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CONTENT

TSINTSIFAS – ȘAHIN'S INEQUALITY

By Daniel Sitaru – Romania

Abstract: In this paper we connect two famous relationships in any triangle, both published in American Mathematical Monthly.

Keywords: Tsintsifas; Șahin

Main result: If $x, y, z > 0$ then in acute ΔABC the following relationship holds:

$$\frac{x}{y+z}\cdot a+\frac{y}{z+x}\cdot b+\frac{z}{x+y}\cdot c\geq\sqrt{3r(4R+r)}$$

Lemma 1 (TSINTSIFAS' INEQUALITY): If $x, y, z > 0$ then in acute ΔABC holds:

$$\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2\geq 2F\sqrt{3}$$

Proof: $\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2=$

$$=\frac{(x+y+z)a^2-(y+z)a^2}{y+z}+\frac{(x+y+z)b^2-(z+x)b^2}{z+x}-\frac{(x+y+z)c^2-(x+y)c^2}{x+y}$$

$$=(x+y+z)\left(\frac{a^2}{y+z}+\frac{b^2}{z+x}+\frac{c^2}{x+y}\right)-(a^2+b^2+c^2)=$$

$$=\left(\frac{x+y}{2}+\frac{y+z}{2}+\frac{z+x}{2}\right)\left(\frac{a^2}{y+z}+\frac{b^2}{z+x}+\frac{c^2}{x+y}\right)-(a^2+b^2+c^2)\geq$$

$$\overset{CBS}{\geq}\left(\sqrt{\frac{x+y}{2}\cdot\frac{a^2}{x+y}}+\sqrt{\frac{y+z}{2}\cdot\frac{b^2}{y+z}}+\sqrt{\frac{z+x}{2}\cdot\frac{c^2}{z+x}}\right)^2-(a^2+b^2+c^2)=$$

$$=\left(\frac{a}{\sqrt{2}}+\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}\right)^2-(a^2+b^2+c^2)=\frac{1}{2}(a+b+c)^2-(a^2+b^2+c^2)=$$

$$=\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{2}=\frac{2(s^2+r^2+4Rr)-2(s^2-r^2-4Rr)}{2}=$$

$$=s^2+r^2+4Rr-s^2+r^2+4Rr=2r^2+8Rr=2r(r+4R)\overset{DOUCET}{\geq}$$

$$\geq 2r\cdot s\sqrt{3}=2F\sqrt{3}$$

Equality holds for $a=b=c$ and $x=y=z$.

Observation: $\left(\sqrt{a}+\sqrt{b}\right)^2=a+b+2\sqrt{ab}>a+b>c=\left(\sqrt{c}\right)^2$

$\left(\sqrt{a}+\sqrt{b}\right)^2>\left(\sqrt{c}\right)^2\Rightarrow\sqrt{a}+\sqrt{b}>\sqrt{c}$ and analogous: $\sqrt{b}+\sqrt{c}>\sqrt{a}$; $\sqrt{c}+\sqrt{a}>\sqrt{b}$

hence: $\sqrt{a},\sqrt{b},\sqrt{c}$ can be sides in a triangle

Lemma 2 (MEHMET ȘAHIN'S IDENTITY): Let a, b, c – be sides in a triangle. The triangle formed with sides $\sqrt{a},\sqrt{b},\sqrt{c}$ has area $\Delta=\frac{1}{2}\sqrt{r(4R+r)}$

Proof: $\Delta\overset{HERON}{=}\sqrt{\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\cdot\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{2}\cdot\frac{\sqrt{b}+\sqrt{c}-\sqrt{a}}{2}\cdot\frac{\sqrt{c}+\sqrt{a}-\sqrt{b}}{2}}=$

$$= \frac{1}{4}\sqrt{\left(\left(\sqrt{a}+\sqrt{b}\right)^2-\left(\sqrt{c}\right)^2\right)\left(\left(\sqrt{c}\right)^2-\left(\sqrt{a}-\sqrt{b}\right)^2\right)} =$$
$$= \frac{1}{4}\sqrt{(a+b+2\sqrt{ab}-c)(c-a-b+2\sqrt{ab})} =$$
$$= \frac{1}{4}\sqrt{\left(2\sqrt{ab}+(a+b-c)\right)\left(2\sqrt{ab}-(a+b-c)\right)} = \frac{1}{4}\sqrt{4ab-(a+b-c)^2} =$$
$$= \frac{1}{4}\sqrt{4ab-a^2-b^2-c^2-2ab+2bc+2ca} = \frac{1}{4}\sqrt{2(ab+bc+ca)-(a^2+b^2+c^2)} =$$
$$= \frac{1}{4}\sqrt{2s^2+2r^2+8Rr-2s^2+2r^2+8Rr} = \frac{1}{4}\sqrt{4r^2+16Rr} = \frac{1}{2}\sqrt{r(4R+r)}$$

Back to the main problem:

We apply Tsintsifas' inequality for the triangle with sides: $\sqrt{a},\sqrt{b},\sqrt{c}$:

$$\frac{x}{y+z}\cdot\left(\sqrt{a}\right)^2+\frac{y}{z+x}\cdot\left(\sqrt{b}\right)^2+\frac{z}{x+y}\cdot\left(\sqrt{c}\right)^2 \ge 2\sqrt{3}\Delta$$
$$\frac{x}{y+z}\cdot a+\frac{y}{z+x}\cdot b+\frac{z}{x+y}\cdot c \ge 2\sqrt{3}\cdot\frac{1}{2}\sqrt{r(4R+r)}$$
$$\frac{x}{y+z}\cdot a+\frac{y}{z+x}\cdot b+\frac{z}{x+y}\cdot c \ge \sqrt{3r(4R+r)}$$

Equality holds for $a=b=c$ and $x=y=z$.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE **– www.ssmrmh.ro**

SPECIAL LIMITS AND SUMS-(II)

By Florică Anastase-Romania

Abstract: *In this paper are presented some calculation techniques on special class of limits and sums.*

Theorem 1. Let $(a_n)_{n\ge1},(b_n)_{n\ge1}$ be sequences of real numbers such that

$$\lim_{n\to\infty} a_n = a, \lim_{n\to\infty}(b_{n+1}+b_{n+2}+\cdots+b_{2n}) = b, b_n > 0, \forall n\in\mathbb{N}$$

Prove that: $\lim_{n\to\infty}(a_{n+1}b_{n+1}+a_{n+2}b_{n+2}+\cdots+a_{2n}b_{2n}) = ab$

Proof. Let $c_n = b_{n+1}+b_{n+2}+\cdots+b_{2n}$. Observe that $(a_n)_{n\ge1}$ and $(c_n)_{n\ge1}$ converges, then are bounded. Thus, exists $M_1>0$ and $M_2>0$ such that $|a_n|\le M_1$ and $|c_n|\le M_2$, $\forall n\in\mathbb{N}$.

Now, $\forall\varepsilon>0, \exists n_\varepsilon\in\mathbb{N}$ such that $\forall n\ge n_\varepsilon$ we have $|a_n-a|\le\frac{\varepsilon}{M+|a|}$ and $|c_n-b|\le\frac{\varepsilon}{M+|a|}$.

Denoting $d_n = a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + \cdots + a_{2n}b_{2n}$, it follows that

$$d_n = (a_{n+1} - a)b_{n+1} + \cdots + (a_{2n} - a)b_{2n} + ac_n =$$

$$= (a_{n+1} - a)b_{n+1} + \cdots + (a_{2n} - a)b_{2n} + a(c_n - b) + ab;$$

$$|d_n - ab| \le |a_{n+1} - a||b_{n+1}| + \cdots + |a_{2n} - a||b_{2n}| + |a||c_n - b|$$

For $n \ge n_\varepsilon$ we have: $|d_n - ab| < \frac{\varepsilon}{M+|a|}(b_{n+1} + \cdots + b_{2n}) + |a|\frac{\varepsilon}{M+|a|} \le \varepsilon$

Hence, $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that $\forall n \ge n_\varepsilon$ we have $|d_n - ab| < \varepsilon$. So,

$$\lim_{n\to\infty}(a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + \cdots + a_{2n}b_{2n}) = ab.$$

Application 1. Find: $\boldsymbol{\Omega = \lim_{n\to\infty}\left(\sin\frac{\pi}{n+1} + \sin\frac{\pi}{n+2} + \cdots + \sin\frac{\pi}{2n}\right)}$

Solution. If in Theorem 1 we take $a_n = n \cdot \sin\frac{\pi}{n}$ and $b_n = \frac{1}{n}$, we have:

$$n \cdot \sin\frac{\pi}{n} \to \pi, \qquad \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \gamma_{2n} - \gamma_n + \log 2n - \log n$$

$\because \gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$, γ *—Euler Mascheroni Constant*

$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \to \log 2$. Hence, from Theorem 1:

$$\Omega = \lim_{n\to\infty}\left(\sin\frac{\pi}{n+1} + \sin\frac{\pi}{n+2} + \cdots + \sin\frac{\pi}{2n}\right) = \log 2$$

Application 2. Find: $\boldsymbol{\Omega = \lim_{n\to\infty}\left[(n+1)\tan^2\frac{\pi}{n+1} + (n+2)\tan^2\frac{\pi}{n+2} + \cdots + 2n\tan^2\frac{\pi}{2n}\right]}$

Solution. If in Theorem 1 we take $a_n = n^2 \cdot \tan^2\frac{\pi}{n}$ and $b_n = \frac{1}{n}$ we have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 \cdot \tan^2\frac{\pi}{n} = \pi^2 \cdot \lim_{n\to\infty}\left(\frac{n^2}{\pi^2} \cdot \tan^2\frac{\pi}{n}\right) = \pi^2$$

$b_{n+1} + b_{n+2} + \cdots + b_{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \to \log 2$. Hence, from Theorem 1:

$$\Omega = \lim_{n\to\infty}\left[(n+1)\tan^2\frac{\pi}{n+1} + (n+2)\tan^2\frac{\pi}{n+2} + \cdots + 2n\tan^2\frac{\pi}{2n}\right]$$

Application 3. Prove that:

$$\textbf{(i)} \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{\frac{n}{n+1}} + \left(\frac{n+1}{n+2}\right)^{\frac{n+1}{n+2}} + \cdots + \left(\frac{2n-1}{2n}\right)^{\frac{2n-1}{2n}}\right] = \frac{1}{2}$$

$$\textbf{(ii)} \lim_{n\to\infty}\left[\frac{1}{n+1+\sqrt{n(n+1)}}+\cdots+\frac{1}{2n+\sqrt{(2n-1)2n}}\right]=\frac{1}{2}\log 2$$

Theorem 2: Let $p,q\in\mathbb{R}, p>1$ and $q\neq 0$, $f:(-1,\infty)\to\mathbb{R}$, continuous function such that $\lim_{x\to 0}\frac{f(x)}{x}=1$. Prove that:

$$\lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\left(1+f\left(\frac{i^{p-1}}{n^p}\right)\right)^q\right)=\frac{q}{p}$$

Proof. $\lim_{x\to 0}\frac{f(x)}{x}=1 \Leftrightarrow \forall n\in\mathbb{N}, \exists\xi_n>0$

such that $1-\xi_n\leq\frac{f(x)}{x}\leq 1+\xi_n \Leftrightarrow (1-\xi_n)x\leq f(x)\leq(1+\xi_n)x \Leftrightarrow \lim_{x\to 0}f(x)=0$

Then, it follows that $\lim_{x\to 0}\frac{(1+f(x))^q-1}{x}=q$

$$(q-\xi_n)\frac{i^{p-1}}{n^p}\leq\left(1+f\left(\frac{i^{p-1}}{n^p}\right)\right)^q-1\leq(q+\xi_n)\frac{i^{p-1}}{n^p}\Leftrightarrow$$

$$(q-\xi_n)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}\leq\sum_{i=1}^{n}\left(1+f\left(\frac{i^{p-1}}{n^p}\right)\right)^q-n\leq(q+\xi_n)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{p-1}=\int_0^1 x^{p-1}dx=\frac{1}{p}, \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(1+f\left(\frac{i^{p-1}}{n^p}\right)\right)^q-n\right)=\frac{q}{p}$$

Application 4. For $n,p\in\mathbb{N}, p\geq 2, n\geq p$ find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\cdot\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\sin\left(\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right]$$

Solution. In Theorem 1, we take

$$a_n=\sum_{i=1}^{n}\left(\sqrt[p]{1+\sin\left(\frac{i^{p-1}}{n^p}\right)}-1\right) \text{ and } b_n=\frac{1}{n}$$

$$\sum_{k=1}^{n}\left[\frac{1}{n+k}\cdot\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\sin\left(\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right]=\sum_{k=1}^{n}a_{n+k}b_{n+k}$$

$$b_{n+1}+b_{n+2}+\cdots+b_{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\to\log 2$$

Now, using Theorem 2, we have: $\lim_{x\to 0}\frac{(1+\sin x)^{\frac{1}{p}}-1}{x}=\frac{1}{p}\Leftrightarrow \forall n\in\mathbb{N},\exists \xi_n>0$

such that: $\frac{1}{p}-\xi_n\le \frac{\left(1+\sin\left(\frac{i^{p-1}}{n^p}\right)\right)^{\frac{1}{p}}-1}{\frac{i^{p-1}}{n^p}}\le \frac{1}{p}+\xi_n$

$$\left(\frac{1}{p}-\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}\le\sum_{i=1}^{n}\sqrt[p]{1+\sin\left(\frac{i^{p-1}}{n^p}\right)}-n\le\left(\frac{1}{p}+\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}$$

$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{p-1}=\int_0^1 x^{p-1}dx=\frac{1}{p}$. So, it follows that:

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\sqrt[p]{1+\sin\left(\frac{i^{p-1}}{n^p}\right)}\right)=\frac{1}{p^2}$. Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\cdot\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\sin\left(\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right]=\frac{1}{p^2}\cdot\log 2$$

Application 5. For $\boldsymbol{n,p\in\mathbb{N},p\ge 2,n\ge p}$ find

$$\boldsymbol{\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\cdot\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\tan\left(\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right]}$$

Solution. In Theorem 1, we take $a_n=\sum_{i=1}^{n}\left(\sqrt[p]{1+\tan\left(\frac{i^{p-1}}{n^p}\right)}-1\right)$ and $b_n=\frac{1}{n}$

$$\left[\frac{1}{n+k}\cdot\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\tan\left(\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right]=\sum_{k=1}^{n}a_{n+k}b_{n+k}$$

$$b_{n+1}+b_{n+2}+\cdots+b_{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\to\log 2$$

Now, using Theorem 2, we have: $\lim_{x\to 0}\frac{(1+\tan x)^{\frac{1}{p}}-1}{x}=\frac{1}{p}\Leftrightarrow \forall n\in\mathbb{N},\exists \xi_n>0$

$$\frac{1}{p}-\xi_n\le\frac{\left(1+\tan\left(\frac{i^{p-1}}{n^p}\right)\right)^{\frac{1}{p}}-1}{\frac{i^{p-1}}{n^p}}\le\frac{1}{p}+\xi_n$$

$\left(\frac{1}{p}-\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}\le\sum_{i=1}^{n}\sqrt[p]{1+\tan\left(\frac{i^{p-1}}{n^p}\right)}-n\le\left(\frac{1}{p}+\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}$. Hence:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{p-1}}{n^p} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{p-1} = \int_0^1 x^{p-1}dx = \frac{1}{p}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(-n + \sum_{i=1}^{n}\sqrt[p]{1+\tan\left(\frac{i^{p-1}}{n^p}\right)}\right) = \frac{1}{p^2}$$

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\cdot\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\tan\left(\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right] = \frac{1}{p^2}\cdot\log 2$$

Application 6. Find: $\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\left(\frac{1}{\sqrt{(n+k)^2+i}}\cdot\sin\left(1+\frac{i}{n+k}\right)\right)\right]$

Solution. In Theorem 1, we take: $a_n = \sum_{i=1}^{n}\left(\frac{1}{\sqrt{n^2+i}}\cdot\sin\left(1+\frac{i}{n}\right)\right)$ and $b_n = \frac{1}{n}$

$$b_{n+1} + b_{n+2} + \cdots + b_{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \to \log 2$$

$$\because \sum_{i=1}^{n}\sin(1+ia) = \frac{\sin\frac{na}{2}\cdot\sin\left(1+\frac{a}{2}+\frac{na}{2}\right)}{\sin\frac{a}{2}} \overset{not.}{=} u_n, \forall n\in\mathbb{N}^*$$

Because: $\sin\left(1+\frac{i}{n}\right) > 0, \forall k\in\{1,2,\dots,n\}$ we get,

$\frac{u_n}{\sqrt{n^2+n}} \le a_n = \sum_{i=1}^{n}\left(\frac{1}{\sqrt{n^2+i}}\cdot\sin\left(1+\frac{i}{a}\right)\right) \le \frac{u_n}{\sqrt{n^2+1}}, \forall n\in\mathbb{N}^*$. So, for $a = \frac{1}{n}$ it follows,

$$\frac{\sin\frac{1}{2}\cdot\sin\frac{3n+1}{2n}}{\sqrt{n^2+n}\cdot\sin\frac{1}{2n}} \le a_n \le \frac{\sin\frac{1}{2}\cdot\sin\frac{3n+1}{2n}}{\sqrt{n^2+1}\cdot\sin\frac{1}{2n}}; \forall n\in\mathbb{N}^*, \lim_{n\to\infty} a_n = 2\sin\frac{1}{2}\cdot\sin\frac{3}{2}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\left(\frac{1}{\sqrt{(n+k)^2+i}}\cdot\sin\left(1+\frac{i}{n+k}\right)\right)\right] = 2\sin\frac{1}{2}\cdot\sin\frac{3}{2}\cdot\log 2$$

Application 7. For $n,p\in\mathbb{N}, p\ge 2, n\ge p$ **find**

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\log\left(1+\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right]$$

Solution. In Theorem 1, we take: $a_n = \sum_{i=1}^{n}\left(\sqrt[p]{1+\log\left(1+\frac{i^{p-1}}{n^p}\right)}-1\right)$ and $b_n = \frac{1}{n}$

$b_{n+1} + b_{n+2} + \cdots + b_{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \to \log 2$. Now, using Theorem 2, we get

$\lim_{x\to 0}\frac{(1+\log(1+x))^{\frac{1}{p}}-1}{x}=\frac{1}{p}\Leftrightarrow \forall n\in\mathbb{N},\exists\xi_n>0$ such that

$\frac{1}{p}-\xi_n\le\frac{\left(1+\log\left(1+\frac{i^{p-1}}{n^p}\right)\right)^{\frac{1}{p}}-1}{\frac{i^{p-1}}{n^p}}\le\frac{1}{p}+\xi_n$ and then

$$\left(\frac{1}{p}-\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}\le\sum_{i=1}^{n}\sqrt[p]{1+\log\left(1+\frac{i^{p-1}}{n^p}\right)}-n\le\left(\frac{1}{p}+\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{p-1}=\int_0^1 x^{p-1}dx=\frac{1}{p}$$

$$\lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\sqrt[p]{1+\log\left(1+\frac{i^{p-1}}{n^p}\right)}\right)=\frac{1}{p^2}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\left(\sqrt[p]{1+\log\left(1+\frac{i^{p-1}}{(n+k)^p}\right)}-1\right)\right]=\frac{1}{p^2}\log 2$$

Application 8. For $n,p\in\mathbb{N},p\ge 2,n\ge p$ find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\left(\sqrt[p]{e^{\frac{i^{p-1}}{(n+k)^p}}}-1\right)\right]}$$

Solution. In Theorem 1, we take: $a_n=\sum_{i=1}^{n}\left(\sqrt[p]{e^{\frac{i^{p-1}}{n^p}}}-1\right)$ and $b_n=\frac{1}{n+k}$

$b_{n+1}+b_{n+2}+\cdots+b_{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\to\log 2$. Now, from Theorem 2, we get

$\lim_{x\to 0}\frac{(e^x-1)^{\frac{1}{p}}-1}{x}=\frac{1}{p}\Leftrightarrow\forall n\in\mathbb{N},\exists\xi_n>0$ such that $\frac{1}{p}-\xi_n\le\frac{\left(e^{\frac{i^{p-1}}{n^p}}-1\right)^{\frac{1}{p}}-1}{\frac{i^{p-1}}{n^p}}\le\frac{1}{p}+\xi_n$

$$\left(\frac{1}{p}-\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}\le\sum_{i=1}^{n}\sqrt[p]{e^{\frac{i^{p-1}}{n^p}}}-n\le\left(\frac{1}{p}+\xi_n\right)\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{p-1}}{n^p}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{p-1}=\int_0^1 x^{p-1}dx=\frac{1}{p}$$

$$\lim_{n\to\infty}\left(-n+\sum_{i=1}^{n}\sqrt[p]{e^{\frac{i^{p-1}}{n^p}}}\right)=\frac{1}{p^2},\qquad \Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\left(\sqrt[p]{e^{\frac{i^{p-1}}{(n+k)^p}}}-1\right)\right]=\frac{1}{p^2}\log 2$$

Application 9. For $p>0$ find: $\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\frac{i^p}{(n+k)^{p+1}+i}\right]$

Solution. In Theorem 1, we take $a_n=\sum_{i=1}^{n}\frac{i^p}{n^{p+1}+i}=\frac{1}{n}\cdot\sum_{i=1}^{n}\frac{\left(\frac{i}{n}\right)^p}{1+\frac{i}{n^{p+1}}}$ and $b_n=\frac{1}{n}$

$$\frac{1}{1+\frac{1}{n^p}}\cdot\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p\le x_n\le\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p,\forall n\ge 1,\ \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\left(\frac{k}{n}\right)^p=\int_0^1 x^p\,dx=\frac{1}{p+1}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n+k}\frac{i^p}{(n+k)^{p+1}+i}\right]=\frac{1}{p+1}\log 2$$

Application 10. For $a,b,p,q\in\mathbb{N}$ such that $p(q-b)=a+1$, find

$$\boldsymbol{\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n}i^a\sin^p\left(\frac{i^b}{(n+k)^q}\right)\right]}$$

Solution. If in Theorem 1 we take:$a_n=\sum_{i=1}^{n}i^a sin^p\left(\frac{i^b}{n^q}\right)$ and $b_n=\frac{1}{n}$

$$b_{n+1}+b_{n+2}+\cdots+b_{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\to\log 2$$

Now, using Theorem 2, we get

$$a_n=\sum_{i=1}^{n}i^a sin^p\left(\frac{i^b}{n^q}\right)=\sum_{i=1}^{n}\left(\frac{sin\left(\frac{i^b}{n^q}\right)}{\frac{i^b}{n^q}}\right)^p\frac{i^{a+bp}}{n^{pq}}=\sum_{i=1}^{n}\left(\frac{sin\left(\frac{i^b}{n^q}\right)}{\frac{i^b}{n^q}}\right)^p\frac{i^{a+bp}}{n^{a+bp+1}}\Leftrightarrow$$

$$\forall n\in\mathbb{N},\exists\xi_n>0\text{ such that}1-\xi_n\le\left(\frac{sin\left(\frac{i^b}{n^q}\right)}{\frac{i^b}{n^q}}\right)^p\le 1+\xi_n$$

$$(1-\xi_n)\sum_{i=1}^{n}\frac{i^{a+bp}}{n^{a+bp+1}}\le a_n\le(1+\xi_n)\sum_{i=1}^{n}\frac{i^{a+bp}}{n^{a+bp+1}}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{a+bp}}{n^{a+bp+1}}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{a+bp}=\int_0^1 x^{a+bp}dx=\frac{1}{a+bp+1}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} i^a sin^p\left(\frac{i^b}{n^q}\right)=\frac{1}{a+bp+1}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{1}{n+k}\sum_{i=1}^{n} i^a \sin^p\left(\frac{i^b}{(n+k)^q}\right)\right]=\frac{1}{a+bp+1}\cdot\log 2$$

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FEW OUTSTANDING LIMITS-(III)

By Florică Anastase-Romania

First section.

Theorem 1. **Let be $f:[0,1]\to\mathbb{R}$ integrable function. Then:**

$$i)\ \forall a\in[0,1],\ \lim_{n\to\infty} n\cdot\int_0^a x^n f(x)dx=0$$

$$ii)\ \text{if } f \text{ is continuous function},\ \lim_{n\to\infty} n\cdot\int_0^1 x^n f(x)dx=f(1)$$

Proof.

$i)$ We have: $\left|n\int_0^a x^n f(x)dx\right|\le n\int_0^a x^n|f(x)|dx\le\frac{na^{n+1}}{n+1}\cdot M$, where $M:=\underbrace{sup}_{x\in[0,a]}|f(x)|$

So, $\lim_{n\to\infty} n\cdot\int_0^a x^n f(x)dx=0$

$ii)$ For $0<a<1$ we have: $n\cdot\int_0^1 x^n f(x)dx=n\cdot\int_0^a x^n f(x)dx+n\cdot\int_a^1 x^n f(x)dx$

Let be $\epsilon > 0$. Because f –continuous function in $x_0 = 1$, then $\exists a_\epsilon \in [0,1]$ such that $\forall x \in [a(\epsilon), 1]$ we have $|f(x) - f(1)| < \frac{\epsilon}{4}$. From (i) it follows that $\exists n_1(\epsilon) \in \mathbb{N}^*$ such that $\forall n \geq n_1(\epsilon)$ we have:

$$\left|n \cdot \int_0^{a(\epsilon)} x^n f(x)dx\right| < \frac{\epsilon}{2}.$$

But: $$\left|n \cdot \int_{a(\epsilon)}^1 x^n f(x)dx - f(1)\right| = n\left|\int_{a(\epsilon)}^1 x^n\big(f(x) - f(1)\big)dx + \left(\frac{1 - a(\epsilon)^{n+1}}{n+1} - \frac{1}{n}\right)f(1)\right|$$

$$\leq n\int_{a(\epsilon)}^1 x^n|f(x) - f(1)|dx + \frac{n}{n+1}\left|\frac{1}{n} - a(\epsilon)^{n+1}\right||f(1)| <$$

$$< \frac{\epsilon}{4} \cdot \frac{n}{n+1}(1 - a(\epsilon)^{n+1}) + \frac{n}{n+1}\left|\frac{1}{n} - a(\epsilon)^{n+1}\right||f(1)| <$$

$$< \frac{\epsilon}{4} + \left|\frac{1}{n} - a(\epsilon)^{n+1}\right||f(1)|$$

Because $\left|\frac{1}{n} - a(\epsilon)^{n+1}\right||f(1)| \underset{n\to\infty}{\longrightarrow} 0$ $\exists n_2(\epsilon) \in \mathbb{N}^*$ such that $\forall n \geq n_2(\epsilon)$ we have:

$$\left|\frac{1}{n} - a(\epsilon)^{n+1}\right||f(1)| < \frac{\epsilon}{4}$$

Let be $n \geq n = \max\{n_1(\epsilon), n_2(\epsilon)\}$.Thus,

$$\left|n \cdot \int_0^1 x^n f(x)dx - f(1)\right| \leq n \cdot \left|\int_0^{a(\epsilon)} x^n f(x)fx\right| + \left|n \cdot \int_{a(\epsilon)}^1 x^n f(x)dx - f(1)\right| <$$

$$< \frac{\epsilon}{2} + \left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) = \epsilon$$

Therefore, $$\lim_{n\to\infty} n \cdot \int_0^1 x^n f(x)dx = f(1)$$

Application 1. **Find:**

$$\boldsymbol{a)}\ \lim_{n\to\infty} n \cdot \int_0^{\frac{\pi}{4}} \tan^n x\, dx$$

$$\boldsymbol{b)}\ \lim_{n\to\infty} n \cdot \int_a^b \left(e^{\frac{x}{n+x}} - 1\right)dx;\ 0 < a < b.$$

Solution.

