt{2n}} \cdot 2^{2n-\frac{1}{2}} \cdot n^{\frac{1}{2}} \cdot \left(n+\frac{1}{2}\right)^n \cdot (n+1)^{-n-\frac{1}{2}}\right)^{\frac{1}{n}}$$

$$= \left(\left(\frac{e}{2^2 n}\right)^{\frac{1}{2}} \cdot 2^{2n} \cdot \left(\frac{n+\frac{1}{2}}{n+1}\right)^n \cdot \left(\frac{n}{n+1}\right)^{\frac{1}{2}}\right)^{\frac{1}{n}} = 2^2\left(\frac{e}{4n}\right)^{\frac{1}{2n}} \cdot \left(\frac{1+\frac{1}{2n}}{1+\frac{1}{n}}\right) \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2n}}$$

$$\text{As } n \to \infty: A = \lim_{n\to\infty} \frac{\Omega}{\beta^{\frac{1}{n}}(n,n)} = \lim_{n\to\infty} 2^2\left(\frac{e}{4n}\right)^{\frac{1}{2n}}\left(\frac{1+\frac{1}{2n}}{1+\frac{1}{n}}\right)\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2n}}$$

$$= 4 \cdot 1 \cdot 1 \cdot 1 = 4$$

Solurion 2 by Hikmat Mammadov-Azerbaijan

$$\text{Note: } \Gamma(x) = \sqrt{2\pi} e^{x\ln x - x - \frac{1}{2}\ln x + O\left(\frac{1}{x^2}\right)}\left(1 + O\left(\frac{1}{x}\right)\right)$$

$$\therefore \int_0^\infty x^{2n} e^{-x^2} dx \underset{dx=\frac{1}{2\sqrt{t}}dt}{\overset{t=x^2}{=}} \int_0^\infty t^n e^{-t} \frac{1}{2\sqrt{t}} dt = \int_0^\infty t^{n-\frac{1}{2}} e^{-t} dt = \frac{1}{2}\Gamma\left(n+\frac{1}{2}\right)$$

$$\therefore \int_0^\infty x^{2n+1} e^{-x^2} \overset{t=x^2}{=} \int_0^\infty t^{n+\frac{1}{2}} e^{-t} \frac{1}{2\sqrt{t}} dt = \frac{1}{2}\int_0^\infty t^n e^{-t} dt = \frac{1}{2}\Gamma(n+1)$$

$$\Omega = \lim_{n\to\infty}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{e^{\left(n+\frac{1}{2}\right)\ln\left(n+\frac{1}{2}\right)-\left(n+\frac{1}{2}\right)-\frac{1}{2}\ln\left(n+\frac{1}{2}\right)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n}\right)\right)}{e^{(n+1)\ln(n+1)-(n+1)-\frac{1}{2}\ln(n+1)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n}\right)\right)}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} e^{\ln\left(n+\frac{1}{2}\right)-\ln(n+1)} = \lim_{n\to\infty} e^{\ln\left(\frac{1+\frac{1}{2n}}{1+\frac{1}{n}}\right)} = e^0 = 1$$

$$\beta(n,n) = \frac{\Gamma(n)\cdot\Gamma(n)}{\Gamma(n+n)} = \frac{\Gamma(n)^2}{\Gamma(2n)} \Rightarrow \text{Generally, } \beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$$\Omega = \lim_{n\to\infty}\beta(n,n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{e^{2n\ln(n)-2n-\ln(n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n^2}\right)\right)}{e^{2n\ln(2n)-2n-\frac{1}{2}\ln(2n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n^2}\right)\right)}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} e^{2\ln(2)-2\ln(2n)} = e^{\ln\left(\frac{1}{4}\right)} = \frac{1}{4}, \qquad \beta(2n,2n) = \frac{\Gamma(2n)^2}{\Gamma(4n)}$$

$$\Omega = \lim_{n\to\infty} \beta(2n,2n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{e^{4n\ln(2n)\ -4n-\ln(2n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n^2}\right)\right)}{e^{4n\ln(4n)-4n-\frac{1}{2}\ln(4n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n^2}\right)\right)}\right)$$

$$= \lim_{n\to\infty} e^{4n\ln(2n)-4n\ln(4n)} = e^{\ln\left(\frac{1}{16}\right)} = \frac{1}{16}$$

$$\therefore \lim_{n\to\infty}\frac{\beta(n,n)^{\frac{1}{n}}}{\beta(2n,2n)^{\frac{1}{n}}} = \lim_{n\to\infty}\frac{\Omega}{\beta(n,n)^{\frac{1}{n}}} = \frac{\frac{1}{4}}{\frac{1}{16}} = \frac{1}{\frac{1}{4}} = 4$$

1967. Find:

$$\Omega(s,t) = \lim_{x\to\infty}\left((x+1)^s\left(\left(\Gamma(x+2)^{\frac{t}{x+1}}\right) - (x)^s\left(\left(\Gamma(x+1)^{\frac{t}{x}}\right)\right)\right); s,t \in \mathbb{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\Gamma(x) = \sqrt{2\pi}e^{x\log x-\frac{1}{2}\log x+O\left(\frac{1}{x^2}\right)}\left(1+O\left(\frac{1}{x}\right)\right)$$

$$\Omega(s,t) = e^{\frac{t}{2(x+1)}\log(2\pi)+s\log(x+1)+t\left(1+\frac{1}{2(x+1)}\right)\log(x+2)-t-\frac{t}{x+1}+O\left(\frac{1}{x^2}\right)}\left(1+O\left(\frac{1}{x^2}\right)\right) -$$

$$-e^{\frac{t}{2x}\log(2\pi)+s\log(x)+t\left(1+\frac{1}{2x}\right)\log(x+2)-t-\frac{t}{x}+O\left(\frac{1}{x^2}\right)}\left(1+O\left(\frac{1}{x^2}\right)\right)$$

$$\log(x+1) = \log x + \frac{1}{x} + O\left(\frac{1}{x^2}\right), \qquad \log(x+2) = \log x + \frac{2}{x} + O\left(\frac{1}{x^2}\right)$$

$$\Omega(s,t) = \left(1+O\left(\frac{1}{x}\right)\right)e^{-t}x^{t+s}\left(e^{\frac{t}{2(x+1)}\log(2\pi)+\frac{t}{2(x+1)}\log x+\frac{t+s}{x}+O\left(\frac{1}{x^2}\right)} - e^{\frac{t}{2x}\log(2\pi)+\frac{t}{2x}\log x+O\left(\frac{1}{x^2}\right)}\right)$$

$$= \left(1+O\left(\frac{1}{x}\right)\right)e^{-t}x^{t+s}\left(\left(1+\frac{t}{2(x+1)}\log(2\pi)+\frac{t}{2(x+1)}\log x+\frac{t+s}{x}\right)-\right.$$

$$\left.-\left(1+\frac{t}{2x}\log(2\pi)+\frac{t}{2x}\log x+O\left(\frac{\log x}{x}\right)^2\right)\right) =$$

$$= \left(1 + O\left(\frac{1}{x}\right)\right) e^{-t} x^{t+s} \left(\frac{t+s}{x} - \frac{t}{2x(x+1)} \log x - \frac{t}{2x(x+1)} \log(2\pi) + O\left(\frac{\log x}{x}\right)^2\right) =$$

$$= \left(1 + O\left(\frac{1}{x}\right)\right) e^{-t} x^{t+s} \left(\frac{t+s}{x} + O\left(\frac{\log x}{x}\right)^2\right) = \left(1 + O\left(\frac{1}{x}\right)\right) e^{-t} x^{t+s-1} \left(t + s + O\left(\frac{\log^2 x}{2}\right)\right)$$

$$\lim_{n\to\infty} \Omega(s,t) = \begin{cases} 0; \ t+s<1 \\ e^{-t}; t+s=1 \\ \infty; \ t+s>1 \end{cases}$$

1968. Find:

$$\Omega = \lim_{n\to\infty} e^{5n+1} \tan^5 n \left(\int_0^n e^{5x}(\tan^4 x + \tan^5 x + \tan^6 x) dx\right)^{-1}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Pham Duc Nam-Vietnam

$$\int e^{5x}(\tan^4 x + \tan^5 x + \tan^6 x) dx = \int e^{5x} \tan^4 x \,(1 + \tan x + \tan^2 x) dx \overset{t=\tan x}{=}$$

$$= \int e^{5\tan^{-1} t} t^4 \left(\frac{t}{t^2+1} + 1\right) dt = \int e^{5\tan^{-1} t} t^4 \, dt + \int \frac{t^5 e^{5\tan^{-1} t}}{t^2+1} dt \overset{IBP}{=}$$

$$\Rightarrow \int e^{5\tan^{-1} t} t^4 \, dt + \int \frac{t^5 e^{5\tan^{-1} t}}{t^2+1} dt = \frac{1}{5} t^5 e^{5\tan^{-1} t} - \int \frac{t^5 e^{5\tan^{-1} t}}{t^2+1} dt + \int \frac{t^5 e^{5\tan^{-1} t}}{t^2+1} dt$$

$$= \frac{1}{5} t^5 e^{5\tan^{-1} t} + C = \frac{1}{5} \tan^5(x e^{5x}) + C$$

$$\int_0^n e^{5x}(\tan^4 x + \tan^5 x + \tan^6 x) dx = \frac{1}{5} \tan^5(n e^{5n})$$

$$\Omega = \lim_{n\to\infty} e^{5n+1} \tan^5 n \left(\int_0^n e^{5x}(\tan^4 x + \tan^5 x + \tan^6 x) dx\right)^{-1} = 5 \lim_{n\to\infty} \frac{e^{5n+1} \tan^5 n}{\tan^5(n e^{5n})}$$

$$= 5e$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\int_0^n e^{5x}(\tan^4 x+\tan^5 x+\tan^6 x)dx \overset{y=\tan x}{=} \int_0^{\tan n} e^{5\tan^{-1}y}(y^4+y^5+y^6)\cdot\frac{1}{1+y^2}dy=$$

$$=\int_0^{\tan n} e^{5\tan^{-1}y}y^4dy+\int_0^{\tan n} e^{5\tan^{-1}y}\frac{y^5}{1+y^2}dy=$$

$$=\frac{1}{5}\int_0^{\tan n} e^{5\tan^{-1}y}dy^5+\int_0^{\tan n} e^{5\tan^{-1}y}\frac{y^5}{1+y^2}dy$$

$$=\frac{1}{5}y^5e^{5\tan^{-1}y}\Big|_0^{\tan n}-\int_0^{\tan n} e^{5\tan^{-1}y}\frac{y^5}{1+y^2}dy+\int_0^{\tan n} e^{5\tan^{-1}y}\frac{y^5}{1+y^2}dy=$$

$$=\frac{1}{5}\tan^5(ne^{5n})$$

$$\Omega=\lim_{n\to\infty} e^{5n+1}\tan^5 n\left(\int_0^n e^{5x}(\tan^4 x+\tan^5 x+\tan^6 x)dx\right)^{-1}=$$

$$=5\lim_{n\to\infty}\frac{e^{5n+1}\tan^5 n}{\tan^5(ne^{5n})}=5e\lim_{n\to\infty} e^{5(n-\tan^{-1}(\tan n))}$$

$$\tan^{-1}(\tan n)=n \text{ for } -\frac{\pi}{2}<n<\frac{\pi}{2}$$

$$\Omega=5e \text{ if } n \text{ tends to a number in } \left(-\frac{\pi}{2};\frac{\pi}{2}\right).$$

1969. Find:

$$\Omega=\lim_{n\to\infty}\frac{\log\left(e\cdot(n+1)^{\frac{n+1}{n}}\right)\cdot\log\left(e\cdot(n)^{\frac{n+2}{n+1}}\right)}{\log(e^n(n+1)\cdot n^n)\cdot\log(e\cdot(n+1)\cdot n^{n+1})}$$

Proposed by Daniel Sitaru – Romania

Solution by Asmat Qatea-Afghanistan

$$p_1=\lim_{n\to\infty}\frac{\ln\left(e(n+1)^{\frac{n+1}{n}}\right)}{\ln(e^n\cdot(n+1)\cdot n^n)}=\lim_{n\to\infty}\underbrace{\frac{1+\left(1+\frac{1}{n}\right)\ln(n+1)}{n+\ln(n+1)+n\ln(n)}}_{Hopital}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{n+1}+\frac{\frac{n}{n+1}-\ln(n+1)}{n^2}}{1+\frac{1}{n+1}+\ln n+1}=\frac{0}{\infty}=0$$

$$p_2=\lim_{n\to\infty}\frac{\ln\left(e(n)^{\frac{n+2}{n+1}}\right)}{\ln(e\cdot(n+1)\cdot n^{n+1})}=\lim_{n\to\infty}\underbrace{\frac{1+\left(1+\frac{1}{n+1}\right)\ln(n)}{1+\ln(n+1)+(n+1)\ln(n)}}_{Hopital}=$$

$$=\frac{\frac{1}{n}+\frac{\frac{n+1}{n}-\ln n}{(n+1)^2}}{\frac{1}{n+1}+\ln(n)+\frac{n+1}{n}}=\frac{0}{\infty}=0$$

$$\Omega=p_1\cdot p_2=0$$

1970.

$$\sum_{k=0}^{\infty}\frac{2^{2k}}{(2k+1)^2({}^{2k}C_k)}\left(\frac{\pi}{2}-\frac{(2k)!!}{(2k+1)!!}\right)=2\pi G-\frac{7}{2}\zeta(3)$$

Here G is the Catalan Constant

Proposed by Kaushik Mahanta-Assam-India

Solution by Ajenikoko Gbolahan-Nigeria

$$\Psi=\sum_{k=0}^{\infty}\frac{2^{2k}}{(2k+1)^2\binom{2n}{n}}\left[\frac{\pi}{2}-\frac{(2k)!!}{(2k+1)!!}\right]$$

$$\Psi_1=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{2^{2k}}{(2k+1)^2\binom{2n}{n}}$$

$$\frac{2^{2k}}{(2k+1)^2\binom{2k}{k}}=\int_0^{\frac{\pi}{2}}\frac{\sin^{2k+1}(x)}{2k+1}dx\ \text{ summing up}$$

$$\Psi_1=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{2^{2k}}{(2k+1)^2\binom{2k}{k}}=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\tanh^{-1}(\sin x)\,dx$$

$$\Psi_1=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\ln(1+\sin x)-\ln(1-\sin x)\,dx$$

$$\Phi_1 = \int_0^{\frac{\pi}{2}} \ln(1+\sin x)\,dx = \int_0^{\frac{\pi}{2}} \ln(2) + 2\ln\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$$

$$\Phi_1 \overset{x\to\frac{\pi}{2}+\frac{\pi}{4}}{=} \frac{\pi}{2}\ln(2) + 4\left\{\int_0^{\frac{\pi}{2}} \ln\sin x\,dx - \int_0^{\frac{\pi}{4}} \ln\sin x\,dx\right\}$$

$$\Phi_1 = \frac{\pi}{2}\ln(2) + 4\left\{\frac{\pi}{2}\ln(2) - \left(-\frac{\pi}{4}\ln(2)\frac{G}{2}\right)\right\} = 2G - \frac{\pi}{2}\ln(2)$$

$$\Phi_2 = \int_0^{\frac{\pi}{2}} \ln(1-\sin x)\,dx = \frac{\pi}{2}\ln(2) + 4\int_0^{\frac{\pi}{4}} \ln\sin x\,dx$$

$$\Phi_2 = \frac{\pi}{2}\ln(2) + 4\left\{-\frac{\pi}{4}\ln(2) - \frac{G}{2}\right\} = -2G - \frac{\pi}{2}\ln(2)$$

$$\Phi = \Phi_1 - \Phi_2 = 2G - \frac{\pi}{2}\ln(2) - \left(-2G - \frac{\pi}{2}\ln(2)\right) = 4G, \qquad \Psi_1 = \frac{\pi}{4}\times 4G = \pi G$$

$$\Psi_2 = \sum_{k=0}^{\infty} \frac{2^{2k}(2k)!!}{(2k+1)^2(2k+1)!!\binom{2k}{k}} = \sum_{k=0}^{\infty} \frac{16^k}{(2k+1)^3\binom{2k}{k}^2}$$

$$\Omega = \sum_{k=1}^{\infty} \frac{x^k}{\binom{2k}{k}} = \frac{x}{4-x} + \frac{4\sqrt{x}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)}{\sqrt{4-x}(4-x)} \left(x \to 16x^2 \; and \; integrate \; [0,y]\right)$$

$$\Omega = \sum_{k=1}^{\infty} \frac{16^k y^{2k+1}}{(2k+1)\binom{2k}{k}} = -y + \frac{\sin^{-1}(2y)}{2\sqrt{1-4y^2}}$$

re-indexing, divide both sides by y and integrate

$$\Omega = \sum_{k=0}^{\infty} \frac{16^k y^{2k+1}}{(2k+1)^2\binom{2k}{k}} = \int_0^y \frac{\sin^{-1}(2y)}{2y\sqrt{1-4y^2}}dz$$

$$\Omega = \sum_{k=0}^{\infty} \frac{16^k y^{2k}}{(2k+1)^2\binom{2k}{k}} = \frac{1}{y}\int_0^y \frac{\sin^{-1}(2y)}{2y\sqrt{1-4y^2}}dz$$

$$\Omega = \sum_{k=0}^{\infty} \frac{16^k\left(y(1-y)\right)^k}{(2k+1)^2\binom{2k}{k}} = \frac{1}{\sqrt{y(1-y)}}\int_0^{\sqrt{y(1-y)}} \frac{\sin^{-1}(2y)}{2y\sqrt{1-4y^2}}dy$$

$$\Psi_2=\sum_{k=0}^{\infty}\frac{16^k}{(2k+1)^3\binom{2k}{k}^2}=\int_0^1\frac{1}{\sqrt{y(1-y)}}\int_0^{\sqrt{y(1-y)}}\frac{\sin^{-1}(2y_1)}{2y_1\sqrt{1-4y_1^2}}dydy_1$$

$$\Psi_2=\int_0^{\frac{1}{2}}\frac{\sin^{-1}(2y_1)\,2\sin^{-1}\left(\sqrt{1-4y_1^2}\right)}{2y_1\sqrt{1-4y_1^2}}\qquad (2y_1=\sin x)$$

$$\Psi_2=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}x\csc dx-\int_0^{\frac{\pi}{2}}x^2\csc x,\qquad \Psi_2=\pi G-\left(2\pi G-\frac{7}{2}\zeta(3)\right)=\frac{7}{2}\zeta(3)-\pi G$$

$$\Psi=\Psi_1-\Psi_2=\pi G-\left(\frac{7}{2}\zeta(3)-\pi G\right)=2\pi G-\frac{7}{2}\zeta(3)$$