$a)$ $$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx = \int_0^{\frac{\pi}{4}} \cos^2 x \cdot \left(\frac{\tan^n x}{\cos^2 x}\right)dx = \frac{1}{2(n+1)} + \frac{2}{n+1}\int_0^{\frac{\pi}{4}} \sin^2 x \cdot \tan^n x\, dx =$$

$$= \frac{1}{2(n+2)} + \frac{2}{n+1} \cdot \sin^2 \xi \cdot I_n$$

Therefore, $$\lim_{n\to\infty} n \cdot \int_0^{\frac{\pi}{4}} \tan^n x\, dx = \lim_{n\to\infty} n \cdot I_n = \frac{1}{2}$$

$b)$ Because $e^u = 1 + u + \frac{u^2}{2}e^{\theta u}, 0 < \theta < 1$, we have:

$$n \cdot \int_a^b \left(e^{\frac{x}{n+x}} - 1\right)dx = n\left(b - a - n \cdot \log\left|\frac{b+n}{a+n}\right|\right) + \frac{n\xi^2}{2(n+\xi)^2}\int_a^b e^{\frac{x}{n+x}}dx$$

Therefore,

$$\lim_{n\to\infty} n \cdot \int_a^b \left(e^{\frac{x}{n+x}} - 1\right)dx = \frac{1}{2}(b^2 - a^2)$$

Application 2. **Find:**

$$\boldsymbol{\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n^3+k}}}$$

Solution.

$$\text{Let } f(x) = \sqrt{\frac{k}{n^3+k}} \text{ and } g(x) = \frac{k}{n^3+k}, n\in\mathbb{N}^*, g'(x) = \frac{n^3}{(n^3+x)^2} > 0,$$

Then g –increasing thus, f –increasing. So, $f(k)\le f(x)\le f(k+1), \forall x\in[k,k+1]$

$$f(k)\le\int_k^{k+1} f(x)\,dx\le f(k+1), \forall k\in\{1,2,\dots,n-1\}$$

Hence,

$$f(1)+f(2)+\dots+f(n-1)\le\int_1^n f(x)\,dx\le f(2)+f(3)+\dots+f(n)$$

Let us denote: $a_n = \sum_{k=1}^{n}\sqrt{\frac{k}{n^3+k}}$, then we have:

$$a_n-\sqrt{\frac{n}{n^3+n}}\le\int_1^n f(x)\,dx\le a_n-\sqrt{\frac{1}{n^3+1}}; (*)$$

On the other hand,

$$\int_1^n f(x)dx = \int_1^n\sqrt{\frac{x}{n^3+x}}dx\le\int_1^n\frac{\sqrt{x}dx}{\sqrt{n^3+1}} = \frac{1}{\sqrt{n^3+1}}\cdot\frac{2x^{\frac{3}{2}}}{3}\Bigg|_1^n = \frac{2}{3}\cdot\frac{n^{\frac{3}{2}}-1}{\sqrt{n^3+1}}\xrightarrow[n\to\infty]{}\frac{2}{3}$$

$$\int_1^n f(x)dx\ge\int_1^n\frac{\sqrt{x}dx}{\sqrt{n^3+n}} = \frac{1}{\sqrt{n^3+n}}\cdot\frac{2}{3}\left(n^{\frac{3}{2}}-1\right)\xrightarrow[n\to\infty]{}\frac{2}{3}$$

$$\text{Therefore, } \Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n^3+k}} = \frac{2}{3}$$

Second section.

Application 3. **Find:**

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\lim_{m\to\infty}\left(1+\sum_{k=1}^{n}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)\right)^{\frac{m!}{n!}}\right)}$$

Solution. We have:

$$\lim_{m\to\infty}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right) = \lim_{m\to\infty}\left(\frac{\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)}{\tan^{-1}\left(\frac{k!}{m!}\right)}\cdot\frac{\tan^{-1}\left(\frac{k!}{m!}\right)}{\frac{k!}{m!}}\cdot\frac{k!}{m!}\right) = 0$$

$$\lim_{m\to\infty}\left(1+\sum_{k=1}^{n}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)\right)^{\frac{1}{\sum_{k=1}^{n}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)}} = e$$

$$\lim_{m\to\infty} m!\sum_{k=1}^{n}\log\left(1+tan^{-1}\left(\frac{k!}{m!}\right)\right) =$$

$$\lim_{m\to\infty}\left(\frac{\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)}{\tan^{-1}\left(\frac{k!}{m!}\right)}\cdot\frac{\tan^{-1}\left(\frac{k!}{m!}\right)}{\frac{k!}{m!}}\cdot k!\right)=\sum_{k=1}^{n}k!$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(1+\sum_{k=1}^{n}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)\right)^{\frac{m!}{n!}}\right)=$$

$$=\lim_{n\to\infty}\left\{\lim_{m\to\infty}\left[\left(1+\sum_{k=1}^{n}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)\right)^{\frac{1}{\sum_{k=1}^{n}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)}}\right]^{\frac{m!}{n!}\cdot\sum_{k=1}^{n}\log\left(1+\tan^{-1}\left(\frac{k!}{m!}\right)\right)}\right\}=$$

$$=e^{\lim_{n\to\infty}\frac{1}{n!}\cdot\left(\sum_{k=1}^{n}k!\right)}\overset{L.C-S}{\cong}e^{\lim_{n\to\infty}\frac{(n+1)!}{(n+1)!-n!}}=e$$

Application 4. **Find:**

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(1+\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)\right)^{\frac{3(m!)^2}{n^3}}\right)}$$

Solution. We have:

$$\lim_{m\to\infty}\left(1+\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)\right)^{\frac{1}{\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)}}=e$$

$$\lim_{m\to\infty}\frac{3\cdot(m!)^2\cdot\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)}{n^3}=\lim_{m\to\infty}\left(\frac{3}{n^3}\cdot\sum_{k=1}^{n}\left(\frac{\sin\left(\frac{k}{m!}\right)}{\frac{k}{m!}}\right)^2\cdot k^2\right)=\frac{3}{n^3}\cdot\sum_{k=1}^{n}k^2$$

$$=\frac{3\cdot n(n+1)(2n+1)}{6n^3}$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(1+\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)\right)^{\frac{3(m!)^2}{n^3}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(1+\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)\right)^{\frac{1}{\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)}}\right)^{\frac{(m!)^2}{2n^3}\cdot\sum_{k=1}^{n}\sin^2\left(\frac{k}{m!}\right)}=e$$

Application 5. **Find:**

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\prod_{k=1}^{n}\left(1+\tan\left(\frac{k!}{m!}\right)\right)\right)^{\frac{m!}{n!}}\right)}$$

Solution.

$$\lim_{m\to\infty}m!\cdot\sum_{k=1}^{n}\log\left(1+tan\left(\frac{k!}{m!}\right)\right)=\lim_{m\to\infty}\left(\sum_{k=1}^{n}\frac{\log\left(1+\tan\left(\frac{k!}{m!}\right)\right)}{\tan\left(\frac{k!}{m!}\right)}\cdot\tan\frac{\left(\frac{k!}{m!}\right)}{\frac{k!}{m!}}\cdot k!\right)=\sum_{k=1}^{n}k!$$

$$\Omega = \lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\prod_{k=1}^{n}\left(1+\tan\left(\frac{k!}{m!}\right)\right)\right)\right)^{\frac{m!}{n!}} =$$

$$= e^{\lim\limits_{n\to\infty}\left(\lim\limits_{m\to\infty}\frac{m!}{n!}\log\left(\prod_{k=1}^{n}\left(1+tan\left(\frac{k!}{m!}\right)\right)\right)\right)} = e^{\lim\limits_{n\to\infty}\frac{1}{n!}\cdot\sum_{k=1}^{n}k!} \overset{L.C-S}{=} e^{\lim\limits_{n\to\infty}\frac{(n+1)!}{(n+1)!-n!}} = e$$

Application 6. **Find:**

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(1+\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{n}}$$

Solution.

$$\forall n\in\mathbb{N}^*, n\geq 2: \sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n} = \frac{1}{2}cot\frac{\pi}{2n}$$

Let be $z = cos\frac{\pi}{n} + isin\frac{\pi}{n}$, then:

$$\sum_{k=1}^{n-1}ksin\frac{k\pi}{n} = Im(z+2z^2+3z^3+\cdots(n-1)z^{n-1}), z^n = -1 \Rightarrow$$

$$z+2z^2+3z^3+\cdots(n-1)z^{n-1} = \frac{(n-1)z^{n+1}-nz^n+z}{(z-1)^2} = \frac{(1-n)z+n+z}{(z-1)^2} =$$

$$= \frac{n-(n-2)z}{1-2sin^2\frac{\pi}{2n}+2isin\frac{\pi}{2n}cos\frac{\pi}{2n}-1} = \frac{n-(n-2)z}{-4sin^2\frac{\pi}{2n}\left(cos\frac{\pi}{n}+isin\frac{\pi}{n}\right)} =$$

$$= \frac{n-2}{4sin^2\frac{\pi}{2n}} - \frac{n}{4sin^2\frac{\pi}{2n}}\left(cos\frac{\pi}{n}-isin\frac{\pi}{n}\right) \Rightarrow$$

$$\sum_{k=1}^{n-1}ksin\frac{k\pi}{n} = Im\left(\sum_{k=1}^{n-1}kz^k\right) = \frac{nsin\frac{\pi}{n}}{4sin^2\frac{\pi}{2n}} = \frac{n}{2}cot\frac{\pi}{2n} \Rightarrow \sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n} = \frac{1}{2}cot\frac{\pi}{2n}$$

$$\lim_{n\to\infty}log\left(1+\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{n} = \lim_{n\to\infty}nlog\left(1+2tan\frac{\pi}{2n}\right) =$$

$$= \lim_{n\to\infty}\frac{log\left(1+2tan\frac{\pi}{2n}\right)}{2tan\frac{\pi}{2n}}\cdot\frac{2tan\frac{\pi}{2n}}{\frac{\pi}{2n}}\cdot\frac{\pi}{2n}\cdot n = \pi$$

$$\Omega = \lim_{n\to\infty}\left(1+\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{n} = e^{\pi}$$

Application 7. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)}}$$

Solution.

$$\because \sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}, \forall n\in\mathbb{N}, n\geq 3$$

$$(1+z)^m=\sum_{l=0}^{m}\binom{m}{l}z^l, m\in\mathbb{N}^*;(1)$$

Let $z=cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n}, k=\overline{1,n}\Rightarrow 1+z=1+cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n}=$

$$=2cos\frac{k\pi}{n}\left(cos\frac{k\pi}{n}+isin\frac{k\pi}{n}\right)\Rightarrow$$

$$2^m cos^m\frac{k\pi}{n}\left(cos\frac{mk\pi}{n}+isin\frac{mk\pi}{n}\right)=\sum_{l=0}^{m}\binom{m}{l}\left(cos\frac{2lk\pi}{n}+isin\frac{2lk\pi}{n}\right)\Rightarrow$$

$$2^m cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\sum_{l=0}^{m}\binom{m}{l}cos\frac{2lk\pi}{n}, k=\overline{1,n}\Rightarrow$$

$$2^m\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\sum_{l=0}^{m}\binom{m}{l}\sum_{k=1}^{n}cos\frac{2ik\pi}{n}=$$

$$=\binom{m}{0}\sum_{k=1}^{n}1+\sum_{i=1}^{n}\binom{m}{i}\sum_{k=1}^{n}cos\frac{2lk\pi}{n};(2)$$

$$\because\sum_{k=1}^{n}a^{k-1}cos(k\theta)=\frac{a^{n+1}cos(n\theta)-a^n cos(n+1)\theta+cos\theta-a}{a^2-2acos\theta+1}; a=1,\theta=\frac{2l\pi}{n}\Rightarrow$$

$$\sum_{k=1}^{n}cos\frac{2lk\pi}{n}=\frac{cos2l\pi-cos\frac{(n+1)2l\pi}{n}+cos\frac{2l\pi}{n}-1}{2-2cos\frac{2l\pi}{n}}=0,\forall l=\overline{1,m}; m<n;(3)$$

From (2), (3) it follows that:

$$2^m\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=n\binom{m}{0}\Rightarrow\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\frac{n}{2^m}$$

For $m=n-1$, it follows that: $\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)=\frac{n}{2^{n-1}}$

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)}=\lim_{n\to\infty}\sqrt[n]{\frac{n}{2^{n-1}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{n+1}{2^n}\cdot\frac{2^{n-1}}{n}=\frac{1}{2}$$

Application 8. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}\right)}$$

Solution.

$$\forall n\in\mathbb{N}^*, n\ge 2:\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}=\frac{1}{2}cot\frac{\pi}{2n}$$

Let be $z=cos\frac{\pi}{n}+isin\frac{\pi}{n}$, then:

$$\sum_{k=1}^{n-1}ksin\frac{k\pi}{n}=Im(z+2z^2+3z^3+\cdots(n-1)z^{n-1}), z^n=-1\Rightarrow$$

$$z+2z^2+3z^3+\cdots(n-1)z^{n-1}=\frac{(n-1)z^{n+1}-nz^n+z}{(z-1)^2}=\frac{(1-n)z+n+z}{(z-1)^2}=$$

$$=\frac{n-(n-2)z}{1-2sin^2\frac{\pi}{2n}+2isin\frac{\pi}{2n}cos\frac{\pi}{2n}-1}=\frac{n-(n-2)z}{-4sin^2\frac{\pi}{2n}\left(cos\frac{\pi}{n}+isin\frac{\pi}{n}\right)}=$$

$$=\frac{n-2}{4sin^2\frac{\pi}{2n}}-\frac{n}{4sin^2\frac{\pi}{2n}}\left(cos\frac{\pi}{n}-isin\frac{\pi}{n}\right)\Rightarrow$$

$$\sum_{k=1}^{n-1}ksin\frac{k\pi}{n}=Im\left(\sum_{k=1}^{n-1}kz^k\right)=\frac{nsin\frac{\pi}{n}}{4sin^2\frac{\pi}{2n}}=\frac{n}{2}cot\frac{\pi}{2n}\Rightarrow\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}=\frac{1}{2}cot\frac{\pi}{2n}$$

$$\Omega=\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}=\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{2}\cdot\left(\frac{1}{2}cot\frac{\pi}{2n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}$$

$$=\lim_{x\to\frac{\pi}{2n}}\left(\frac{cotx}{cot\frac{\pi}{2n}}\right)^{cot(2nx)}=\lim_{x\to\frac{\pi}{2n}}\left(1+\frac{cotx-cot\frac{\pi}{2n}}{cot\frac{\pi}{2n}}\right)^{cot(2nx)}=$$

$$=\lim_{x\to\frac{\pi}{2n}}\left(1+\frac{cotx-cot\frac{\pi}{2n}}{cot\frac{\pi}{2n}}\right)^{\frac{cot\frac{\pi}{2n}}{cotx-cot\frac{\pi}{2n}}\cdot\frac{\left(cotx-cot\frac{\pi}{2n}\right)cot(2nx)}{cot\frac{\pi}{2n}}}=$$

$$=e^{\lim_{x\to\frac{\pi}{2n}}cot(2nx)\cdot\frac{sin\left(\frac{\pi}{2n}-x\right)}{sinx\cdot sin\frac{\pi}{2n}}\cdot tan\frac{\pi}{2n}}=e^{-\frac{1}{nsin\frac{\pi}{n}}}=$$

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{1}{2}(cotx)\left(\sum_{k=1}^{n-1}\frac{k}{n}sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{tan(2nx)}}\right)=\lim_{n\to\infty}e^{-\frac{\frac{\pi}{n}}{sin\frac{\pi}{n}}\cdot\frac{1}{\pi}}=\frac{1}{\sqrt[\pi]{e}}$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE- **www.ssmrmh.ro**

ABOUT A RMM INEQUALITY-(XII)

By Marin Chirciu-Romania

1) If $a,b,c>0$ then:

$$\sum\frac{a}{3b+\sqrt[7]{ab^6}}\ge\frac{3}{4}$$

Proposed by Daniel Sitaru-Romania

Solution: Using the means inequality we obtain:

$$LHS=\sum\frac{a}{3b+\sqrt[7]{ab^6}}\overset{AM-GM}{\ge}\sum\frac{a}{3b+\frac{a+6b}{7}}=\sum\frac{7a}{a+27b}=7\sum\frac{a^2}{a^2+27ab}\overset{Bergstrom}{\ge}$$

$$\ge7\cdot\frac{(\sum a)^2}{\sum(a^2+27ab)}\overset{(1)}{\ge}\frac{3}{4}=RHS\text{, where (1)}\Leftrightarrow7\cdot\frac{(\sum a)^2}{\sum(a^2+27ab)}\ge\frac{3}{4}\Leftrightarrow$$

$$\Leftrightarrow 28\left(\sum a\right)^2 \ge 3\sum(a^2+27ab) \Leftrightarrow$$

$$\Leftrightarrow 28\left(\sum a^2+2\sum bc\right) \ge 3\sum a^2+81\sum bc \Leftrightarrow 28\sum a^2+56\sum bc \ge$$

$$\ge 3\sum a^2+81\sum bc \Leftrightarrow 25\sum a^2 \ge 25\sum bc \Leftrightarrow \sum a^2 \ge \sum bc \Leftrightarrow \sum(b-c)^2 \ge 0$$

Equality holds if and only if $a=b=c$.**Remark:** The problem can be developed.

2) If $a,b,c>0$ and $\lambda \ge \frac{1}{2}, n \in \mathbb{N}, n \ge 2$ then:

$$\sum \frac{a}{\lambda b+\sqrt[n]{ab^{n-1}}} \ge \frac{3}{\lambda+1}$$

Marin Chirciu

Solution: Using the means inequality we obtain:

$$LHS=\sum \frac{a}{\lambda b+\sqrt[n]{ab^{n-1}}} \overset{AM-GM}{\ge} \sum \frac{a}{\lambda b+\frac{a+(n-1)b}{n}} = \sum \frac{na}{a+(\lambda n+n-1)b} =$$

$$= n\sum \frac{a^2}{a^2+(\lambda n+n-1)ab} \overset{Bergstrom}{\ge} n\cdot \frac{(\sum a)^2}{\sum(a^2+(\lambda n+n-1)ab)} \overset{(1)}{\ge} \frac{3}{\lambda+1} = RHS, \text{ where}$$

$$(1) \Leftrightarrow n\cdot \frac{(\sum a)^2}{\sum(a^2+(\lambda+n-1)ab)} \ge \frac{3}{\lambda+1} \Leftrightarrow n(\lambda+1)(\sum a)^2 \ge 3\sum(a^2+(\lambda n+n-1)ab) \Leftrightarrow$$

$$\Leftrightarrow n(\lambda+1)\left(\sum a^2+2\sum bc\right) \ge 3\sum a^2+3(\lambda n+n-1)\sum bc \Leftrightarrow$$

$$\Leftrightarrow n(\lambda+1)\sum a^2+2n(\lambda+1)\sum bc \ge 3\sum a^2+3(\lambda n+n-1)\sum bc \Leftrightarrow$$

$\Leftrightarrow (\lambda n+n-3)\sum a^2 \ge (\lambda n+n-3)\sum bc$, which follows from $(\lambda n+n-3) \ge 0$, true from $\lambda \ge \frac{1}{2}, n \in \mathbb{N}, n \ge 2$ and $\sum a^2 \ge \sum bc \Leftrightarrow \sum(b-c)^2 \ge 0$

Equality holds if and only if $a=b=c$.

Note: For $\lambda=3, n=7$ we obtain the problem proposed by Daniel Sitaru in RMM 12/2020

Reference: ROMANIAN MATHEMATICAL MAGAZINE-**www.ssmrmh.ro**

THE CONTRAHARMONIC MEAN AND CONNECTIONS

Daniel Sitaru, Claudia Nănuți – Romania

Abstract: In this paper is presented the contraharmonic mean with properties and a few connections with the other means.

Let be $a_1, a_2, \dots, a_n > 0; n \in \mathbb{N}^*$. Define:

$$C(a_1, a_2, \dots, a_n) = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1 + a_2 + \cdots + a_n}$$

Property 1: In these conditions $C(a_1, a_2, \dots, a_n)$ is a mean.

Proof: Let be $m = \min_{1 \le i \le n} a_i$; $M = \max_{1 \le i \le n} a_i$

$$a_1^2 + a_2^2 + \cdots + a_n^2 = a_1 \cdot a_1 + a_2 \cdot a_2 + \cdots a_n \cdot a_n \ge$$

$$\ge ma_1 + ma_2 + \cdots + ma_n = m(a_1 + a_2 + \cdots + a_n),\quad \frac{a_1^2+a_2^2+\cdots+a_n^2}{a_1+a_2+\cdots+a_n} \ge m \quad (1)$$

$$a_1^2 + a_2^2 + \cdots + a_n^2 = a_1 \cdot a_1 + a_2 \cdot a_2 + \cdots + a_n \cdot a_n \le$$

$$\le Ma_1 + Ma_2 + \cdots + Ma_n = M(a_1 + a_2 + \cdots + a_n),\quad \frac{a_1^2+a_2^2+\cdots+a_n^2}{a_1+a_2+..+a_n} \le M \quad (2)$$

By (1); (2): $m \le C(a_1, a_2, \dots, a_n) \le M \quad (3)$

$$C(a_1, a_2, \dots, a_n) = C(a, a, \dots, a) = \frac{na^2}{na} = a,\ C(a_1, a_2, \dots, a_n) = a \quad (4)$$

By (3); (4)⇒ $C(a_1, a_2, \dots, a_n)$ is a mean.

Recall: The harmonic mean: $H(a,b) = \frac{2ab}{a+b}$, The geometric mean: $G(a,b) = \sqrt{ab}$

The logarithmic mean: $L(a,b) = \begin{cases} a; a = b \\ \frac{b-a}{\log b - \log a}; a \ne b \end{cases}$, The arithmetic mean: $A(a,b) = \frac{a+b}{2}$

The generalized mean: $M(a,b) = \sqrt[3]{\frac{a^3+b^3}{2}}$, the quadratic mean: $Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}$

It is known that:

$$m \le H(a,b) \le G(a,b) \le L(a,b) \le M(a,b) \le A(a,b) \le Q(a,b) \le M \quad \textbf{(5)}$$

Property 2: $Q(a_1, a_2, \dots, a_n) \le C(a_1, a_2, \dots, a_n)$ (6)

Proof: (6) ⇔ $\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \le \frac{a_1^2+a_2^2+\cdots+a_n^2}{a_1+a_2+\cdots+a_n}$, $\frac{a_1^2+a_2^2+\cdots+a_n^2}{n} \le \frac{(a_1^2+a_2^2+\cdots+a_n^2)^2}{(a_1+a_2+\cdots+a_n)^2}$

$$(a_1 + a_2 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + \cdots + a_n^2)$$

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 2\sum_{1 \le i < j \le n} a_i a_j \le n(a_1^2 + a_2^2 + \cdots + a_n^2)$$

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) - 2\sum_{1 \le i < j \le n} a_i a_j \ge 0,\ \sum_{1 \le i < j \le n} (a_i - a_j)^2 \ge 0$$

In these conditions (5) can be written:

$$\boldsymbol{m \le H(a,b) \le G(a,b) \le L(a,b) \le M(a,b) \le A(a,b) \le Q(a,b) \le C(a,b) \le M}$$

Property 3: If $x > 0$ then: $C(xa_1, xa_2, \dots, xa_n) = xC(a_1, a_2, \dots, a_n)$

Proof: $C(xa_1, xa_2, \dots, xa_n) = \frac{(xa_1)^2+(xa_2)^2+\cdots+(xa_n)^2}{xa_1+xa_2+\cdots+xa_n} =$

$$= \frac{x^2(a_1^2 + a_2^2 + \cdots + a_n^2)}{x(a_1 + a_2 + \cdots + a_n)} = x \cdot \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1 + a_2 + \cdots + a_n} = xC(a_1, a_2, \dots, a_n)$$

Property 4: $C(a,b) = 2A(a,b) - H(a,b)$

Proof: $2A(a,b) - H(a,b) = 2 \cdot \frac{a+b}{2} - \frac{2ab}{a+b} = \frac{(a+b)^2-2ab}{a+b} = \frac{a^2+b^2}{a+b} = C(a,b)$

Property 5: $A\big(H(a,b), C(a,b)\big) = A(a,b)$

Proof:

$$A\big(H(a,b), C(a,b)\big) = \frac{\frac{2ab}{a+b} + \frac{a^2+b^2}{a+b}}{2} = \frac{2ab + a^2 + b^2}{2(a+b)} = \frac{(a+b)^2}{2(a+b)} = \frac{a+b}{2} = H(a,b)$$

Property 6: $G\big(A(a,b), C(a,b)\big) = Q(a,b)$

Proof: $G\big(A(a,b), C(a,b)\big) = \sqrt{A(a,b) \cdot C(a,b)} =$

$$= \sqrt{\frac{a+b}{2} \cdot \frac{a^2+b^2}{a+b}} = \sqrt{\frac{a^2+b^2}{2}} = Q(a,b)$$

Property 7: $G\big(A(a,b)H(a,b)\big) = G(a,b)$

Proof: $G\big(A(a,b), H(a,b)\big) = \sqrt{A(a,b) \cdot H(a,b)} = \sqrt{\frac{a+b}{2} \cdot \frac{2ab}{a+b}} = \sqrt{ab} = G(a,b)$

Property 8: $\frac{L(a^2,b^2)}{L(a,b)} = A(a,b)$

Proof:

$$\frac{L(a^2,b^2)}{L(a,b)} = \frac{\frac{b^2-a^2}{\log b^2 - \log a^2}}{\frac{b-a}{\log b - \log a}} = \frac{(b-a)(b+a)}{2(\log b - \log a)} \cdot \frac{\log b - \log a}{b-a} = \frac{a+b}{2} = A(a,b)$$

Property 9:

$$\sqrt{\frac{L(a,b)}{L\left(\frac{1}{a}, \frac{1}{b}\right)}} = G(a,b)$$

Proof:

$$\sqrt{\frac{L(a,b)}{L\left(\frac{1}{a},\frac{1}{b}\right)}}=\sqrt{\frac{b-a}{\log b-\log a}\cdot\frac{\log\frac{1}{b}-\log\frac{1}{a}}{\frac{1}{b}-\frac{1}{a}}}=$$

$$=\sqrt{\frac{b-a}{\log b-\log a}\cdot\frac{ab}{a-b}\cdot(-\log b+\log a)}=\sqrt{ab}=G(a,b)$$

Property 10:

$$\frac{L\left(\frac{1}{a},\frac{1}{b}\right)}{L\left(\frac{1}{a^2},\frac{1}{b^2}\right)}=H(a,b)$$

Proof: $\frac{L\left(\frac{1}{a},\frac{1}{b}\right)}{L\left(\frac{1}{a^2},\frac{1}{b^2}\right)}=\frac{\frac{\frac{1}{b}-\frac{1}{a}}{\log\frac{1}{b}-\log\frac{1}{a}}}{\frac{1}{b^2}-\frac{1}{a^2}}=\left(\frac{1}{b}-\frac{1}{a}\right)\cdot\frac{1}{-\log b+\log a}\cdot\frac{a^2b^2}{a^2-b^2}\cdot(-2\log b+2\log a)=$

$$=\frac{a-b}{ab}\cdot\frac{1}{\log b-\log a}\cdot\frac{2a^2b^2}{(a-b)(a+b)}(\log b-\log a)=\frac{2ab}{a+b}=H(a,b)$$

Property 11: $\boldsymbol{C(a,b)+C(b,c)+C(c,a)\ge 3G(a,b,c)}$

Proof:

$$C(a,b)\ge G(a,b)\Rightarrow\sum_{cyc}C(a,b)\ge\sum_{cyc}G(a,b)=\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\overset{AM-GM}{\ge}$$

$$\ge 3\cdot\sqrt[3]{(\sqrt{ab})\cdot(\sqrt{bc})\cdot(\sqrt{ca})}=3\sqrt[3]{abc}=3G(a,b,c)$$

Property 12

$$\frac{C^2(a,b)}{C^2(b,c)}+\frac{C^2(b,c)}{C(c,a)}+\frac{C^2(c,a)}{C(a,b)}\ge\frac{C(a,b)}{C(b,c)}+\frac{C(b,c)}{C(c,a)}+\frac{C(c,a)}{C(a,b)}\quad \textbf{(7)}$$

Proof: Denote: $u=C(a,b),v=C(b,c),w=C(c,a)$

$$(7)\Leftrightarrow\sum_{cyc}\frac{u^2}{v^2}\ge\sum_{cyc}\frac{u}{v}\Leftrightarrow\sum_{cyc}\frac{u^2}{v^2}(uvw)^2\ge\sum_{cyc}\frac{u}{v}(uvw)^2\Leftrightarrow\sum_{cyc}u^4w^2\ge(uvw)^2\sum_{cyc}\frac{u}{v}$$

$$\sum_{cyc}u^4w^2=\frac{1}{6}\sum_{cyc}(6u^4w^2)=\frac{1}{6}\sum_{cyc}(4u^4w^2+u^4w^2+u^4w^2)=$$

$$=\frac{1}{6}\sum_{cyc}(4u^4w^2+v^4y^2+w^4v^2)\overset{AM-GM}{\ge}\frac{1}{6}\sum_{cyc}6\cdot\sqrt[6]{(u^4w^2)^4\cdot v^4\cdot u^2\cdot w^4v^2}=$$

$$= \sum_{cyc} \sqrt[6]{u^{18} \cdot v^6 \cdot w^{12}} = \sum_{cyc} u^3 v w^2 = \sum_{cyc} (uvw)^2 \cdot \frac{u}{v} = (uvw)^2 \sum_{cyc} \frac{u}{v}$$

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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-(XV)

By Marin Chirciu-Romania

1) In ΔABC, I – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\left(\frac{R_a}{a}\right)^2 + \left(\frac{R_b}{b}\right)^2 + \left(\frac{R_c}{c}\right)^2 \geq 1$$

Proposed by Marian Ursărescu – Romania

Solution: We prove:

Lemma 1: 2) In ΔABC, I – incenter, R_a – circumradii of ΔIBC. Prove that:

$$R_a = 2R \sin \frac{A}{2}$$

Proof: Using the formula $S = \frac{abc}{4R}$ in ΔIBC we obtain:

$$R_a = \frac{IB \cdot IC \cdot BC}{4S_{\Delta IBC}} = \frac{IB \cdot IC \cdot a}{4\frac{IB \cdot IC \cdot \sin(BIC)}{2}} = \frac{a}{2 \cdot \cos\frac{A}{2}} = \frac{2R \sin A}{2 \cdot \cos\frac{A}{2}} = \frac{2R \cdot 2\sin\frac{A}{2}\cos\frac{A}{2}}{2 \cdot \cos\frac{A}{2}} = 2R \cdot \sin\frac{A}{2}$$

Lemma 2: 3) In ΔABC, I – incenter, R_a – circumradii of ΔIBC. Prove that:

$$\sum \left(\frac{R_a}{a}\right)^2 = \frac{1}{4}\left[1 + \left(\frac{4R + r}{p}\right)^2\right]$$

Proof: Using the formula $R_a = 2R \sin\frac{A}{2}$ we obtain:

$$\sum \left(\frac{R_a}{a}\right)^2 = \sum \left(\frac{2R \sin\frac{A}{2}}{a}\right)^2 = \sum \left(\frac{2R \sin\frac{A}{2}}{2R \sin a}\right)^2 = \sum \left(\frac{\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}\right)^2 =$$

$$= \sum \left(\frac{\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}\right)^2 = \sum \left(\frac{1}{2\cos\frac{A}{2}}\right)^2 = \frac{1}{4}\sum \frac{1}{\cos^2\frac{A}{2}} = \frac{1}{4}\left[1 + \left(\frac{4R + r}{p}\right)^2\right]$$

which follows from the identity in triangle:

$$\sum \frac{1}{\cos^2\frac{A}{2}} = 1 + \left(\frac{4R + r}{p}\right)^2$$

Let's get back to the main problem. Using the Lemma the inequality can be written:

$$\frac{1}{4}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\ge 1\Leftrightarrow 1+\left(\frac{4R+r}{p}\right)^2\ge 4\Leftrightarrow (4R+r)^2\ge 3p^2 \text{ (Doucet's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark: The inequality can be strengthened.