1971. Find a closed form:

$$\Omega=\int_0^1\int_0^1(x^3+2x^2y)\,ln\left(1+\frac{1}{x+y}\right)dxdy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\int_0^1(x^3+2x^2y)\ln\left(1+\frac{1}{x+y}\right)dxdy$$

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^3+2x^2y}{x+y+z}dzdydx\quad (y=ux\quad z=vx)$$

$$\Omega=\int_0^1x^4\int_0^{\frac{1}{x}}\int_0^{\frac{1}{x}}\frac{1+2u}{1+u+v}dudvdx\overset{\text{symmetry}}{=}\int_0^1x^4\int_0^{\frac{1}{x}}\int_0^{\frac{1}{x}}\frac{1+2v}{1+u+v}dudvdx$$

$$2\Omega=\int_0^1x^4\int_0^{\frac{1}{x}}\int_0^{\frac{1}{x}}\frac{2+2u+2v}{1+u+v}dudvdx=2\int_0^1x^4\int_0^{\frac{1}{x}}\int_0^{\frac{1}{x}}dudvdx$$

$$\Omega=\int_0^1x^2dx=\frac{1}{3}$$

1972. Solve for real numbers:

$$\frac{1}{x}\left(e^{\frac{\pi}{2}}+\int_1^x\frac{(t+1)^2}{t^2+1}\cdot e^{2arctant}dt\right)=e^{\frac{2\sqrt{3}}{3}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Igor Soposki-Skopje-Macedonia

$$\int_1^x \frac{(t+1)^2}{t^2+1}\cdot e^{2arctant}dt = \int_1^x \frac{t^2+2t+1}{t^2+1}\cdot e^{2arctant}dt =$$

$$= \int_1^x e^{2arctant}dt + \int_1^x \frac{2t}{t^2+1}\cdot e^{2arctant}dt \overset{IBP}{=}$$

$$= xe^{2arctanx} - e^{\frac{\pi}{2}} - \int_1^x \frac{2t}{t^2+1}\cdot e^{2arctant}dt + \int_1^x \frac{2t}{t^2+1}\cdot e^{2arctant}dt =$$

$$= xe^{2arctanx} - e^{\frac{\pi}{2}}$$

$$\frac{1}{x}\left(e^{\frac{\pi}{2}} + \int_1^x \frac{(t+1)^2}{t^2+1}\cdot e^{2arctant}dt\right) = e^{\frac{2\sqrt{3}}{3}}$$

$$\frac{1}{x}\left(e^{\frac{\pi}{2}} + xe^{2arctanx} - e^{\frac{\pi}{2}}\right) = e^{\frac{2\sqrt{3}}{3}}, \quad e^{2arctanx} = e^{\frac{2\sqrt{3}}{3}} \to 2arctanx = \frac{2\sqrt{3}}{3} \to x = \frac{\pi}{6}$$

Solution 2 by Asmat Qatea-Afghanistan

$$\frac{1}{x}\left(e^{\frac{\pi}{2}} + \int_1^x \frac{(t+1)^2}{t^2+1}\cdot e^{2arctant}dt\right) = e^{\frac{2\sqrt{3}}{3}}$$

$$\frac{1}{x}\left(e^{\frac{\pi}{2}} + \int_1^x e^{2arctant}dt + \int_1^x \frac{2t}{t^2+1}\cdot e^{2arctant}dt\right) = e^{\frac{2\sqrt{3}}{3}}$$

$$\frac{1}{x}\left(e^{\frac{\pi}{2}} + \int_1^x (te^{2arctant})'dt\right) = e^{\frac{2\sqrt{3}}{3}}$$

$$\frac{1}{x}\left(e^{\frac{\pi}{2}} + xe^{2arctanx} - e^{\frac{\pi}{2}}\right) = e^{\frac{2\sqrt{3}}{3}}, \quad e^{2arctanx} = e^{\frac{2\sqrt{3}}{3}} \to 2arctanx = \frac{2\sqrt{3}}{3} \to x = \frac{\pi}{6}$$

1973. Find:

$$\Omega = \int \frac{x^3 + x - arctanx}{(x^2+1)(x^2 - arctan^2x)}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ose Favour-Nigeria

$$\mathbf{I}=\int\frac{\mathbf{x^3+x-atan\,x}}{\mathbf{(x^2+1)(x^2-atan^2x)}}\mathbf{dx}\overset{x=\tan(y)}{\widehat{=}}\int\frac{\mathbf{tan\,y\,sec^2y-y}}{\mathbf{tan^2y-y^2}}\mathbf{dy}$$

$$\mathbf{I}\overset{u=\tan^2y-y^2}{\widehat{=}}\int\frac{\mathbf{tan\,y\,sec^2y-y}}{\mathbf{u}}\cdot\frac{\mathbf{du}}{\mathbf{2(tan\,y\,sec^2y-y)}}=\frac{1}{2}\int\frac{1}{\mathbf{u}}\mathbf{du}$$

$$\mathbf{I}=\frac{1}{2}\ln u+C=\frac{1}{2}\ln(\tan^2y-y^2)+C=\frac{1}{2}\ln(x^2-atan^2x)+C$$

Solution 2 by Pham Duc Nam-Vietnam

Let $t=arctanx, x=tant, dt=\frac{dx}{1+x^2}$

$$\Omega=\int\frac{tan^3t+tant-t}{tan^2t-t^2}dt\,, u=tan^2t-t^2,\qquad \frac{1}{2}du=(tan^3t+tant-t)dt$$

$$\Omega=\frac{1}{2}\int\frac{1}{u}du=\frac{1}{2}ln(tan^2t-t^2)+C=\frac{1}{2}ln(x^2-arctan^2x)+C$$

Solution 3 by Saboor Halimi-Afghanistan

Let u be: $x^2-\arctan^2(x)$

$$du=2x-\frac{2\arctan(x)}{1+x^2}dx\Rightarrow x-\frac{\arctan(x)}{1+x^2}dx=\frac{du}{2}$$

$$\frac{1}{2}\int\frac{1}{u}du=\frac{1}{2}\ln|u|\Rightarrow\frac{1}{2}\ln|x^2-\arctan^2(x)|+C$$

Solution 4 by Yen Tung Chung-Taichung-Taiwan

$$\int\frac{x^3+x-\arctan x}{(x^2+1)(x^2-\arctan^2x)}dx=\underbrace{\int\frac{1}{x^2-\arctan^2x}\cdot\left(x-\frac{\arctan x}{x^2+1}\right)dx}_{let\ u=x^2-\arctan^2x\Rightarrow du=2\left(x-\frac{\arctan x}{x^2+1}\right)dx}$$

$$=\int\frac{1}{u}\cdot\frac{1}{2}du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln|x^2-\arctan^2x|+C$$

1974. Find a closed form:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^9-x^3y^3z^3}{(1-x^2y^2z^2)(x^3+y^3+z^3)}dxdydz$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^9-x^3y^3z^3}{(1-x^2y^2z^2)(x^3+y^3+z^3)}dxdydz$$

$$3\Omega=\int_0^1\int_0^1\int_0^1\frac{x^9+y^9+z^9-3x^3y^3z^3}{(1-x^2y^2z^2)(x^3+y^3+z^3)}dxdydz$$

$$\Omega=\frac{1}{3}\int_0^1\int_0^1\int_0^1\frac{x^6+y^6+z^6-x^3y^3-x^3z^3-y^3z^3}{1-x^2y^2z^2}dxdydz$$

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^6-x^3y^3}{1-x^2y^2z^2}dxdydz=\underbrace{\int_0^1\int_0^1\int_0^1\frac{x^6}{1-x^2y^2z^2}dxdydz}_{A}-\underbrace{\int_0^1\int_0^1\int_0^1\frac{x^3y^3}{1-x^2y^2z^2}dxdydz}_{B}$$

$$A=\sum_{n=0}^{\infty}\int_0^1 z^{2n}\int_0^1 y^{2n}\int_0^1 x^{2n+6}dxdydz=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2(2n+7)}$$

$$A=\frac{1}{36}\sum_{n=0}^{\infty}\left(\frac{1}{2n+7}-\frac{1}{2n+1}\right)+\frac{1}{6}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{1}{72}\psi\left(\frac{1}{2}\right)-\frac{1}{72}\psi\left(\frac{7}{2}\right)+\frac{1}{24}\psi^{(1)}\left(\frac{1}{2}\right)$$

$$A=-\frac{23}{540}+\frac{\pi^2}{48}$$

$$B=\sum_{n=0}^{\infty}\int_0^1 z^{2n}\int_0^1 y^{2n+3}\int_0^1 x^{2n+3}dxdydz=\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+4)^2}$$

$$B=\frac{1}{18}\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n+2}\right)-\frac{1}{12}\sum_{n=0}^{\infty}\frac{1}{(n+2)^2}=\frac{1}{18}\psi(2)-\frac{1}{18}\psi\left(\frac{1}{2}\right)-\frac{1}{12}\psi^{(1)}(2)$$

$$B=\frac{1}{18}+\frac{1}{9}\log 2+\frac{1}{12}-\frac{\pi^2}{72}=\frac{5}{36}+\frac{1}{9}\log 2-\frac{\pi^2}{72}$$

$$\int_0^1\int_0^1\int_0^1\frac{x^9-x^3y^3z^3}{(1-x^2y^2z^2)(x^3+y^3+z^3)}dxdydz=\frac{5\pi^2}{144}-\frac{1}{9}\log 2-\frac{49}{270}$$

1975. Let $(x_n)_{n\ge1}$ be a sequence such that $x_1=2, x_2=16$ and

$$x_n^{n^2+3n+2}\cdot x_{n+2}^{n^2+n}=x_{n+1}^{2n^2+4n},\forall n\ge 1.\text{ Find:}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt[k]{x_k}}{k\binom{2k}{k}}$$

Proposed by Ruxandra Daniela Tonilă-Romania

Solution 1 by proposer

$$x_n^{n^2+3n+2} \cdot x_{n+2}^{n^2+n} = x_{n+1}^{2n^2+4n} \Leftrightarrow x_n^{(n+1)(n+2)} \cdot x_{n+2}^{n(n+1)} = x_{n+1}^{2n(n+2)}$$

$$\Leftrightarrow x_n^{\frac{1}{n}} \cdot x_{n+2}^{\frac{1}{n+2}} = x_{n+1}^{\frac{2}{n+1}} \Leftrightarrow \frac{\log x_n}{n} + \frac{\log x_{n+2}}{n+2} = \frac{2\log x_{n+1}}{n+1}$$

Let $(y_n)_{n\geq 1}$ be a sequence such that $y_n = \frac{\log x_n}{n}$, with $y_1 = \log 2\,, y_2 = 2\log 2$.

Then:

$$y_n + y_{n+2} = 2y_{n+1}$$

We have:

$$1 + r^2 = 2r \Leftrightarrow (r-1)^2 = 0 \Leftrightarrow r = 1$$

$$\Rightarrow y_n = c_1 r^n + n c_2 r^n = c_1 + n c_2$$

$$\begin{cases} y_1 = c_1 + c_2 = \log 2 \\ y_2 = c_1 + 2c_2 = 2\log 2 \end{cases} \Rightarrow \begin{cases} c_2 = \log 2 \\ c_1 = 0 \end{cases}$$

Therefore, $y_n = n\log 2 \Leftrightarrow x_n = 2^{n^2}, \forall\, n \geq 1$

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt[k]{2^{k^2}}}{k\binom{2k}{k}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^k}{k}\cdot\frac{k!\cdot k!}{(2k)!} = \lim_{n\to\infty}\sum_{k=1}^{n}2^k\cdot\frac{(k-1)!\cdot k!}{(2k)!}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}2^k\cdot\frac{\Gamma(k)\cdot\Gamma(k+1)}{\Gamma(2k+1)} = \lim_{n\to\infty}\sum_{k=1}^{n}2^k\cdot B(k,k+1)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}2^k\cdot\int_0^1 x^{k-1}(1-x)^k\,dx = \lim_{n\to\infty}\int_0^1\sum_{k=1}^{n}2^k\cdot x^{k-1}(1-x)^k\,dx$$

$$= \int_0^1\sum_{k=1}^{\infty}\frac{[2x(1-x)]^k}{x}\,dx = \int_0^1\frac{1}{x}\cdot\left(\frac{1}{1-2x(1-x)}-1\right)dx$$

$$\left(\because \sum_{k=1}^{\infty}x^k = \frac{1}{1-x}-1, \forall\,|x|<1\right)$$

$$\Rightarrow \Omega = \int_0^1\frac{1}{x}\cdot\frac{2x(1-x)}{2x^2-2x+1}\,dx = -\int_0^1\frac{2x-2}{2x^2-2x+1}\,dx$$

$$= -\frac{1}{2}\left(\int_0^1\frac{4x-2}{2x^2-2x+1}\,dx - 2\int_0^1\frac{1}{2x^2-2x+1}\,dx\right)$$

$$= -\frac{1}{2}\left(\log(2x^2-2x+1)\Big|_0^1 - \int_0^1 \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx\right)$$

$$= -\frac{1}{2}\cdot\left(-2\cdot\arctan 2\left(x-\frac{1}{2}\right)\Big|_0^1\right) = \arctan(2x\text{-}1)\Big|_0^1$$

Hence, $\Omega = \frac{\pi}{2}$

Solution 2 by Remus Florin Stanca-Romania

$$x_n^{n^2+3n+2}\cdot x_{n+2}^{n^2+n} = x_{n+1}^{2n^2+4n} \Rightarrow (n^2+3n+2)\ln x_n + (n^2+n)\ln(x_{n+2}) =$$

$$= (2n^2+4n)\ln x_{n+1}\Big|\cdot\frac{1}{n(n+1)(n+2)}$$

$$\Leftrightarrow \frac{\ln x_n}{n} + \frac{\ln x_{n+2}}{n+2} = \frac{2\ln x_{n+1}}{n+1}, \text{ let } \frac{\ln x_n}{n} = y_n \Rightarrow y_n + y_{n+2} =$$

$$= 2y_{n+1} \Rightarrow y_{n+2} - y_{n+1} = y_{n+1} - y_n = \cdots = y_2 - y_1 =$$

$$= \ln 4 - \ln 2 = \ln 2 \Rightarrow y_n = y_1 + (n-1)\cdot\ln 2 = n\ln 2 \Rightarrow \frac{\ln x_n}{n} = n\ln 2 \Rightarrow$$

$$x_n = 2^{n^2} \Rightarrow$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^k}{k\cdot C_{2k}^k};\frac{1}{kC_{2k}^k} = \frac{1}{k\cdot\frac{(2k)!}{k!\cdot k!}} = \frac{\Gamma(k)\Gamma(k+1)}{\Gamma(2k+1)} = B(k,k+1) =$$

$$= \int_0^1 x^{k-1}(1-x)^k\,dx \Rightarrow$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\int_0^1 \frac{1}{x}\cdot 2(1-x)\cdot x\cdot\lim_{n\to\infty}\frac{\left(2x(1-x)\right)^{n+1}-1}{2x(1-x)-1} = \int_0^1 \frac{2(1-x)}{2x^2-2x+1}dx =$$

$$= -\frac{1}{2}\left(\int_0^1 \frac{4x-2}{2x^2-2x+1}dx - 2\int_0^1\frac{1}{2x^2-2x+1}dx\right) =$$

$$= -\frac{1}{2}\ln|2x^2-2x+1|\Bigg| + \int_0^1 \frac{1}{\left(x\sqrt{2}-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}dx = \frac{\pi}{4} \cdot\!= \frac{\pi}{2}$$

1976. Find:

$$\Omega = \lim_{x\to\infty}\left(log^2 x\cdot\int_0^1 \frac{t\cdot x^t}{(1+x^t)^2}dt\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

$$\int_0^1 \frac{t\cdot x^t}{(1+x^t)^2}dt = -\frac{1}{(x+1)lnx}+\frac{1}{lnx}\int_0^1\frac{1+x^t-x^t}{1+x^t}dt =$$

$$= -\frac{1}{(x+1)lnx}+\frac{1}{lnx}\int_0^1\left(1-\frac{x^t}{1+x^t}\right)dt = -\frac{1}{(x+1)lnx}+\frac{1}{lnx}\left(1-\frac{ln(x+1)}{lnx}+\frac{ln2}{lnx}\right)$$

$$\Omega = \lim_{x\to\infty}\left(ln^2 x\cdot\int_0^1\frac{t\cdot x^t}{(1+x^t)^2}dt\right) =$$

$$= \lim_{x\to\infty}\left(ln^2 x\cdot\left(-\frac{1}{(x+1)lnx}+\frac{1}{lnx}\left(1-\frac{ln(x+1)}{lnx}+\frac{ln2}{lnx}\right)\right)\right) =$$

$$= \lim_{x\to\infty}\left(-\frac{lnx}{(x+1)}+lnx-ln(x+1)+ln2\right) = ln2+\lim_{x\to\infty} ln\frac{x}{x+1} = ln2$$

Solution 2 by Pham Duc Nam-Vietnam

$$\int_0^1\frac{tx^t}{(1+x^t)^2}dt = \frac{1}{\ln x}\int_0^1\frac{t}{(1+x^t)^2}d(x^t+1), \begin{cases} u = t \\ dv = \frac{1}{(1+x^t)^2}d(x^t+1)\end{cases} \Rightarrow \begin{cases} du = dt \\ v = -\frac{1}{1+x^t}\end{cases}$$

$$\Rightarrow \int_0^1\frac{tx^t}{(1+x^t)^2}dt = -\frac{t}{\ln x\,(1+x^t)}\Big|_0^1+\frac{1}{\ln x}\int_0^1\frac{1}{1+x^t}dt$$

$$= -\frac{1}{\ln x\,(1+x)}+\frac{1}{\ln^2 x}\int_0^1\frac{1}{x^t(x^t+1)}d(x^t)$$

$$= -\frac{1}{\ln x\,(1+x)}+\frac{1}{\ln^2 x}\int_0^1\frac{1}{x^t}d(x^t)-\frac{1}{\ln^2 x}\int_0^1\frac{1}{x^t+1}d(x^t)$$

$$= -\frac{1}{\ln x\,(1+x)}+\frac{1}{\ln^2 x}\left(\ln\frac{x^t}{x^t+1}\right)\Big|_0^1$$

$$= -\frac{1}{\ln x\,(1+x)}+\frac{1}{\ln^2 x}\left(\ln\frac{x}{x+1}+\ln 2\right)$$

$$*\Rightarrow \Omega = \lim_{x\to\infty}\left(\ln^2 x \cdot \int_0^1 \frac{tx^t}{(1+x^t)^2}dt\right) = \lim_{x\to\infty}\left(\ln^2 x\left(-\frac{1}{\ln x\,(1+x)} + \frac{1}{\ln^2 x}\left(\ln\frac{x}{x+1} + \ln 2\right)\right)\right)$$

$$= \underbrace{\lim_{x\to\infty}\left(-\frac{\ln x}{1+x}\right)}_{=0} + \lim_{x\to\infty}\left(\ln\frac{x}{x+1} + \ln 2\right) = \ln 2$$

Solution 3 by Rana Ranino-Setif-Algerie

$$\Omega = \lim_{x\to\infty}\left(\log^2 x \int_0^1 \frac{tx^t}{(1+x^t)^2}dt\right)$$

$$x^t = y \qquad t = \frac{\log y}{\log x} \qquad dt = \frac{dy}{y\log x}$$

$$\lim_{x\to\infty}\int_1^x \frac{\log y}{(1+y)^2}dy \overset{y\to\frac{1}{y}}{\cong} -\int_0^1 \frac{\log y}{(1+y)^2}dy \overset{IBP}{\cong} \left[\log(y+1) - \frac{y\log y}{y+1}\right]_0^1 = \log 2$$

Solution 4 by Adrian Popa-Romania

$$\int_0^1 \frac{t\cdot x^t}{(1+x^t)^2}dt = -\frac{1}{(x+1)\ln x} + \frac{1}{\ln x}\int_0^1 \frac{1+x^t-x^t}{1+x^t}dt =$$

$$= -\frac{1}{(x+1)\ln x} + \frac{1}{\ln x}\int_0^1\left(1 - \frac{x^t}{1+x^t}\right)dt =$$

$$= -\frac{1}{(x+1)\ln x} + \frac{1}{\ln x}\left(1 - \frac{\ln(x+1)}{\ln x} + \frac{\ln 2}{\ln x}\right)$$

$$\Omega = \lim_{x\to\infty}\left(\ln^2 x\cdot\int_0^1 \frac{t\cdot x^t}{(1+x^t)^2}dt\right) =$$

$$= \lim_{x\to\infty}\left(\ln^2 x\cdot\left(-\frac{1}{(x+1)\ln x} + \frac{1}{\ln x}\left(1 - \frac{\ln(x+1)}{\ln x} + \frac{\ln 2}{\ln x}\right)\right)\right) =$$

$$= \lim_{x\to\infty}\left(-\frac{\ln x}{(x+1)} + \ln x - \ln(x+1) + \ln 2\right) = \ln 2 + \lim_{x\to\infty}\ln\frac{x}{x+1} = \ln 2$$

1977. Find:

$$\Omega=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin 2x\cdot\cos 4x\cdot\cos 8x}{\sin^5\left(\frac{\pi}{4}-x\right)+\cos^5\left(\frac{\pi}{4}-x\right)}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$\Omega=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin(-2x)\cos(-4x)\cos(-8x)}{\sin^5\left(\frac{\pi}{4}+x\right)+\cos^5\left(\frac{\pi}{4}+x\right)}dx=$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{-\sin(2x)\cos(4x)\cos(8x)}{\cos^5\left(\frac{\pi}{2}-\left(\frac{\pi}{4}+x\right)\right)+\sin^5\left(\frac{\pi}{2}-\left(\frac{\pi}{4}+x\right)\right)}dx=$$

$$=-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin 2x\cos 4x\cos 8x}{\cos^5\left(\frac{\pi}{4}-x\right)+\sin^5\left(\frac{\pi}{4}-x\right)}dx=-\Omega\Rightarrow 2\Omega=0\Rightarrow\Rightarrow\Omega=0$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin(2x)\cdot\cos(4x)\cdot\cos(8x)}{\sin^4\left(\frac{\pi}{4}-x\right)+\cos^5\left(\frac{\pi}{4}-x\right)}dx$$

$$\cos^5\left(\frac{\pi}{4}-x\right)=\frac{5}{8}\cos\left(\frac{\pi}{4}-x\right)+\frac{5}{16}\cos\left[3\left(\frac{\pi}{4}-x\right)\right]+\frac{1}{16}\cos\left[5\left(\frac{\pi}{4}-x\right)\right]$$

$$\sin^5\left(\frac{\pi}{4}-x\right)=\frac{5}{8}\sin\left(\frac{\pi}{4}-x\right)-\frac{5}{16}\sin\left[3\left(\frac{\pi}{4}-x\right)\right]+\frac{1}{16}\sin\left[5\left(\frac{\pi}{4}-x\right)\right]$$

$$\cos^5\left(\frac{\pi}{4}-x\right)+\sin^5\left(\frac{\pi}{4}-x\right)$$

$$=\frac{5}{8}\left(\frac{1}{\sqrt{2}}\cos(x)+\frac{1}{\sqrt{2}}\sin(x)+\frac{1}{\sqrt{2}}\cos(x)-\frac{1}{\sqrt{2}}\sin(x)\right)$$

$$+\frac{5}{16}\left(-\frac{1}{\sqrt{2}}\cos(3x)+\frac{1}{\sqrt{2}}\sin(3x)-\frac{1}{\sqrt{2}}\sin(3x)-\frac{1}{\sqrt{2}}\cos(3x)\right)$$

$$+\frac{1}{16}\left(-\frac{1}{\sqrt{2}}\cos(5x)-\frac{1}{\sqrt{2}}\sin(5x)-\frac{1}{\sqrt{2}}\cos(5x)+\frac{1}{\sqrt{2}}\sin(5x)\right)$$

$$=\frac{5\sqrt{2}}{8}\cos(x)-\frac{5\sqrt{2}}{16}\cos(3x)-\frac{\sqrt{2}}{16}\cos(5x)\Rightarrow\text{even function}$$

The integrand is odd due to $\sin(2x)=\Omega=0$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin(2x)\cdot\cos(4x)\cdot\cos(8x)}{\sin^5\left(\frac{\pi}{4}-x\right)+\cos^5\left(\frac{\pi}{4}-x\right)}dx\Rightarrow$$

$$\Rightarrow 4\sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin(2x)\cdot\cos(4x)\cdot\cos(8x)}{(\cos(x)-\sin(x))^5+(\cos(x)+\sin(x))^5}dx$$

Let $(x)=\frac{\sin(2x)\cdot\cos(4x)\cdot\cos(8x)}{(\cos(x)-\sin(x))^5+(\cos(x)+\sin(x))^5}$, replace x by $-x$

$$f(-x)=-\frac{\sin(2x)\cdot\cos(4x)\cdot\cos(8x)}{(\cos(x)+\sin(x))^5+(\cos(x)-\sin(x))^5}\Rightarrow f(-x)=-f(x)$$

That means $f(x)$ is an odd function and we know,

$$\int_{-a}^{+a}f(x)\,dx=\begin{cases}2\int_0^a f(x)\,dx, if\ f(-x)=f(x)\\ 0,\quad if\ f(-x)=-f(x)\end{cases}$$

$$\Omega=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin(2x)\cdot\cos(4x)\cdot\cos(8x)}{\sin^5\left(\frac{\pi}{4}-x\right)+\cos^5\left(\frac{\pi}{4}-x\right)}dx=0$$

1978. Find:

$$\Omega=\int\frac{sinx+4cosx}{5(e^{-x}+sinx)+3cosx}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammad Rostami-Afghanistan

$$\Omega=\int\frac{sinx+4cosx}{5(e^{-x}+sinx)+3cosx}dx=$$

$$=\int\frac{5(e^{-x}+sinx)+3cosx+cosx-5e^{-x}-4sinx}{5(e^{-x}+sinx)+3cosx}dx=$$

$$=\int\left(1+\frac{-5e^{-x}+5cosx-3sinx-sinx-4cosx}{5(e^{-x}+sinx)+3cosx}\right)dx=$$

$$=\int dx+\int\frac{-5e^{-x}+5cosx-3sinx}{5(e^{-x}+sinx)+3cosx}dx-\Omega$$

$$2\Omega=x+ln(5(e^{-x}+sinx)+3cosx)+C$$

$$\Omega=\frac{x}{2}+\frac{1}{2}ln(5(e^{-x}+sinx)+3cosx)+C$$

Solution 2 by Tapas Das-India

$$u = 5(1 + e^x sinx) + 3e^x cosx, \qquad du = e^x(2sinx + 8cosx)dx$$

$$\frac{1}{2}du = e^x(sinx + 4cosx)dx$$

$$\Omega = \int \frac{sinx + 4cosx}{5(e^{-x} + sinx) + 3cosx}dx = \int \frac{e^x(sinx + 4cosx)}{5(1 + e^x sinx) + 3e^x cosx}dx =$$

$$= \int \frac{1}{2u}du = \frac{1}{2}log|u| + C = \frac{1}{2}log|5(1 + e^x sinx) + 3e^x cosx| + C$$

Solution 3 by Saboor Halimi-Afghanistan

$$m = 5(1 + e^x sinx) + 3e^x cosx, \qquad dm = e^x(2sinx + 8cosx)dx$$

$$\frac{1}{2}dm = e^x(sinx + 4cosx)dx$$

$$\Omega = \int \frac{sinx + 4cosx}{5(e^{-x} + sinx) + 3cosx}dx = \int \frac{e^x(sinx + 4cosx)}{5(1 + e^x sinx) + 3e^x cosx}dx =$$

$$= \int \frac{1}{2m}dm = \frac{1}{2}log|m| + C = \frac{1}{2}log|5(1 + e^x sinx) + 3e^x cosx| + C$$

1979. Find a closed form:

$$\Omega = \int_0^1 \int_0^1 \frac{dxdy}{(x + y)\sqrt{(1 - x)(1 - y)}}$$

Proposed by Abdul Mukhtar-Nigeria

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1 \int_0^1 \frac{1}{\sqrt{1 - x}} \frac{1}{x + y} \frac{1}{\sqrt{1 - y}} dxdy$$

$$\Omega \overset{\substack{x=1-x \\ y=1-y}}{\cong} \int_0^1 \int_0^1 \frac{1}{\sqrt{x}} \left(\frac{1}{2 - x - y}\right) \frac{1}{\sqrt{x}} dxdy \overset{\substack{x=x^2 \\ y=y^2}}{\cong} 4 \int_0^1 \int_0^1 \frac{1}{(2 - y^2) - x^2} dxdy$$

Polar coordinates substitution : $\begin{cases} x = r\cos\theta & \left(0 < \theta < \frac{\pi}{4} \quad 0 < r < \frac{1}{\cos\theta}\right) \\ y = r\sin\theta & \left(\frac{\pi}{4} < \theta < \frac{\pi}{2} \quad 0 < r < \frac{1}{\sin\theta}\right) \end{cases}$ $\quad dxdy = rdrd\theta$

$$\Omega = 4\int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\theta}} \frac{r}{2 - r^2} drd\theta + 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\theta}} \frac{r}{2 - r^2} drd\theta$$

$$\Omega = -2\int_0^{\frac{\pi}{4}}[\log(2-r^2)]_0^{\frac{1}{\cos\theta}}d\theta - 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}[\log(2-r^2)]_0^{\frac{1}{\sin\theta}}d\theta$$

$$\Omega = -2\underbrace{\int_0^{\frac{\pi}{4}}\log\left(1-\frac{1}{2\cos^2\theta}\right)d\theta}_{\theta=\frac{\pi}{4}-\theta} - 2\underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\log\left(1-\frac{1}{2\sin^2\theta}\right)d\theta}_{\theta=\frac{\pi}{4}+\theta}$$

$$\Omega = -4\underbrace{\int_0^{\frac{\pi}{4}}\log\left(\frac{\sin 2\theta}{1+\sin 2\theta}\right)d\theta}_{\theta=\frac{\pi}{4}-\theta} = -4\int_0^{\frac{\pi}{4}}\log\left(\frac{\cos 2\theta}{1+\cos 2\theta}\right)d\theta \overset{\varphi=2\theta}{\cong} -2\int_0^{\frac{\pi}{2}}\log\left(\frac{\cos\varphi}{1+\cos\varphi}\right)d\varphi$$

$$\Omega = -2\int_0^{\frac{\pi}{2}}\log(\cos\varphi)\,d\varphi + 2\int_0^{\frac{\pi}{2}}\log\left(2\cos^2\frac{\varphi}{2}\right)d\varphi$$

$$\Omega = 2\pi\log 2 + 8\int_0^{\frac{\pi}{4}}\log(\cos\varphi)\,d\varphi = 2\pi\log 2 + 8\left(\frac{G}{2}-\frac{\pi}{4}\log 2\right)$$

$$\int_0^1\int_0^1 \frac{1}{\sqrt{1-x}}\frac{1}{x+y}\frac{1}{\sqrt{1-y}}dxdy = 4G$$