4) In ΔABC, I – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2\ge\frac{1}{4}\left(5-\frac{2r}{R}\right)$$

Marin Chirciu

Solution : Using the Lemma we obtain:

$$\sum\left(\frac{R_a}{a}\right)^2=\frac{1}{4}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\ge\frac{1}{4}\left[1+\frac{(4R+r)^2}{R(4R+r)^2}\right]=\frac{1}{4}\left[1+\frac{2(2R-r)}{R}\right]=\frac{1}{4}\left(\frac{5R-2r}{R}\right)$$

$=\frac{1}{4}\left(5-\frac{2r}{R}\right)$, which follows from Blundon-Gerretsen inequality: $p^2\le\frac{R(4R+r)^2}{2(2R-r)}$

Equality holds if and only if the triangle is equilateral.

Remark: Inequality 4) is stronger than inequality 1).

5) In ΔABC, I – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2\ge\frac{1}{4}\left(5-\frac{2r}{R}\right)\ge 1$$

Marin Chirciu

Solution: See inequality 4) and $\frac{1}{4}\left(5-\frac{2r}{R}\right)\ge 1\Leftrightarrow R\ge 2r$, (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

Remark: Let's find an inequality of opposite sense.

6) In ΔABC, I – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2\le\frac{1}{4}\left(2+\frac{R}{r}\right)$$

Marin Chirciu

Solution: Using the Lemma the inequality can be written:

$$\frac{1}{4}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\le\frac{1}{4}\left[1+\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{1}{4}\left(1+\frac{R+r}{r}\right)=\frac{1}{4}\left(2+\frac{R}{r}\right)$$

which follows from Gerretsen inequality: $p^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$

Equality holds if and only if the triangle is equilateral. **Remark.** We can write the double inequality:

7) In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\frac{1}{4}\left(5-\frac{2r}{R}\right) \le \left(\frac{R_a}{a}\right)^2 + \left(\frac{R_b}{b}\right)^2 + \left(\frac{R_c}{c}\right)^2 \le \frac{1}{4}\left(2+\frac{R}{r}\right)$$

Marin Chirciu

Solution: RHS inequality:

Using the Lemma the inequality can be written:

$$\frac{1}{4}\left[1+\left(\frac{4R+r}{p}\right)^2\right] \le \frac{1}{4}\left[1+\frac{(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] = \frac{1}{4}\left(1+\frac{R+r}{r}\right) = \frac{1}{4}\left(2+\frac{R}{r}\right)$$

which follows from Gerretsen inequality: $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}$.

Equality holds if and only if the triangle is equilateral.

LHS inequality: Using the Lemma we obtain:

$$\sum\left(\frac{R_a}{a}\right)^2 = \frac{1}{4}\left[1+\left(\frac{4R+r}{p}\right)^2\right] \ge \frac{1}{4}\left[1+\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right] = \frac{1}{4}\left[1+\frac{2(2R-r)}{R}\right] =$$

$= \frac{1}{4}\left(\frac{5R-2r}{R}\right) = \frac{1}{4}\left(5-\frac{2r}{R}\right)$, which follows from Blundon – Gerretsen inequality:

$p^2 \le \frac{R(4R+r)^2}{2(2R-r)}$.Equality holds if and only if the triangle is equilateral.

Above, we've used Blundon-Gerretsen inequality:

$$\frac{r(4R+r)^2}{R+r} \le 16Rr - 5r^2 \le p^2 \le \frac{R(4R+r)^2}{2(2R-r)} \le 4R^2 + 4Rr + 3r^2$$

Remark: We can write the inequalities:

8) In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$1 \le \frac{1}{4}\left(5-\frac{2r}{R}\right) \le \left(\frac{R_a}{a}\right)^2 + \left(\frac{R_b}{b}\right)^2 + \left(\frac{R_c}{c}\right)^2 \le \frac{1}{4}\left(2+\frac{R}{r}\right)$$

Solution: See the above inequalities.Equality holds if and only if the triangle is equilateral.

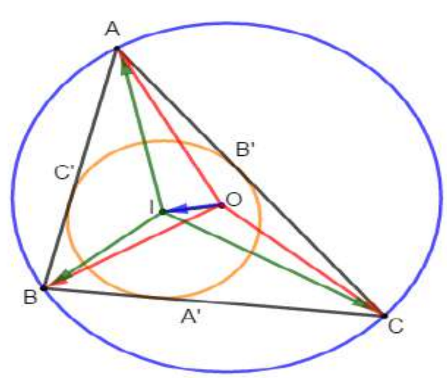
Reference:

ROMANIAN MATHEMATICAL MAGAZINE-**www.ssmrmh.ro**

VECTORIAL GEOMETRY-I

By Florică Anastase-Romania

"In memory of my colleague teacher ION CHEŞCĂ"



Let $\Delta ABC, I$ –incenter, O-circumcenter, the following relationship holds:

$$\overrightarrow{AA'} = \frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{b+c}$$

$$\overrightarrow{\boldsymbol{AI}}=\frac{\boldsymbol{b}\overrightarrow{\boldsymbol{AB}}+\boldsymbol{c}\overrightarrow{\boldsymbol{AC}}}{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}},\ \overrightarrow{\boldsymbol{OI}}=\frac{\boldsymbol{a}\overrightarrow{\boldsymbol{OA}}+\boldsymbol{b}\overrightarrow{\boldsymbol{OB}}+\boldsymbol{c}\overrightarrow{\boldsymbol{OC}}}{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}$$

Proof. From $\frac{A'B}{A'C}=\frac{c}{b}\Rightarrow\overrightarrow{A'B}=-\frac{c}{b}\overrightarrow{A'C}=-\frac{c}{b}\left(\overrightarrow{A'B}+\overrightarrow{BC}\right)\Rightarrow\overrightarrow{A'B}=-\frac{c}{b+c}\overrightarrow{BC}$, then

$$\overrightarrow{AA'}=\overrightarrow{AB}+\overrightarrow{BA'}\Rightarrow\overrightarrow{AA'}=\overrightarrow{AB}+\frac{c}{b+c}\left(\overrightarrow{AC}-\overrightarrow{AB}\right)\Rightarrow\overrightarrow{\boldsymbol{AA'}}=\frac{\boldsymbol{b}\overrightarrow{\boldsymbol{AB}}+\boldsymbol{c}\overrightarrow{\boldsymbol{AC}}}{\boldsymbol{b}+\boldsymbol{c}};(\mathbf{1})$$

How $\left(\overrightarrow{AI};\overrightarrow{AA'}\right)$ are collinear, we must to find $x\in\mathbb{R}$ such that $\overrightarrow{AI}=x\left(b\overrightarrow{AB}+c\overrightarrow{AC}\right)$ and analogously, $y\in\mathbb{R}$ such that $\overrightarrow{BI}=y\left(c\overrightarrow{BC}+a\overrightarrow{BA}\right)$.

We have: $\overrightarrow{BI}=\overrightarrow{BA}+\overrightarrow{AI}$;

$$(\textbf{\textit{Chasles identity}})\Rightarrow\overrightarrow{AI}=\overrightarrow{AB}+\overrightarrow{BI}\Rightarrow\overrightarrow{AI}=y\left(c\overrightarrow{BC}+a\overrightarrow{BA}\right)+\overrightarrow{AB};(2)$$

$$y\left(a\overrightarrow{BC}+a\overrightarrow{BA}\right)+\overrightarrow{AB}=x\left(b\overrightarrow{AB}+c\overrightarrow{AB}+c\overrightarrow{BC}\right)\Leftrightarrow(-1+ya+xb+xc)\overrightarrow{AB}=(cx-cy)\overrightarrow{BC}$$

How, $\left(\overrightarrow{AB};\overrightarrow{BC}\right)$ cannot be collinear, it follows that $bx+cx+ay=1$ and $cx-cy=0$, then

$$x=y=\frac{1}{a+b+c},\qquad\overrightarrow{\boldsymbol{AI}}=\frac{\boldsymbol{b}\overrightarrow{\boldsymbol{AB}}+\boldsymbol{c}\overrightarrow{\boldsymbol{AC}}}{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}};(\mathbf{3})$$

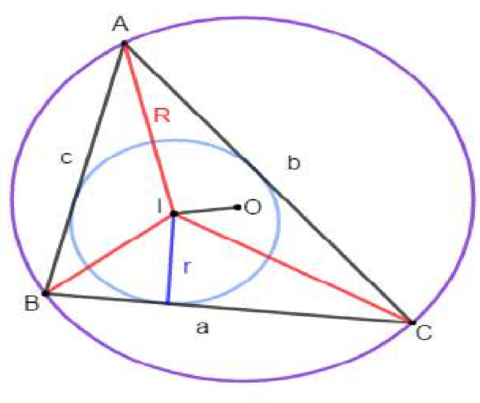
Let $\{I\}=AA'\cap BB'\cap CC'$, where AA',BB',CC' internal bisectors of $\angle BAC,\angle CBA$ and $\angle ACB$. From $\frac{IA'}{IA}=\frac{BA'}{BA}\Leftrightarrow\frac{IA'}{IA}=-\frac{\frac{ac}{b+c}}{c}\Leftrightarrow\frac{IA'}{IA}=-\frac{a}{a+c}$. Therefore,

$$\overrightarrow{OI}=\frac{\overrightarrow{OA'}+\frac{a}{b+c}\overrightarrow{OA}}{1+\frac{a}{b+c}}=\frac{(b+c)\overrightarrow{OA'}+a\overrightarrow{OA}}{a+b+c};(4)$$

From (1),(2),(3),(4) it follows that:

$$\overrightarrow{OA'}=\frac{\overrightarrow{OB}+\frac{c}{b}\overrightarrow{OC}}{1+\frac{c}{b}}=\frac{b\overrightarrow{OB}+c\overrightarrow{OC}}{b+c}\Rightarrow$$

$$\overrightarrow{\boldsymbol{OI}}=\frac{\boldsymbol{a}\overrightarrow{\boldsymbol{OA}}+\boldsymbol{b}\overrightarrow{\boldsymbol{OB}}+\boldsymbol{c}\overrightarrow{\boldsymbol{OC}}}{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}$$



Now, let's proof Euler's inequality.

Let ΔABC, I –incenter and O –circumcenter be origin to position vectors. Then,

$$\overrightarrow{OI}=\frac{a\overrightarrow{OA}+b\overrightarrow{OB}+c\overrightarrow{OC}}{a+b+c}$$

Squaring that identity, it follows

$$OI^2 = \frac{R^2(a^2+b^2+c^2) - 2\left(ab\overrightarrow{OA}\cdot\overrightarrow{OB} + bc\overrightarrow{OB}\cdot\overrightarrow{OC} + ca\overrightarrow{OC}\cdot\overrightarrow{OA}\right)}{(a+b+c)^2}$$

$$\overrightarrow{OA}\cdot\overrightarrow{OB} = OA\cdot OB\cdot\cos\left(\widehat{AOB}\right) = R\cdot R\frac{2R^2-c^2}{2R^2} = R^2 - \frac{c^2}{2}$$

$$OI^2 = \frac{R^2(a+b+c)^2 - abc(a+b+c)}{2(a+b+c)^2} = R^2 - \frac{abc}{2s} = R^2 - 2Rr = R(R-2r)$$

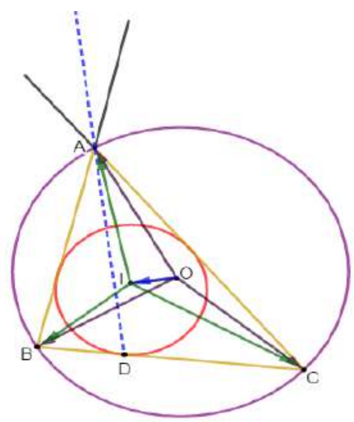
How, $OI^2 \geq 0$ we get: $\boldsymbol{R \geq 2r}$ $(\boldsymbol{Euler})$. Now, squaring identity $\overrightarrow{AI} = \frac{b\overrightarrow{AB}+c\overrightarrow{AC}}{a+b+c}$, we obtain

$$\left(\overrightarrow{AI}\right)^2 = \left(\frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{a+b+c}\right)^2 \Leftrightarrow$$

$$AI^2 = \frac{b^2c^2 + c^2b^2 + 2bc\cdot bc\cdot\cos C}{(b+c)^2} = \frac{2b^2c^2 + bc(b^2+c^2-a^2)}{(b+c)^2} =$$

$$= \frac{bc(2bc+b^2+c^2-a^2)}{(b+c)^2} = \frac{bc(b+c-a)(b+c+a)}{(b+c)^2} = \frac{bc\cdot 2(s-a)\cdot 2s}{(b+c)^2}$$

Hence: $\boldsymbol{AA'} = \frac{\boldsymbol{2}}{\boldsymbol{b+c}}\sqrt{\boldsymbol{bcs(s-a)}}$



Now, in ΔABC suppose that $b > c$ and let AD –external bisector of $\angle CAB, D \in (BC)$, we have:

$$\overrightarrow{AD} = \frac{b\overrightarrow{AB} - c\overrightarrow{AC}}{b-c} \Rightarrow \overrightarrow{AD}^2 = \left(\frac{b\overrightarrow{AB} - c\overrightarrow{AC}}{b-c}\right)^2 \Leftrightarrow$$

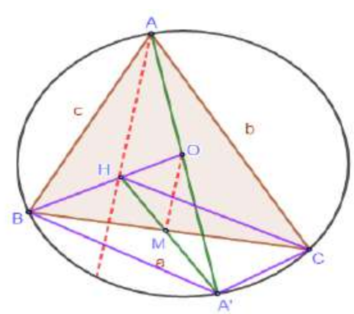
$$AD^2 = \frac{b^2c^2 + c^2b^2 - 2b^2c^2\cdot\cos A}{(b-c)^2} =$$

$$= \frac{2b^2c^2 - bc(b^2+c^2-a^2)}{(b-c)^2} =$$

$$= \frac{bc[a^2-(b-c)^2]}{(b-c)^2} = \frac{bc(a-b+c)(a+b-c)}{(b-c)^2} =$$

$$= \frac{bc(2s-2b)(2s-2c)}{(b-c)^2} = \frac{4bc(s-b)(s-c)}{(b-c)^2}$$

Hence: $\boldsymbol{AD} = \frac{\boldsymbol{2}}{\boldsymbol{b-c}}\sqrt{\boldsymbol{bc(s-b)(s-c)}}$



In ΔABC, O –circumcenter, H –orthocenter the following relationship holds:

$$\boldsymbol{a})\ \overrightarrow{\boldsymbol{HA}} + \overrightarrow{\boldsymbol{HB}} + \overrightarrow{\boldsymbol{HC}} = 2\overrightarrow{\boldsymbol{HO}}$$

$$\boldsymbol{b})\ \overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} + \overrightarrow{\boldsymbol{OC}} = \overrightarrow{\boldsymbol{OH}}\ (\boldsymbol{Sylvester})$$

Proof. Let $OM \perp BC$, then M –middle point of $[BC]$. In $\Delta AHA'$, $[OM]$ is middle line and $AH \perp BC$, $[AH]$ –altitude in ΔABC, O –middle point of $[AA']$, then M is middle point of $[HA']$. Therefore, $BHCA'$ –is parallelogram and $2\overrightarrow{HO} = \overrightarrow{HA} + \overrightarrow{HA'}$.

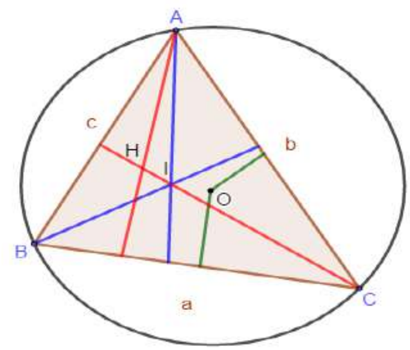
Now, from $\overrightarrow{HA'} = \overrightarrow{HB} + \overrightarrow{HC}$ it follows that $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} = 2\overrightarrow{HO}$ and from $2\overrightarrow{OM} = \overrightarrow{AH}$, $\overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OH}$, then $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OA} + 2\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.

In ΔABC the following relationship holds:

$$\boldsymbol{a)\ OH^2 = R^2(1 - 8\cos A\cos B\cos C)}$$

$$\boldsymbol{b)\ OH^2 = 9R^2 - (a^2 + b^2 + c^2)}$$

$$\boldsymbol{c)\ OI^2 = R^2 - \frac{abc}{a + b + c}}$$



Proof.

Using **Sylvester** identity: $\overrightarrow{r_H} = \overrightarrow{r_A} + \overrightarrow{r_B} + \overrightarrow{r_C}$ and squaring, we get:

$$\overrightarrow{r_H}^2 = \overrightarrow{r_A}^2 + \overrightarrow{r_B}^2 + \overrightarrow{r_C}^2 + 2(\overrightarrow{r_A} \cdot \overrightarrow{r_B} + \overrightarrow{r_B} \cdot \overrightarrow{r_C} + \overrightarrow{r_C} \cdot \overrightarrow{r_A})$$

$$\overrightarrow{r_A} \cdot \overrightarrow{r_B} = R^2 \cos 2C\ ;\ (\mu(\widehat{AOB}) = \mu(\widehat{AB}) = 2\mu(C)$$

$\overrightarrow{r_B} \cdot \overrightarrow{r_C} = R^2 \cos 2A$; $\overrightarrow{r_C} \cdot \overrightarrow{r_A} = R^2 \cos 2B$

Hence:

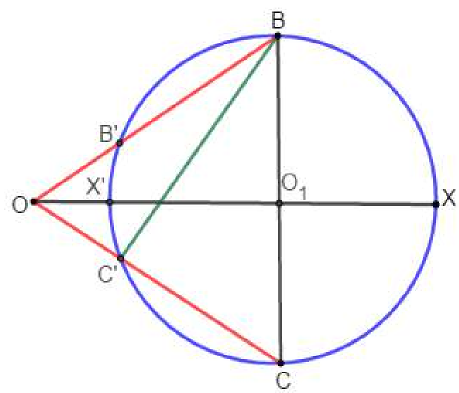
$OH^2 = 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C)$.

Now, using identity $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$, we get:

$$OH^2 = R^2(1 - 8\cos A\cos B\cos C)$$

Using Law of cosines, we have:

$OH^2 = 9R^2 - (a^2 + b^2 + c^2)$.



$$\overrightarrow{OB} \cdot \overrightarrow{OC} = OB \cdot OC \cdot \cos(\widehat{BOC}) = OC(OB \cdot \cos(\widehat{BOC}) = OC \cdot OC' = OX' \cdot OX =$$

$$= (d - R_1)(d + R_1) = d^2 - R_1^2, \text{ where } d = OO_1 \text{ and } R_1 = \frac{BC}{2}.$$

If A is middle point of $[BC]$, then $\overrightarrow{r_B} \cdot \overrightarrow{r_C} = OA'^2 - \frac{a^2}{2} = R^2 - \frac{a^2}{4} - \frac{a^2}{4} = R^2 - \frac{a^2}{2}$. Analogously, $\overrightarrow{r_A} \cdot \overrightarrow{r_B} = R^2 - \frac{c^2}{2}$ and $\overrightarrow{r_C} \cdot \overrightarrow{r_A} = R^2 - \frac{b^2}{2}$. Hence,

$$\overrightarrow{r_H}^2 = 3R^2 + 2\left(R^2 - \frac{a^2}{2} + R^2 - \frac{b^2}{2} + R^2 - \frac{c^2}{2}\right) \Leftrightarrow OH^2 = 9R^2 - (a^2 + b^2 + c^2)$$

Now, squaring in identity $\overrightarrow{r_I} = \frac{a\overrightarrow{r_A}+b\overrightarrow{r_B}+c\overrightarrow{r_C}}{a+b+c}$ and from $\overrightarrow{r_A}^2 = \overrightarrow{r_B}^2 = \overrightarrow{r_C}^2 = R^2$, $\overrightarrow{r_B}\cdot\overrightarrow{r_C} = R^2 - \frac{a^2}{2}$, $\overrightarrow{r_A}\cdot\overrightarrow{r_B} = R^2 - \frac{c^2}{2}$ and $\overrightarrow{r_C}\cdot\overrightarrow{r_A} = R^2 - \frac{b^2}{2}$ it follows that:

$$OI^2 = R^2 - \frac{abc}{a+b+c}$$

How $OH^2 \geq 0$, then $9R^2 - (a^2+b^2+c^2) \geq 0$ and $\boldsymbol{a^2+b^2+c^2 \leq 9R^2\ (Leibniz)}$.

Application 1: In ΔABC, I −incentre, the following relationship holds:

$$\boldsymbol{AI\cdot\frac{a^2}{w_a} + BI\cdot\frac{b^2}{w_b} + CI\cdot\frac{c^2}{w_c} \leq 12\sqrt{2}r\cdot\left(2R^2+\frac{F}{3\sqrt{3}}\right)}$$

Solution. Using bisector theorem, it follows that:

$$CB_1 = \frac{ab}{c+a}, \frac{BI}{BI_1} = \frac{a}{\frac{ab}{c+a}} = \frac{c+a}{b}$$

Also, $\frac{BB_1}{IB} = \frac{BI+IB_1}{IB} = 1 + \frac{IB_1}{IB} = 1 + \frac{b}{c+a} = \frac{a+b+c}{c+a} \Rightarrow \frac{BI}{BB_1} = \frac{c+a}{a+b+c} \Leftrightarrow \frac{BI}{w_b} = \frac{c+a}{a+b+c}$. Similarly,

$$\frac{AI}{w_a} = \frac{b+c}{a+b+c}, \frac{CI}{w_c} = \frac{a+b}{a+b+c}$$

$$AI\cdot\frac{a^2}{w_a} + BI\cdot\frac{b^2}{w_b} + CI\cdot\frac{c^2}{w_c} = \frac{a^2(b+c)+b^2(c+a)+c^2(a+b)}{a+b+c} =$$

$$= \frac{(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)}{a+b+c}$$

Applying Cauchy-Schwarz inequality, we have

$$(a^4+b^4+c^4)(b^2+c^2+a^2) \geq (a^2b+b^2c+c^2a)^2 \Leftrightarrow$$

$$a^2b+b^2c+c^2a \leq \sqrt{a^2+b^2+c^2}\cdot\sqrt{a^4+b^4+c^4}$$

$$ab^2+bc^2+ca^2 \leq \sqrt{a^2+b^2+c^2}\cdot\sqrt{a^4+b^4+c^4}$$

$$AI\cdot\frac{a^2}{w_a} + BI\cdot\frac{b^2}{w_b} + CI\cdot\frac{c^2}{w_c} \leq \frac{2\sqrt{a^2+b^2+c^2}\cdot\sqrt{a^4+b^4+c^4}}{a+b+c}$$
$$= \frac{\sqrt{a^2+b^2+c^2}\cdot\sqrt{a^4+b^4+c^4}}{s}$$

But: $a^2+b^2+c^2 \leq 9R^2\ (Leibniz)$ and $a^4+b^4+c^4 \leq 2(a^2+b^2+c^2)^2$, then

$$\sqrt{a^2+b^2+c^2}\cdot\sqrt{a^4+b^4+c^4} \leq 3R\cdot\sqrt{2}(a^2+b^2+c^2); (1)$$

Now, we want to prove that: $a^2+b^2+c^2 \leq 8R^2 + \frac{4F}{3\sqrt{3}}$; $(Nakajima's\ inequality)$; (2)

But: $a^2 + b^2 + c^2 = 2s^2 - 2r(4R + r)$, then $a^2 + b^2 + c^2 \le 8R^2 + \frac{4}{3\sqrt{3}}F \Leftrightarrow$

$$2s^2 - 8Rr - 2r^2 \le 8R^2 + \frac{4}{3\sqrt{3}} \Leftrightarrow s^2 \le 4R^2 + 4Rr + r^2 + \frac{2F}{3\sqrt{3}}$$

From $s^2 \le 4R^2 + 4Rr + 3r^2$ $(Gerretsen)$, we must to prove that $\frac{F}{3\sqrt{3}} \ge r^2 \Leftrightarrow \frac{rs}{3\sqrt{3}} \ge r^2$.

From $s^2 \ge 16Rr - 5r^2$ $(Gerretsen)$, it is suffices to prove $16Rr - 5r^2 \ge 27r^2 \Leftrightarrow$

$R \ge 2r$ $(Euler)$. From (1),(2) it follows that:

$$AI \cdot \frac{a^2}{w_a} + BI \cdot \frac{b^2}{w_b} + CI \cdot \frac{c^2}{w_c} \le \frac{3\sqrt{2}R}{s} \cdot \left(8R^2 + \frac{4}{3\sqrt{3}}F\right) \Leftrightarrow$$

$$AI \cdot \frac{a^2}{w_a} + BI \cdot \frac{b^2}{w_b} + CI \cdot \frac{c^2}{w_c} \le \frac{12\sqrt{2}R}{s} \cdot \left(2R^2 + \frac{F}{3\sqrt{3}}\right); \left(\frac{R}{s} = F\right) \Leftrightarrow$$

$$AI \cdot \frac{a^2}{w_a} + BI \cdot \frac{b^2}{w_b} + CI \cdot \frac{c^2}{w_c} \le 12\sqrt{2}r \cdot \left(2R^2 + \frac{F}{3\sqrt{3}}\right)$$

Application 2: In ΔABC, I –incentre, O –circumcentre, G –centroid. Prove that:

$$\left(\sum_{cyc} IA\right)\left(\sum_{cyc} OA\right)\left(\sum_{cyc} GA\right) < (a + b)(b + c)(c + a)$$

Daniel Sitaru

Solution. From Visschers's theorem (1902) in any triangle, the sum of the segments that unite a point $M \in Int(\Delta ABC)$ is less then, the sum of any two sides of the triangle.

$$MA + MB + MC < a + b; MA + MB + MC < b + c; MA + MB + MC < c + a$$

$$(MA + MB + MC)^3 < (a + b)(b + c)(c + a) \Leftrightarrow \sum_{cyc} MA < \sqrt[3]{(a + b)(b + c)(c + a)}; (*)$$

$$M = A \Rightarrow \sum_{cyc} IA < \sqrt[3]{(a + b)(b + c)(c + a)}; (1)$$

$$M = O \Rightarrow \sum_{cyc} OA < \sqrt[3]{(a + b)(b + c)(c + a)}; (2)$$

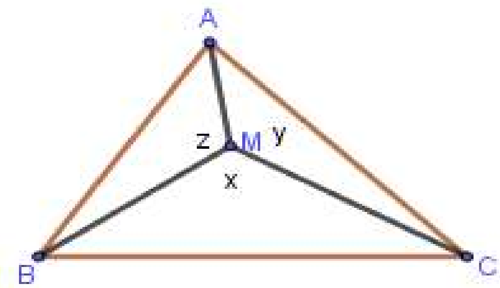
$$M = G \Rightarrow \sum_{cyc} GA < \sqrt[3]{(a + b)(b + c)(c + a)}; (3)$$

By multiplying (1),(2),(3) it follows that:

$$\left(\sum_{cyc} IA\right)\left(\sum_{cyc} OA\right)\left(\sum_{cyc} GA\right) < (a+b)(b+c)(c+a)$$

Application 3: In $\Delta ABC, M \in Int(\Delta ABC)$. Prove that:

$$[BMC]\cdot\overrightarrow{MA} + [AMC]\cdot\overrightarrow{MB} + [AMB]\cdot\overrightarrow{MC} = \vec{0}.$$



Solution. Let $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ —versors have some direction with $\overrightarrow{MA}, \overrightarrow{MB}$ and $\overrightarrow{MC}$, respectively.