1980. Find a closed form:

$$\Omega = \int_0^{\frac{\pi}{3}} ln^3\left(\frac{2tanx}{\sqrt{3}+tanx}\right)dx$$

Proposed by Jalam Hauwa'u-Nigeria

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^{\frac{\pi}{3}}\log^3\left(\frac{2\tan x}{\sqrt{3}+\tan x}\right)dx$$

$$\Omega \overset{x\to\frac{\pi}{3}-x}{\cong} \int_0^{\frac{\pi}{3}}\log^3\left(\frac{\sqrt{3}-\tan x}{\sqrt{3}+\tan x}\right)dx \overset{\tan x=y\sqrt{3}}{\cong} \sqrt{3}\int_0^1\frac{\log^3\left(\frac{1-y}{1+y}\right)}{1+3y^2}dy$$

$$\Omega \overset{y\to\frac{1-y}{1+y}}{\cong} \frac{\sqrt{3}}{2}\int_0^1\frac{\log^3 y}{y^2-y+1}dy = \frac{\sqrt{3}}{2}\int_0^1\frac{(1+y)(1-y^3)\log^3 y}{1-y^6}dy$$

$$\Omega \overset{y^6=t}{\cong} \frac{1}{864\sqrt{3}}\int_0^1\frac{\left(t^{\frac{1}{6}-1}+t^{\frac{1}{3}-1}-t^{\frac{2}{3}-1}-t^{\frac{5}{6}-1}\right)\log^3 t}{1-t}dt$$

$$\Omega \overset{y^6=t}{=} \frac{1}{864\sqrt{3}}\int_0^1 \frac{\left(t^{\frac{1}{6}-1}+t^{\frac{1}{3}-1}-t^{\frac{2}{3}-1}-t^{\frac{5}{6}-1}\right)\log^3 t}{1-t}dt$$

$$\Omega = \frac{1}{864\sqrt{3}}\left(\psi^{(3)}\left(\frac{5}{6}\right)-\psi^{(3)}\left(\frac{1}{6}\right)+\psi^{(3)}\left(\frac{2}{3}\right)-\psi^{(3)}\left(\frac{1}{3}\right)\right)$$

$$\psi^{(3)}\left(\frac{5}{6}\right)+\psi^{(3)}\left(\frac{1}{6}\right)=80\pi^4 \qquad \psi^{(3)}\left(\frac{2}{3}\right)+\psi^{(3)}\left(\frac{1}{3}\right)=\frac{16\pi^4}{3}$$

$$\int_0^{\frac{\pi}{3}}\log^3\left(\frac{2\tan x}{\sqrt{3}+\tan x}\right)dx=\frac{8\pi^4}{81\sqrt{3}}-\frac{1}{432\sqrt{3}}\left(\psi^{(3)}\left(\frac{1}{6}\right)+\psi^{(3)}\left(\frac{1}{3}\right)\right)$$

1981.

$$\sum_{k=1}^{n}\cos\left(\frac{k\pi}{n}\right)\left[\psi\left(\frac{k}{2n}\right)-\psi\left(\frac{k}{2n}+\frac{1}{2}\right)\right]=2n\ln\left[2\sin\left(\frac{\pi}{2n}\right)\right]$$

Where $\psi(*)$ is digamma function

Proposed by Asmat Qatea-Afghanistan

Solution by Hamza Djahel-Algerie

$$S(n)=\sum_{k=1}^{n}\cos\left(\frac{k\pi}{n}\right)\left[\psi\left(\frac{k}{2n}\right)-\psi\left(\frac{k}{2n}+\frac{1}{2}\right)\right]=2n\ln\left[2\sin\left(\frac{\pi}{2n}\right)\right]$$

$$\sum_{k=1}^{n}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right)=\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right)-\sum_{k=n+1}^{\infty}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right)$$

$$=\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right)-\sum_{k=1}^{\infty}\cos\left(\frac{k+n}{n}\pi\right)\psi\left(\frac{k+n}{2n}\right)$$

$$=\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{2n}\right)\psi\left(\frac{k}{2n}\right)+\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}+\frac{1}{2}\right)$$

$$=\sum_{k=1}^{\infty}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right)+\sum_{k=1}^{n}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}+\frac{1}{2}\right)+\sum_{k=n+1}^{\infty}\cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}+\frac{1}{2}\right)$$

$$\Rightarrow \sum_{k=1}^{n} \cos\left(\frac{k\pi}{n}\right)\left[\psi\left(\frac{k}{2n}\right) - \psi\left(\frac{k}{2n}+\frac{1}{2}\right)\right] =$$

$$= \sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right) + \sum_{k=n+1}^{\infty} \cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}+\frac{1}{2}\right)$$

$$\Rightarrow S(n) = \sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right) + \sum_{k=1}^{\infty} \cos\left(\frac{k+n}{n}\pi\right)\psi\left(\frac{k+n}{2n}+\frac{1}{2}\right)$$

$$\Rightarrow S(n) = \sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}\right) - \sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{n}\right)\psi\left(\frac{k}{2n}+1\right)$$

$$\Rightarrow S(n) = \sum_{k=1}^{\infty} \cos\left(\frac{k\pi}{n}\right)\left[\psi\left(\frac{k}{2n}\right) - \psi\left(\frac{k}{2n}+1\right)\right] =$$

$$= -\sum_{k=1}^{\infty} \frac{2n\cos\left(\frac{k\pi}{n}\right)}{k} = 2n\ln\left(2\sin\left(\frac{1}{2n}\right)\right)$$

Notice:

$$-\sum_{k=1}^{n} \frac{\cos(kx)}{k} = \ln\left(2\sin\left(\frac{x}{2}\right)\right);\ \psi(x+1) - \psi(x) = \frac{1}{x}$$

1982. Prove the below integral

$$\int_0^\infty \int_0^\infty \frac{\log_e^2(xy) - \log_e^2(x) - \log_e^2(y)}{\sqrt{y}(1+x^3)(1+y)^2(1+x^2)} dxdy = \frac{444\pi\zeta(2)}{432}$$

Where $\zeta(s), \mathcal{R}(s) > 1$ is the Euler-Riemann zeta function.

Proposed by Ankush Kumar Parcha-India

Solution by proposer

$$\text{Say } A = \int_0^\infty \int_0^\infty \frac{\log_e^2(xy) - \log_2^2(x) - \log_e^2(y)}{\sqrt{y}(1+x^3)(1+y)^2(1+x^2)} dxdy \Rightarrow$$

$$\Rightarrow \int_0^\infty \int_0^\infty \frac{\log_e^2(x) + \log_e^2(y) + 2\log_e(x)\log_e(y) - \log_e^2(x) - \log_e^2(y)}{2\sqrt{y}(1+y)^2}\left[\frac{1+x}{1+x^2} + \frac{1-x-x^2}{1+x^3}\right] dxdy$$

$$A = \int_0^\infty \int_0^\infty \frac{\log_e(x)\log_e(y)}{\sqrt{y}(1+y)^2}\left[\frac{1+x}{1+x^2} + \frac{1-x-x^3}{1+x^3}\right] dxdy \Rightarrow$$

$$\Rightarrow \int_0^\infty\int_0^\infty \frac{(1+x)\log_e(x)\log_e(y)}{\sqrt{y}(1+y)^2(1+x^2)}dxdy + \int_0^\infty\int_0^\infty \frac{(1-x-x^2)\log_e(x)\log_e(y)}{\sqrt{y}(1+y)^2(1+x^3)}dxdy$$

$$A = \frac{\partial}{\partial a}\left[\int_0^\infty \frac{y^{a-1}}{\sqrt{y}(1+y)^2}dy\right]_{a=1}\left[\begin{array}{c}\int_0^1 \frac{(1+x)\log_e(x)}{1+x^2}dx + \int_0^\infty \frac{(1+x)\log_e(x)}{1+x^2}dx + \\ + \int_0^1 \frac{(1-x-x^2)\log_e(x)}{1+x^3}dx + \int_0^\infty \frac{(1-x-x^2)\log_e(x)}{1+x^3}dx\end{array}\right]$$

$$A = \frac{\partial}{\partial a}\left[\int_0^\infty \frac{y^{a-\frac{1}{2}-1}}{(1+y)^2}\right]_{a=1}\left[\begin{array}{c}\int_0^1 \frac{(1+x)\log_e(x)}{1+x^2}dx - \int_{\frac{1}{z}=\infty}^{\frac{1}{z}=1} \frac{\left(\frac{1+z}{z}\right)\log_e(z)}{\left(\frac{1+z^2}{z^2}\right)}\frac{dz}{z^2} + \\ + \int_0^1 \frac{(1-x-x^2)\log_e(x)}{1+x^3}dx - \int_{\frac{1}{z}=\infty}^{\frac{1}{z}=1} \frac{\left(\frac{z^2-z-1}{z^2}\right)\log_e(z)}{\frac{1+z^3}{z^3}}\frac{dz}{z^2}\end{array}\right]$$

$$A = \frac{\partial}{\partial a}\left[\int_0^\infty \frac{y^{a-\frac{1}{2}-1}}{(1+y)^{a-\frac{1}{2}+\frac{5}{2}-a}}dy\right]\left[\begin{array}{c}-\int_0^1 \frac{\log_e(x)}{1+x^2}dx + 2\int_0^1 \frac{x\log_e(x)}{1+x^2} - \\ -\int_0^1 \frac{\log_e(x)}{x}dx + \int_0^1 \frac{\log_e(x)}{1+x^2}dx + \\ +2\int_0^1 \frac{(1-x-x^2)\log_e(x)}{1+x^3}dx + \int_0^1 \frac{\log_e(x)}{x}dx\end{array}\right]$$

$$\frac{A}{2} = \frac{\partial}{\partial a}\left[\beta\left(a-\frac{1}{2},\frac{5}{2}-a\right)\right]_{a=1}\left[\sum_{n=0}^{n=\infty}(-1)^n\int_0^1 x^{2n+1}\log_e(x)\,dx + \sum_{n=0}^{n=\infty}(-1)^n\int_0^1 x^{3n}\log_e(x)\,dx\right]$$

$$\left(\because \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \beta(m,n)\right)$$

$$\frac{A}{2} = \frac{\partial}{\partial a}\left[\frac{\Gamma\left(a-\frac{1}{2}\right)\Gamma\left(\frac{5}{2}-a\right)}{\Gamma(2)}\right]_{a=1}\left[\begin{array}{c}\frac{1}{9}\sum_{n=1}^{n=\infty}\frac{(-1)^{n+1}}{n^2} - \frac{1}{9}\sum_{n=0}^{n=\infty}\frac{(-1)^{n+1}}{\left(n+\frac{2}{3}\right)^2} - \\ -\frac{1}{4}\sum_{n=1}^{n=\infty}\frac{(-1)^{n+1}}{n^2} + \frac{1}{9}\sum_{n=0}^{n=\infty}\frac{(-1)^{n+1}}{\left(n+\frac{1}{3}\right)^2}\end{array}\right]$$

$$\left(\because \int_0^1 x^m \log_e^n(x)\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, n > -1. m \neq -1\right)$$

$$\frac{A}{2}=\left[\beta\left(a-\frac{1}{2},\frac{5}{2}-a\right)\left(\psi^{(0)}\left(a-\frac{1}{2}\right)-\psi^{(0)}\left(\frac{5}{2}-a\right)\right)\right]_{a=1}\left[\begin{array}{c}\frac{1}{36}\left(\zeta\left(2,\frac{4}{6}\right)-\zeta\left(2,\frac{1}{6}\right)\right)-\\ -\frac{1}{36}\left(\zeta\left(2,\frac{5}{6}\right)-\zeta\left(2,\frac{2}{6}\right)\right)-\\ -\frac{5}{36}\eta(2)\end{array}\right]$$

$$\frac{A}{2}=\beta\left(\frac{1}{2},\frac{3}{2}\right)\left(\psi^{(0)}\left(\frac{1}{2}\right)-\psi^{(0)}\left(\frac{3}{2}\right)\right)\left[\frac{1}{36}\left(\begin{array}{c}\psi^{(1)}\left(\frac{2}{3}\right)+\psi^{(1)}\left(\frac{1}{3}\right)-\\ -\psi^{(1)}\left(\frac{1}{6}\right)-\psi^{(1)}\left(\frac{5}{6}\right)-5\eta(2)\end{array}\right)\right]$$

$$\left(\because \psi^{(n)}(1-z)=(-1)^n\psi^{(n)}(z)+(-1)^n\pi\frac{d^n}{dx^n}\cot(\pi z)\right)$$

$$\frac{A}{2}=\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(\psi^{(0)}\left(\frac{1}{2}\right)-\psi^{(0)}\left(\frac{3}{2}\right)\right)\left[\frac{1}{36}\left(\pi^2\csc^2\left(\frac{\pi}{3}\right)-\pi^2\csc^2\left(\frac{\pi}{6}\right)-5\eta(2)\right)\right]$$

$$\left(\because \psi^{(n)}(1+z)=\psi^{(n)}(z)+(-1)^n n!\, z^{-n-1}\right)$$

$$\frac{A}{2}=\frac{\pi}{2}(-2)\left(\frac{16\pi^2-48\pi^2-5\pi^2}{12\cdot 36}\right)\Rightarrow\frac{37\pi\zeta(2)}{72}=\frac{222\pi\zeta(2)}{432}$$

$$A=\int_0^\infty\int_0^\infty\frac{\log_e^2(xy)-\log_e^2(x)-\log_e^2(y)}{\sqrt{y}(1+x^3)(1+y)^2(1+x^2)}dxdy=\frac{444\pi\zeta(2)}{432}$$

1983. Find:

$$I=\int_0^1\int_0^1\int_0^1\frac{yz^2}{1-(xyz)^3}dxdydz$$

Proposed by Ankush Kumar Parcha-India

Solution by Pham Duc Nam-Vietnam

$$*\ \text{Recall: }\frac{1}{1-x}=\sum_{n=0}^{+\infty}x^n\ (|x|<1)$$

$$\Rightarrow I=\int_0^1\int_0^1\int_0^1 yz^2\sum_{n=0}^{+\infty}(xyz)^{3n}\,dxdydz=\int_0^1\int_0^1\int_0^1\sum_{n=0}^{+\infty}x^{3n}\,y^{3n+1}z^{3n+2}\,dxdydz$$

$$=\sum_{n=0}^{+\infty}\frac{1}{(3n+1)(3n+2)(3n+3)}=\sum_{n=0}^{+\infty}\frac{\Gamma(3n+1)}{\Gamma(3n+4)}=\sum_{n=0}^{+\infty}\frac{1}{2}B(3n+1,3)$$

$$=\frac{1}{2}\sum_{n=0}^{+\infty}\int_0^1 x^{3n}(1-x)^2dx=\frac{1}{2}\int_0^1\frac{(1-x)^2}{1-x^3}dx=\frac{1}{2}\int_0^1\frac{1-x}{x^2+x+1}dx$$

$$=\frac{1}{2}\int_0^1\frac{-\frac{1}{2}(2x+1)+\frac{3}{2}}{x^2+x+1}dx$$

$$=\frac{1}{2}\left(-\frac{1}{2}\int_0^1\frac{2x+1}{x^2+x+1}dx+\frac{3}{2}\int_0^1\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx\right)$$

$$=-\frac{1}{4}ln(x^2+x+1)\Big|_0^1+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{\sqrt{3}}arctan\frac{2x+1}{\sqrt{3}}\Big|_0^1$$

$$=-\frac{1}{4}ln\,3+\frac{\sqrt{3}}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{12}\pi-\frac{1}{4}ln\,3=\frac{\pi}{4\sqrt{3}}-\frac{ln\,3}{4}=\frac{\pi-\sqrt{3}\,ln\,3}{4\sqrt{3}}$$

1984. Find:

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{sin[ln(tan\,x)]}{ln(tan\,x)}dx$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Pham Duc Nam-Vietnam

$$*\,Let: t=-ln(tan\,x)\Rightarrow tan\,x=e^{-t}, dt=-\frac{dx}{cos\,x\,sin\,x}=-\frac{dx}{cos^2x\,tan\,x}$$

$$=-\frac{(1+tan^2x)}{tan\,x}dx\Rightarrow dx=\frac{-e^{-t}}{1+e^{-2t}}dt$$

$$\Rightarrow\Omega=\int_0^{+\infty}\frac{sin\,t}{t}\cdot\frac{e^{-t}}{1+e^{-2t}}dt=\int_0^{+\infty}\frac{sin\,t}{t}\cdot\frac{e^t}{1+e^{2t}}dt=\frac{1}{2}\int_0^{+\infty}\frac{sin\,t}{t\cosh t}dt=\frac{1}{4}\int_{-\infty}^{+\infty}\frac{sin\,t}{t\cosh t}dt$$

$$*\,Define: I(k)=\frac{1}{4}\int_{-\infty}^{+\infty}\frac{sin(kt)}{t\cosh t}dt\ (k\geq 0)\Rightarrow I'(k)=\frac{1}{4}\int_{-\infty}^{+\infty}\frac{\partial}{\partial k}\left(\frac{sin(kt)}{t\cosh t}\right)dt$$

$$=\frac{1}{4}\int_{-\infty}^{+\infty}\frac{cos(kt)}{\cosh t}dt=\frac{1}{4}\Re\int_{-\infty}^{+\infty}\frac{e^{ikt}}{\cosh t}dt$$

$*\int_{-\infty}^{+\infty}\frac{e^{ikt}}{\cosh t}dt$, Define a rectangular closed contour, counter-clockwise direction: C

$$=[-R,R]\cup[R,R+i\pi]\cup[R+i\pi,-R+i\pi]\cup[-R+i\pi,-R](R\to+\infty)$$

$And: f(z)=\frac{e^{ikz}}{\cosh z}$, has only one pole (order 1) $z=\frac{i\pi}{2}$ inside the contour.

$$\Rightarrow\int_C\frac{e^{ikz}}{\cosh z}dz=\int_{-R}^{R}\frac{e^{ikt}}{\cosh t}dt+\int_R^{R+i\pi}\frac{e^{ikz}}{\cosh z}dz+\int_{R+i\pi}^{-R+i\pi}\frac{e^{ikz}}{\cosh z}dz+\int_{-R+i\pi}^{-R}\frac{e^{ikz}}{\cosh z}dz$$

$$*\ By: ML\ inequality \Rightarrow \int_R^{R+i\pi}\frac{e^{ikz}}{\cosh z}dz=\int_{-R+i\pi}^{-R}\frac{e^{ikz}}{\cosh z}dz=0 \text{ when } R\to+\infty$$

$$*\int_{R+i\pi}^{-R+i\pi}\frac{e^{ikz}}{\cosh z}dz, let: z=u+i\pi\Rightarrow\int_{R+i\pi}^{-R+i\pi}\frac{e^{ikz}}{\cosh z}dz$$

$$=\int_R^{-R}\frac{e^{ik(u+i\pi)}}{\cosh(u+i\pi)}du=\int_R^{-R}\frac{e^{iku-\pi k}}{-\cosh u}du=e^{-\pi k}\int_{-R}^{R}\frac{e^{iku}}{\cosh u}du$$

$$=e^{-\pi k}\int_{-R}^{R}\frac{e^{ikt}}{\cosh t}dt$$

$$*\int_C\frac{e^{ikz}}{\cosh z}dz=2\pi i\,Res\left(f(z),z=\frac{i\pi}{2}\right)=2\pi i\lim_{z\to\frac{i\pi}{2}}\left(z-\frac{i\pi}{2}\right)\frac{e^{ikz}}{\cosh z}$$

$$=2\pi i\lim_{z\to\frac{i\pi}{2}}2\left(z-\frac{i\pi}{2}\right)\frac{e^ze^{ikz}}{-\left(e^{2\left(z-\frac{i\pi}{2}\right)}-1\right)}=2\pi i\left(-ie^{\frac{-\pi k}{2}}\right)=2\pi e^{\frac{-\pi k}{2}}$$

$$\Rightarrow 2\pi e^{\frac{-\pi k}{2}}=e^{-\pi k}\int_{-R}^{R}\frac{e^{ikt}}{\cosh t}dt+\int_{-R}^{R}\frac{e^{ikt}}{\cosh t}dt\Rightarrow\int_{-R}^{R}\frac{e^{ikt}}{\cosh t}dt$$

$$=\frac{2\pi e^{\frac{-\pi k}{2}}}{e^{-\pi k}+1}, \text{ taking real part and let } R\to+\infty\Rightarrow\int_{-\infty}^{+\infty}\frac{\cos(kt)}{\cosh t}=\frac{2\pi e^{\frac{-\pi k}{2}}}{e^{-\pi k}+1}$$

$$\Rightarrow I'(k)=\frac{\pi e^{\frac{-\pi k}{2}}}{2(e^{-\pi k}+1)}, \text{ integrating both sides}\Rightarrow I(k)=\frac{\pi}{2}\int\frac{e^{\frac{-\pi k}{2}}}{e^{-\pi k}+1}dk$$

$$=\frac{\pi}{2}.\frac{2}{\pi}\arctan e^{\frac{\pi k}{2}}+C=\arctan e^{\frac{\pi k}{2}}+C, let: k=0\Rightarrow C=-\frac{\pi}{4}$$

$$\Rightarrow I(k)=\arctan e^{\frac{\pi k}{2}}-\frac{\pi}{4}\Rightarrow\Omega=I(1)=\arctan e^{\frac{\pi}{2}}-\frac{\pi}{4}=\arctan e^{\frac{\pi}{2}}-\arctan(1)$$

$$=\arctan\left(\tanh\frac{\pi}{4}\right)=\frac{1}{2}\arctan\left(\sinh\frac{\pi}{2}\right)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\sin[\ln(\tan x)]}{\ln(\tan x)}dx$$

$$\Omega\overset{\tan x=e^{-t}}{\triangleq}\frac{1}{2}\int_0^\infty\frac{\sin t}{t\cosh t}dt=\frac{1}{2}\int_0^1\int_0^\infty\frac{\cos(yt)}{\cosh t}dtdy$$

$$\Omega=\frac{\pi}{4}\int_0^1\operatorname{sech}\left(\frac{\pi y}{2}\right)dy\overset{x=\frac{\pi y}{2}}{\triangleq}\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{dx}{\cosh x}=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cosh x}{1+\sinh^2 x}dx$$

$$\Omega\overset{u=\sinh x}{\triangleq}\frac{1}{2}\int_0^{\sinh\left(\frac{\pi}{2}\right)}\frac{du}{1+u^2}=\frac{1}{2}\arctan\left[\sinh\left(\frac{\pi}{2}\right)\right]$$

1985. Prove the below closed form:

$$\int_0^\infty \frac{\log(1+x^2)}{(1+x^2)^3}dx = \frac{\pi}{32}(12\log 2 - 7)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^\infty \frac{\log(1+x^2)}{(1+x^2)^3}dx \overset{(x=\tan y)}{=} -2\int_0^{\frac{\pi}{2}} \cos^4 y \log(\cos y)\,dy \overset{(IBP)}{=}$$

$$= -2\left[\frac{3}{8}y\log(\cos y)\Big|_0^{\frac{\pi}{2}} + \frac{1}{32}\overbrace{\int_0^{\frac{\pi}{2}} \sin y \tan y\,dy}^{-\frac{\pi}{2}} + \frac{1}{4}\overbrace{\int_0^{\frac{\pi}{2}} \sin(2y)\tan y\,dy}^{\frac{\pi}{2}} + \frac{3}{8}\int_0^{\frac{\pi}{2}} y\tan y\,dy\right]$$

$$= -2\left[\frac{7\pi}{64} + \frac{3}{8}\left(y\log(\cos y)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} y\tan y\,dy\right)\right]$$

$$= -\frac{7\pi}{32} - \frac{3}{4}\left[y\log(\cos y)\Big|_0^{\frac{\pi}{2}} + \left(-y\log(\cos y)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \log(\cos y)\,dy\right)\right]$$

$$= -\frac{7\pi}{32} - \frac{3}{4}\int_0^{\frac{\pi}{2}} \log(\cos y)\,dy = -\frac{7\pi}{32} - \frac{3}{4}\cdot\left(-\frac{\pi}{2}\log 2\right)$$

$$= -\frac{7\pi}{32} + \frac{3\pi}{8}\log 2 = \frac{\pi}{32}(12\log 2 - 7)$$

1986. Prove that:

$$\int_0^1 \frac{\sin(n\log x)}{1-x}dx = \frac{1}{2n} - \frac{\pi}{2}\coth(\pi n)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Santiago Alvarez-Quito-Ecuador

$$\Omega = \int_0^1 \frac{\sin(n\log x)}{1-x}dx = \Im\left(\int_0^1 \frac{e^{i\ln(x^n)}}{1-x}dx\right) = \Im\left(\sum_{k=0}^{\infty}\int_0^1 x^{in+k}\,dx\right) =$$

$$= \Im\left(\sum_{k=0}^{\infty}\frac{1}{in+k+1}\right) = \Im\left(\sum_{k=0}^{\infty}\frac{in-k+1}{-n^2-(k+1)^2}\right) = -n\sum_{k=1}^{\infty}\frac{1}{n^2+k^2} = \frac{1}{2n} - \frac{\pi}{2}\coth(n\pi)$$

1987. Find:

$$\Omega = \int_0^1 \frac{\arctan x \, . \ln(x^2+1)}{(x^2+1)^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$$\begin{cases} u = \ln(x^2+1) \\ dv = \dfrac{\arctan x}{(x^2+1)^2}dx \end{cases}$$

$$\Rightarrow \begin{cases} du = \dfrac{2x}{x^2+1} \\ v = \dfrac{1}{4}\arctan^2 x + \dfrac{1}{4(x^2+1)} + \dfrac{x\arctan x}{2(x^2+1)} \end{cases} (Sub: u = \arctan x \text{ then } IBP)$$

$$\Rightarrow \Omega = \ln(x^2+1)\left(\frac{1}{4}\arctan^2 x + \frac{1}{4(x^2+1)} + \frac{x\arctan x}{2(x^2+1)}\right)\Bigg|_0^1$$

$$-2\int_0^1 \frac{x}{x^2+1}\left(\frac{1}{4}\arctan^2 x + \frac{1}{4(x^2+1)} + \frac{x\arctan x}{2(x^2+1)}\right)dx$$

$$= \ln 2\left(\frac{\pi^2}{64} + \frac{1}{8} + \frac{\pi}{16}\right) - \frac{1}{2}\int_0^1 \frac{x\arctan^2 x}{x^2+1}dx - \frac{1}{2}\int_0^1 \frac{x}{(x^2+1)^2}dx - \int_0^1 \frac{x^2\arctan x}{(x^2+1)^2}dx$$

$$* -\frac{1}{2}\int_0^1 \frac{x}{(x^2+1)^2}dx = \frac{1}{4(x^2+1)}\Bigg|_0^1 = -\frac{1}{8}$$

$$* \int_0^1 \frac{x^2\arctan x}{(x^2+1)^2}dx, let: t = \arctan x \Rightarrow dt = \frac{dx}{x^2+1} \Rightarrow \int_0^1 \frac{x^2\arctan x}{(x^2+1)^2}dx$$

$$= \int_0^{\frac{\pi}{4}} t\sin^2 t\,dt = \int_0^{\frac{\pi}{4}} t\left(\frac{1-\cos 2t}{2}\right)dt \xrightarrow{IBP} \frac{\pi^2}{64} - \frac{\pi}{16} + \frac{1}{8}$$

$$* \int_0^1 \frac{x\arctan^2 x}{x^2+1}dx, let: t = \arctan x \Rightarrow dt = \frac{dx}{x^2+1} \Rightarrow \int_0^1 \frac{x\arctan^2 x}{x^2+1}dx$$

$$= \int_0^{\frac{\pi}{4}} t^2\tan t\,dt \xrightarrow{IBP} -\ln(\cos t)t^2\Big|_0^{\frac{\pi}{4}}$$

$$+2\int_0^{\frac{\pi}{4}} t\ln(\cos t)dt = \frac{\pi^2}{32}\ln 2 + 2\int_0^{\frac{\pi}{4}} t\ln(\cos t)dt$$

$$* 2\int_0^{\frac{\pi}{4}} t\ln(\cos t)dt = 2\int_0^{\frac{\pi}{4}} t\left(-\ln 2 - \sum_{k=1}^{+\infty}(-1)^k\frac{\cos(2kt)}{k}\right)dt$$

$$= -2\ln 2\int_0^{\frac{\pi}{4}} t dt - 2\int_0^{\frac{\pi}{4}} t\sum_{k=1}^{+\infty}(-1)^k\frac{\cos(2kt)}{k}dt$$

$$= -\frac{\pi^2}{16}\ln 2 - 2\int_0^{\frac{\pi}{4}}\sum_{k=1}^{+\infty}(-1)^k\frac{t\cos(2kt)}{k}dt$$

$$*\ 2\int_0^{\frac{\pi}{4}}\sum_{k=1}^{+\infty}(-1)^k\frac{t\cos(2kt)}{k}dt\xrightarrow{IBP}\sum_{k=1}^{+\infty}\frac{(-1)^k}{4k^3}\left(\pi k\sin\frac{\pi k}{2}+2\cos\frac{\pi k}{2}-2\right)$$

$$=\frac{\pi}{4}\sum_{k=1}^{+\infty}\frac{(-1)^k\sin\frac{\pi k}{2}}{k^2}+\sum_{k=1}^{+\infty}\frac{(-1)^k\cos\frac{\pi k}{2}}{2k^3}-\sum_{k=1}^{+\infty}\frac{(-1)^k}{2k^3}$$

$$\oplus\ \frac{\pi}{4}\sum_{k=1}^{+\infty}\frac{(-1)^k\sin\frac{\pi k}{2}}{k^2}=\frac{\pi}{4}\sum_{j=0}^{+\infty}\frac{(-1)^{j+1}}{(2j+1)^2}=-\frac{\pi}{4}\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)^2}=-\frac{\pi}{4}G$$

$$\oplus\ \sum_{k=1}^{+\infty}\frac{(-1)^k\cos\frac{\pi k}{2}}{2k^3}=\frac{1}{16}\sum_{j=1}^{+\infty}\frac{(-1)^j}{j^3}=\frac{1}{16}\left(-\frac{3}{4}\zeta(3)\right)=-\frac{3}{64}\zeta(3)$$

$$\oplus\ \sum_{k=1}^{+\infty}\frac{(-1)^k}{2k^3}=-\frac{3}{8}\zeta(3)$$

$$\Rightarrow 2\int_0^{\frac{\pi}{4}}t\ln(\cos t)dt=-\frac{\pi^2}{16}\ln 2-\left(-\frac{\pi}{4}G-\frac{3}{64}\zeta(3)+\frac{3}{8}\zeta(3)\right)=\frac{\pi}{4}G-\frac{21}{64}\zeta(3)-\frac{\pi^2}{16}\ln 2$$

$$\Rightarrow\int_0^1\frac{x\arctan^2 x}{x^2+1}dx=\frac{\pi^2}{32}\ln 2+\frac{\pi}{4}G-\frac{21}{64}\zeta(3)-\frac{\pi^2}{16}\ln 2=\frac{\pi}{4}G-\frac{21}{64}\zeta(3)-\frac{\pi^2}{32}\ln 2$$

$$\Rightarrow\Omega=\ln 2\left(\frac{\pi^2}{64}+\frac{1}{8}+\frac{\pi}{16}\right)-\frac{1}{8}-\left(\frac{\pi^2}{64}-\frac{\pi}{16}+\frac{1}{8}\right)-\frac{1}{2}\left(\frac{\pi}{4}G-\frac{21}{64}\zeta(3)-\frac{\pi^2}{32}\ln 2\right)$$

$$=\ln 2\left(\frac{\pi^2}{32}+\frac{1}{8}+\frac{\pi}{16}\right)-\frac{\pi^2}{64}+\frac{\pi}{16}-\frac{\pi}{8}G+\frac{21}{128}\zeta(3)-\frac{1}{4}$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega\overset{x=\tan y}{\widehat{=}}-2\int_0^{\frac{\pi}{4}}y\cos^2 y\log(\cos y)\,dy$$

$$\Omega\overset{IBP}{\widehat{=}}-\left[\left(\frac{y^2}{2}+\frac{y\sin(2y)}{2}+\frac{\cos(2y)}{4}\right)\log(\cos y)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\left(\frac{y^2}{2}+\frac{y\sin(2y)}{2}+\frac{\cos(2y)}{4}\right)\tan y\,dy$$

$$\Omega=\left(\frac{\pi^2}{64}+\frac{\pi}{16}\right)\log 2-\frac{1}{2}\int_0^{\frac{\pi}{4}}y^2\tan y\,dy-\int_0^{\frac{\pi}{4}}y\sin^2 y\,dy-\frac{1}{4}\int_0^{\frac{\pi}{4}}\cos(2y)\tan y\,dy$$

$$\int_0^{\frac{\pi}{4}}y^2\tan y\,dy=[-y^2\log(\cos y)]_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}y\log(\cos y)\,dy=\frac{\pi^2}{32}\log 2+2\left(\frac{\pi G}{8}-\frac{21}{128}\zeta(3)-\frac{\pi^2}{32}\log 2\right)$$

$$\int_0^{\frac{\pi}{4}}y^2\tan y\,dy=\frac{\pi G}{4}-\frac{21}{64}\zeta(3)-\frac{\pi^2}{32}\log 2$$

$$\int_0^{\frac{\pi}{4}}y\sin^2 y\,dy=\left[\frac{y^2}{4}-\frac{y\sin(2y)}{4}-\frac{\cos(2y)}{8}\right]_0^{\frac{\pi}{4}}=\frac{\pi^2}{64}-\frac{\pi}{16}+\frac{1}{8}$$

$$\int_0^{\frac{\pi}{4}}\cos(2y)\tan y\,dy=\int_0^{\frac{\pi}{4}}(2\cos^2 y-1)\tan y\,dy=\int_0^{\frac{\pi}{4}}(\sin(2y)-\tan y)dy=\frac{1}{2}-\frac{1}{2}\log 2$$