Let $x = \mu(\widehat{BMC}), y = \mu(\widehat{AMC})$ and $z = \mu(\widehat{AMB})$ respectively.

We have:

$$[BMC]\cdot\overrightarrow{MA} + [AMC]\cdot\overrightarrow{MB} + [AMB]\cdot\overrightarrow{MC} = \vec{0}$$

$$\frac{MB\cdot MC\cdot\sin x}{2}\cdot\overrightarrow{MA} + \frac{MC\cdot MA\cdot\sin y}{2}\cdot\overrightarrow{MB} + \frac{MB\cdot MA\cdot\sin z}{2}\cdot\overrightarrow{OC} = \vec{0}$$

How $\overrightarrow{MA} = MA\cdot\overrightarrow{e_1}; \overrightarrow{MB} = MB\cdot\overrightarrow{e_2}$ and $\overrightarrow{MC} = MC\cdot\overrightarrow{e_3}$, we must to prove that:

$$\overrightarrow{e_1}\cdot\sin x + \overrightarrow{e_2}\cdot\sin y + \overrightarrow{e_3}\cdot\sin z = \vec{0}$$

Let $\Delta A_1B_1C_1$ such that the sides are parallels with MC, MA, MB and applying Law of sinus, we get: $A_1B_1 = 2R\cdot\sin z \Rightarrow \overrightarrow{A_1B_1} = 2R\overrightarrow{e_3}\sin z$ $(and\ analogs)$.Therefore,

$$\vec{0} = \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} + \overrightarrow{C_1A_1} = 2R(\overrightarrow{e_1}\cdot\sin x + \overrightarrow{e_2}\cdot\sin y + \overrightarrow{e_3}\cdot\sin z)$$

Application 4: In $\Delta ABC, G \in Int(\Delta ABC)$. Prove that if exist the point $M \in (ABC)$ such that: $3\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}$ then $G-$ is centroid.

Solution. Let be the points $\{D\} = AG\cap BC, \{E\} = BG\cap AC, \{F\} = CG\cap AC.$

Let us denote $\frac{BD}{DC} = x, \frac{CE}{EA} = y$ and $\frac{AF}{FB} = z$, then applying Van Aubel's theorem, we get:

$$\frac{AG}{GD} = z + \frac{1}{y} \Rightarrow \frac{AG}{AD} = \frac{yz+1}{yz+y+1}$$

Now, from Ceva's theorem, we have $xyz = 1$. For all point $M \in Int(\Delta ABC)$, we have:

$$\overrightarrow{MG} = \frac{yz+1}{yz+y+z}\overrightarrow{MD} + \frac{y}{yz+y+1}\overrightarrow{MA} =$$

$$= \frac{yz+1}{yz+y+1}\left(\frac{x}{1+x}\overrightarrow{MC} + \frac{1}{1+x}\overrightarrow{MB}\right) + \frac{y}{yz+y+1}\overrightarrow{MA} =$$

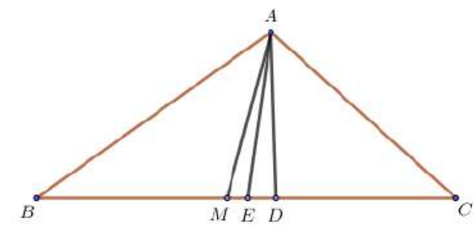
$$= \frac{1+x}{x+xy+1}\cdot\frac{x}{1+x}\overrightarrow{MC} + \frac{1}{x+xy+1}\overrightarrow{MB} + \frac{xy}{x+xy+1}\overrightarrow{MA}$$

On the other hand, we have: $3\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}$. Therefore, $x = y = z = 1 \Rightarrow$ G –centroid.

Application 5: In $\Delta ABC, \mu(\widehat{A}) = 90^\circ$, M is middle point of $(BC), AD$ –altitude, AE –internal bisector. Prove that:

$$\overrightarrow{AE} = \left(\frac{a}{b+c}\right)^2 \overrightarrow{AD} + \left[1 - \left(\frac{a}{b+c}\right)^2\right] \overrightarrow{AB}$$

Solution.



From bisector theorem, we have $\frac{BC}{c} = \frac{CE}{b} = \frac{a}{b+c}$ and from $b^2 = a \cdot CD$ it follows that:

$$\frac{ME}{MD} = \frac{BE - BM}{CM - CD} = \frac{\frac{a}{2} - \frac{ac}{b+c}}{\frac{b^2}{a} - \frac{a}{2}} = \frac{ab + ac - 2ac}{2(b+c)} \cdot \frac{2a}{2b^2 - a^2} =$$

$$= \frac{a^2(b-c)}{(b+c)(2b^2 - a^2)} = \frac{a^2}{(b+c)^2}, \qquad \overrightarrow{ME} = \left(\frac{a}{b+c}\right)^2 \cdot \overrightarrow{MD}$$

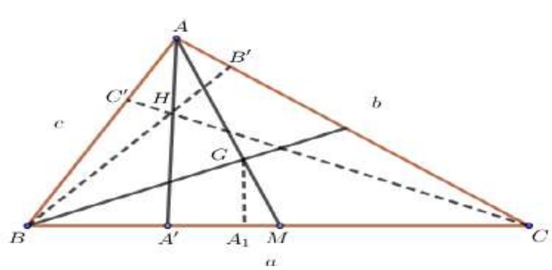
$$\overrightarrow{AE} - \overrightarrow{AM} = \overrightarrow{ME} = \left(\frac{a}{b+c}\right)^2 \cdot \overrightarrow{MD} = \left(\frac{a}{b+c}\right)^2 \left(\overrightarrow{AD} - \overrightarrow{AM}\right)$$

$$\overrightarrow{AE} = \left(\frac{a}{b+c}\right)^2 \overrightarrow{AD} + \left[1 - \left(\frac{a}{b+c}\right)^2\right] \overrightarrow{AB}$$

Application 6: Let A_1, B_1, C_1 be projection to centroid G in ΔABC. Prove that:

$$a^2 \cdot \overrightarrow{GA_1} + b^2 \cdot \overrightarrow{GB_2} + c^2 \cdot \overrightarrow{GC_1} = \vec{0}.$$

Solution.



Let us denote M middle point of BC and AA' –altitude. We have $\overrightarrow{GA_1} = \frac{1}{3}\overrightarrow{AA'}$.

But $BA' + A'C = a$ then, $A'C\left(1 + \frac{BA'}{A'C}\right) = a$. Hence, $\frac{A'C}{a} = \frac{1}{1+\frac{BA'}{A'C}}$. Denote $\frac{BA'}{A'C} = k$, it follows that: $\overrightarrow{AA'} = \frac{\overrightarrow{AB}+k\cdot\overrightarrow{AC}}{1+k} = \frac{A'C}{a}\cdot\overrightarrow{AB} + \frac{A'B}{a}\cdot\overrightarrow{AC}$

$$\overrightarrow{BB'} = \frac{CB'}{b}\cdot\overrightarrow{BA} + \frac{AB'}{b}\cdot\overrightarrow{BC},\qquad \overrightarrow{CC'} = \frac{C'A}{c}\cdot\overrightarrow{CB} + \frac{C'B}{c}\cdot\overrightarrow{CA};\ (BB' \perp AC, CC' \perp AB).$$

$$\Delta AA'C \sim \Delta BB'C \Rightarrow \frac{A'C}{B'C} = \frac{b}{a}; \frac{BA'}{BC'} = \frac{c}{a}; \frac{AC'}{AB'} = \frac{b}{c}.$$

Now, G –centroid, namely $\overrightarrow{GA_1} = \frac{1}{3}\overrightarrow{AA'} \Rightarrow a^2\cdot\overrightarrow{GA_1} = \frac{a}{3}\left(A'C\cdot\overrightarrow{AB} + A'B\cdot\overrightarrow{AC}\right)$. Therefore,

$$a^2\cdot\overrightarrow{GA_1} + b^2\cdot\overrightarrow{GB_2} + c^2\cdot\overrightarrow{GC_1} = \frac{a}{3}\left(A'C\cdot\overrightarrow{AB} + A'B\cdot\overrightarrow{AC}\right) + \frac{b}{3}\left(B'C\cdot\overrightarrow{CA} + C'A\cdot\overrightarrow{CB}\right) =$$

$$= \frac{1}{3}\left[(a\cdot A'C - b\cdot B'C)\overrightarrow{AB} + (b\cdot B'A - c\cdot C'A)\overrightarrow{BC} + (c\cdot C'B - a\cdot A'B)\overrightarrow{CA}\right] = \vec{0}$$

Application 7: In ΔABC, I –incentre, G –centroid. Prove that $IG \| BC$ if and only if $b + c = 2a$.

Solution. It is well-known that: $(a + b + c)\overrightarrow{MI} = a\overrightarrow{MA} + b\overrightarrow{MB} + c\overrightarrow{MC}, \forall M \in (ABC)$

Taking $M = G$ and from $\overrightarrow{GA} = -\frac{2}{3}\overrightarrow{AA'}$, $\overrightarrow{GB} = -\frac{2}{3}\overrightarrow{BB'}$, $\overrightarrow{GC} = -\frac{2}{3}\overrightarrow{CC'}$, where A', B', C' are middle points of BC, CA, AB respectively. Hence,

$$(a + b + c)\overrightarrow{GI} = -\frac{2}{3}\left(a\overrightarrow{AA'} + b\overrightarrow{BB'} + c\overrightarrow{CC'}\right) =$$

$$= -\frac{1}{3}\left[a\left(\overrightarrow{AB} + \overrightarrow{AC}\right) + b\left(\overrightarrow{BA} + \overrightarrow{BC}\right) + c\left(\overrightarrow{CA} + \overrightarrow{CB}\right)\right] =$$

$$= \frac{1}{3}\left[(2a - b - c)\overrightarrow{AB} + (b + a - 2c)\overrightarrow{BC}\right]$$

So, $IG \| BC$ if and only if $\overrightarrow{IG}, \overrightarrow{BC}$ have same direction, hence $2a - b - c = 0$.

Application 8: In ΔABC, points $P, Q \in (ABC)$ such that $\beta\overrightarrow{AB} + \gamma\overrightarrow{BP} + \overrightarrow{PC} = 0$ and

$$\overrightarrow{AQ} + \alpha\overrightarrow{QB} + \overrightarrow{BC} = 0, \alpha, \beta, \gamma \in \mathbb{R}, \alpha, \gamma \neq 1$$

Prove that *A, P, Q* are collinear if and only *if* $\alpha + \gamma = \beta + 1$.

Solution: $\overrightarrow{AQ} + \alpha\overrightarrow{QB} + \overrightarrow{BC} = 0 \Leftrightarrow \left(\overrightarrow{AQ} + \overrightarrow{QB} + \overrightarrow{BC}\right) = (\alpha - 1)\overrightarrow{BQ} \Leftrightarrow \overrightarrow{AC} = (\alpha - 1)\overrightarrow{BQ}$

$$\overrightarrow{AQ} = \overrightarrow{AB} + \overrightarrow{BQ} = \overrightarrow{AB} + \frac{1}{\alpha - 1}\overrightarrow{AC};\ (1)$$

$$\beta\overrightarrow{AB} + \gamma\overrightarrow{BP} + \overrightarrow{PC} = 0 \Leftrightarrow \beta\overrightarrow{AB} + \gamma\left(\overrightarrow{BA} + \overrightarrow{AP}\right) + \overrightarrow{PC} = 0 \Leftrightarrow$$

$$(\beta-\gamma)\overrightarrow{AB}+\gamma\overrightarrow{AP}+\overrightarrow{PC}=0\Leftrightarrow$$

$$(\beta-\gamma)\overrightarrow{AB}+(\gamma-1)\overrightarrow{AP}+\overrightarrow{AC}=0\Leftrightarrow$$

$$\overrightarrow{AP}=-\frac{1}{\gamma-1}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right)=\frac{1}{1-\gamma}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right);\quad (2)$$

From (1) and (2) A, P, Q are collinear if and only if exist $\lambda\in\mathbb{R}$ such that

$$\overrightarrow{AP}=\lambda\overrightarrow{AQ}\Leftrightarrow\frac{1}{1-\gamma}\left((\beta-\gamma)\overrightarrow{AB}+\overrightarrow{AC}\right)=\lambda\left(\overrightarrow{AB}+\frac{1}{\alpha-1}\overrightarrow{AC}\right)\Leftrightarrow$$

$$\begin{cases}\dfrac{\beta-\gamma}{1-\gamma}=\lambda\\ \dfrac{1}{1-\gamma}=\dfrac{\lambda}{\alpha-1}\end{cases}\Leftrightarrow\begin{cases}\beta-\gamma=\lambda(1-\gamma)\\ \alpha-1=\lambda(1-\gamma)\end{cases}\Leftrightarrow\alpha+\gamma=\beta+1$$

Application 9: In ΔABC; $M,E\in(AB)$; $N,F\in(AC)$ such that $\overrightarrow{AE}=m\overrightarrow{EB},\overrightarrow{AF}=n\overrightarrow{FC},\overrightarrow{MO}=p\overrightarrow{ON}$ and $\frac{MB}{MA}=\frac{NA}{NC}=\lambda$; $m,n,p,\lambda\in\mathbb{R}^*$; $p\neq-1,\lambda\neq1$; $m\cdot p=1$.

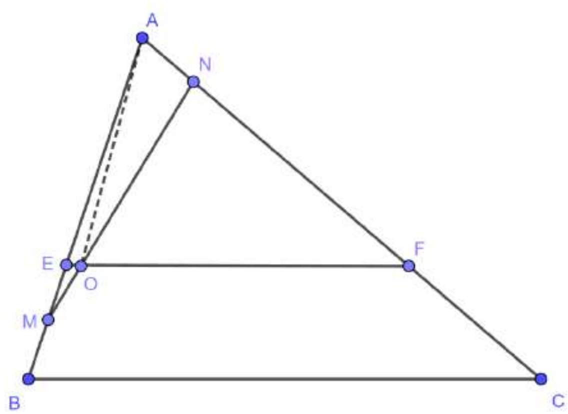
Prove that: E,O,F are collinear if and only if $p=n$.

Solution.

$$\frac{MB}{MA}=\frac{NA}{NC}=\lambda\Rightarrow\begin{cases}\overrightarrow{MB}=\lambda\overrightarrow{NA}\\ \overrightarrow{MA}=\lambda\overrightarrow{NC}\end{cases}$$

$$\Rightarrow\begin{cases}\overrightarrow{MA}=-\dfrac{1}{1-\lambda}\overrightarrow{AB}\\ \overrightarrow{AN}=-\dfrac{\lambda}{1-\lambda}\overrightarrow{AC}\end{cases};\quad (1)$$

$$\begin{cases}\overrightarrow{AE}=m\overrightarrow{EB}\\ \overrightarrow{AF}=n\overrightarrow{FC}\end{cases}\Rightarrow\begin{cases}\overrightarrow{AB}=\dfrac{m+1}{m}\overrightarrow{AE}\\ \overrightarrow{AC}=\dfrac{n+1}{n}\overrightarrow{AF}\end{cases};\quad (2)$$



$$\overrightarrow{MO}=p\overrightarrow{ON}\Rightarrow\overrightarrow{MA}+\overrightarrow{AO}=p\left(\overrightarrow{OA}+\overrightarrow{AN}\right),(1+p)\overrightarrow{AO}=\overrightarrow{AM}+p\overrightarrow{AN};\quad (3)$$

From (1),(3) we get: $\overrightarrow{AO}=\frac{1}{1+p}\cdot\frac{1}{1-\lambda}\overrightarrow{AB}-\frac{p}{1+p}\cdot\frac{\lambda}{1-\lambda}\overrightarrow{AC}$; (4)

From (2),(4) we have: $\overrightarrow{AO}=\frac{1}{1+p}\cdot\frac{1}{1-\lambda}\cdot\frac{m+1}{m}\overrightarrow{AE}-\frac{p}{1+p}\cdot\frac{\lambda}{1-\lambda}\cdot\frac{n+1}{n}\overrightarrow{AF}$

E,O,F are collinear if and only if $\frac{1}{1+p}\cdot\frac{1}{1-\lambda}\cdot\frac{m+1}{m}-\frac{p}{1+p}\cdot\frac{\lambda}{1-\lambda}\cdot\frac{n+1}{n}=1\Leftrightarrow$

$$n(m+1)-\lambda mp(n+1)=mn(p+1)(1-\lambda)\Leftrightarrow$$

$$n-mp\lambda=mnp-\lambda mn\Leftrightarrow n(1-mp)=m\lambda(p-n)\overset{mp=1}{\Longleftrightarrow}p=n.$$

Application 10: In ΔABC, H –orthocenter, M, N, P middle points of $(BC), (CA), (AB)$ respectively and $A_1 \in (AH), B_1 \in (BH), C_1 \in (CH)$ such that $\frac{AA_1}{A_1H} = \frac{BB_1}{B_1H} = \frac{CC_1}{C_1H}$.

Prove that the lines A_1M, B_1N, C_1P are concurrences.

Solution: From Sylvester identity, we have: $\overrightarrow{r_H} = \overrightarrow{r_A} + \overrightarrow{r_B} + \overrightarrow{r_C}$. Let us denote: $\frac{AA_1}{A_1H} = \frac{BB_1}{B_1H} = \frac{CC_1}{C_1H} = k$, so $\overrightarrow{r_{A_1}} = \frac{\overrightarrow{r_A}+k\overrightarrow{r_H}}{1+k} = \frac{1}{1+k}\overrightarrow{r_A} + \frac{k}{1+k}\overrightarrow{r_H} = \overrightarrow{r_A} + \frac{k}{1+k}\overrightarrow{r_B} + \frac{k}{1+k}\overrightarrow{r_C}$

Let's consider the point $Q \in (A_1M)$ such that $\frac{A_1Q}{QM} = l$. Hence: $\overrightarrow{r_Q} = \frac{1}{1+l}\overrightarrow{r_{A_1}} + \frac{l}{1+l}\overrightarrow{r_M} =$

$$= \frac{1}{1+l}\left(\overrightarrow{r_A} + \frac{k}{1+k}\overrightarrow{r_B} + \frac{k}{1+k}\overrightarrow{r_C}\right) + \frac{1}{1+l}\left(\frac{1}{2}\overrightarrow{r_B} + \frac{1}{2}\overrightarrow{r_C}\right) =$$

$$= \frac{1}{1+l}\left[\overrightarrow{r_A} + \left(\frac{k}{1+k} + \frac{l}{2(1+l)}\right)\overrightarrow{r_B} + \left(\frac{k}{1+k} + \frac{l}{2(1+l)}\right)\overrightarrow{r_C}\right]$$

$\frac{k}{1+k} + \frac{l}{2(1+l)} = 1 \Leftrightarrow l = \frac{2}{k-1}$. Hence, $\overrightarrow{r_Q} = \frac{k-1}{k+1}(\overrightarrow{r_A} + \overrightarrow{r_B} + \overrightarrow{r_C})$.

References: [1]. ROMANIAN MATHEMATICAL MAGAZINE**--www.ssmrmh.ro.**

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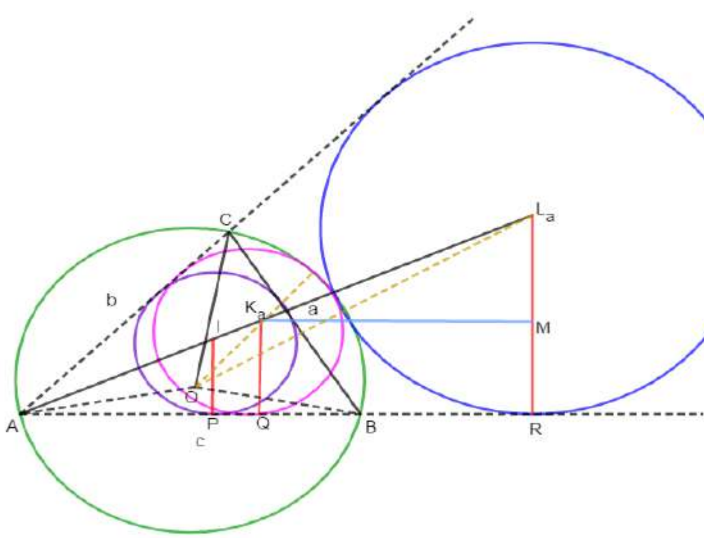
METRIC RELATIONSHIPS FOR MIXTILINIAR INCIRCLES AND EXCIRCLES

By Thanasis Gakopoulos-Greece



ΔABC: R –circumradii, r –inradius, F –area, $K_aQ = r_a$ radius of A mixtilinear incircle, $L_aR = R_a$ –radius of A mixtilinear excircle.

Plagiogonal system: $AB = Ax; AC = Ay, A(0,0), B(c,0), C(0,b)$

$$K_a(k,k), O(o_1,o_2), L_a(l,l), \begin{cases} o_1 = \dfrac{c - b\cdot\cos A}{2\cdot\sin^2 A}; (1) \\ o_2 = \dfrac{b - c\cdot\cos A}{2\cdot\sin^2 A}; (2) \end{cases}; r_a = k\cdot\sin A; (3); R = \frac{a}{2\sin A}; (4)$$

Is $(R - r_a)^2 = OK_a^2 \Rightarrow R^2 - 2Rr_a + r_a^2 = OK_a^2$; and from (3) $\Rightarrow$

$$\frac{a^2}{4\cdot\sin^2 A} - 2\cdot\frac{a}{2\cdot\sin A}\cdot k\cdot\sin A + k^2\cdot\sin^2 A = OK_a^2$$

$$k^2\cdot\sin^2 A + a\cdot k + \frac{a^2}{4\cdot\sin^2 A} = (o_1 - k)^2 + (o_2 - k)^2 - 2(o_1 - k)(o_2 - k)\cdot\cos A$$

From (1),(2) it follows $k = \frac{-a+b+c}{(1+\cos A)^2}$; (5)

From (3),(5) it follows $r_a = \frac{-a+b+c}{(1+\cos A)^2}\cdot\sin A \Rightarrow r_a = (-a+b+c)\cdot\frac{\tan\frac{A}{2}}{1+\cos A}$; (6). Similarly,

$$l = \frac{a+b+c}{(1+\cos A)^2}, R_a = (a+b+c)\cdot\frac{\tan\frac{A}{2}}{1+\cos A}; (7)$$

From (6),(7) it follows that $R_a - r_a = 2a\cdot\frac{\tan\frac{A}{2}}{1+\cos A}$; (8)

In ΔK_aL_aM: $\sin\frac{A}{2} = \frac{R_a - r_a}{K_aL_a}$ and from (8) we get:

$$(L_aK_a)^2 = \frac{4a^2}{(1+\cos A)^2\cdot\cos^2\frac{A}{2}} = \frac{8a^3}{(1+\cos A)^3} \Rightarrow \left(\frac{K_aL_a}{a}\right)^2 = \left(\frac{2}{1+\cos A}\right)^3 \Rightarrow$$

$$\frac{K_aL_a}{a} = \left(\frac{2}{1+\cos A}\right)^{\frac{3}{2}} \Rightarrow K_aL_a = \left(\frac{bc}{s(s-a)}\right)^{\frac{3}{2}}$$

$$\prod_{cyc}\left(\frac{K_aL_a}{a}\right) = \left[\frac{8}{(1+\cos A)(1+\cos B)(1+\cos C)}\right]^{\frac{3}{2}} = \left[\frac{a^2b^2c^2}{s^3(s-a)(s-b)(s-c)}\right]^{\frac{3}{2}} \Rightarrow$$

$$\prod_{cyc}\left(\frac{K_aL_a}{a}\right) = \left(\frac{4R}{s}\right)^3 \Rightarrow \prod_{cyc}(K_al_a) = 256\cdot\frac{R^4r}{s^2}$$

$$\begin{cases} r_a = (-a+b+c)\cdot\dfrac{\tan\frac{A}{2}}{1+\cos A} \\ r = \dfrac{bc}{a+b+c}\cdot\sin A \end{cases} \Rightarrow \frac{r_a}{r} = \frac{-a+b+c}{\frac{bc}{a+b+c}}\cdot\frac{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cdot\frac{1}{1+\cos A} \Rightarrow$$

$$\frac{r_a}{r} = \frac{(b+c)^2 - a^2}{2bc}\cdot\frac{1}{\cos^2\frac{A}{2}}\cdot\frac{1}{1+\cos 2A} \Rightarrow \frac{r_a}{r} = \frac{1}{\cos^2\frac{A}{2}} \Rightarrow \frac{r_a}{r} = \frac{bc}{s(s-a)}$$

$$\begin{cases} R_a = (a+b+c)\cdot\dfrac{\tan\frac{A}{2}}{1+\cos A} \\ r = \dfrac{bc}{a+b+c}\cdot\sin A \end{cases} \Rightarrow \frac{R_a}{r} = \frac{a+b+c}{\frac{bc}{a+b+c}}\cdot\frac{1}{\cos^2\frac{A}{2}}\cdot\frac{1}{1+\cos A} \Rightarrow$$

$$\frac{R_a}{r} = \frac{a+b+c}{-a+b+c}\cdot\frac{1}{\cos^2\frac{A}{2_a}} \Rightarrow \frac{R_a}{r} = \frac{s}{s-a}\cdot\frac{bc}{s(s-a)} \Rightarrow \frac{R_a}{r} = \frac{1}{\cos^2\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}$$

$$\Rightarrow \frac{R_a}{r}=\frac{bc}{(s-a)^2};\ \frac{R_a}{r_a}=\frac{1}{\tan\frac{B}{2}\cdot\tan\frac{C}{2}}\Rightarrow\frac{R_a}{r_a}=\frac{s}{s-b}$$

$$\prod_{cyc} r_a=\frac{a^2b^2c^2\cdot r^3}{s^2\cdot s(s-a)(s-b)(s-c)}\Rightarrow\prod_{cyc} r_a=\frac{16R^2r^3}{s^2}$$

$$\prod_{cyc} R_a=\frac{a^2b^2c^2\cdot r^3}{(s-a)^2(s-b)^2(s-c)^2}\Rightarrow\prod_{cyc} R_a=16R^2r$$

$$\prod_{cyc}\frac{R_a}{r_a}=\frac{16R^2r\cdot s^2}{16R^2r^3}\Rightarrow\prod_{cyc}\frac{R_a}{r_a}=\frac{s^2}{r^2}=\frac{s^4}{F^2}$$

Resume:

$$\frac{r_a}{r}=\frac{bc}{s(s-a)};(and\ analogs),\qquad \frac{R_a}{r}=\frac{bc}{(s-a)^2};(and\ analogs)$$

$$\frac{R_a}{r_a}=\frac{s}{s-a};(and\ analogs)$$

$$\prod_{cyc} r_a=16R^2\cdot\frac{r^3}{s^2};\ \prod_{cyc} R_a=16R^2r;\ \prod_{cyc}\frac{R_a}{r_a}=\frac{s^2}{r^2}=\frac{s^4}{F^2}$$

$$\frac{K_aL_a}{a}=\left[\frac{bc}{s(s-a)}\right]^{\frac{3}{2}};(and\ analogs)$$

$$\prod_{cyc}\left(\frac{K_aL_a}{a}\right)=\left(\frac{4R}{s}\right)^3;\ \prod_{cyc}(K_aL_a)=256\cdot\frac{R^4r}{s^2}=256\cdot\frac{R^4F}{s^3}$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE**- -www.ssmrmh.ro**

ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(VII)

By Bogdan Fuştei-Romania

In ΔABC the following relationship holds:

$\boldsymbol{s_a=\frac{2bc}{b^2+c^2}\ (and\ analogs)\ m_a-s_a=\frac{m_a(b-c)^2}{b^2+c^2}\le\frac{1}{2}|b-c|}$. If $b=c$ we have equality.