1988. **Prove that:**

$$\int_0^{\frac{\pi}{4}} \tanh^{-1}\left(\frac{\csc x - \sec x}{\csc x + \sec x}\right)dx = \frac{G}{2}$$

where G is Catalan's constant.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Asmat Qatea-Afghanistan

$$\Omega = \int_0^{\frac{\pi}{4}} \tanh^{-1}\left(\frac{\csc x - \sec x}{\csc x + \sec x}\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{\cos x}{\sin x}\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \log(\cot x)\,dx = \frac{G}{2}$$

$$\because \tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\because \tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1}\left(\frac{\csc x - \sec x}{\csc x + \sec x}\right) = \frac{1}{2}\log\left(\frac{1 + \frac{\csc x - \sec x}{\csc x + \sec x}}{1 - \frac{\csc x - \sec x}{\csc x + \sec x}}\right) =$$

$$= \frac{1}{2}\log\left(\frac{2\csc x}{2\sec x}\right) = \frac{1}{2}\log(\cot x) = -\frac{1}{2}\log(\tan x)$$

$$\Omega = \int_0^{\frac{\pi}{4}} \tanh^{-1}\left(\frac{\csc x - \sec x}{\csc x + \sec x}\right)dx = -\frac{1}{2}\int_0^{\frac{\pi}{4}} \log(\tan x)\,dx = -\frac{1}{2}\cdot(-G) = \frac{G}{2}$$

Solution 3 by Toubal Fethi-Algerie

$$\because \tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\Omega = \int_0^{\frac{\pi}{4}} \tanh^{-1}\left(\frac{\csc x - \sec x}{\csc x + \sec x}\right)dx = -\frac{1}{2}\int_0^{\frac{\pi}{4}} \log(\tan x)\,dx = \frac{1}{4}\left(-2\int_0^{\frac{\pi}{4}} \log(\tan x)\,dx\right)$$

We use Fourier series of $x \to \log(\tan x)$:

$$\Omega=\frac{1}{4}\left(4\int_0^{\frac{\pi}{4}}\frac{\cos(2(2k-1)x)}{2k-1}dx\right)=\sum_{k=1}^{\infty}\frac{1}{2k-1}\int_0^{\frac{\pi}{4}}\cos(2(2k-1)x)\,dx=$$

$$=\sum_{k=1}^{\infty}\frac{1}{2k-1}\left[\frac{\sin(2(2k-1)x)}{2(2k-1)}\right]_0^{\frac{\pi}{4}}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin\left(k\pi-\frac{\pi}{2}\right)}{(2k-1)^2}=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos(k\pi)}{(2k-1)^2}=$$

$$=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}=-\frac{1}{2}\cdot(-G)=\frac{G}{2}$$

1989. Find:

$$\Omega=\int_0^{\frac{\pi}{2}}\sqrt[3]{\tan x}\log(\tan x)\,dx$$

Proposed by Kader Tapsoba-Burkina Faso

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\lim_{s\to 0}\frac{\partial}{\partial s}\int_0^{\frac{\pi}{2}}(\sin x)^{s+\frac{1}{3}}(\cos x)^{-s-\frac{1}{3}}\,dx=\frac{1}{2}\lim_{s\to 0}\frac{\partial}{\partial s}B\left(\frac{s}{2}+\frac{2}{3},\frac{1}{3}-\frac{s}{2}\right)$$

$$\because B(z,1-z)=\frac{\pi}{\sin(\pi z)}$$

$$\Omega=\frac{\pi}{2}\lim_{s\to 0}\frac{\partial}{\partial s}\frac{1}{\sin\left(\frac{\pi s}{2}+\frac{2\pi}{3}\right)}=-\frac{\pi^2}{4}\lim_{s\to 0}\frac{\cos\left(\frac{\pi s}{2}+\frac{2\pi}{3}\right)}{\sin^2\left(\frac{\pi s}{2}+\frac{2\pi}{3}\right)}=-\frac{\pi^2}{4}\frac{\cos\left(\frac{2\pi}{3}\right)}{\sin^2\left(\frac{2\pi}{3}\right)}=\frac{\pi^2}{6}$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{2}}\sqrt[3]{\tan x}\log(\tan x)\,dx=\frac{\pi^2}{6}$$

Solution 2 by Bamidele Oluwatosin-Nigeria

$$\Omega_p=\int_0^{\frac{\pi}{2}}\tan^p x\,dx=\int_0^{\frac{\pi}{2}}\sin^p x\cos^{-p}x\,dx=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{2\Gamma(1)}=\frac{\pi}{2}\csc\left(\pi\left(\frac{p+1}{2}\right)\right)$$

$$\frac{\partial(\Omega_p)}{\partial p}=\int_0^{\frac{\pi}{2}}\tan^p x\log(\tan x)\,dx=-\frac{\pi^2}{4}\cot\left(\pi\left(\frac{p+1}{2}\right)\right)\csc\left(\pi\left(\frac{p+1}{2}\right)\right)$$

$$\Omega=\frac{\partial\left(\Omega_{\frac{1}{3}}\right)}{\partial p}=\int_0^{\frac{\pi}{2}}\sqrt[3]{\tan x}\log(\tan x)\,dx=-\frac{\pi^2}{4}\cot(\pi\left(\frac{y_3+1}{2}\right)\csc\left(\pi\left(\frac{y_3+1}{2}\right)\right)=\frac{\pi^2}{6}$$

1990. Prove that:

$$\int_0^1 \frac{\sin(n\log x)\log(1-x)\log x}{x}dx = \frac{1}{n^3} - \frac{\pi\coth(n\pi)}{2n^2} - \frac{\pi^2\operatorname{csch}^2(n\pi)}{2n}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1 \frac{\sin(n\log x)\log(1-x)\log x}{x}dx \overset{t=-\log x}{=}$$

$$= \int_0^\infty t\sin(nt)\log(1-e^{-t})\,dt = -\sum_{k=1}^\infty \frac{1}{k}\int_0^\infty t\sin(nt)\,e^{-kt}\,dt =$$

$$= \sum_{k=1}^\infty \frac{1}{k}\frac{\partial}{\partial n}\int_0^\infty \cos(nt)\,e^{-kt}\,dt = \sum_{k=1}^\infty \frac{1}{k}\frac{\partial}{\partial n}\left(\frac{k}{k^2+n^2}\right) = -\sum_{k=1}^\infty \frac{2n}{(n^2+k^2)^2}$$

Using identity: $\sum_{k=-\infty}^\infty \frac{1}{k^2+n^2} = \frac{\pi\coth(n\pi)}{n}$ derivative both sides w.r.to n

$$\sum_{k=-\infty}^\infty \frac{2n}{(n^2+k^2)^2} = \frac{\pi\coth(n\pi)}{n^2} + \frac{\pi^2\operatorname{csch}^2(n\pi)}{n}$$

$$\frac{2}{n^3} + 2\sum_{k=1}^\infty \frac{2n}{(n^2+k^2)^2} = \frac{\pi\coth(n\pi)}{n^2} + \frac{\pi^2\operatorname{csch}^2(n\pi)}{n}$$

$$\sum_{k=1}^\infty \frac{2n}{(n^2+k^2)^2} = \frac{\pi\coth(n\pi)}{2n^2} + \frac{\pi^2\operatorname{csch}^2(n\pi)}{2n} - \frac{1}{n^3}$$

Therefore,

$$\int_0^1 \frac{\sin(n\log x)\log(1-x)\log x}{x}dx = \frac{1}{n^3} - \frac{\pi\coth(n\pi)}{2n^2} - \frac{\pi^2\operatorname{csch}^2(n\pi)}{2n}$$

1991.

$$If\ 0 < a \le b\ then:$$

$$\int_a^b\int_a^b\int_a^b (x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)dxdydz \ge 3(b-a)^3(a^2+ab+b^2)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\int_a^b\int_a^b\int_a^b\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{x^2}{y}\right)dxdydz$$

$$=\sum_{cyc}\int_a^b x^2dx\int_a^b dy\int_a^b dz+\sum_{cyc}\int_a^b x^3\int_a^b\frac{dy}{y}\int_a^b dz+\sum_{cyc}\int_a^b x^2dx\int_a^b zdz\int_a^b\frac{dy}{y}$$

$$=(b-a)^2(b^3-a^3)+3\cdot\frac{(b-a)(b^4-a^4)}{4}ln\left(\frac{b}{a}\right)+\frac{(b^3-a^3)(b^2-a^2)}{2}ln\left(\frac{b}{a}\right)$$

$$=(b-a)^3(b^2+ba+a^2)+\frac{3}{4}(b-a)^3(b+a)(b^2+a^2)\cdot\frac{ln\left(\frac{b}{a}\right)}{b-a}+\frac{(b-a)^3(b+a)(b^2+ba+a^2)}{2}\cdot\frac{ln\left(\frac{b}{a}\right)}{b-a}$$

$$\ge(b-a)^3(b^2+ba+a^2)+\frac{3}{4}(b-a)^3(b+a)(b^2+a^2)\cdot\frac{2}{b+a}+\frac{(b-a)^3(b+a)(b^2+ba+a^2)}{2}\cdot\frac{2}{b+a}$$

$$\left[\begin{array}{c}let\ f(x)=\frac{1}{x}\ for\ all\ x>0,f''(x)=\frac{2}{x^3}>0\ for\ all\ x>0\\ hence\ f\ is\ convex\ \therefore\ applying\ Hermite-Hadamard\\ \frac{2}{a+b}\le\frac{1}{b-a}\int_a^b\frac{dx}{x}\le\frac{\frac{1}{a}+\frac{1}{b}}{2}\end{array}\right]$$

$$=(b-a)^3(b^2+ba+a^2)+\frac{3(b-a)^3(b^2+a^2)}{2}+(b-a)^3(b^2+ba+a^2)$$

$$we\ need\ to\ prove,2(b-a)^3(b^2+ba+a^2)+\frac{3(b-a)^3(b^2+a^2)}{2}\ge3(b-a)^3(b^2+ba+a^2)\Leftrightarrow2(b^2+ba+a^2)+\frac{3(b^2+a^2)}{2}\ge3(b^2+ba+a^2)\Leftrightarrow\frac{3(b^2+a^2)}{2}\ge b^2+ba+a^2\Leftrightarrow b^2+a^2\ge2ab,which\ is\ true\ by\ A.M\ge G.M$$

$$\therefore\int_a^b\int_a^b\int_a^b\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{x^2}{y}\right)dxdydz\ge3(b-a)^3(b^2+ba+a^2)(proved)$$

Equality holds for $a = b$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Lemma}: \ \textit{If } x,y,z > 0 \textit{ then}: \ \frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x} \ge \frac{3(x^2+y^2+z^2)}{x+y+z} \quad (1)$$

$$\textit{Proof}: \textit{We have}: \ (1) \Leftrightarrow \sum_{cyc}\left(\frac{x^2}{y}-2x+y\right) \ge \frac{3(x^2+y^2+z^2)}{x+y+z}-(x+y+z)$$

$$\Leftrightarrow \sum_{cyc}\frac{(x-y)^2}{y} \ge \frac{2(x^2+y^2+z^2)-2(xy+yz+zx)}{x+y+z} = \sum_{cyc}\frac{(x-y)^2}{x+y+z}.$$

$$\textit{Which is true because } x+y+z > y \quad (\textit{and analogs}).$$

$$\textit{Now we have}: \int_a^b\int_a^b\int_a^b (x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)dxdydz \ge$$

$$\overset{Lemma}{\ge} \int_a^b\int_a^b\int_a^b 3(x^2+y^2+z^2)dxdydz = 3\int_a^b dy\int_a^b dz\int_a^b 3x^2dx =$$

$$= 3(b-a)^2(b^3-a^3) = 3(b-a)^3(a^2+ab+b^2), \ \textit{as desired}.$$

Equality holds for $a = b$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x} \ge \frac{3(x^2+y^2+z^2)}{x+y+z} \Leftrightarrow$$

$$\Leftrightarrow (x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) \ge 3(x^2+y^2+z^2) \Leftrightarrow \sum_{cyc}\frac{x^3}{y}+\sum_{cyc}\frac{xy^2}{z} \ge 2\sum_{cyc}x^2$$

$$\sum_{cyc}\frac{x^3}{y}+\sum_{cyc}\frac{xy^2}{z} = \sum_{cyc}\frac{x^4}{yx}+\sum_{cyc}\frac{x^2y^2}{xz} \overset{BERGSTROM}{\ge} \frac{1}{\sum_{cyc}xy}\left(\left(\sum_{cyc}x^2\right)^2+\left(\sum_{cyc}xy\right)^2\right) =$$

$$= \frac{1}{\sum_{cyc}xy}\left(\sum_{cyc}x^2\right)^2+\sum_{cyc}xy \ge \ 2\sum_{cyc}x^2 \Leftrightarrow$$

$$\Leftrightarrow\left(\sum_{cyc}x^2\right)^2-2\sum_{cyc}x^2\sum_{cyc}xy+\left(\sum_{cyc}xy\right)^2\ge 0\Leftrightarrow\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)^2\ge 0$$

Equality holds for: $x=y=z$.

$$\int_a^b\int_a^b\int_a^b(x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)dxdydz\ge$$

$$\overset{Lemma}{\ge}\int_a^b\int_a^b\int_a^b 3(x^2+y^2+z^2)dxdydz=3\int_a^b dy\int_a^b dz\int_a^b 3x^2dx=$$

$$=3(b-a)^2(b^3-a^3)=3(b-a)^3(a^2+ab+b^2)$$

1992. *For $a,b>0$ prove that:*

$$\int_a^b\int_a^b\sqrt{x^2+y^2}dxdy\ge\frac{\sqrt{8}}{27}\left[7a^3+7b^3-9ab(a+b)+4\sqrt{a^3b^3}\right]$$

Proposed by Asmat Qatea-Afghanistan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma:

$$If\ x,y>0\ then:\ \sqrt{x^2+y^2}\ge\frac{\sqrt{2}}{3}\left[2(x+y)-\sqrt{xy}\right].$$

Proof: After squaring both sides, the inequality is equivalent to:

$$\frac{(x-y)^2+8\sqrt{xy}.\left(\sqrt{x}-\sqrt{y}\right)^2}{9}\ge 0\ which\ is\ true\ for\ any\ x,y>0.$$

Using the lemma we get:

$$\int_a^b\int_a^b\sqrt{x^2+y^2}dxdy\ge\frac{\sqrt{2}}{3}\int_a^b\int_a^b\left[2(x+y)-\sqrt{xy}\right]dxdy$$

$$=\frac{\sqrt{2}}{3}\left[4\int_a^b dy\int_a^b xdx-\left(\int_a^b\sqrt{x}dx\right)^2\right]=$$

$$=\frac{\sqrt{2}}{3}\left[4(b-a)\left(\frac{b^2}{2}-\frac{a^2}{2}\right)-\left(\frac{2\sqrt{b^3}}{3}-\frac{2\sqrt{a^3}}{3}\right)^2\right]=\frac{\sqrt{8}}{27}\left[7a^3+7b^3-9ab(a+b)+4\sqrt{a^3b^3}\right].$$

$$Equality\ holds\ iff\ a=b.$$

1993. Find:

$$\Omega=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8(n-k+1)^2-4n+4k-5}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Naren Bhandari-Bajura-Nepal

$$\Omega=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8(n-k+1)^2-4n+4k-5}\right)\right)\right)$$

We can observe that for $k=n-k+1$ the sum becomes

$$\Omega=\sum_{k=1}^{\infty}\left(\sum_{k=n-k+1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)\right)\right)$$

which further can be decomposed into two infinite arctangent sum le.

$$\Omega=\left(\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{k^2-k-1}\right)\right)\left(\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)\right)$$

Now, note that:

$$\lim_{M\to\infty}\sum_{k=1}^{M}\tan^{-1}\left(\frac{3}{k^2-k-1}\right)=\lim_{M\to\infty}\sum_{k=1}^{M}\tan^{-1}\left(\frac{(k+1)+(2-k)}{1+(2-k)(k+1)}\right)$$

$$=\lim_{M\to\infty}\sum_{k=1}^{M}(\tan^{-1}(k+1)+\tan^{-1}(2-k))$$

Since sum is telescoping sum and gives us the partial sum as

$$\Omega_1=\lim_{M\to\infty}(\tan^{-1}(M+4)+\tan^{-1}(M+5)+\tan^{-1}(M+6))-\tan^{-1}(0)=\frac{3\pi}{2}$$

as $k+1+2-k=3$

Since $8k^2-4k-1$ cannot be factored into two linear factors so that sum becomes telescoping to make it multiple and divide by any number now

$$\Omega_2=\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2}{8k^2-4k-1}\right)=\lim_{N\to\infty}\sum_{k=1}^{N}\tan^{-1}\left(\frac{10}{40k^2-20k-5}\right)$$

As

$$40k^2-20k-5=4(k+1)+(6k+1)(6k-5)$$

$$10=\frac{6k+1}{2(k+1)}-\frac{6k-5}{2k}$$

thus

$$\Omega_2=\sum_{k=1}^{N}\tan^{-1}\left(\frac{\frac{6k+1}{2(k+1)}-\frac{6k-5}{2k}}{1+\frac{(6k+1)(6k-5)}{4k(k+1)}}\right)=\sum_{k=1}^{N}\left(\tan^{-1}\left(\frac{6k+1}{2(k+1)}\right)-\tan^{-1}\left(\frac{6k-5}{2k}\right)\right)$$

Observe that Ω_2 is telescoping sum giving us the partial sum as

$$\Omega_2=\lim_{N\to\infty}\left(\left(\tan^{-1}\left(\frac{6N+1}{2(N+1)}\right)\right)-\tan^{-1}\left(\frac{1}{2}\right)\right)=\tan^{-1}(3)-\tan^{-1}\left(\frac{1}{2}\right)$$

Therefore,

$$\Omega=\frac{\pi}{2}\left(\tan^{-1}(3)-\tan^{-1}\left(\frac{1}{2}\right)\right)=\frac{\pi}{2}\tan^{-1}(1)=\frac{3\pi^2}{8}$$

Note: The principle branch

$$-\frac{\pi}{2}\le\tan^{-1}x\le\frac{\pi}{2}\ \textit{thus}$$

$$-\pi\le\tan^{-1}x+\tan^{-1}y\le\pi\ .\ \textit{As}$$

$$\Omega_1=\frac{3\pi}{2}>\pi.\ \textit{Therefore,}$$

$$\Omega_1=\frac{3\pi}{2}-\pi=\frac{\pi}{2}.\ \textit{So, the answer is}\ \frac{\pi^2}{8}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\tan^{-1}\left(\frac{3}{k^2-k-1}\right)\tan^{-1}\left(\frac{2}{8(n-k+1)^2-4n+4k-5}\right)\right)\right)=$$

$$=\left(\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{3}{k^2-k+1}\right)\right)\left(\sum_{m=0}^{\infty}\tan^{-1}\left(\frac{2}{8(m+1)^2-4m-5}\right)\right)=\Omega_1\cdot\Omega_2$$

$$\Omega_1=\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{(k+1)-(k-2)}{1+(k+1)(k-2)}\right)=\sum_{k=1}^{\infty}\left(\tan^{-1}(k+1)-\tan^{-1}(k-2)\right)=$$

$$= -\left(\tan^{-1}(-1) + \tan^{-1}(0) + \tan^{-1}(1)\right) + \lim_{k\to\infty} \tan^{-1}(k+1) =$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2} + 0 + \frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\frac{2}{8m^2 + 12m + 3} = \frac{\frac{b(m+1)}{c(m+1)+k} - \frac{bm}{cm+k}}{1 + \frac{bm}{cm+k}\cdot\frac{b(m+1)}{c(m+1)+k}}$$

$$\frac{1}{4m^2 + 6m + \frac{3}{2}} = \frac{b(m+1)(cm+k) - bm(c(m+1)+k)}{(cm+k)(c(m+1)+k) + b^2m(m+1)} =$$

$$= \frac{bk}{m(m+1)(b^2+c^2) + k^2 + ck(2m+1)} =$$

$$= \frac{1}{\frac{b^2+c^2}{bk}m^2 + \frac{b^2+c^2+2ck}{bk}m + \frac{k^2+ck}{bk}}$$

Evaluating the coefficients $\frac{b^2+c^2}{bk} = 4$ *and* $\frac{b^2+c^2+ck}{bk} = 6$

$$\frac{b^2+c^2}{bk} + \frac{2c}{b} = 4 + \frac{2c}{b} = 6 \Rightarrow a = b.$$

$$\frac{k^2+ck}{bk} = \frac{3}{2} \Rightarrow \frac{k}{b} + 1 = \frac{3}{2} \Rightarrow \frac{k}{b} = \frac{1}{2} \Rightarrow b = 2k$$

$$\Rightarrow \frac{b^2+c^2}{bk} = 4 = \frac{4k^2+4k^2}{2k^2} = 4k^2 \Rightarrow k = 1 \text{ and } b = c = 2.$$

$$\frac{\frac{2(m+1)}{2(m+1)+1} - \frac{2m}{2m+1}}{1 + \frac{2(m+1)}{2(m+1)+1}\cdot\frac{2m}{2m+1}} = 2\frac{(m+1)(2m+1) - m(2m+3)}{(2m+1)(2m+3) + 4m(m+1)} =$$

$$= 2\frac{2m^2 + 3m + 1 + 2m^2 - 3m}{4m^2 + 8m + 3 + 4m^2 + 4m}$$

Thus,

$$\Omega_2 = \sum_{m=0}^{\infty}\left(\tan^{-1}\left(\frac{2(m+1)}{2(m+1)+1}\right) - \tan^{-1}\left(\frac{2m}{2m+1}\right)\right) =$$

$$\lim_{M\to\infty} \tan^{-1}\left(\frac{2(M+1)}{2(M+1)+1}\right) - \tan^{-1} 0 = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\Rightarrow \Omega_1=\frac{\pi}{2},\Omega_2=\frac{\pi}{4}\Rightarrow \Omega=\frac{\pi^2}{8}.$$

1994. ***Find:***

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k} i\left(k-i+\frac{1}{2}\right)\right]^{-2}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Pham Duc Nam-Vietnam

$$*\sum_{i=1}^{k} i\left(k-i+\frac{1}{2}\right)=\sum_{i=1}^{k}\left(ki-i^2+\frac{1}{2}i\right)=k\frac{k(k+1)}{2}-\frac{1}{6}k(k+1)(2k+1)+\frac{1}{4}k(k+1)$$

$$=\frac{1}{12}k(k+1)(6k-2(2k+1)+3)=\frac{1}{12}k(k+1)(2k+1)$$

$$\Rightarrow\left[\sum_{i=1}^{k} i\left(k-i+\frac{1}{2}\right)\right]^{-2}=\left(\frac{1}{12}k(k+1)(2k+1)\right)^{-2}=\frac{144}{\left(k(k+1)(2k+1)\right)^2}$$

$$*\frac{1}{\left(k(k+1)(2k+1)\right)^2}=\frac{1}{k^2}+\frac{1}{(k+1)^2}+6\left(\frac{1}{k+1}-\frac{1}{k}\right)+\frac{16}{(2k+1)^2}$$

$$\Rightarrow \Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k} i\left(k-i+\frac{1}{2}\right)\right]^{-2}=144\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+\frac{1}{(k+1)^2}+6\left(\frac{1}{k+1}-\frac{1}{k}\right)+\frac{16}{(2k+1)^2}\right)$$

$$*\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\ \text{(Basel Problem)}$$

$$*\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{1}{k^2}-1=\frac{\pi^2}{6}-1$$

$$*\sum_{k=1}^{\infty}6\left(\frac{1}{k+1}-\frac{1}{k}\right)=-6\ \text{(Telescoping series)}$$

$$*\sum_{k=1}^{\infty}\frac{16}{(2k+1)^2}=16\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}-1\right)$$

$$Known: \sum_{n \text{ is even}} \frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{(k)^2} \Rightarrow \sum_{n \text{ is odd}} \frac{1}{n^2} = \frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{(k)^2} = \frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{16}{(2k+1)^2} = 16\left(\frac{\pi^2}{8} - 1\right) = 2\pi^2 - 16$$

$$\Rightarrow \Omega = \lim_{n\to\infty} \sum_{k=1}^{n}\left[\sum_{i=1}^{k} i\left(k - i + \frac{1}{2}\right)\right]^{-2} = 144\sum_{k=1}^{\infty}\left(\frac{1}{k^2} + \frac{1}{(k+1)^2} + 6\left(\frac{1}{k+1} - \frac{1}{k}\right) + \frac{16}{(2k+1)^2}\right)$$

$$= 144\left(\frac{\pi^2}{6} + \frac{\pi^2}{6} - 1 - 6 + 2\pi^2 - 16\right) = 144\left(\frac{7\pi^2}{3} - 23\right) = 48(7\pi^2 - 69)$$

1995. Prove that:

$$\sqrt{18\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}} = \pi$$

Proposed by Toubal Fethi-Algerie

Solution by Rana Ranino-Setif-Algerie

$$\because (\sin^{-1} z)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2z)^{2n}}{n^2\binom{2n}{n}}$$

$$z = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = 2\left(\sin^{-1}\left(\frac{1}{2}\right)\right)^2 = \frac{\pi^2}{18}$$

$$\sqrt{18\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}} = \pi$$

1996. Find:

$$\Omega = \int_0^1 \sum_{n=1}^{\infty} \frac{(2x)^n}{n^2\binom{2n}{n}} dx$$

Proposed by Le Thu-Vietnam

Solution 1 by Rana Ranino-Setif-Algerie

$$Using\ identity: (\sin^{-1} z)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2z)^{2n}}{n^2\binom{2n}{n}}:$$

$$\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2\binom{2n}{n}}=\sum_{n=1}^{\infty}\frac{\left(2\sqrt{\frac{x}{2}}\right)^{2n}}{n^2\binom{2n}{n}}=2\left(\sin^{-1}\left(\sqrt{\frac{x}{2}}\right)\right)^2$$

$$\Omega=\int_0^1\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2\binom{2n}{n}}dx=2\int_0^1\left(\sin^{-1}\left(\sqrt{\frac{x}{2}}\right)\right)^2dx\overset{x=2\sin^2 y}{=}4\int_0^{\frac{\pi}{4}}y^2\sin(2y)\,dy=$$

$$=\left[-2y^2\cos 2y+2y\sin 2y+\cos 2y\right]_0^{\frac{\pi}{4}}=\frac{\pi}{2}-1$$

$$\textit{Therefore,}\qquad \Omega=\int_0^1\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2\binom{2n}{n}}dx=\frac{\pi}{2}-1$$

Solution 2 by Peter Oladele-Akure-Nigeria

$$\Omega=\int_0^1\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2\binom{2n}{n}}dx=\int_0^1\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2\left(\frac{2^{2n}\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}\right)}dx=\int_0^1\sum_{n=1}^{\infty}\frac{\sqrt{\pi}\Gamma(n+1)(2x)^n}{2^{2n}n^2\Gamma\left(n+\frac{1}{2}\right)}dx=$$

$$=\int_0^1\sum_{n=1}^{\infty}\frac{\sqrt{\pi}(2^{n-2n})x^n\Gamma(n)}{n^2\Gamma\left(n+\frac{1}{2}\right)}dx=\int_0^1\sum_{n=1}^{\infty}\frac{x^n\Gamma(n)}{2^n\Gamma\left(n+\frac{1}{2}\right)}dx=$$

$$=2\int_0^1\left(\sin^{-1}\left(\sqrt{\frac{x}{2}}\right)\right)^2dx\overset{u=\sqrt{\frac{x}{2}}}{=}8\int_0^{\frac{\sqrt{2}}{2}}u\cdot(\sin^{-1}u)^2\,du\overset{IBP}{=}$$

$$=8\left[\frac{u^2\cdot(\sin^{-1}u)^2}{2}\right]_0^{\frac{\sqrt{2}}{2}}-8\int_0^{\frac{\sqrt{2}}{2}}\frac{u^2\cdot(\sin^{-1}u)^2}{\sqrt{1-u^2}}du\overset{v-\sin^{-1}u}{=}$$

$$=8\left(\frac{\pi^2}{16}-\left[2\int_0^{\frac{\pi}{4}}v\,dv-\frac{1}{2}\int_0^{\frac{\pi}{4}}v\cos 2v\,dv\right]\right)=8\left(\frac{\pi^2}{16}-\left[\frac{\pi^2}{16}-\frac{\pi}{16}+\frac{2}{16}\right]\right)=\frac{\pi-2}{16}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$(\sin^{-1}x)^2=\frac{1}{2}\sum_{n=1}^{\infty}\frac{4^n}{\binom{2n}{n}}\frac{x^{2n}}{n^2}$$

$$\Omega=\int_0^1\sum_{n=1}^{\infty}\frac{(2x)^n}{n^2\binom{2n}{n}}dx=2\int_0^1\left(\sin^{-1}\left(\sqrt{\frac{x}{2}}\right)\right)^2dx\overset{u=\sqrt{\frac{x}{2}}}{=}8\int_0^{\frac{\sqrt{2}}{2}}u\cdot(\sin^{-1}u)^2\,du\overset{t=\sin^{-1}u}{=}$$

$$= 8\int_0^{\frac{\pi}{4}} t^2 \cdot \sin t \cos t\, dt = 4\int_0^{\frac{\pi}{4}} t^2 d(\sin 2t) =$$

$$= -2t\cos(2t)\Big|_0^{\frac{\pi}{4}} + 4\int_0^{\frac{\pi}{4}} t\cos(2t)\, dt = 2\int_0^{\frac{\pi}{4}} t\, d(\sin 2t) =$$

$$= 2t\sin(2t)\Big|_0^{\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}} \sin(2t)\, dt = \frac{\pi}{2} + \cos(2t)\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{2} - 1$$

$$\textit{Therefore},\qquad \Omega = \int_0^1 \sum_{n=1}^{\infty} \frac{(2x)^n}{n^2\binom{2n}{n}} dx = \frac{\pi}{2} - 1$$

1997. Prove that:

$$\sum_{n=1}^{\infty} (-1)^n \left(\operatorname{csch}^2\left(\frac{\pi n}{2}\right) + \operatorname{sech}^2\left(\frac{\pi n}{2}\right)\right) = -\frac{1}{3}$$