If $b\neq c\Rightarrow\frac{m_a(b-c)^2}{b^2+c^2}<\frac{1}{2}|b-c|\Leftrightarrow\frac{m_a|b-c|}{b^2+c^2}<\frac{1}{2}\Leftrightarrow 2m_a|b-c|<b^2+c^2$

$\Leftrightarrow 2m_a|b-c|<|b^2-c^2|$ true from $|b^2-c^2|<b^2+c^2$.

So, we have a new inequality: $\boldsymbol{\frac{1}{2}|b-c|\ge m_a-s_a\ (and\ analogs);(1)}$

$$\frac{1}{2}\sum_{cyc}|b-c|=\max\{a,b,c\}-\min\{a,b,c\}\Rightarrow\boldsymbol{\max\{a,b,c\}-\min\{a,b,c\}}$$

$$\boldsymbol{\ge\sum_{cyc}(m_a-s_a);(2)}$$

But $\begin{cases}|b-c|\ge n_a-g_a\\ \frac{1}{2}|b-c|\ge m_a-s_a\end{cases}\Rightarrow\boldsymbol{\frac{3}{2}|b-c|\ge n_a+m_a-g_a-s_a\ (and\ analogs);(3)}$

Adding these up relations, we get:

$$\mathbf{max\{a,b,c\}-min\{a,b,c\}\ge\frac{1}{3}\cdot\sum_{cyc}(n_a+m_a-g_a-s_s)\,;(4)}$$

$$\frac{3}{2}|b-c|\ge n_a+m_a-g_a-s_a; n_a+g_a\ge 2m_a\Rightarrow n_a\ge 2m_a-g_a$$

$$\frac{3}{2}|b-c|\ge 2m_a-g_a+m_a-g_a-s_a=3m_a-2g_a-s_a$$

$$\Rightarrow\mathbf{\frac{3}{2}|b-c|\ge\frac{3}{2}m_a-2g_a-s_a\ (and\ analogs);(5)}$$

Adding these up relations, we get:

$$\mathbf{max\{a,b,c\}-min\{a,b,c\}\ge\frac{1}{3}\cdot\sum_{cyc}(3m_a-2g_a-s_a)\,;(6)}$$

$$\max\{a,b,c\}-\min\{a,b,c\}\ge\frac{1}{3}\cdot(n_a+n_b+n_c)+\frac{1}{3}\cdot\sum_{cyc}(m_a-g_a-s_a)$$

But $n_a+n_b+n_c\ge s\sqrt{4-\frac{2r}{R}}$ then:

$$\max\{a,b,c\}-\min\{a,b,c\}\ge\frac{1}{3}\cdot s\sqrt{4-\frac{2r}{R}}+\frac{1}{3}\cdot\sum_{cyc}(m_a-g_a-s_a)$$

So, it follows that:

$$\mathbf{3(max\{a,b,c\}-min\{a,b,c\})\ge s\sqrt{4-\frac{2r}{R}}+\sum_{cyc}(m_a-g_a-s_a)\,;(7)}$$

$$s^2=n_a^2+2r_ah_a\Rightarrow\frac{s^2}{h_a^2}=\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a};\ a\cdot h_a=2sr\Rightarrow\frac{a}{2r}=\frac{s}{h_a}$$

$$\Rightarrow\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a}(and\ analogs)$$

$$r_br_c=s(s-a)=\frac{(a+b+c)(b+c-a)}{4}=\frac{(b+c)^2-a^2}{4}$$

$$a^2=(b+c)^2-4r_br_c;\ \frac{a^2}{4r^2}=\frac{(b+c)^2}{4r^2}-\frac{r_br_c}{r^2},\qquad\frac{n_a^2}{h_a^2}+\frac{2r_a}{h_a}=\frac{(b+c)^2}{4r^2}-\frac{r_br_c}{r^2}$$

$$\frac{r}{2R}\cdot\frac{r_a}{h_a}=\frac{r_a-r}{4R}=\sin^2\frac{A}{2}\Rightarrow\frac{r_a}{h_a}=\frac{r_a-r}{2r}\ (and\ analogs)$$

$$bc=r_br_c+rr_a;\ \frac{(b+c)^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{r_a-r}{r}+\frac{r_br_c}{r^2}$$

$$\frac{(b+c)^2}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{bc-r^2}{r^2}\Rightarrow 1+\frac{(b+c)^2-4bc}{4r^2}=\frac{n_a^2}{h_a^2}$$

So, it follows that: $\mathbf{\frac{n_a^2}{h_a^2}=1+\frac{(b-c)^2}{4r^2}\ (and\ analogs);(8)}$

$$\frac{(b-c)^2}{4}\ge\frac{(n_a+m_a-g_a-s_a)^2}{9}\Bigg|\cdot\left(\frac{1}{r^2}+1\right)\Rightarrow$$

$$1+\frac{(b-c)^2}{4r^2}\ge\frac{9r^2+(n_a+m_a-g_a-s_a)^2}{9r^2},\qquad\frac{n_a^2}{h_a^2}\ge\frac{9r^2+(n_a+m_a-g_a-s_a)^2}{9r^2}$$

So, it follows that:

$$\frac{n_a}{h_a} \ge \frac{\sqrt{9r^2 + (n_a + m_a - g_a - s_a)^2}}{3r}; \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \Rightarrow$$

$$\mathbf{3 \ge \sum_{cyc} \frac{\sqrt{9r^2 + (n_a + m_a - g_a - s_a)^2}}{n_a}; (9)}$$

$$\frac{a^2}{4r^2} = \frac{n_a^2}{h_a^2} + \frac{2r_a}{h_a} \ge \frac{9r^2 + (n_a + m_a - g_a - s_a)^2}{9r^2} + \frac{r_a - r}{r}$$

$$\frac{a^2}{4r^2} \ge \frac{9r^2 + (n_a + m_a - g_a - s_a)^2 + 9rr_a - 9r^2}{9r^2}$$

$$\frac{a^2}{4r^2} \ge \frac{9rr_a + (m_a + n_a - g_a - s_a)^2}{9r^2}, \frac{9r^2}{4r^2} \ge \frac{9rr_a + (m_a + n_a - g_a - s_a)^2}{a^2}$$

$$\mathbf{\frac{3}{2} \ge \frac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{a}\ (and\ analogs); (10)}$$

Summing, we get:

$$\mathbf{\frac{9}{2} \ge \sum_{cyc} \frac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{a}; (11)}$$

$$\frac{3}{2}a \ge \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2} \Rightarrow$$

$$\frac{3}{2}(a + b + c) \ge \sum_{cyc} \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2}$$

$$\mathbf{3s \ge \sum_{cyc} \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2}; (12)}$$

$$s^2 = n_a^2 + 2r_a h_a \Rightarrow 2r_a h_a = s^2 - n_a^2 = (s + n_a)(s + n_a)$$

$$s - n_a = \frac{2r_a h_a}{s + n_a} \Rightarrow s = n_a + \frac{2r_a h_a}{s + n_a}\ (and\ analogs) \Rightarrow 3s = n_a + n_b + n_c + \sum_{cyc} \frac{2r_a h_a}{s + n_a}$$

So, it follows that:

$$\mathbf{n_a + n_b + n_c + \sum_{cyc} \frac{2r_a h_a}{s + n_a} \ge \sum_{cyc} \sqrt{9rr_a + (n_a + m_a - g_a - s_a)^2}; (13)}$$

$$\begin{cases} \dfrac{s}{h_a} = \dfrac{a}{2r} = \dfrac{n_a}{h_a} + \dfrac{2r_a}{s + n_a} \\ \dfrac{a}{2r} \ge \dfrac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{3r} \end{cases} \Rightarrow$$

$$\mathbf{\frac{n_a}{h_a} + \frac{r_a}{s + n_a} \ge \frac{\sqrt{9rr_a + (m_a + n_a - g_a - s_a)^2}}{3r}; (14)}$$

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}; \sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1; \tan\frac{A}{2} = \frac{r_a}{s}$$

$$\sin A = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}} \cdot \frac{\frac{1}{\cos^2\frac{A}{2}}}{\frac{1}{\cos^2\frac{A}{2}}} = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}}, \sin A = \frac{2sr_a}{s^2 + r_a^2}; s^2 = n_a^2 + 2r_a h_a$$

$$\frac{1}{\sin A} = \frac{n_a^2 + r_a^2 + 2r_a h_a}{2sr_a} \ge \frac{2n_a r_a + 2r_a h_a}{2sr_a} = \frac{n_a + h_a}{s}$$

So, we have:

$$\frac{1}{\sin A} \ge \frac{n_a + h_a}{s}\ (and\ analogs); 2F = bc \cdot \sin A$$

$$\Rightarrow bc = \frac{2F}{\sin A} \ge \frac{2F(n_a + h_a)}{s} \ge 2r(n_a + h_a), bc = 2Rh_a \ge 2r(n_a + h_a) \Rightarrow \frac{R}{r} \ge \frac{n_a + h_a}{h_a}$$

$$\Rightarrow \frac{R - r}{r} \ge \frac{\sqrt{9r^2 + (m_a + n_a - g_a - s_a)^2}}{3r}$$

$$\mathbf{3(R - r) \ge \sqrt{9r^2 + (m_a + n_a - g_a - s_a)^2}; (15)}$$

$$9R^2 - 18Rr + 9r^2 \ge 9r^2 + (n_a + m_a - g_a - s_a)^2, 9R(R - 2r) \ge (n_a + m_a - g_a - s_a)^2$$

So, we get:

$$\mathbf{9R(R - 2r) \ge (n_a + m_a - g_a - s_a)^2; (16)}$$

Now, using : $m_a - h_a \ge \frac{(b-c)^2}{2a}$ $(and\ analogs)$

$$\frac{a(m_a - h_a)}{2r^2} \ge \frac{(b - c)^2}{4r^2}; \frac{(b - c)^2}{4r^2} = \frac{n_a^2}{h_a^2} - 1\ (and\ analogs)$$

$$\frac{a(m_a - h_a)}{2r^2} \ge \frac{n_a^2 - h_a^2}{h_a^2} \Rightarrow \frac{h_a^2}{2r^2}(m_a - h_a) \ge \frac{n_a^2 - h_a^2}{a}$$

$$\frac{h_a}{2r^2}(m_a - h_a) \ge \frac{n_a^2 - h_a^2}{2F} = \frac{n_a^2 - h_a^2}{2sr}, \frac{s}{r}(m_a - h_a) \ge \frac{(n_a - h_a)(n_a + h_a)}{h_a}$$

$$\frac{s}{r} \cdot \frac{m_a - h_a}{n_a + h_a} \ge \frac{n_a - h_a}{h_a} = \frac{n_a}{h_a} - 1 \Rightarrow \frac{s}{r} \cdot \frac{m_a - h_a}{n_a + h_a} \ge \frac{n_a}{h_a} - 1$$

So, we get:

$$\frac{s}{r} \cdot \frac{m_a - h_a}{n_a + h_a} \ge \frac{\sqrt{9r^2 + (m_a + n_a - g_a - s_a)^2} - 3r}{3r}$$

$$\mathbf{\frac{m_a - h_a}{n_a + h_a} \ge \frac{\sqrt{9r^2 + (m_a + n_a - g_a - s_a)^2} - 3r}{3s}; (17)}$$

$$\mathbf{\sum_{cyc} \frac{m_a - h_a}{n_a + h_a} \ge \sum_{cyc} \frac{\sqrt{9r^2 + (m_a + n_a - g_a - s_a)^2} - 3r}{3s}; (18)}$$

$$n_a^2 = s^2 - 2r_a h_a; \frac{n_a^2}{h_a} = \frac{s^2}{h_a} - 2r_a \Rightarrow \sum_{cyc} \frac{n_a^2}{h_a} = \frac{s^2}{r} - 2(4R + r)$$

$$\sum_{cyc} \frac{n_a^2}{h_a} = \frac{s^2 - 2r(4R + r)}{r}; \frac{n_a}{h_a} = \frac{n_a}{\sqrt{h_a}} \cdot \frac{1}{\sqrt{h_a}}, \sum_{cyc} \frac{n_a}{h_a} \overset{CBS}{\le} \sqrt{\left(\frac{n_a^2}{h_a} + \frac{n_b^2}{h_b} + \frac{n_c^2}{h_c}\right)\left(\frac{1}{h_a} + \frac{1}{h_c} + \frac{1}{h_c}\right)}$$

$$\mathbf{\sum_{cyc} \frac{n_a}{h_a} \ge \frac{1}{3r} \cdot \sum_{cyc} \sqrt{9r^2 + (n_a + m_a - g_a - s_a)^2}; (19)}$$

$$\mathbf{\sqrt{s^2 - 2r(4R + r)} \ge \frac{1}{3} \cdot \sum_{cyc} \sqrt{9r^2 + (n_a + m_a - g_a - s_a)^2}; (20)}$$

But: $s^2 \le 4R^2 + 4Rr + 3r^2$ $(Gerretsen) \Rightarrow$

$$\sum_{cyc} \frac{n_a}{h_a} \le \sqrt{\frac{4R^2 + 4Rr + 3r^2 - 8Rr - 2r^2}{r^2}} = \sqrt{\frac{(2R - r)^2}{r^2}}$$

$$\mathbf{\sum_{cyc} \frac{n_a}{h_a} \le \frac{2R - r}{r} \Rightarrow 3(2R - r) \ge \sum_{cyc} \sqrt{9r^2 + (n_a + m_a - g_a - s_a)^2}; (21)}$$

We known that: $n_a g_a \geq m_a w_a$, $n_a + g_a \geq 2m_a$ and $\frac{n_a g_a(n_a+g_a)}{2w_a} \geq m_a^2$.

But: $m_a^2 = r_b r_c + \frac{1}{4}(b-c)^2 \Rightarrow \frac{n_a g_a(n_a + g_a)}{2w_a} - r_b r_c \geq \frac{1}{4}(b-c)^2$

$$\sqrt{\frac{n_a g_a(n_a + g_a)}{2w_a} - r_b r_c} \geq \frac{1}{2}|b-c|$$

Summing, we get:

$$\sum_{cyc}\sqrt{\frac{n_a g_a(n_a + g_a)}{2w_a} - r_b r_c} \geq \max\{a,b,c\} - \min\{a,b,c\};\ (22)$$

$$\sum_{cyc}\sqrt{\frac{n_a g_a(n_a + g_a)}{2w_a} - r_b r_c} \geq \sum_{cyc}(m_a - s_a);\ (23)$$

$$\sum_{cyc}\sqrt{\frac{n_a g_a(n_a + g_a)}{2w_a} - r_b r_c} \geq \frac{1}{3}\cdot\sum_{cyc}(n_a + m_a - g_a - s_a);\ (24)$$

$$\sum_{cyc}\sqrt{\frac{n_a g_a(n_a + g_a)}{2w_a} - r_b r_c} \geq \frac{1}{3}\cdot\sum_{cyc}(3m_a - 2g_a - s_a);\ (25)$$

$$\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} = \frac{2R - r}{r} \geq \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c}$$

$$\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \geq \frac{1}{3r}\cdot\sum_{cyc}\sqrt{9r^2 + (n_a + m_a - g_a - s_a)^2};\ (26)$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE- **www.ssmrmh.ro**

A NEW PROOF FOR EULER'S INEQUALITY

By Neculai Stanciu-Romania

Let ABC be a triangle with angles A, B, C in radians, R –circumradius and r –inradius.

We consider the function: $f:(0,\pi) \to \mathbb{R}, f(x) = \log\left(\sin\frac{x}{2}\right) - \log x$

$$f'(x) = \frac{1}{2}\cot\frac{x}{2} - \frac{1}{x}, \qquad f''(x) = -\frac{1}{4\sin^2\frac{x}{2}} + \frac{1}{x^2} = \frac{\left(\sin\frac{x}{2} + \frac{x}{2}\right)\left(\sin\frac{x}{2} - \frac{x}{2}\right)}{x^2\sin^2\frac{x}{2}}$$

Because $0 < \sin\frac{x}{2} < \frac{x}{2}, \forall x \in (0,\pi)$ it results $f''(x) > 0$, so f –is concave on $(0,\pi)$.

From Jensen's inequality we deduce that:

$$f(A)+f(B)+f(C)\le 3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)$$

$$\Leftrightarrow \log\left(\frac{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{ABC}\right)\le\log\left(\frac{3}{2\pi}\right)^3\Leftrightarrow ABC\ge\frac{8\pi^3\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{27}$$

Using $\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{r}{4R}$ we obtain $ABC\ge\frac{2\pi^3 r}{27R}$. Hence,

$$\pi=A+B+C\ge 3\cdot\sqrt[3]{ABC}\ge 3\cdot\sqrt[3]{\frac{2\pi^3 r}{27R}}=\pi\cdot\sqrt[3]{\frac{2r}{R}}$$

$$1\ge\sqrt[3]{\frac{2r}{R}}\Leftrightarrow R\ge 2r\ (Euler)$$

ABOUT ȚIU-LEUENBERGER'S INEQUALITY

By D.M. Bătinețu-Giurgiu-Romania

Abstract: *This inequality was published by Constantin Ionescu-Țiu in REVISTA DE MATEMATICĂ ȘI FIZICĂ in 1953. Independently F. Leuenberger published in Elem. Math. in 1961 the same inequality. We will call this inequality:* ***ȚIU-LEUENBERGER'S INEQUALITY.***

ȚIU-LEUENBERGER'S inequality:

In any ΔABC the following relationship holds:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{\sqrt{3}}{R};\quad (1)$$

Proof. We have: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}=\frac{ab+bc+ca}{4RF}\overset{Gordon}{\ge}\frac{4\sqrt{3}F}{4RF}=\frac{\sqrt{3}}{R}$

Generalization. If $m\ge 0$ the in any triangle ABC the following relationship holds

$$\frac{1}{a^{m+1}}+\frac{1}{b^{m+1}}+\frac{1}{c^{m+1}}\ge\frac{\left(\sqrt{3}\right)^{1-m}}{R^{m+1}};\quad (2)$$

Proof. We have:

$$\frac{1}{a^{m+1}}+\frac{1}{b^{m+1}}+\frac{1}{c^{m+1}}=\frac{(ab)^{m+1}+(bc)^{m+1}+(ca)^{m+1}}{(abc)^{m+1}}=$$

$$=\frac{(ab)^{m+1}+(bc)^{m+1}+(ca)^{m+1}}{(4RF)^{m+1}}\overset{Radon}{\ge}\frac{(ab+bc+ca)^{m+1}}{3^m(4RF)^{m+1}}\overset{Gordon}{\ge}$$

$$\overset{Gordon}{\ge}\frac{\left(4\sqrt{3}F\right)^{m+1}}{3^m(4RF)^{m+1}}=\frac{\left(\sqrt{3}\right)^{m+1}}{3^m\cdot R^{m+1}}=\frac{\left(\sqrt{3}\right)^{1-m}}{R^{m+1}}$$

If $m = 0$ then (2) becomes (1).

Note by editor: A simple proof for Gordon's inequality:

In any ΔABC the following relationship holds: $ab + bc + ca \geq 4\sqrt{3}F$

Proof. $ab + bc + ca = s^2 + r^2 + 4Rr \overset{Gerretsen}{\geq} 16Rr - 5r^2 + r^2 + 4Rr =$

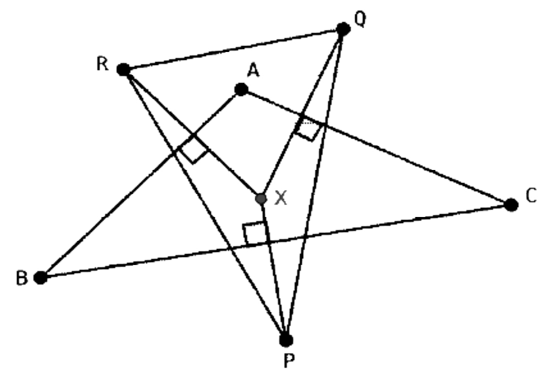
$$= 20Rr - 4r^2 \overset{Euler}{\geq} 20Rr - 2Rr = 18Rr \overset{Mitrinovic}{\geq} 18 \cdot \frac{2}{3\sqrt{3}} \cdot sr =$$

$$= \frac{12sr}{\sqrt{3}} = \frac{12\sqrt{3}F}{3} = 4\sqrt{3}F.$$

METRIC RELATIONSHIPS IN ȘAHIN'S TRIANGLE

By Daniel Sitaru – Romania

Abstract: In this article are proved a few metric relationships in a geometrical configuration created by the mathematician **Mehmet Șahin** from **Ankara – Turkiye**.



Theorem (Mehmet Șahin)

Let ΔABC be an acute triangle and $X \in Int\ (\Delta ABC)$ such that $XP \perp BC; XQ \perp AC;$ $XR \perp AB; XP = BC; XQ = AC; XR = AB$ (such in above figure). In these conditions:

1. $QR = 2m_a, RP = 2m_b, PQ = 2m_c$,($m_a, m_b, m_c$ – medians in the original ΔABC)

2. $[PQR] = 3F$,($[PQR]$ – area; F – area of the original ΔABC)

3. $m_{a'} = \frac{3a}{2}; m_{b'} = \frac{3b}{2}; m_{c'} = \frac{3c}{2}$,($m_{a'}, m_{b'}, m_{c'}$ - medians in ΔPQR; a, b, c – sides of original ΔABC)

4. $R^* = \frac{8}{3} \cdot \frac{m_a m_b m_c R}{abc}$,($R^*, R$ – circumradii of $\Delta PQR, \Delta ABC$)

5. $R^* \leq \frac{\sqrt{3}}{4} \cdot \frac{R^3}{r^2}$, (r – inradii of ΔABC)

6. $aR_a + bR_b + cR_c \leq 9R^2$,**(**$R_a, R_b, R_c$ **– circumradii of** $\Delta XQR, \Delta XRP, \Delta XPQ$**)**

7. $R^* = \frac{R_aR_bR_c}{3R^2}$

8. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{m_a+m_b+m_c+a+b+c}{F}$, **(**$r_1, r_2, r_3$ **– inradii of** $\Delta XQR, \Delta XRP, \Delta XPQ$**)**

Proof (Daniel Sitaru)

1. In ΔXQR by cosine law: $QR^2 = XQ^2 + XR^2 - 2XQ \cdot XR \cdot \cos(\pi - A)$

($ARXQ$ is cyclic quadrilateral, $\mu(\sphericalangle XRA) = \mu(XQA) = \frac{\pi}{2}$)

$$QR^2 = b^2 + c^2 - 2bc\cos(\pi - A), \quad QR^2 = b^2 + c^2 + 2bc\cos A$$

$$QR^2 = b^2 + c^2 + 2bc \cdot \frac{b^2 + c^2 - a^2}{2bc}, \quad QR^2 = 2(b^2 + c^2) - a^2$$

$$QR^2 = 4 \cdot \frac{2(b^2 + c^2) - a^2}{4}, \quad QR^2 = 4m_a^2 \Rightarrow QR = 2m_a$$

Analogous: $RP = 2m_b, PQ = 2m_c$

2. $[PQR] = [XPQ] + [XQR] + [XRP] = \frac{1}{2}XP \cdot XQ \cdot \sin(\sphericalangle PXQ) +$

$$+\frac{1}{2}XQ \cdot XR \cdot \sin(\sphericalangle XQR) + \frac{1}{2}XR \cdot XP\sin(\sphericalangle RXP) =$$

$$= \frac{1}{2}bc\sin(\pi - A) + \frac{1}{2}ca\sin(\pi - B) + \frac{1}{2}ab\sin(\pi - C) =$$

$$= \frac{1}{2}bc\sin A + \frac{1}{2}ca\sin B + \frac{1}{2}ab\sin C = F + F + F = 3F$$

3. Denote: $a' = QR = 2m_a, b' = RP = 2m_b, c' = PQ = 2m_c$

$$m_{a'}^2 = \frac{1}{2}(b'^2 + c'^2) - \frac{1}{4}a'^2 = \frac{1}{2}(4m_b^2 + 4m_c^2) - \frac{1}{4} \cdot 4m_a^2 =$$

$$= 2m_b^2 + 2m_c^2 - m_a^2 =$$

$$= 2\left(\frac{1}{2}(a^2 + c^2) - \frac{1}{4}b^2\right) + 2\left(\frac{1}{2}(a^2 + b^2) - \frac{1}{4}c^2\right) - \frac{1}{2}(b^2 + c^2) + \frac{1}{4}a^2 =$$

$$= a^2 + c^2 - \frac{1}{2}b^2 + a^2 + b^2 - \frac{1}{2}c^2 - \frac{1}{2}b^2 - \frac{1}{2}c^2 + \frac{1}{4}a^2 = 2a^2 + \frac{1}{4}a^2 = \frac{9a^2}{4}$$

$m_{a'}^2 = \frac{9a^2}{4} \Rightarrow m_{a'} = \frac{3a}{2}$. Analogous: $m_{b'} = \frac{3b}{2}; m_{c'} = \frac{3c}{2}$

4. $R^* = \frac{a'b'c'}{4[PQR]} = \frac{2m_a \cdot 2m_b \cdot 2m_c}{4 \cdot 3F} = \frac{2m_am_bm_c}{3 \cdot \frac{abc}{4R}} = \frac{8}{3} \cdot \frac{m_am_bm_cR}{abc}$

5. We will use the known inequalities:

$$m_a \le 2R\cos^2\frac{A}{2}; m_b \le 2R\cos^2\frac{B}{2}; m_c \le 2R\cos^2\frac{C}{2}$$

$$\frac{8}{3}\cdot\frac{m_a m_b m_c R}{abc} \le \frac{8R}{3abc}\cdot 2R\cos^2\frac{A}{2}\cdot 2R\cos^2\frac{B}{2}\cdot 2R\cos^2\frac{C}{2} =$$

$$= \frac{64R^4}{3abc}\cdot\frac{s(s-a)}{bc}\cdot\frac{s(s-b)}{ca}\cdot\frac{s(s-c)}{ab} = \frac{64R^4 s^2\cdot F^2}{3(abc)^3} = \frac{64R^4 s^2\cdot F^2}{3\cdot 16R^2F^2\cdot 4RF}$$

$$= \frac{4R^2s^2}{3\cdot 4RF} = \frac{R^2s^2}{3R\cdot rs} = \frac{RS}{3r} \overset{MITRINOVIC}{\le} \frac{R\cdot\frac{3\sqrt{3}}{2}R}{3r} = \frac{\sqrt{3}R^2}{2r} =$$

$$= \frac{\sqrt{3}R^3}{2rR} \overset{EULER}{\le} \frac{\sqrt{3}R^3}{2r\cdot 2r} = \frac{\sqrt{3}}{4}\cdot\frac{R^3}{r^2}$$

6. $R_a = \frac{XQ\cdot XR\cdot RQ}{4[XQR]} = \frac{b\cdot c\cdot 2m_a}{4F} = \frac{\frac{2F}{\sin A}\cdot 2m_a}{4F} = \frac{m_a}{\sin A} = \frac{m_a}{\frac{a}{2R}} = \frac{2Rm_a}{a}$

$$a\cdot R_a + b\cdot R_b + c\cdot R_c = 2R(m_a + m_b + m_c) \le 2R\cdot\frac{9R}{2} = 9R$$

(The inequality: $m_a + m_b + m_c \le \frac{9R}{2}$ is known)

7. $\frac{R_aR_bR_c}{3R^2} = \frac{\frac{2Rm_a}{a}\cdot\frac{2Rm_b}{b}\cdot\frac{2Rm_c}{c}}{3R^2} = \frac{8R^3}{3R^2}\cdot\frac{m_am_bm_c}{abc} = \frac{8}{3}\cdot\frac{m_am_bm_cR}{abc} = R^*$

8. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{\frac{2[XRQ]}{b+c+2m_a}} + \frac{1}{\frac{2[XPR]}{c+a+2m_b}} + \frac{1}{\frac{2[XQP]}{a+b+2m_c}} =$

$$= \frac{b+c+2m_a}{2F} + \frac{c+a+2m_b}{2F} + \frac{a+b+2m_c}{2F} =$$

$$= \frac{2(b+c+a+m_a+m_b+m_c)}{2F} = \frac{a+b+c+m_a+m_b+m_c}{F}$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE **– www.ssmrmh.ro**

A SIMPLE PROOF FOR WILKER'S INEQUALITY

By Daniel Sitaru – Romania

WILKER'S INEQUALITY: If $0 < x < \frac{\pi}{2}$ then:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2$$

Proof: Let be $f:\left(0,\frac{\pi}{2}\right) \to \mathbb{R}, f(x) = 2x^2 + x\sin 2x - 4\sin^2 x$

$$f'(x)=4x+\sin 2x+2x\cos 2x-4\sin 2x,\ f'(x)=4x+2x\cos 2x-3\sin 2x$$

$$f'(x)=2x(2+2x)-3\sin 2x,\ f'(x)=(2+\cos 2x)\left(2x-\frac{3\sin 2x}{2+\cos 2x}\right)$$

$$sgn\ f'(x)=sgn\ \left(2x-\frac{3\sin 2x}{2+\cos 2x}\right) \text{ because } 2+\cos 2x>0$$

$$\text{Let be } g:\left(0,\frac{\pi}{2}\right)\to\mathbb{R};\ g(x)=2x-\frac{3\sin 2x}{2+\cos 2x}$$

$$g'(x)=2-\frac{6\cos 2x\,(2+\cos 2x)-3\sin 2x\,(-2\sin 2x)}{(2+\cos 2x)^2}$$

$$g'(x)=\frac{2(2+\cos 2x)^2-6\cos 2x\,(2+\cos 2x)-6\sin^2 2x}{(2+\cos 2x)^2}$$

$$g'(x)=\frac{8+8\cos 2x+2\cos^2 2x-12\cos 2x-6\cos^2 2x-6\sin^2 2x}{(2+\cos 2x)^2}$$

$$g'(x)=\frac{8-4\cos 2x+2\cos^2 2x-6}{(2+\cos 2x)^2},\ g'(x)=\frac{2(\cos^2 2x-1)^2}{(2+\cos 2x)^2}\geq 0$$

$$g \text{ increasing on } \left(0,\frac{\pi}{2}\right)\Rightarrow g(x)>\lim_{\substack{x\to 0\\ x>0}} g(x)=0$$

$$g(x)>0\Rightarrow f'(x)=(2+\cos 2x)g(x)>0$$

$$f \text{ increasing on } \left(0,\frac{\pi}{2}\right)\Rightarrow f(x)>\lim_{\substack{x\to 0\\ x>0}} f(x)=0,\ \ f(x)>0;(\forall)x\in\left(0,\frac{\pi}{2}\right)$$

$$2x^2+x\sin 2x-4\sin^2 x>0,\ x^2+x\sin x\cos x-2\sin^2 x>0$$

$$\left(\frac{x}{\sin x}\right)^2+x\left(\frac{\cos x}{\sin x}\right)-2>0,\ \left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE – **www.ssmrmh.ro**

DINCĂ'S REFINEMENT FOR IONESCU-NESBITT'S INEQUALITY

By Daniel Sitaru – Romania

If $a,b,c>0$ then:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)}\geq\frac{3}{2} \quad \textbf{(1)}$$

Marian Dincă

Proof:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ac+bc}\overset{BERGSTROM}{\geq}\frac{(a+b+c)^2}{2(ab+bc+ca)}$$

Denote $S_1=a+b+c;S_2=ab+bc+ca$

$$S_1^2-2S_2=a^2+b^2+c^2>0, S_1^2-2S_2>0\Rightarrow S_1^2>2S_2\Rightarrow x=\frac{S_1^2}{S_2}>2$$

$$\frac{(a+b+c)^2}{2(ab+bc+ca)}\ge\frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)}\Leftrightarrow\Leftrightarrow\frac{S_1^2}{2S_2}\ge\frac{3\sqrt{3(S_1^2-2S_2)}}{2S_1}$$

$$S_1^3\ge 3S_2\sqrt{3(S_1^2-2S_2)}, S_1^6\ge 9S_2^2\left(3(S_1^2-2S_2)\right)$$

$$S_1^6\ge 27S_1^2S_2^2-54S_2^3, \frac{S_1^6}{S_2^3}-\frac{27S_1^2}{S_2}+54\ge 0$$

$$x^3-27x+54\ge 0, x>2>0$$

$$x^3+54=x^3+27+27\overset{AM-GM}{\ge}3\sqrt[3]{x^3\cdot 27\cdot 27}=27x$$

$$x^3-27x+54\ge 0$$

$$\frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)}\ge\frac{3}{2}\Leftrightarrow\sqrt{3(a^2+b^2+c^2)}\ge a+b+c$$

$$3(a^2+b^2+c^2)\ge(a+b+c)^2$$

$$a^2+b^2+c^2\ge ab+bc+ca, \frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)\ge 0$$

Equality holds in (1) for $a=b=c$.**Observation:** If a,b,c are sides in a triangle then (1) can be written: $\frac{a}{2s-a}+\frac{b}{2s-b}+\frac{c}{2s-c}\ge\frac{2s^2}{s^2+r^2+4Rr}\ge\frac{3\sqrt{6(s^2-r^2-4Rr)}}{4s}\ge\frac{3}{2}$

Reference: ROMANIAN MATHEMATICAL MAGAZINE **– www.ssmrmh.ro**

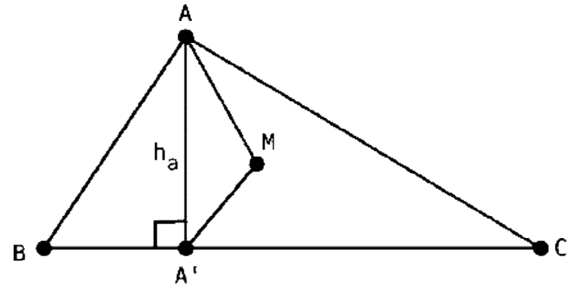
A SIMPLE PROOF FOR SCHREIBER'S INEQUALITY

By Daniel Sitaru, Claudia Nănuți – Romania

ABSTRACT. In this paper it is given a simple proof for Schreiber's inequality in triangle published first time in 1935. **Keywords:** Schreiber, Erdos - Mordell

SCHREIBER'S INEQUALITY: If $M\in Int\,(\Delta ABC)$ then: $MA+MB+MC\ge 6r$ (1)

Proof.



$$MA + MA' \geq h_a \Rightarrow MA' \geq h_a - MA \quad (2).$$ Analogous:

$$MB' \geq h_b - MB \quad (3), \quad MC' \geq h_c - MC \quad (4)$$

By Erdos-Mordell theorem: $MA + MB + MC \geq 2(MA' + MB' + MC') \geq$

$$\overset{(2);(3);(4)}{\geq} 2(h_a + h_b + h_c) - 2(MA + MB + MC)$$

$$3(MA + MB + MC) \geq 2(h_a + h_b + h_c)$$

$$MA + MB + MC \geq \frac{2}{3}(h_a + h_b + h_c) = \frac{2}{3}\left(\frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c}\right) =$$

$$= \frac{4F}{3} \cdot \frac{ab + bc + ca}{abc} = \frac{4F}{3} \cdot \frac{s^2 + r^2 + 4Rr}{4RF} =$$

$$= \frac{s^2 + r^2 + 4Rr}{3R} \geq 6r \Leftrightarrow s^2 + r^2 + 4Rr \geq 18Rr \Leftrightarrow s^2 \geq 14Rr - r^2$$

By Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2 \geq 14Rr - r^2 \Leftrightarrow$

$$2Rr \geq 4r^2 \Leftrightarrow R \geq 2r \quad \text{(EULER)}$$

Equality holds for M – center of an equilateral triangle.

Reference:ROMANIAN MATHEMATICAL MAGAZINE **– www.ssmrmh.ro**

TRIGONOMETRIC INTEGRAL INEQUALITIES

By Florică Anastase-Romania

Application 1. If $n \in \mathbb{N}, n \geq 2$ then:

$$\frac{n}{n+2} + \int_0^1 \left(\tan^{-1}(x^n)\right)^2 dx \geq 2\int_0^1 \tan^{-1}(x^n)\sqrt[n]{\tan^{-1}x}\,dx$$

Solution.

$$\tan x \geq x, \forall x \in [0,1] \Rightarrow \tan^{-1}x \leq x, \forall x \in [0,1]$$

$$\left(\sqrt[n]{x} - \tan^{-1}(x^n)\right)^2 \geq 0$$

$$\sqrt[n]{x^2} - 2\sqrt[n]{x}\tan^{-1}(x^n) + \left(\tan^{-1}(x^n)\right)^2 \geq 0$$

$$\sqrt[n]{x^2} + \left(\tan^{-1}(x^n)\right)^2 \geq 2\sqrt[n]{x}\tan^{-1}(x^n) \geq 2\sqrt[n]{\tan^{-1}x}\,\tan^{-1}(x^n)$$

$$\int_0^1 \sqrt[n]{x^2}dx + \int_0^1 \left(\tan^{-1}(x^n)\right)^2 dx \geq 2\int_0^1 \sqrt[n]{\tan^{-1}x}\,\tan^{-1}(x^n)dx$$

$$\frac{n}{n+2} + \int_0^1 \left(\tan^{-1}(x^n)\right)^2 dx \geq 2\int_0^1 \tan^{-1}(x^n)\sqrt[n]{\tan^{-1}x}\,dx\,; n \in \mathbb{N}, n \geq 2$$

Application 2. If $a > 1$, then:

$$\int_a^{a+1} \ln(\operatorname{arctg}(x+1))\,dx \leq \ln\left(\frac{\operatorname{arctg}^{a+2}(a+2)}{\operatorname{arctg}^{a+1}(a+1)}\right) - \operatorname{arctg}\left(\frac{1}{a^2 + 3a + 3}\right)$$

Solution. Let $f:[a,b]\to[f(a),f(b)]$ invertible and with derivative continuous. Then:

$$\int_a^b f(x)dx+\int_{f(a)}^{f(b)} f^{-1}(y)dy=bf(b)-af(a)$$

Let $f:[a,a+1]\to[f(a),f(a+1)], f(x)=ln\big(arctg(x+1)\big)$

$$f^{-1}(y)=tg(e^y)-1$$

$$\int_a^{a+1} ln\big(arctg(x+1)\big)\,dx+\int_{ln(arctg(a+1))}^{ln(arctg(a+2))}(tge^y-1)dy=$$

$$=(a+1)\,ln\big(arctg(a+2)\big)$$

$$-a\,ln\big(arctg(a+1)\big)=ln\left(\frac{arctg^{a+1}(a+2)}{arctg^{a}(a+1)}\right)\quad(1)$$

$$\int_a^{a+1} ln\big(arctg(x+1)\big)\,dx=ln\left(\frac{arctg^{a+1}(a+2)}{arctg^{a}(a+1)}\right)-\int_{ln(arctg(a+1))}^{ln(arctg(a+2))}(tge^y-1)dy$$

$$\therefore tg(e^y)\ge e^y$$

$$\int_{ln(arctg(a+1))}^{ln(arctg(a+2))}(tge^y-1)dy\ge\int_{ln(arctg(a+1))}^{ln(arctg(a+2))}(e^y-1)dy=$$

$$=e^{ln\,arctg(a+2))}-e^{ln\,arctg(a+1)}-(ln\,arctg(a+2)-ln\,arctg(a+1))=$$

$$=arctg(a+2)-arctg(a+1)-ln\left(\frac{lnarctg(a+2)}{lnarctg(a+1)}\right)=$$

$$=arctg\left(\frac{1}{a^2+3a+3}\right)-ln\left(\frac{lnarctg(a+2)}{lnarctg(a+1)}\right)\quad(2)$$

From (1),(2) it follows that:

$$\int_a^{a+1} ln\big(arctg(x+1)\big)\,dx\le$$

$$\le ln\left(\frac{arctg^{a+1}(a+2)}{arctg^{a}(a+1)}\right)+ln\left(\frac{lnarctg(a+2)}{lnarctg(a+1)}\right)-arctg\left(\frac{1}{a^2+3a+3}\right)$$

$$\int_a^{a+1} ln\big(arctg(x+1)\big)\,dx\le ln\left(\frac{arctg^{a+2}(a+2)}{arctg^{a+1}(a+1)}\right)-arctg\left(\frac{1}{a^2+3a+3}\right)$$

Application 3. If $a,b>0$, then:

$$\int_0^{\frac{\pi}{4}}\frac{dx}{(x+1)(a^2cos^2x+b^2sin^2x)}<\frac{1}{ab(\pi+4)}\left(\pi\frac{b}{a}+4tan^{-1}\left(\frac{b}{a}\right)\right)$$

Solution: *Theorem (Bonnet-Weierstrass):*

If $f:[a,b]\to R$ decreasing function of C^1class and $g:[a,b]\to R$ continuous function, then $\exists c\in[a,b]$ such that:

$$\int_a^b f(x)g(x)dx=f(a)\int_a^c g(x)dx+f(b)\int_c^b g(x)dx$$

Proof: Let $h:[a,b]\to R, h(x)=f(x)-f(b)$ decreasing and $h(x)\ge 0,\forall x\in[a,b]$.

From second M.V.T. $\exists c\in[a,b]$ such that:

$$\int_a^b g(x)h(x)dx=h(a)\int_a^c g(x)dx$$

$$\int_a^b g(x)\big(f(x)-f(b)\big)dx = \big(f(a)-f(b)\big)\int_a^c g(x)dx$$

$$\int_a^b f(x)g(x)dx =$$

$$= f(b)\int_a^b g(x)dx + \big(f(b)-f(a)\big)\int_a^c g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

q.e.d.

Let $f,g:\left[0,\frac{\pi}{4}\right]\to R, g(x)=\frac{1}{a^2cos^2x+b^2sin^2x}, f(x)=\frac{1}{x+1}, f'(x)=-\frac{1}{(x+1)^2}<0$ then f is decreasing.

$$G(x)=\int\frac{dx}{a^2cos^2x+b^2sin^2x}=$$

$$=\int\frac{1}{a^2+b^2tan^2x}\cdot\frac{dx}{cos^2x}=\frac{1}{b^2}\int\frac{dt}{\left(\frac{a}{b}\right)^2+t^2}=\frac{1}{ab}tan^{-1}\left(\frac{btanx}{a}\right)+C$$

Then $\exists c\in\left[0,\frac{\pi}{4}\right]$ for which:

$$\int_0^{\frac{\pi}{4}}\frac{dx}{(x+1)(a^2cos^2x+b^2sin^2x)}=f(0)\big(G(c)-G(0)\big)+f\left(\frac{\pi}{4}\right)\big(G(b)-G(c)\big)=$$

$$=\frac{1}{ab}tan^{-1}\left(\frac{btanc}{a}\right)+\frac{1}{\frac{\pi}{4}+1}\cdot\frac{1}{ab}\left(tan^{-1}\frac{b}{a}-tan^{-1}\left(\frac{b}{a}tanc\right)\right)=$$

$$=\frac{1}{ab(\pi+4)}\left(\pi tan^{-1}\left(\frac{b}{a}tanc\right)+4tan^{-1}\frac{b}{a}\right)$$

$$\because tan^{-1}x<x,\forall x>0\to tan^{-1}\left(\frac{b}{a}tanc\right)<\frac{b}{a}tan\,c<\frac{b}{a}tan\frac{\pi}{4}=\frac{b}{a}$$

$$\int_0^{\frac{\pi}{4}}\frac{dx}{(x+1)(a^2cos^2x+b^2sin^2x)}=\frac{1}{ab(\pi+4)}\left(\pi tan^{-1}\left(\frac{b}{a}tanc\right)+4tan^{-1}\left(\frac{b}{a}\right)\right)<$$

$$<\frac{1}{ab(\pi+4)}\left(\pi\frac{b}{a}+4tan^{-1}\left(\frac{b}{a}\right)\right)$$

Application 4. Prove that:

$$\int_0^1\frac{tan^{-1}x}{x\sqrt{1-x^2}}dx=log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}log(sinx)\,dx$$

Solution.

$\because \prod_{k=1}^{n-1}sin\frac{k\pi}{n}=\frac{\sqrt{n}}{2^{n-1}}$. *Let:* $x_k, k=1,2,..,2n$ *the roots of the unity.*

$$x_k=cos\frac{k\pi}{2n}+isin\frac{k\pi}{2n}, k=1,2,\dots,2n$$

$$x^{2n}-1=\prod_{k=1}^{2n}(x-x_k)\overset{x_{1,2}=\pm1-roots}{\triangleq}(x^2-1)\prod_{k=1}^{n-1}(x-x_k)(x-\overline{x_k})$$

$$= (x^2-1)\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)$$

$$\Rightarrow x^{2n-2}+x^{2n-4}+\dots+x^2+1=\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right)\overset{x=1}{\Longrightarrow}$$

$$n=\prod_{k=1}^{n-1}\left(2-2\cos\frac{k\pi}{n}\right)=\prod_{k=1}^{n-1}\left(4\sin^2\frac{k\pi}{2n}\right)$$

$$n=2^{2(n-1)}\cdot\sin^2\frac{\pi}{2n}\cdot\sin^2\frac{2\pi}{2n}\cdot\ldots\cdot\sin^2\frac{(n-1)\pi}{2n}$$

$$2^{n-1}\cdot\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\ldots\cdot\sin\frac{(n-1)\pi}{2n}=\sqrt{n}\Rightarrow\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=\frac{\sqrt{n}}{2^{n-1}}$$

$$\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx=\frac{1}{2}\int_0^{\pi}\log(\sin x)\,dx=\frac{\pi}{2}\int_0^1\log(\sin\pi x)\,dx=$$

$$=\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=1}^{n-1}\log\left(\sin\frac{k\pi}{n}\right)=\lim_{n\to\infty}\frac{\pi}{2n}\log\left(\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}\right)=$$

$$=\lim_{n\to\infty}\frac{\pi}{2n}\log\left(\frac{\sqrt{n}}{2^{n-1}}\right)=\frac{\pi}{2}\lim_{n\to\infty}\frac{\log\sqrt{n}-(n-1)\log 2}{n}=-\frac{\pi}{2}\log 2;\ (1)$$

$$F(y)=\int_0^1\frac{\tan^{-1}xy}{x\sqrt{1-x^2}}dx\Rightarrow F'(x)=\int_0^1\frac{dx}{(1+x^2y^2)\sqrt{1-x^2}}=\int_0^{\frac{\pi}{2}}\frac{dx}{1+y^2\cos^2 t}$$

$$=\frac{1}{\sqrt{1+y^2}}\tan^{-1}\left(\frac{\tan t}{\sqrt{1+y^2}}\right)=\frac{\pi}{2\sqrt{1+y^2}}\Rightarrow$$

$$F(y)=\frac{\pi}{2}\log\left(y+\sqrt{1+y^2}\right)+C,\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log\left(1+\sqrt{2}\right);\ (2)$$

From (1), (2) we get:

$$\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}}\log(\sin x)\,dx$$

Application 5. If $0<a<b<\frac{\pi}{2}$ then:

$$\frac{3(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-\sin 4(a+b)}}<3\int_a^b\frac{dx}{\sqrt[3]{1-\cos 4x}}<\cot(2a)-\cot(2b)+\frac{\pi}{4}$$

Solution.

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=1-\cos 4x\text{ –continuous}$$

$$g:(0,\infty)\to(0,\infty), g(x)=\frac{1}{\sqrt[3]{x}}=x^{-\frac{1}{3}}, g'(x)=-\frac{1}{3}x^{-\frac{4}{3}}<0, g''(x)=\frac{4}{9}x^{-\frac{7}{3}}>0\Rightarrow$$

g –convex function.Applying Jensen integral inequality, we get:

$$g\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\le\frac{1}{b-a}\int_a^b g\big(f(x)\big)dx\Leftrightarrow\frac{1}{\sqrt[3]{\frac{1}{b-a}\int_a^b f(x)dx}}\le\frac{1}{b-a}\int_a^b\frac{dx}{\sqrt[3]{f(x)}}\Leftrightarrow$$

$$\frac{1}{\sqrt[3]{\frac{1}{b-a}\int_a^b(1-\cos 4x)dx}} \le \frac{1}{b-a}\int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}} \Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{\frac{1}{b-a}\left(b-a-\frac{\sin 4b-\sin 4a}{4}\right)}} \le \int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}} \Leftrightarrow$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{4}\cdot\frac{\sin 4b-\sin 4a}{b-a}}} \le \int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}} \Leftrightarrow \frac{b-a}{\sqrt[3]{1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}}} \le \int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}}$$

$$u(t)=\frac{\sin t}{t}, t\in\left(0,\frac{\pi}{2}\right) \Rightarrow u'(t)=\frac{t\cos t-\sin t}{t^2}$$

$$v(t)=t\cos t-\sin t \Rightarrow v'(t)=-t\sin t<0, \forall t\in\left(0,\frac{\pi}{2}\right) \Rightarrow v(t)<v(0)=0$$

$$\Rightarrow u'(t)<0 \Rightarrow u(t)=\frac{\sin t}{t}-\text{decreasing} \Rightarrow \frac{\sin 2(a+b)}{2(a+b)}<\frac{\sin 2(b-a)}{2(b-a)} \Rightarrow$$

$$1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}<1-\frac{1}{2}\cdot\frac{\sin 2(b+a)\cos 2(a+b)}{b+a}; (*)$$

$$\frac{b-a}{\sqrt[3]{1-\frac{1}{2}\cdot\frac{\sin 2(b-a)\cos 2(a+b)}{b-a}}}=$$

$$=\frac{b-a}{\sqrt[3]{1-\frac{\sin 2(b-a)}{2(b-a)}\cdot\cos 2(a+b)}} \overset{(*)}{\ge} \frac{b-a}{\sqrt[3]{1-\frac{\sin 2(b+a)}{2(b+a)}\cdot\cos 2(a+b)}}=$$

$$=\frac{b-a}{\sqrt[3]{1-\frac{\sin 4(a+b)}{4(a+b)}}}=\frac{(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-\sin 4(a+b)}}$$

$$\frac{(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-\sin 4(a+b)}} \le \int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}}; \quad (1)$$

$$1+\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x} \overset{AGM}{\ge} 3\sqrt[3]{\frac{1}{\sin^2 x\cos^2 x}}=\frac{6}{\sqrt[3]{2(4\sin^2 x\cos^2 x)}}=$$

$$=\frac{6}{\sqrt[3]{2\sin^2 2x}}=\frac{6}{\sqrt[3]{1-\cos 4x}}$$

$$6\int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}}<\int_a^b dx+\int_a^b \frac{1}{\cos^2 x}dx+\int_a^b \frac{1}{\sin^2 x}dx \Leftrightarrow$$

$$\int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}}<\frac{1}{6}[(b-a)+(\tan b-\tan a)+(\cot a-\cot b)]<$$

$$<\frac{1}{6}(\tan b-\tan a)\left(1+\frac{1}{\tan a\tan b}\right)+\frac{\pi}{12}=\frac{1}{6}\left(\frac{\sin b}{\cos b}-\frac{\sin a}{\cos a}\right)\cdot\frac{1+\tan a\tan b}{\tan a\tan b}+\frac{\pi}{12}=$$

$$=\frac{1}{6}\cdot\frac{\sin(b-a)}{\cos a\cos b}\cdot\frac{\cos a\cos b+\sin a\sin b}{\sin a\sin b}+\frac{\pi}{12}=$$

$$=\frac{1}{6}\cdot\frac{4sin(b-a)cos(b-a)}{4sina\ sinb\ cosa\ cosb}+\frac{\pi}{12}=\frac{1}{6}\cdot\frac{2sin(2b-2a)}{sin2a\ sin2b}+\frac{\pi}{12}=$$

$$=\frac{1}{3}\cdot\frac{sin2b\ cos\ 2a-sin2a\ cos2b}{sin2a\ sin2b}+\frac{\pi}{12}=\frac{1}{3}\left(\frac{cos2a}{sin2a}-\frac{cos2b}{sin2b}\right)+\frac{\pi}{12}=$$

$$=\frac{1}{3}(cot2a-cot2b)+\frac{\pi}{12}$$

$$3\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}<cot2a-cot2b+\frac{\pi}{4};\quad (2)$$

From (1), (2) it follows that:

$$\frac{3(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-sin4(a+b)}}<3\int_a^b\frac{dx}{\sqrt[3]{1-cos4x}}<\cot(2a)-\cot(2b)+\frac{\pi}{4}$$

Application 6

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{sinx+cosx}dx>\frac{3}{\pi}\sqrt{\frac{\pi}{3}log3}$$

Solution.

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{sinx+cosx}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{sinx}\cdot\frac{1}{1+tanx}dx\overset{Cebyshev}{\geq}\quad(*)$$

Let $f,g:\left[\frac{\pi}{6},\frac{\pi}{3}\right]\to R, f(x)=\frac{1}{sinx}, g(x)=\frac{1}{1+tanx}$ decreasing functions.

$$(*)\geq\frac{6}{\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{sinx}dx\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{1+tanx}dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{1+tanx}dx\overset{tanx=t;dx=\frac{dt}{1+t^2}}{=}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{dt}{(1+t^2)(1+t)}=$$

$$=\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1}{1+t}dt+\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{1}{1+t^2}dt-\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{t}{1+t^2}dt=\frac{1}{4}\left(\frac{\pi}{3}+log3\right)$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{sinx}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+tan\frac{x}{2}}{2tan\frac{x}{2}}dx=log\left(tan\frac{x}{2}\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=log\frac{1+\sqrt{3}}{3-\sqrt{3}}>0$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{cotx}{sinx+cosx}dx\geq\frac{3}{\pi}\cdot\frac{1}{2}\left(\frac{\pi}{3}+log3\right)\cdot log\frac{1+\sqrt{3}}{3-\sqrt{3}}\overset{AM-GM}{>}\frac{3}{\pi}\sqrt{\frac{\pi}{3}log3}$$

Application 7.

$$\int_0^{\frac{\pi}{4}} \frac{\sin(2x)}{sinx + cosx} dx < \frac{2-\sqrt{2}}{\pi}\left(log2 + \frac{\pi}{2}\right)$$

Solution. Let $f(x) = \frac{1}{1+tanx}$ decreasing and $g(x) = sinx$ increasing on $\left[0, \frac{\pi}{4}\right]$

Applying Cebyshev's integral inequality, we get:

$$\int_0^{\frac{\pi}{4}} \frac{cosx}{sinx + cosx} dx = \int_0^{\frac{\pi}{4}} \frac{1}{1 + tanx} dx \overset{tanx=t, dx=\frac{dt}{1+t^2}}{\cong} \int_0^1 \frac{1}{(1+t^2)(1+t)} dt =$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+t} dt + \frac{1}{2}\int_0^1 \frac{1-t}{1+t^2} dt = \frac{1}{4}\left(log2 + \frac{\pi}{2}\right)$$

$$\int_0^{\frac{\pi}{4}} \frac{sin2x}{sinx + cosx} dx = 2\int_0^{\frac{\pi}{4}} sinx \cdot \frac{cosx}{sinx + cosx} dx \le 2\frac{4}{\pi}\left(\int_0^{\frac{\pi}{4}} sinx dx\right)\left(\int_0^{\frac{\pi}{4}} \frac{1}{1 + tanx} dx\right)$$

$$< \frac{2-\sqrt{2}}{\pi}\left(log2 + \frac{\pi}{2}\right)$$

Application 8. If $\pi < a \le b < \frac{3\pi}{2}$, then:

$$\int_a^b \frac{\left(\frac{1+\sin(sinx)}{sinx}\right)^{\frac{1}{1+cotx}} \cdot \left(\frac{1+\sin(cosx)}{cosx}\right)^{\frac{1}{1+tanx}}}{(1 + sin(sinx))sinx + (1 + sin(cosx))cosx} dx \le b - a$$

Solution:

$$Let\ f: \left(\pi, \frac{3\pi}{2}\right) \to R, f(x) = log\left(\frac{x}{1 + sinx}\right), f'(x) = \frac{1}{x} - \frac{cosx}{1 + sinx}, f''(x) = \frac{x^2 - sinx - 1}{x^2(1 + sinx)}$$

Let $h(x) = x^2 - sinx - 1, h'(x) = 2x - cosx, h''(x) = 2 + \sin x > 0; \forall x \in \left(\pi, \frac{3\pi}{2}\right)$

$$h'(x) > h'(\pi) = 2\pi + 1 > 0 \Rightarrow h(x) > h(\pi) > \pi^2 - 1 \Rightarrow f''(x) > 0; \forall x \in \left(\pi, \frac{3\pi}{2}\right)$$

$$f\left(\frac{sin^2x + cos^2x}{sinx + cosx}\right) \le \frac{sinx f(sinx) + cosx f(cosx)}{sinx + cosx} \leftrightarrow$$

$$log\left(\frac{1}{(sinx + cosx)\left(1 + sin\left(\frac{1}{sinx+cosx}\right)\right)}\right) \le \frac{log\left(\left(\frac{sinx}{1+\sin(sinx)}\right)^{sinx}\left(\frac{cosx}{1+\sin(cosx)}\right)^{cosx}\right)}{sinx + cosx}$$

Hence,

$$\left(\frac{1 + \sin(sinx)}{sinx}\right)^{\frac{sinx}{sinx+cosx}}\left(\frac{1 + \sin(cosx)}{cosx}\right)^{\frac{cosx}{sinx+cosx}} \le$$

$$\le (sinx + cosx)\left(1 + sin\left(\frac{1}{sinx + cosx}\right)\right) \overset{sinx\ is\ \underset{for\ x\in\left(\pi,\frac{3\pi}{2}\right)}{convex}}{\lesssim}$$

$$\le (sinx + cosx)\left(1 + \frac{sinx \cdot sin(sinx) + cosx \cdot sin(cosx)}{sinx + cosx}\right)$$

$$\left(\frac{1+\sin(sinx)}{sinx}\right)^{\frac{1}{1+cotx}}\left(\frac{1+\sin(cosx)}{cosx}\right)^{\frac{1}{1+tanx}}\le$$
$$\le(1+sin(sinx))sinx+(1+sin(cosx))cosx$$
$$\frac{\left(\frac{1+sin\,(sinx)}{sinx}\right)^{\frac{1}{1+cotx}}\cdot\left(\frac{1+sin\,(cosx)}{cosx}\right)^{\frac{1}{1+tanx}}}{(1+sin(sinx))sinx+(1+sin(cosx))cosx}\le 1$$

Application 9.

If 0 $< a \le b < 1$, **then:**

$$\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{b+asinx}dx\cdot\int_0^{\frac{\pi}{4}}\frac{1}{b+asinx}dx\ge\frac{\pi}{4}\left(\frac{1}{b}-\frac{\sqrt{2}}{2b+a\sqrt{2}}\right)$$

Solution. Let: $f,g:\left[0,\frac{\pi}{4}\right]\to R, f(x)=\frac{a+bsinx}{b+asinx}, g(x)=\frac{1}{b+asinx}$ derivable with $f'(x)=\frac{(b^2-a^2)cosx}{(b+asinx)^2}>0, g'(x)=-\frac{acosx}{(b+asinx)^2}<0\to f$ is increasing and g decreasing

$$\overset{Chebychev's}{\Rightarrow}\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{b+asinx}dx\cdot\int_0^{\frac{\pi}{4}}\frac{1}{b+asinx}dx\ge\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{(b+asinx)^2}dx\quad (i)$$

$$I=\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{(b+asinx)^2}dx=\frac{b}{a}\int_0^{\frac{\pi}{4}}\frac{\frac{a^2}{b}-b+(b+asinx)}{(b+asinx)^2}dx=$$

$$=\frac{a^2-b^2}{a}\int_0^{\frac{\pi}{4}}\frac{dx}{(b+asinx)^2}+\frac{b}{a}\int_0^{\frac{\pi}{4}}\frac{dx}{b+asinx}\quad (ii)$$

Let $t=\frac{cosx}{b+asinx}\to dt=-\frac{b}{a}\left(\frac{1}{b+asinx}+\frac{a^2-b^2}{b(b+asinx)^2}\right)dx\to$

$$t=-\frac{b}{a}\int_0^{\frac{\pi}{4}}\frac{dx}{b+asinx}-\frac{a^2-b^2}{b}\int_0^{\frac{\pi}{4}}\frac{dx}{(b+asinx)^2}\quad (iii)$$

From $(ii),(iii)$ we get: $I=\frac{-cosx}{b+asinx}\Big|_0^{\frac{\pi}{4}}=\frac{1}{b}-\frac{\sqrt{2}}{2b+a\sqrt{2}}$

So: $$\int_0^{\frac{\pi}{4}}\frac{a+bsinx}{b+asinx}dx\cdot\int_0^{\frac{\pi}{4}}\frac{1}{b+asinx}dx\ge\frac{\pi}{4}\left(\frac{1}{b}-\frac{\sqrt{2}}{2b+a\sqrt{2}}\right)$$

Application 10. If 0 $< a < b \le \frac{\pi}{2}$, **then:**

$$\frac{1}{b-a}\cdot\int_a^b\left(\frac{\int_0^x\frac{sint}{1+cost}dt}{\int_0^x log\left(\frac{1+sint}{1+cost}\right)dt}\right)dx<\frac{a}{b}\cdot\frac{log2}{\int_0^a log\left(\frac{1+sint}{1+cost}\right)dt}$$

Solution:

First: $$\int_0^{\frac{\pi}{2}}\frac{sint}{1+cost}dt=\int_0^{\frac{\pi}{2}}\frac{(-cost)'}{1+cost}dt=log2$$

Let functions: $f,g:\left[0,\frac{\pi}{2}\right]\to R, f(x)=\frac{sinx}{1+cosx}$,

$$F(x)=\int_0^x f(x)dx, g(x)=log\left(\frac{1+sinx}{1+cosx}\right) \text{ and } G(x)=\int_0^x g(x)dx$$

How: $F''(x)=f'(x)=\frac{1}{1+cosx}>0$,

$$G''(x)=g'(x)=\frac{1+cosx+sinx}{(1+sinx)(1+cosx)}>0, \forall x\in\left[0,\frac{\pi}{2}\right]\to$$

F,G are convexe$\to$ $\forall\tau\in[0,1]$ and $p,q\in\mathbb{R}$ such that:

$$F\big((1-\tau)p+\tau q\big)\le(1-\tau)F(p)+\tau F(q), \text{for } p=0, q=x_2, \tau=\frac{x_1}{x_2}, x_1<x_2\to$$

$\frac{F(x_1)}{x_1}<\frac{F(x_2)}{x_2}\to\frac{F(x)}{x}$ is increasing ($analogous$ $\frac{G(x)}{x}$ is increasing$\to$ $\frac{x}{G(x)}$ $decreasing$)

Applying Chebyshev's inequality, we get:

$$\int_a^b\frac{F(x)}{G(x)}dx=\int_a^b\frac{F(x)}{x}\cdot\frac{x}{G(x)}dx\le\frac{1}{b-a}\cdot\int_a^b\frac{F(x)}{x}dx\cdot\int_a^b\frac{x}{G(x)}dx\le(b-a)\cdot\frac{F(b)}{b}\cdot\frac{a}{G(a)}$$

$$\le\frac{a}{b}\cdot\frac{F\left(\frac{\pi}{2}\right)}{G(a)}=\frac{a}{b}\cdot\frac{log2}{\int_0^a log\left(\frac{1+sint}{1+cost}\right)dt}$$

Application 11. If $0<a\le b<\frac{\pi}{2}$ then:

$$\int_a^b\frac{\tan(\sin x)\tan(\cos x)}{\tan x\tan(\cos x)+\tan(\sin x)}\cdot\frac{\sin^2(\sin x)}{\sin^2 x}dx\ge\frac{5}{3}\tan^{-1}\left(\frac{\sin b-\sin a}{1+\sin a\sin b}\right)$$

Solution.

$$\frac{\tan(\sin x)\tan(\cos x)\sin^2(\sin x)}{(\tan x\tan(\cos x)+\tan(\sin x))\sin^2 x}=\frac{\tan(\sin x)\tan(\cos x)\sin^2(\sin x)}{\frac{\sin x\tan(\cos x)+\cos x\tan(\sin x)}{\cos x}\cdot\sin^2 x}=$$

$$=\frac{\tan(\sin x)\tan(\cos x)}{\sin x\tan(\cos x)+\cos x\tan(\sin x)}\cdot\frac{\cos x\sin^2(\sin x)}{\sin^2 x};(1)$$

Now, from Maclaurin series expansion for $f(x)=\tan x$, we have that:

$$\tan x\ge x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}$$

Hence,

$$(3-x^2)\tan x-3x\ge(3-x^2)\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}\right)-3x=$$

$$=\left(3x+x^3+\frac{6x^5}{15}+\frac{51x^7}{315}\right)-\left(x^3+\frac{x^5}{3}+\frac{2x^7}{15}+\frac{17x^9}{315}\right)-3x=$$

$$=\frac{x^5(21+9x^2-17x^4)}{315}\ge0, (3-x^2)\tan x\ge3x, \forall x\in\left(0,\frac{\pi}{2}\right)$$

For $x\to\sin x$ and $x\to\cos x$ it follows that:

$$\frac{\sin x}{\tan(\sin x)}+\frac{\cos x}{\tan(\cos x)}\le\frac{3-\sin^2 x}{3}+\frac{3-\cos^2 x}{3}=\frac{5}{3}, \forall x\in\left(0,\frac{\pi}{2}\right)$$

$$\frac{\sin x\tan(\cos x)+\cos x\tan(\sin x)}{\tan(\sin x)\tan(\cos x)}\le\frac{3}{5}, \forall x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow$$

$$\frac{\tan(\sin x)\tan(\cos x)}{\sin x\tan(\cos x)+\cos x\tan(\sin x)}\geq\frac{5}{3};(2)$$

$$x\in\left(0,\frac{\pi}{2}\right)\Rightarrow\tan x\geq x\Rightarrow\sin x\geq x\cos x$$

$$\sin^2 x\geq x^2\cos^2 x\Rightarrow\sin^2 x\geq x^2(1-\sin^2 x)\Rightarrow\sin^2 x\,(1+x^2)\geq x^2$$

$$\sin^2 x\geq\frac{x^2}{1+x^2}$$

Putting $x=\sin x$ it follows that:

$$\sin^2(\sin x)\geq\frac{\sin^2 x}{1+\sin^2 x}\Rightarrow\frac{\sin^2(\sin x)}{\sin^2 x}\geq\frac{1}{1+\sin^2 x}\Rightarrow$$

$$\frac{\cos x\sin^2(\sin x)}{\sin^2 x}\geq\frac{\cos x}{1+\sin^2 x};(3)$$

From (1),(2) and (3) it follows that:

$$\int_a^b\frac{\tan(\sin x)\tan(\cos x)}{\tan x\tan(\cos x)+\tan(\sin x)}\cdot\frac{\sin^2(\sin x)}{\sin^2 x}dx\geq\frac{5}{3}\int_a^b\frac{\cos x}{1+\sin^2 x}dx=$$

$$=\frac{5}{3}\tan^{-1}(\sin x)\Big|_a^b=\frac{5}{3}(\tan^{-1}(\sin b)-\tan^{-1}(\sin a))=\frac{5}{3}\tan^{-1}\left(\frac{\sin b-\sin a}{1+\sin a\sin b}\right)$$

Application 12. If $0<a\leq b<\frac{\pi}{2}$ then:

$$\int_a^b\left(\frac{(2+\cos^2 x)^2}{\cos^2(\sin x)}+\frac{(2+\sin^2 x)^2}{\cos^2(\cos x)}\right)\frac{\cos x\sin^2(\sin x)}{\sin^2 x}dx\geq 21\tan^{-1}\left(\frac{\sin b-\sin a}{1+\sin a\sin b}\right)$$

Solution.

For $x\in\left(0,\frac{\pi}{2}\right)\Rightarrow\tan x\geq x\Rightarrow\sin x\geq x\cos x$

$$\sin^2 x\geq x^2\cos^2 x\Rightarrow\sin^2 x\geq x^2(1-\sin^2 x)\Rightarrow\sin^2 x\,(1+x^2)\geq x^2$$

$$\sin^2 x\geq\frac{x^2}{1+x^2}$$

Putting $x=\sin x$ it follows that:

$$\sin^2(\sin x)\geq\frac{\sin^2 x}{1+\sin^2 x}\Rightarrow\frac{\sin^2(\sin x)}{\sin^2 x}\geq\frac{1}{1+\sin^2 x}\Rightarrow$$

$$\frac{\cos x\sin^2(\sin x)}{\sin^2 x}\geq\frac{\cos x}{1+\sin^2 x};(1)$$

Now, from Maclaurin series expansion for $f(x)=\tan x$, we have that:

$$\tan x\geq x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}$$

Hence,

$$(3-x^2)\tan x-3x\geq(3-x^2)\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}\right)-3x=$$

$$=\left(3x+x^3+\frac{6x^5}{15}+\frac{51x^7}{315}\right)-\left(x^3+\frac{x^5}{3}+\frac{2x^7}{15}+\frac{17x^9}{315}\right)-3x=$$

$$=\frac{x^5(21+9x^2-17x^4)}{315}\geq 0$$

Hence,

$$(3-x^2)\tan x\geq 3x,\forall x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow\tan x\geq\frac{3x}{3-x^2},\forall x\in\left(0,\frac{\pi}{2}\right)$$

Integrating (1) we have:

$$\int_0^x \tan t\,dt \ge \int_0^x \frac{3t}{3-t^2}dt \Rightarrow -\log(\cos x) \ge -\frac{3}{2}(\log(3-x^2)-\log 3)$$

$$\cos^2 x \le \left(\frac{3-x^2}{3}\right)^3, \forall x \in \left(0,\frac{\pi}{2}\right) \Leftrightarrow \frac{(3-x^2)^2}{\cos^2 x} \ge 9+3x^2, \forall x \in \left(0,\frac{\pi}{2}\right); (2)$$

From (2) we get:

$$\frac{(2+\cos^2 x)^2}{\cos^2(\sin x)} + \frac{(2+\sin^2 x)^2}{\cos^2(\cos x)} \ge 9+3\sin^2 x+9+3\cos^2 x = 21; (3)$$

From (1), (2) and (3) it follows that:

$$\left(\frac{(2+\cos^2 x)^2}{\cos^2(\sin x)} + \frac{(2+\sin^2 x)^2}{\cos^2(\cos x)}\right)\frac{\cos x\sin^2(\sin x)}{\sin^2 x} \ge 21\cdot\frac{\cos x}{1+\sin^2 x}$$

Therefore,

$$\int_a^b \left(\frac{(2+\cos^2 x)^2}{\cos^2(\sin x)} + \frac{(2+\sin^2 x)^2}{\cos^2(\cos x)}\right)\frac{\cos x\sin^2(\sin x)}{\sin^2 x}dx \ge 21\int_a^b \frac{\cos x}{1+\sin^2 x}dx =$$

$$= 21\tan^{-1}(\sin x)|_a^b = 21(\tan^{-1}(\sin b)-\tan^{-1}(\sin a)) = 21\tan^{-1}\left(\frac{\sin b-\sin a}{1+\sin a\sin b}\right)$$

Application 13. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{\cos x\,dx}{(a^2+\sin^2 x)(b^2+\sin^2 x)} \le \frac{1}{2ab(a+b)}\log\left(\frac{b(ab+\sin^2 a)}{a(ab+\sin^2 b)}\right)$$

Solution.

$$(a^2+x^2)(b^2+x^2) = a^2b^2+(a^2+b^2)x+x^4 \overset{AM-GM}{\ge} a^2b^2+2abx^2+x^4 =$$
$$= (ab+x^2)^2; (1)$$

$$(a^2+x^2)(b^2+x^2) = a^2b^2+x^4+(a^2+b^2)x^2 \overset{AM-GM}{\ge} 2abx^2+(a^2+b^2)x^2 =$$
$$= (a+b)^2x^2; (2)$$

Multiplying (1) and (2) we obtain:

$$(a^2+x^2)(b^2+x^2) \ge (ab+x^2)(a+b)x$$

Hence,

$$\frac{1}{(a^2+x^2)(b^2+x^2)} \le \frac{1}{(ab+x^2)(a+b)x} = \frac{1}{ab(a+b)}\left(\frac{1}{x}-\frac{x}{ab+x^2}\right)$$

Putting $x = \sin x$ it follows that:

$$\frac{1}{(a^2+\sin^2 x)(b^2+\sin^2 x)} \le \frac{1}{ab(a+b)}\left(\frac{1}{\sin x}-\frac{\sin x}{ab+\sin^2 x}\right); \forall x \in \left(0,\frac{\pi}{2}\right); (1)$$

$$\frac{\cos x}{(a^2+\sin^2 x)(b^2+\sin^2 x)} \le \frac{1}{ab(a+b)}\left(\frac{\cos x}{\sin x}-\frac{\sin x\cos x}{ab+\sin^2 x}\right)$$

Hence,

$$\int_a^b \frac{\cos x\,dx}{(a^2+\sin^2 x)(b^2+\sin^2 x)} \le \frac{1}{ab(a+b)}\int_a^b \frac{\cos x}{\sin x}dx - \frac{1}{ab(a+b)}\int_a^b \frac{\sin x\cos x}{ab+\sin^2 x}dx$$

$$= \frac{1}{ab(a+b)}\log(\sin x)|_a^b - \frac{1}{2ab(a+b)}\log(ab+\sin^2 x)|_a^b =$$

$$= \frac{1}{2ab(a+b)}(\log(\sin b)-\log(\sin a)-\log(ab+\sin^2 b)+\log(ab+\sin^2 a)) =$$

$$= \frac{1}{2ab(a+b)}\log\left(\frac{b(ab+\sin^2 a)}{a(ab+\sin^2 b)}\right)$$

Application 14.If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \left(\frac{\cot^2 x \sin^4(\sin x) + \sin^2 x}{\cos x \sin^2(\sin x) + \sin^2 x}\right)^2 dx \ge \tan^{-1}\left(\frac{\sin b - \sin a}{1 + \sin a \sin b}\right)$$

Solution.

$$\text{For } x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \tan x \ge x \Rightarrow \sin x \ge x \cos x$$

$$\sin^2 x \ge x^2 \cos^2 x \Rightarrow \sin^2 x \ge x^2(1 - \sin^2 x) \Rightarrow \sin^2 x\,(1 + x^2) \ge x^2$$

$$\sin^2 x \ge \frac{x^2}{1 + x^2}$$

Putting $x = \sin x$ it follows that:

$$\sin^2(\sin x) \ge \frac{\sin^2 x}{1 + \sin^2 x} \Rightarrow \frac{\sin^2(\sin x)}{\sin^2 x} \ge \frac{1}{1 + \sin^2 x} \Rightarrow \frac{\cos x \sin^2(\sin x)}{\sin^2 x} \ge \frac{\cos x}{1 + \sin^2 x}; (1)$$

If $u, v > 0$ then $\left(\frac{u^2+v^2}{u+v}\right)^2 \ge uv$

Putting $u = \frac{\cos x \sin^2(\sin x)}{\sin^2 x}$ and $v = 1$, we get:

$$\left(\frac{\frac{\cos^2 x \sin^4(\sin x)}{\sin^4 x} + 1}{\frac{\cos x \sin^2(\sin x)}{\sin^2 x} + 1}\right)^2 \ge \frac{\cos x \sin^2(\sin x)}{\sin^2 x} \Leftrightarrow$$

$$\left(\frac{\cos^2 x \sin^4(\sin x) + \sin^4 x}{\cos x \sin^2 x \sin^2(\sin x) + \sin^4 x}\right)^2 \ge \frac{\cos x \sin^2(\sin x)}{\sin^2 x} \Leftrightarrow$$

$$\left(\frac{\cot^2 x \sin^4(\sin x) + \sin^2 x}{\cos x \sin^2(\sin x) + \sin^2 x}\right)^2 \ge \frac{\cos x \sin^2(\sin x)}{\sin^2 x}$$

Therefore,

$$\int_a^b \left(\frac{\cot^2 x \sin^4(\sin x) + \sin^2 x}{\cos x \sin^2(\sin x) + \sin^2 x}\right)^2 dx \ge \int_a^b \frac{\cos x \sin^2(\sin x)}{\sin^2 x} dx \overset{(1)}{\ge}$$

$$\ge \int_a^b \frac{\cos x}{1 + \sin^2 x} dx = \tan^{-1}\left(\frac{\sin b - \sin a}{1 + \sin a \sin b}\right)$$

Application 15. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \left(\frac{\cos^2 x + \sin^4(\sin x)}{\sin x\,(\cos x + \sin^2(\sin x))}\right)^2 dx \ge \tan^{-1}\left(\frac{\sin b - \sin a}{1 + \sin a \sin b}\right)$$

Solution.

$$\text{For } x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \tan x \ge x \Rightarrow \sin x \ge x \cos x$$

$$\sin^2 x \ge x^2 \cos^2 x \Rightarrow \sin^2 x \ge x^2(1 - \sin^2 x) \Rightarrow \sin^2 x\,(1 + x^2) \ge x^2$$

$$\sin^2 x \ge \frac{x^2}{1 + x^2}$$

Putting $x = \sin x$ it follows that:

$$\sin^2(\sin x) \ge \frac{\sin^2 x}{1 + \sin^2 x} \Rightarrow \frac{\sin^2(\sin x)}{\sin^2 x} \ge \frac{1}{1 + \sin^2 x} \Rightarrow$$

$$\frac{\cos x \sin^2(\sin x)}{\sin^2 x} \ge \frac{\cos x}{1 + \sin^2 x}; (1)$$

If $u, v > 0$ then $\left(\frac{u^2+v^2}{u+v}\right)^2 \ge uv$ and for $u = \cot x$, $v = \frac{\sin^2(\sin x)}{\sin x}$, we get:

$$\left(\frac{\cos^2 x+\sin^4(\sin x)}{\sin x\,(\cos x+\sin^2(\sin x))}\right)^2=\left(\frac{\frac{\cos^2 x}{\sin^2 x}+\frac{\sin^4(\sin x)}{\sin^2 x}}{\frac{\cos x}{\sin x}+\frac{\sin^2(\sin x)}{\sin x}}\right)^2=$$

$$=\left(\frac{\cot^2 x+\frac{\sin^4(\sin x)}{\sin^2 x}}{\cot x+\frac{\sin^2(\sin x)}{\sin x}}\right)^2\geq \cot x\cdot\frac{\sin^2(\sin x)}{\sin x}=\frac{\cos x\sin^2(\sin x)}{\sin^2 x}\overset{(1)}{\geq}\frac{\cos x}{1+\sin^2 x}$$

Therefore,

$$\int_a^b\left(\frac{\cos^2 x+\sin^4(\sin x)}{\sin x\,(\cos x+\sin^2(\sin x))}\right)^2 dx\geq\int_a^b\frac{\cos x}{1+\sin^2 x}dx=\tan^{-1}\left(\frac{\sin b-\sin a}{1+\sin a\sin b}\right)$$

Application 16. If $0<a\leq b<\frac{\pi}{2}$ then:

$$\int_a^b\frac{(1+x\sin^2 x)(1+x^2\sin x)}{(1+\sin x)(1+x)x^2\cos^2 x}dx\geq\log\left(\frac{1+\tan b}{1+\tan a}\right)$$

Solution. Let $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\sin x+\sin x\tan x-x$ then,

$$f'(x)=\cos x+\sin x+\frac{\sin x}{\cos^2 x}-1=\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)+\frac{\sin x}{\cos^2 x}-1>0;\forall x\in\left(0,\frac{\pi}{2}\right)$$

Hence,

$$\frac{\sin x}{x}\geq\frac{1}{1+\tan x};\forall x\in\left(0,\frac{\pi}{2}\right)$$

Lemma. For all $u,v>0$, holds:

$$\frac{(u^2+v)(v^2+u)}{(1+u)(1+v)}\geq uv$$

Proof.

$$\frac{(u^2+v)(v^2+u)}{(1+u)(1+v)}\geq uv\Leftrightarrow(u^2+v)(v^2+u)\geq uv(1+u)(1+v)\Leftrightarrow$$
$$u^2v^2+u^3+v^3+uv\geq uv(1+u+v+uv)\Leftrightarrow u^3+v^3\geq u^2v+uv^2$$
$$(u+v)(u-v)^2\geq 0;\forall u,v>0$$

Now, let $u=\sin x, v=\frac{1}{x}$ then,

$$\frac{(1+x\sin^2 x)(1+x^2\sin x)}{(1+\sin x)(1+x)x^2\cos^2 x}\geq\frac{\sin x}{x}\geq\frac{1}{1+\tan x};\forall x\in\left(0,\frac{\pi}{2}\right)$$

Therefore,

$$\int_a^b\frac{(1+x\sin^2 x)(1+x^2\sin x)}{(1+\sin x)(1+x)x^2\cos^2 x}dx\geq\log\left(\frac{1+\tan b}{1+\tan a}\right)$$

Application 17. If $0<a\leq b<\frac{\pi}{2}$ then:

$$\int_a^b\frac{(\cot^2 x+\sin^2(\sin x))(\cot x+\sin^4(\sin x))}{(1+\cot x)(1+\sin^2(\sin x))}dx\geq\frac{1}{2}\log\left(\frac{1+\sin^2 b}{1+\sin^2 a}\right)$$

Solution. We have: $\tan x\geq x\geq\sin x;\forall x>0\Rightarrow\sin x\geq x\cdot\cos x,\forall x>0\Rightarrow$

$$\sin^2 x\geq x^2\cdot\cos^2 x,\forall x>0\Rightarrow\sin^2 x\geq\frac{x^2}{1+x^2},\forall x>0\Rightarrow\frac{\sin^2 x}{x}\geq\frac{x}{(1+x^2)},\forall x>0$$

$$\frac{\sin^2(\sin x)}{\sin x}\geq\frac{\sin x}{1+\sin^2 x};\forall x>0$$

Let $u=\cot x, v=\sin^2(\sin x)$ then,

$$\frac{(\cot^2 x + \sin^2(\sin x))(\cot x + \sin^4(\sin x))}{(1+\cot x)(1+\sin^2(\sin x))} \geq \cot x \cdot \sin^2(\sin x), \forall x > 0$$

Therefore,

$$\int_a^b \frac{(\cot^2 x + \sin^2(\sin x))(\cot x + \sin^4(\sin x))}{(1+\cot x)(1+\sin^2(\sin x))} dx \geq \int_a^b \cot x \cdot \sin^2(\sin x)\, dx \geq$$

$$\geq \int_a^b \frac{\sin x \cdot \cos x}{1+\sin^2 x} dx = \frac{1}{2}\log\left(\frac{1+\sin^2 b}{1+\sin^2 a}\right)$$

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ROUTH'S THEOREMS REVISITED

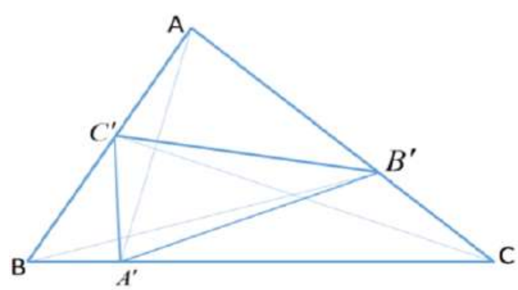
By Neculai Stanciu-Romania

ROUTH'S THEOREM (I)

Let $\Delta ABC, A' \in (BC), B' \in (CA), C' \in (AB), x = \frac{BA'}{A'C}, y = \frac{CB'}{B'A}, z = \frac{AC'}{C'B}$.

If we denote with $[XYZ]$ the area of triangle XYZ, then:

$$[A'B'C'] = \frac{xyz+1}{(x+1)(y+1)(z+1)} \cdot [ABC]$$



Proof. $BA' = \frac{ax}{x+1}, CB' = \frac{by}{y+1}, AC' = \frac{cz}{z+1}, A'C = \frac{a}{x+1}, B'A = \frac{b}{y+1}, C'B = \frac{c}{z+1}$

$$[A'B'C'] = [ABC] - [A'BC'] - [A'CB'] - [B'AC'] =$$

$$= [ABC] - \frac{A'B \cdot C'B \cdot \sin B}{2} - \frac{A'C \cdot B'C \cdot \sin c}{2} - \frac{B'A \cdot C'A \cdot \sin A}{2} =$$

$$= [ABC] - \frac{acx \cdot \sin B}{2(x+1)(z+1)} - \frac{aby \cdot \sin C}{2(x+1)(y+1)} - \frac{bcz \cdot \sin A}{2(y+1)(z+1)} =$$

$$= [ABC]\left(1 - \frac{x}{(x+1)(z+1)} - \frac{y}{(x+1)(y+1)} - \frac{z}{(y+1)(z+1)}\right) =$$

$$= \frac{xyz+1}{(x+1)(y+1)(z+1)} \cdot [ABC]$$

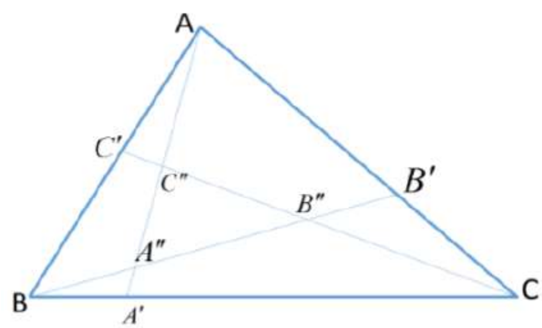
ROUTH'S THEOREM (II)

Let $\Delta ABC, A' \in (BC), B' \in (CA), C' \in (AB), x = \frac{BA'}{A'C}, y = \frac{CB'}{B'A}, z = \frac{AC'}{C'B}$.

$$AA' \cap BB' = \{A''\}, BB' \cap CC' = \{B''\}, CC' \cap AA' = \{C''\}$$

If we denote with $[XYZ]$ the area of triangle XYZ, then

$$[A'B'C'] = \frac{(xyz-1)^2}{(xy+x+1)(z+y+1)(zx+z+1)} \cdot [ABC]$$



Proof. $BA' = \frac{ax}{x+1}, CB' = \frac{by}{y+1}, AC' = \frac{cz}{z+1}, A'C = \frac{a}{x+1}, B'A = \frac{b}{y+1}, C'B = \frac{c}{z+1}$

$$[ABA''] = \frac{AA''}{AA'} \cdot [ABA'] = \frac{AA''}{AA'} \cdot \frac{BA'}{BC} \cdot [ABC] = \frac{AA''}{AA'} \cdot \frac{ax}{(x+1)a} \cdot [ABC]$$

By Menelaus Theorem for $\Delta AA'C$ with transversal $B' - A'' - B'$ we deduce

$$\frac{AB'}{B'C} \cdot \frac{BC}{BA'} \cdot \frac{A'A''}{A''A} = 1 \Leftrightarrow \frac{1}{y} \cdot \frac{x+1}{x} \cdot \frac{A'A''}{A''A} = 1 \Leftrightarrow \frac{A'A''}{A''A} = \frac{xy}{x+1} \Rightarrow \frac{AA''}{AA'} = \frac{x+1}{xy+x+1}$$

$$[ABA''] = \frac{x+1}{xy+x+1} \cdot \frac{ax}{(x+1)a} \cdot [ABC] = \frac{x}{xy+x+1} \cdot [ABC]$$

Analogously, we obtain $[BCB''] = \frac{y}{yz+y+1} \cdot [ABC]; [CAC''] = \frac{z}{zx+z+1} \cdot [ABC]$

$$[A''B''C''] = [ABC] - [ABA''] - [BCB''] - [CAC''] =$$

$$= \frac{(xyz-1)^2}{(xy+x+1)(z+y+1)(zx+z+1)} \cdot [ABC]$$

Application 1. Let $\Delta ABC, A', A'' \in (BC), B', B'' \in (CA), C', C'' \in (AB)$

$$AA' \cap BB' \cap CC' = \{P\}, AA'' \cap BB'' \cap CC'' = \{Q\}, x' = \frac{BA'}{A'C}, y' = \frac{CB'}{B'A},$$

$z' = \frac{AC'}{C'B}, x'' = \frac{BA''}{A''C}, y'' = \frac{CB''}{B''A}, z'' = \frac{AC''}{C''B}$**. Prove that:**

$$\frac{[A'B'C']}{[A''B''C'']} = \frac{(x''+1)(y''+1)(z''+1)}{(x'+1)(y'+1)(z'+1)}$$

Solution. Let $\Delta ABC, A', A'' \in (BC), B', B'' \in (CA), C', C'' \in (AB)$

$$AA' \cap BB' \cap CC' = \{P\}, AA'' \cap BB'' \cap CC'' = \{Q\}, x' = \frac{BA'}{A'C}, y' = \frac{CB'}{B'A}, z' = \frac{AC'}{C'B}$$

$$x'' = \frac{BA''}{A''C}, y'' = \frac{CB''}{B''A}, z'' = \frac{AC''}{C''B}$$

By Routh's theorem we obtain: $[A'B'C'] = \frac{x'y'z'+1}{(x'+1)(y'+1)(z'+1)}[ABC]$,

$$[A''B''C''] = \frac{x''y''z''+1}{(x''+1)(y''+1)(z''+1)}[ABC]$$

From Ceva's theorem we have: $x'y'z' = x''y''z'' = 1$.Hence:

$$\frac{[A'B'C']}{[A''B''C'']} = \frac{(x''+1)(y''+1)(z''+1)}{(x'+1)(y'+1)(z'+1)}$$

Application 2. Let $\Delta ABC, A', A'' \in (BC), B', B'' \in (CA), C', C'' \in (AB)$

$$AA' \cap BB' \cap CC' = \{P\}, AA'' \cap BB'' \cap CC'' = \{Q\}, x' = \frac{BA'}{A'C}, y' = \frac{CB'}{B'A},$$

$z' = \frac{AC'}{C'B}, x'' = \frac{BA''}{A''C}, y'' = \frac{CB''}{B''A}, z'' = \frac{AC''}{C''B}$**.Prove that:**

$$\frac{[A'B'C']}{[A''B''C'']} = \frac{BA'}{BA''} \cdot \frac{CB'}{CB''} \cdot \frac{AC'}{AC''}$$

Solution. $BA' = \frac{ax'}{x'+1}, CB' = \frac{by'}{y'+1}, AC' = \frac{cz'}{z'+1}, A'C = \frac{a}{x'+1}, B'A = \frac{b}{y'+1}, C'B = \frac{c}{z'+1}$

$$B''A = \frac{b}{y''+1}, C''B = \frac{ax''}{x''+1}, CB'' = \frac{by''}{y''+1}, AC'' = \frac{cz''}{z''+1}, A''C = \frac{a}{x''+1},$$

$$B''A = \frac{b}{y''+1}, C''B = \frac{c}{z''+1}$$

$$\frac{BA'}{BA''} = \frac{x'(x''+1)}{x''(x'+1)}, \frac{CB'}{CB''} = \frac{y'(y''+1)}{y''(y'+1)}, \frac{AC'}{AC''} = \frac{z'(z''+1)}{z''(z'+1)}$$

$$\frac{27(x''+1)(y''+1)(z''+1)}{(x'+1)(y'+1)(z'+1)} \le \left(\frac{x'(x''+1)}{x''(x'+1)} + \frac{y'(y''+1)}{y''(y'+1)} + \frac{z'(z''+1)}{z''(z'+1)}\right)$$

From Routh's Theorem we obtain: $[A'B'C'] = \frac{x'y'z'+1}{(x'+1)(y'+1)(z'+1)} \cdot [ABC]$

$$[A''B''C''] = \frac{x''y''z''+1}{(x''+1)(y''+1)(z''+1)} \cdot [ABC]$$

From Ceva's Theorem we have $x'y'z' = x''y''z'' = 1$. Therefore,

$$\frac{[A'B'C']}{[A''B''C'']} = \frac{(x''+1)(y''+1)(z''+1)}{(x'+1)(y'+1)(z'+1)}$$

$$\frac{BA'}{BA''} \cdot \frac{CB'}{CB''} \cdot \frac{AC'}{AC''} = \frac{x'(x''+1)}{x''(x'+1)} + \frac{y'(y''+1)}{y''(x'+1)} + \frac{z'(z''+1)}{z''(z'+1)}$$

$$\frac{[A'B'C']}{[A''B''C'']} = \frac{BA'}{BA''} \cdot \frac{CB'}{CB''} \cdot \frac{AC'}{AC''}$$

Application 3. Let ABC be a triangle, $A' \in (BC), B' \in (CA), C' \in (AB)$,

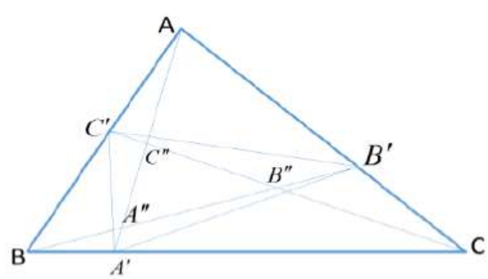
$BA' = A'C, CB' = 2AB', C'A = 3BC'$. If $AA' \cap BB' = \{A''\}, BB' \cap CC' = \{B''\}$,

$CC' \cap AA' = \{C''\}$.Prove that: $\frac{[A'B'C']}{[A''B''C'']} = \frac{147}{50}$

Solution. Denoting $x = \frac{BA'}{A'C} = 1, y = \frac{CB'}{B'A} = 2, z = \frac{AC'}{C'B} = 3$ then by Routh's Theorem

$$[A'B'C'] = \frac{xyz+1}{(x+1)(y+1)(z+1)} \cdot [ABC]$$

$$[A''B''C''] = \frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(zx+z+1)} \cdot [ABC]$$



So, $[A'B'C'] = \frac{7}{24} \cdot [ABC], \{A''B''C''\} = \frac{25}{36 \cdot 7} \cdot [ABC]$. Hence, $\frac{[A'B'C']}{[A''B''C'']} = \frac{147}{50}$.

Application 4. Let ABC be a triangle, $A' \in (BC), B' \in (CA), C' \in (AB)$,

$BA' = A'C, CB' = 2AB', C'A = 3BC'$. If $AA' \cap BB' = \{A''\}, BB' \cap CC' = \{B''\}$,

$CC' \cap AA' = \{C''\}$.Prove that:

$$\frac{[A'B'C']}{[ABC]} = \frac{2}{xy + yz + zx + x + y + z + 2}$$

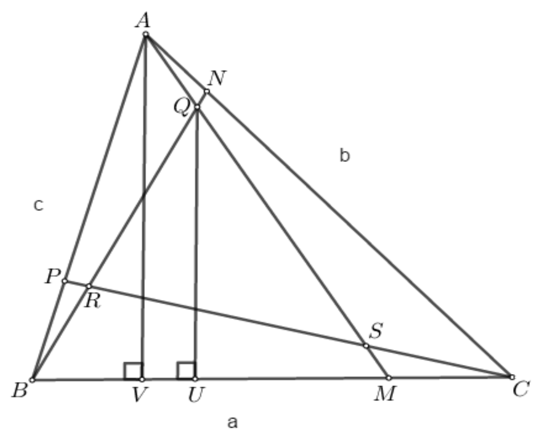
Solution. From the theorem of Routh's we have: $[A'B'C'] = \frac{xyz+1}{(x+1)(y+1)(z+1)} \cdot [ABC]; (1)$

From Ceva's theorem we have $xyz = 1; (2)$. From (1) and (2) it follows that

$$\frac{[A'B'C']}{[ABC]} = \frac{2}{xy + yz + zx + x + y + z + 2}$$

FEW LEMMAS

By Florentin Vișescu-Romania



$$F_{\Delta ABC} = F; F_{\Delta MNP} = F_1; F_{\Delta QRS} = F_2, \frac{AP}{PB} = k; \frac{BM}{MC} = l; \frac{CN}{NA} = m; klm = t$$
$$F_1 = F - (F_{\Delta APM} + F_{\Delta BPM} + F_{\Delta CMN})$$
$$F_{\Delta APN} = \frac{AP \cdot AN \cdot \sin A}{2} = \frac{k}{k+1} \cdot AB \cdot \frac{1}{m+1} \cdot AC \cdot \frac{\sin A}{2} = \frac{k}{k+1} \cdot \frac{1}{m+1} \cdot F$$
$$F_{\Delta BPM} = \frac{l}{l+1} \cdot \frac{1}{k+1} \cdot F; F_{\Delta CMN} = \frac{m}{m+1} \cdot \frac{1}{l+1} \cdot F$$
$$F_1 = F\left(1 - \frac{k}{(k+1)(m+1)} - \frac{l}{(l+1)(k+1)} - \frac{m}{(m+1)(l+1)}\right)$$
$$= \frac{klm+1}{(k+1)(l+1)(m+1)}; (1)$$
$$F = F_{\Delta BNC} + F_{\Delta CPA} + F_{\Delta AMB} - F_{\Delta RNC} - F_{\Delta SAP} - F_{\Delta QBM} + F_2; (2)$$

$$\frac{AN}{NC}\cdot\frac{CB}{BM}\cdot\frac{MQ}{QA}=1\ (Menelaus)\Rightarrow\frac{1}{m}\cdot\frac{l+1}{l}\cdot\frac{MQ}{QA}=1$$

$$\frac{MQ}{QA}=\frac{lm}{l+1};\frac{MQ}{MA}=\frac{lm}{lm+l+1};\frac{MQ}{MA}=\frac{QU}{AV}$$

$$\frac{QU}{AV}=\frac{lm}{lm+l+1};QU=\frac{lm}{lm+l+1}\cdot AV$$

$$F_{\Delta QBM}=\frac{BM\cdot QU}{2}=\frac{lm}{lm+l+1}\cdot\frac{BM\cdot AV}{2};BM=\frac{l}{l+1}\cdot BC$$

$$F_{\Delta QBM}=\frac{l^2m}{(lm+l+1)(l+1)}\cdot\frac{BC\cdot AV}{2}=\frac{l^2m}{(lmn+l+1)(l+1)}$$

$$F_{\Delta ASP}=\frac{k^2l}{(kl+k+1)(k+1)}\cdot F;F_{\Delta PNC}=\frac{m^2k}{(mk+m+1)(m+1)}\cdot F$$

$$F=\frac{m}{m+1}\cdot F+\frac{k}{k+1}\cdot F+\frac{l}{l+1}\cdot F-\frac{l^2m}{(lm+l+1)(l+1)}\cdot F-$$

$$-\frac{k^2k}{(kl+k+1)(k+1)}\cdot F-\frac{m^2k}{(mk+m+1)(m+1)}\cdot F+F_2$$

$$F_2=F\left(1-\frac{m}{m+1}-\frac{k}{k+1}-\frac{l}{l+1}+\frac{l^2m}{(lm+l+1)(l+1)}+\frac{k^2l}{(kl+k+1)(k+1)}+\right.$$
$$\left.+\frac{m^2k}{(km+m+1)(m+1)}\right)$$

$$F_2=F\left(1-\frac{m}{mk+m+1}-\frac{l}{lm+l+1}-\frac{k}{kl+k+1}\right)$$

$$F_2=F\left\{1-\left(\frac{m}{mk+m+1}+\frac{l}{lm+l+1}+\frac{k}{kl+k+1}\right)\right\}$$

Lemma 1. Let $k,l,m\in(0,\infty)$, then:

$$\frac{\boldsymbol{klm+1}}{\boldsymbol{(m+1)(k+1)(l+1)}}\le\frac{\sqrt[3]{\boldsymbol{(klm)^2}}-\sqrt[3]{\boldsymbol{klm}}+\boldsymbol{1}}{\left(\sqrt[3]{\boldsymbol{klm}}+\boldsymbol{1}\right)^2}$$

Equality holds for $k=l=m$.

Proof. Let $f:\mathbb{R}\to\mathbb{R}, f(x)=\log(e^x+1)$, then $f'(x)=\frac{e^x}{e^x+1}=1-\frac{1}{e^x+1}$, $f''(x)=\frac{e^x}{(e^x+1)^2}>0,\forall x\in\mathbb{R}\Rightarrow f$ –convex function, then from Jensen's inequality:

$$f(x)+f(y)+f(z)\ge 3f\left(\frac{x+y+z}{3}\right);\forall x,y,z\in\mathbb{R}$$

$$\log(e^x+1)(e^y+1)(e^z+1)\ge\log\left(e^{\frac{x+y+z}{3}}+1\right)^3;\forall x,y,z\in\mathbb{R}$$

$$(e^x+1)(e^y+1)(e^z+1)\ge\left(e^{\frac{x+y+z}{3}}+1\right)^3;\forall x,y,z\in\mathbb{R}$$

Let $x=\log m$; $y=\log k$; $z=\log l$, then

$$(m+1)(k+1)(l+1)\ge\left(\sqrt[3]{klm}+1\right)^3$$

$$\frac{1}{(m+1)(k+1)(l+1)}\le\frac{1}{\left(\sqrt[3]{klm}+1\right)^3}$$

$$\frac{klm+1}{(m+1)(k+1)(l+1)} \le \frac{klm+1}{\left(\sqrt[3]{klm}+1\right)^3}$$

$$\frac{klm+1}{(m+1)(k+1)(l+1)} \le \frac{\sqrt[3]{(klm)^2}-\sqrt[3]{klm}+1}{\left(\sqrt[3]{klm}+1\right)^2}; (t = klm)$$

$$\frac{klm+1}{(m+1)(k+1)(l+1)} \le \frac{\sqrt[3]{t^2}-\sqrt[3]{t}+1}{\left(\sqrt[3]{t}+1\right)^2}$$

$$F_1 \le F \cdot \frac{\sqrt[3]{t^2}-\sqrt[3]{t}+1}{\left(\sqrt[3]{t}+1\right)^2} = F \cdot \frac{\sqrt[3]{(klm)^2}-\sqrt[3]{klm}+1}{\left(\sqrt[3]{klm}+1\right)^2}$$

Let: $g: (0,\infty) \to \mathbb{R}, g(t) = \frac{t^2-t+1}{(t+1)^2}$, then $g'(t) = \frac{3(t-1)}{(t+1)^3}$

t	0		1		∞
$g'(t)$		$-----$	0	$++++$	
$g(t)$	1	$\searrow$	$\frac{1}{4}$	$\nearrow$	1

$\lim_{t\to 0_+} g(t) = 1; \lim_{t\to\infty} g(t) = 1; g(1) = \frac{1}{4}$. So, $min\, F_1 = \frac{F}{4}$

Equality holds for $k = l = m = 1$.

Lemma 2. Let $k, l, m \in (0,\infty)$, then:

$$\frac{m}{mk+m+1} + \frac{l}{lm+l+1} + \frac{k}{kl+k+1} \ge \frac{3\cdot\sqrt[3]{klm}}{\sqrt[3]{(klm)^2}+\sqrt[3]{klm}+1}$$

Proof. Let be $klm = t$ and let $x, y, z \in (0,\infty)$ such that $m = \frac{x}{y}\cdot\sqrt[3]{t}; k = \frac{z}{x}\cdot\sqrt[3]{t}; l = \frac{y}{z}\cdot\sqrt[3]{t}$.

Inequality can be written as:

$$\frac{\frac{x}{y}\cdot\sqrt[3]{t}}{\frac{x}{y}\cdot\sqrt[3]{t}\cdot\frac{z}{x}\cdot\sqrt[3]{t}+\frac{x}{y}\cdot\sqrt[3]{t}+1} + \frac{\frac{y}{z}\cdot\sqrt[3]{t}}{\frac{y}{z}\cdot\sqrt[3]{t}\cdot\frac{x}{y}\cdot\sqrt[3]{t}+\frac{y}{z}\cdot\sqrt[3]{t}+1} + \frac{\frac{z}{x}\cdot\sqrt[3]{t}}{\frac{z}{x}\cdot\sqrt[3]{t}\cdot\frac{y}{z}\cdot\sqrt[3]{t}+\frac{z}{x}\cdot\sqrt[3]{t}+1} \ge$$

$$\ge \frac{3\cdot\sqrt[3]{t}}{\sqrt[3]{t^2}+\sqrt[3]{t}+1}$$

Hence,

$$\frac{x}{z\sqrt[3]{t^2}+x\sqrt[3]{t}+y} + \frac{y}{x\sqrt[3]{t^2}+y\sqrt[3]{t}+z} + \frac{z}{y\sqrt[3]{t^2}+z\sqrt[3]{t}+x} \ge \frac{3}{\sqrt[3]{t^2}+\sqrt[3]{t}+1}$$

Let $x + y + z = 1$, then:

$$\frac{x}{z\sqrt[3]{t^2}+x\sqrt[3]{t}+y} + \frac{y}{x\sqrt[3]{t^2}+y\sqrt[3]{t}+z} + \frac{z}{y\sqrt[3]{t^2}+z\sqrt[3]{t}+x} =$$

$$\frac{x^2}{xz\sqrt[3]{t^2}+x^2\sqrt[3]{t}+xy} + \frac{y^2}{xy\sqrt[3]{t^2}+y^2\sqrt[3]{t}+yz} + \frac{z^2}{yz\sqrt[3]{t^2}+z^2\sqrt[3]{t}+xz} \overset{Bergstrom}{\ge}$$

$$\ge \frac{(x+y+z)^2}{(xy+yz+zx)\left(\sqrt[3]{t^2}+1\right)+(x^2+y^2+z^2)\sqrt[3]{t}} =$$

$$= \frac{1}{(xy + yz + zx)\left(\sqrt[3]{t^2} - 2\sqrt[3]{t} + 1\right) + \sqrt[3]{t}}$$

So, from $x, y, z \in (0, \infty), x + y + z = 1$, we get: $xy + yz + zx \leq \frac{1}{3}$. Thus,

$$\frac{x}{z\sqrt[3]{t^2} + x\sqrt[3]{t} + y} + \frac{y}{x\sqrt[3]{t^2} + y\sqrt[3]{t} + z} + \frac{z}{y\sqrt[3]{t^2} + z\sqrt[3]{t} + x} \geq \frac{3}{\sqrt[3]{t^2} + \sqrt[3]{t} + 1}$$

Hence,

$$F_2 \leq F\left(1 - \frac{3 \cdot \sqrt[3]{klm}}{\sqrt[3]{(klm)^2} + \sqrt[3]{klm} + 1}\right) = F \cdot \frac{\sqrt[3]{(klm)^2} - 2\sqrt[3]{klm} + 1}{\sqrt[3]{(klm)^2} + \sqrt[3]{klm} + 1}$$

$$F_2 \leq F \cdot \frac{\sqrt[3]{t^2} - 2\sqrt[3]{t} + 1}{\sqrt[3]{t^2} + \sqrt[3]{t} + 1} = F \cdot \frac{\sqrt[3]{(klm)^2} - 2\sqrt[3]{klm} + 1}{\sqrt[3]{(klm)^2} + \sqrt[3]{klm} + 1}$$

Let $h: (0, \infty) \to \mathbb{R}, h(t) = \frac{(t-1)^2}{t^2 + t + 1}, h'(t) = \frac{3(t-1)(t+1)}{(t^2 + t + 1)}$

$$h'(t) = 0 \Rightarrow t = 1$$

t	0		1		∞
$h'(t)$		$- - - - -$	0	$+ + + + +$	
$h(t)$	$\frac{1}{3}$	↘	0	↗	1

$$\lim_{t \to 0_+} h(t) = \frac{1}{3}; \lim_{t \to \infty} h(t) = 1, h(1) = 0$$

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SEQUENCES OF SOLUTIONS FOR GIVED PARAMETRIZED EQUATIONS

By Marian Ursărescu, Florică Anastase-Romania

Abstract: *In this paper are presented a way to find the limit of a sequence defined as solution for a parametrized equation.*

Application 1. For $n \in \mathbb{N}^*, n \geq 3$ let us denote $x(n)$ solution of the equation

$$\boldsymbol{n(n-1)\sin^{n-2} x - n^2 \sin^n x = 0}.$$

Prove that $\sin x(n) = \sqrt{\frac{n-1}{n}}, \forall n \geq 3$ and find $\lim_{n \to \infty} \sin^n x(n)$.

Solution. Let be the function $f_n: \mathbb{R} \to \mathbb{R}, f_n(x) = \sin^n x$, then $f_n'(x) = n \sin^{n-1} x \cdot \cos x$

$$f_n''(x) = n(n-1)\sin^{n-2} x \cdot \cos^2 x + n \sin^{n-1} x \cdot (-\sin x) =$$

$$= n(n-1)\sin^{n-2} x - n(n-1)\sin^n x - n\sin^n x = n(n-1)\sin^{n-2} x - n^2 \sin^n x$$

$x_n = x(n)$ –solution of the equation $n(n-1)\sin^{n-2}x - n^2\sin^n x = 0$

$$\Rightarrow \sin^2 x_n = \frac{n-1}{n} \Rightarrow \sin x_n = \sqrt{\frac{n-1}{n}}$$

$$\lim_{n\to\infty}\sin^n x(n) = \lim_{n\to\infty}(\sin x_n)^n = \lim_{n\to\infty}(1+\sin x_n - 1)^{\frac{1}{\sin x_n - 1}\cdot n(\sin x_n - 1)} =$$

$$= \lim_{n\to\infty} e^{n\left(\sqrt{\frac{n-1}{n}}-1\right)} = \lim_{n\to\infty} e^{\frac{-1}{1+\sqrt{\frac{n-1}{n}}}} = \frac{1}{\sqrt{e}}.$$

Application 2. For $n \in \mathbb{N}^*, n \geq 3$ let us denote $x(n)$ solution of the equation

$x^n - nx + 1 = 0$. Prove that the equation have just two solutions $a_n \in (0,1)$,

$b_n \in (1,\infty)$ and find $\Omega = \lim_{n\to\infty} a_n$.

Solution. Let be the function $f_n: \mathbb{R}_+ \to \mathbb{R}, f_n(x) = x^n - nx + 1$, then

$$f_n'(x) = n(x^{n-1} - 1), \forall x \geq 0$$

$$f_n'(1) = 0 \text{ and } f_n'(x) < 0, \forall x \in [0,1) \text{ and } f_n'(x) > 0, \forall x \in [1,\infty).$$

So, f_n –is decreasing on [0,1] and increasing on $[1,\infty)$. Because f_n –is continuous, decreasing on [0,1] and $f_n(0)\cdot f_n(1) < 0$ hence, f_n –has only a root on the interval (0,1). Now, f_n –continuous, increasing, $f_n(1) < 0$ and $\lim_{n\to\infty} f_n(x) = +\infty$ hence

f_n –has only root on the interval $[1,\infty)$. Now, $a_n \in (0,1)$ and from $f_n(0) > 0$,

$$f_n\left(\frac{2}{n}\right) < 0, \forall n \geq 3 \text{ we get: } a_n \in \left(0, \frac{2}{n}\right) \Rightarrow \lim_{n\to\infty} a_n = 0.$$

Application 3. For $n \in \mathbb{N}$ let us denote $x(n)$ solution of the equation

$x^3 + x - 2 - \frac{1}{n+1} = 0$. Find: $\Omega = \lim_{n\to\infty} n(x(n) - 1)$.

Solution. Let be the function $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 + x + 1$.

$$\lim_{n\to-\infty} f(x) = -\infty; \lim_{n\to\infty} f(x) = +\infty \Rightarrow f - \text{continuous and surjective.}$$

$$f'(x) = 3x^2 + 1 > 0, \forall x \in \mathbb{R} \Rightarrow f - \text{increasing.}$$

So, $f(x) = 3 + \frac{1}{n+1}$ has only a solution $x(n) = x_n$ such that

$x_n^3 + x_n + 1 = 3 + \frac{1}{n+1}$ and applying limit when $n \to \infty$, we get

$$x^3 + x = 2 \Leftrightarrow (x-1)(x^2 + x + 2) = 0 \Rightarrow x = 1 \text{ unique solution.}$$

$$x_n - 1 = \frac{1}{(n+1)(x_n^2 + x_n + 2)} \Rightarrow \Omega = \lim_{n\to\infty} n(x(n) - 1) = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{x_n^2 + x_n + 2} = \frac{1}{4}.$$

Application 4. For $n \in \mathbb{N}^*$ let us denote $x(n)$ solution of the equation

$$e^x + x - 1 - \frac{1}{n} = 0. \text{ Find: } \Omega = \lim_{n\to\infty} n \cdot x(n).$$

Solution. Let be the function $f: \mathbb{R} \to \mathbb{R}, f(x) = e^x + x$, then $f'(x) = e^x + 1 > 0, \forall x \in \mathbb{R}$ hence f –increasing, so f –injective. How, $\lim_{n\to\pm\infty} f(x) = \pm\infty \Rightarrow f$ –has Darboux property, hence f surjective. So, f –bijective for all $n \geq 1, \exists x(n) = x_n \in \mathbb{R}$ such that $f(x_n) = \frac{n+1}{n}$ has an unique solution.How f –continuous function and use that f –invertible, we have:

$nx_n = \frac{f^{-1}\left(\frac{n+1}{n}\right) - f^{-1}(1)}{\frac{n+1}{n}}$. Using theorem of differentiable invertible function, we get:

$$\lim_{n\to\infty} nx_n = \lim_{n\to\infty} \frac{f^{-1}\left(\frac{n+1}{n}\right) - f^{-1}(1)}{\frac{n+1}{n}} = (f^{-1})'_{(1)} = \frac{1}{f'\left(f^{-1}(1)\right)} = \frac{1}{f'(0)} = \frac{1}{2}.$$

Application 5. For $n \in \mathbb{N}$ let us denote $x(n)$ solution of the equation

$$x + \sin x - \frac{1}{n} = 0. \text{ Find: } \Omega = \lim_{n\to\infty} n \cdot x(n).$$

Solution. Let be the function $f: \mathbb{R} \to \mathbb{R}, f(x) = x + \sin x$ then $x - 1 \leq f(x) \leq x + 1$, $\forall x \in \mathbb{R}$. $\lim_{n\to\pm\infty} f(x) = \pm\infty \Rightarrow f$ –has Darboux property, hence f surjective.

$$f'(x) = 1 + \cos x \geq 0, \forall x \in \mathbb{R} \Rightarrow f \text{ –increasing.}$$

So, f –bijective and for all $n \geq 1, \exists x(n) = x_n \in \mathbb{R}$ such that $f(x_n) = \frac{1}{n}$ has an unique solution and f invertible, we have: $x_n = f^{-1}\left(\frac{1}{n}\right) \to f^{-1}(0)$.

Using theorem of differentiable invertible function, we get:

$$\lim_{n\to\infty} nx_n = \lim_{n\to\infty} \frac{f^{-1}\left(\frac{1}{n}\right) - f^{-1}(0)}{\frac{1}{n} - 0} = (f^{-1})'_{(0)} = \frac{1}{f'\left(f^{-1}(0)\right)} = \frac{1}{f'(0)} = \frac{1}{2}.$$

Application 6. Let $\alpha > 1$ fixed. For $\forall n \in \mathbb{N}^*$ denote $k(n) = \min\left\{k \in \mathbb{N} \middle| (n+1)^k \geq \alpha \cdot n^k\right\}$

and $(x_n)_{n\geq1}, x_{n+1} = x_n + \frac{1}{e^{x_n}}$. Find: $\Omega = \lim_{n\to\infty} \frac{k(