Proposed by Lucas Paes Barreto-Brazil

Solution by Asmat Qatea-Afghanistan

$$\Omega = \sum_{n=1}^{\infty} (-1)^n \left(\operatorname{csch}^2\left(\frac{\pi n}{2}\right) + \operatorname{sech}^2\left(\frac{\pi n}{2}\right)\right) = \sum_{n=1}^{\infty} (-1)^n \left(\frac{\sinh^2\left(\frac{\pi n}{2}\right) + \cosh^2\left(\frac{\pi n}{2}\right)}{\sinh^2\left(\frac{\pi n}{2}\right) \cdot \cosh^2\left(\frac{\pi n}{2}\right)}\right)$$

$$\sinh^2 x + \cosh^2 x = \cosh(2x)$$

$$2\sinh x \cosh x = \sinh(2x)$$

$$\Omega = \sum_{n=1}^{\infty} (-1)^n \left(\frac{\cosh(\pi n)}{\frac{1}{4}\sinh^2(\pi n)}\right) = 4\sum_{n=1}^{\infty} (-1)^n \left(\frac{\cosh(\pi n)}{\sinh^2(\pi n)}\right) = 4\left(-\frac{1}{12}\right) = -\frac{1}{3}$$

$$\textit{Because}:\ \frac{1}{x\sinh(\pi x)} - \frac{1}{\pi x^2} = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 + k^2} \Rightarrow$$

$$\frac{1}{\sinh(\pi x)} - \frac{1}{\pi x} = \frac{2}{\pi}\sum_{k=1}^{\infty} (-1)^k \frac{x}{x^2 + k^2}\ \textit{take derivative}:$$

$$\frac{-\pi\cosh(\pi x)}{\sinh^2(\pi x)} + \frac{1}{\pi x^2} = \frac{2}{\pi}\sum_{k=1}^{\infty} (-1)^k \frac{k^2 - x^2}{x^2 + k^2}\left(\textit{multiply by}(-1)^x \textit{ and take } \sum_{k=1}^{\infty}\ \right)$$

$$\sum_{x=1}^{\infty}(-1)^x\frac{\pi\cosh(\pi x)}{\sinh^2(\pi x)}+\sum_{x=1}^{\infty}\frac{(-1)^x}{\pi x^2}=\frac{2}{\pi}\underbrace{\sum_{x=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+x}\frac{k^2-x^2}{x^2+k^2}}_{0}$$

$$Hence:\ -\sum_{x=1}^{\infty}(-1)^x\frac{\pi\cosh(\pi x)}{\sinh^2(\pi x)}-\frac{1}{\pi}\cdot\frac{\pi^2}{12}=0\Rightarrow$$

$$\sum_{x=1}^{\infty}(-1)^x\frac{\cosh(\pi x)}{\sinh^2(\pi x)}=-\frac{1}{12}$$

1998. If we define the sum function:

$$S(n)=\sum_{m=0}^{n}(-1)^{\left[\frac{m+1}{5}\right]}\left(\sin\left(\frac{m\pi}{5}\right)+\cos\left(\frac{m\pi}{5}\right)\right)$$

then prove the sum:

$$\sum_{n=0}^{\infty}\frac{S(5n)}{5^n}=\frac{5}{32}\left(11+\sqrt{5}+\sqrt{2(5+\sqrt{5})}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Asmat Qatea-Afghanistan

$$S(5n)=1+\sum_{k=1}^{5n}(-1)^{\left[\frac{k+1}{5}\right]}\left(\sin\left(\frac{k\pi}{5}\right)+\cos\left(\frac{k\pi}{5}\right)\right)$$

$$\sum_{k=1}^{[n]}f(k)=\sum_{r=1}^{m}\sum_{k=1}^{\left[\frac{n+r-1}{m}\right]}f(mk-r+1)$$

$$S(5n)=1+a_{5k}+a_{5k-1}+a_{5k-2}+a_{5k-3}+a_{5k-4}$$

$$\left[\frac{5n+r-1}{5}\right]=n;r=1,2,3,4,5$$

$$\left[\frac{5k-r+1}{5}\right]=\begin{cases}k, & r=0,1\\ k-1, & r=2,3,4\end{cases}$$

$$\begin{cases}\sin(k\pi+\theta)=(-1)^k\sin\theta\\ \cos(k\pi+\theta)=(-1)^k\cos\theta\end{cases}$$

$$a_{5k}=\sum_{k1}^{n}(-1)^k(\sin(k\pi)+\cos(k\pi))=n$$

$$a_{5k-1}=\sum_{k1}^{n}(-1)^k\left(-(-1)^k\sin\left(\frac{\pi}{5}\right)-(-1)^k\cos\left(\frac{\pi}{5}\right)\right)=n\left(-\sin\left(\frac{\pi}{5}\right)+\cos\left(\frac{\pi}{5}\right)\right)$$

$$a_{5k-2}=\sum_{k1}^{n}(-1)^k\left((-1)^k\sin\left(\frac{2\pi}{5}\right)+(-1)^k\cos\left(\frac{2\pi}{5}\right)\right)=n\left(\sin\left(\frac{2\pi}{5}\right)-\cos\left(\frac{2\pi}{5}\right)\right)$$

$$a_{5k-3}=\sum_{k1}^{n}(-1)^k\left((-1)^k\sin\left(\frac{3\pi}{5}\right)+(-1)^k\cos\left(\frac{3\pi}{5}\right)\right)=n\left(\sin\left(\frac{3\pi}{5}\right)-\cos\left(\frac{3\pi}{5}\right)\right)$$

$$a_{5k-4}=\sum_{k1}^{n}(-1)^k\left((-1)^k\sin\left(\frac{4\pi}{5}\right)+(-1)^k\cos\left(\frac{4\pi}{5}\right)\right)=n\left(\sin\left(\frac{4\pi}{5}\right)-\cos\left(\frac{4\pi}{5}\right)\right)$$

$$\sin\left(\frac{4\pi}{5}\right)=\sin\left(\frac{\pi}{5}\right);\ \cos\left(\frac{4\pi}{5}\right)=-\cos\left(\frac{\pi}{5}\right)$$

$$\sin\left(\frac{3\pi}{5}\right)=\sin\left(\frac{2\pi}{5}\right);\ \cos\left(\frac{3\pi}{5}\right)=-\cos\left(\frac{2\pi}{5}\right)$$

$$S(5n)=1+n+\left(2\sin\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{2\pi}{5}\right)\right)n$$

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x};\ \sum_{n=0}^{\infty}nx^n=\frac{x}{(1-x)^2}$$

$$\sum_{n=0}^{\infty}\frac{S(5n)}{5^n}=\frac{5}{4}+\frac{5}{16}+\frac{5}{8}\left(\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{\pi}{5}\right)\right)$$

$$\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{2(5+\sqrt{5})}}{4};\cos\left(\frac{\pi}{5}\right)=\frac{1+\sqrt{5}}{4}$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{S(5n)}{5^n}=\frac{5}{32}\left(11+\sqrt{5}+\sqrt{2(5+\sqrt{5})}\right)$$

1999. Prove that:

$$S=\sum_{k=0}^{+\infty}\frac{x^{8k}}{(8k)!}=\frac{1}{2}\cos\frac{x}{\sqrt{2}}\cosh\frac{x}{\sqrt{2}}+\frac{1}{4}(\cosh x+\cos x)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$S=\sum_{k=0}^{+\infty}\frac{x^{8k}}{(8k)!}=\frac{1}{2}\cos\frac{x}{\sqrt{2}}\cosh\frac{x}{\sqrt{2}}+\frac{1}{4}(\cosh x+\cos x)$$

$$*\ \mathcal{L}(t^n)=\frac{n!}{s^{n+1}}$$

$$PROVE:\mathcal{L}(t^n)=\int_0^{+\infty}t^ne^{-st}dt,\begin{cases}u=t^n\\dv=e^{-st}dt\end{cases}\Rightarrow\begin{cases}du=nt^{n-1}\\v=-\dfrac{e^{-st}}{s}\end{cases}\Rightarrow\mathcal{L}(t^n)$$

$$=\underbrace{-\frac{t^ne^{-st}}{s}\Big|_0^{+\infty}}_{=0}+\frac{n}{s}\int_0^{+\infty}t^{n-1}e^{-st}dt=\frac{n}{s}\mathcal{L}(t^{n-1})=\frac{n(n-1)}{s^2}\mathcal{L}(t^{n-2})=\ldots=\frac{n!}{s^{n+1}}$$

$$*\ Taking\ Laplace\ Transform:\mathcal{L}(S)=\sum_{k=0}^{+\infty}\left(\mathcal{L}\left(\frac{x^{8k}}{(8k)!}\right)\right)=\sum_{k=0}^{+\infty}\left(\frac{1}{s^{8k+1}}\right)=\frac{s^7}{s^8-1}$$

$$=\frac{s}{4(s^2+1)}+\frac{s^3}{2(s^4+1)}+\frac{1}{8(s-1)}+\frac{1}{8(s+1)}$$

$$*\ Taking\ Inverse\ Laplace\ Transform:S=\mathcal{L}^{-1}\left(\frac{s^7}{s^8-1}\right)$$

$$=\mathcal{L}^{-1}\left(\frac{s}{4(s^2+1)}+\frac{s^3}{2(s^4+1)}+\frac{1}{8(s-1)}+\frac{1}{8(s+1)}\right)$$

$$=\frac{1}{4}\cos x+\frac{1}{8}\underbrace{(e^x+e^{-x})}_{=2\cosh x}+\frac{1}{4\sqrt{2}}\mathcal{L}^{-1}\left(\frac{s\sqrt{2}-1}{s^2-s\sqrt{2}+1}+\frac{s\sqrt{2}+1}{s^2+s\sqrt{2}+1}\right)$$

$$=\frac{1}{4}(\cosh x+\cos x)+\frac{1}{4\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}\left(s-\frac{1}{\sqrt{2}}\right)}{\left(s-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{\sqrt{2}\left(s+\frac{1}{\sqrt{2}}\right)}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right)$$

$$=\frac{1}{4}(\cosh x+\cos x)+\frac{1}{4\sqrt{2}}\left(\sqrt{2}e^{-\frac{1}{\sqrt{2}}x}\cos\frac{x}{\sqrt{2}}+\sqrt{2}e^{\frac{1}{\sqrt{2}}x}\cos\frac{x}{\sqrt{2}}\right)$$

$$=\frac{1}{4}(\cosh x+\cos x)+\frac{1}{4}\cos\frac{x}{\sqrt{2}}\underbrace{\left(e^{-\frac{1}{\sqrt{2}}x}+e^{\frac{1}{\sqrt{2}}x}\right)}_{=2\cosh\frac{x}{\sqrt{2}}}$$

$$=\boxed{\frac{1}{2}\cos\frac{x}{\sqrt{2}}\cosh\frac{x}{\sqrt{2}}+\frac{1}{4}(\cosh x+\cos x)}$$

2000. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n\cdot 27^n}\sum_{i=1}^{n}\sum_{j=1}^{n}3^{i+j}\binom{3n-i-j}{n}\binom{2n-i-j}{n-i}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{n\cdot 27^n}\sum_{i=1}^{n}\sum_{j=1}^{n}3^{i+j}\binom{3n-i-j}{n}\binom{2n-i-j}{n-i}\right)$$

$$2n-i-j \geq 2n-1;\ \ n+1 \geq i+j$$

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i+j\leq n+1}}^{n}\frac{1}{3^{3n-(i+j)}}\cdot\frac{(3n-i-j)!}{(2n-i-j)!\,n!}\cdot\frac{(2n-i-j)!}{(n-1)!\,(n+1-i-j)}\cdot\frac{(2n-1)!}{(2n-1)!}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i+j\leq n+1}}^{n}\frac{1}{3^{3n-(i+j)}}\binom{2n-1}{n}\binom{3n-i-j}{2n-1}$$

$$=\lim_{n\to\infty}\frac{1}{n}\binom{2n-1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\ i+j\leq n+1}}^{n}\frac{1}{3^{3n-(i+j)}}\binom{3n-i-j}{2n-1}$$

$$=\lim_{n\to\infty}\frac{1}{n}\binom{2n-1}{n}\sum_{v=2}^{n+1}(v-1)\cdot\frac{1}{3^{3n-v}}\binom{3n-v}{2n-1}$$

$$\overset{k=v-1}{=}\lim_{n\to\infty}\frac{1}{n}\binom{2n-1}{n}\sum_{k=1}^{n}\frac{k}{3^{3n-k-1}}\binom{3n-k-1}{2n-1}$$

$$\overset{\substack{appendix\\ bellow}}{=}\lim_{n\to\infty}\frac{1}{2n}\binom{2n-1}{n}\frac{1}{3^{3n-2}}(3n-1)\binom{3n-2}{2n-1}$$

$$\Gamma(x)=\sqrt{2\pi}e^{x\log x-x-\frac{1}{2}\log x+O\left(\frac{1}{x^2}\right)}\left(1+O\left(\frac{1}{x}\right)\right)$$

$$v\in\mathbb{Z}_{\geq 1}\Rightarrow\Gamma(z)=(v-1)!$$

$$\Omega=\frac{1}{2}\lim_{n\to\infty}\frac{1}{3^{3n-2}}\cdot\frac{1}{n^2}\cdot\frac{\Gamma(3n)}{\Gamma(n)^3}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1}{3^{3n-2}}\cdot\frac{1}{n^2}\cdot\frac{\sqrt{2\pi}e^{3n\log(3n)-3n-\frac{1}{2}\log(3n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n}\right)\right)}{2\pi\sqrt{2\pi}e^{3n\log(3n)-3n-\frac{1}{2}\log(3n)+O\left(\frac{1}{n^2}\right)}\left(1+O\left(\frac{1}{n}\right)\right)}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1}{3^{3n-2}}\cdot\frac{1}{n^2}\cdot\frac{1}{2\pi}\cdot 3^{3n}\cdot\frac{n}{\sqrt{3}}=0$$

Appendix:

$$\sum_{k=1}^{n}\frac{k}{3^{3n-k-1}}\binom{3n-1-k}{2n-1}\overset{?}{=}\frac{1}{2}\cdot\frac{1}{3^{3n-2}}\binom{3n-2}{2n-1}$$

$$\sum_{k=1}^{n}\binom{3n-1-k}{2n-1}z^k\frac{1}{z\binom{3n-2}{2n-1}}\overset{m=k-1}{=}\sum_{m=0}^{n-1}\frac{\binom{3n-m-2}{2n-1}}{\binom{3n-2}{2n-1}}=\sum_{m=0}^{n-1}z^m\frac{\frac{(3n-m-2)!}{(n-m-1)!}}{\frac{(3n-2)!}{(n-1)!}}=$$

$$=\sum_{m=0}^{n-1}z^m\frac{(n-1)!}{(n-1-m)!}\cdot\frac{1}{\frac{(3n-2)!}{(3n-2-m)!}}=$$

$$=\sum_{m=0}^{n}\frac{z^m}{m!}\cdot\frac{m!\left(-(n-1)\right)\left(-(n-2)\right)\dots\left(-\left(n-1-(m-1)\right)\right)}{\left(-(3n-2)\right)\left(-(3n-3)\right)\dots\left(-\left(3n-2-(m-1)\right)\right)}=$$

$$=\ {}_2F_1(1;-(n-1);-(3n-2);z)$$

$$\frac{d}{dz}\left(\sum_{k=1}^{n}\binom{3n-1-k}{2n-1}z^k\right)=\sum_{k=1}^{n}k\binom{3n-1-k}{2n-1}=\binom{3n-2}{2n-1}\frac{d}{dz}=$$

$$=\ {}_2F_1(1;-(n-1);-(3n-2);z)=$$

$$=\binom{3n-2}{2n-1}\Big(\ {}_2F_1(1;-(n-1);-(3n-2);z)$$

$$+z\frac{n-1}{3n-2}\ {}_2F_1(2;-(n-1);-(3n-2);z)\Big)$$

$$=\binom{3n-2}{2n-1}(1+\frac{n-1}{3n-2}x+\frac{(n-1)(n-2)}{(3n-2)(3n-3)}z^2+\frac{(n-1)(n-2)(n-3)}{(3n-2)(3n-3)(3n-4)}z^3+\cdots$$

$$+\cdots+\frac{n-1}{3n-2}z+2\frac{(n-1)(n-2)}{(3n-2)(3n-3)}z+3\dots)$$

$$=\binom{3n-2}{2n-1}\ {}_2F_1(1;-(n-1);-(3n-2);z)$$

$$\sum_{k=1}^{n}k\binom{3n-1-k}{2n-1}z^{k-1}=\binom{3n-2}{2n-1}\ {}_2F_1(1;-(n-1);-(3n-2);z)$$

$$z=3\Rightarrow\sum_{k=1}^{n}k\binom{3n-1-k}{2n-1}3^{k-1}=\binom{3n-2}{2n-1}\ {}_2F_1(1;-(n-1);-(3n-2);3)$$

$$\sum_{k=1}^{n}\frac{k}{3^{3n-k-1}}\binom{3n-1-k}{2n-1}=\frac{1}{3^{3n-2}}\binom{3n-2}{2n-1}\ {}_2F_1(1;-(n-1);-(3n-2);3)=$$

$$=\frac{1}{3^{3n-2}}\binom{3n-2}{2n-1}\frac{3n-1}{2}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru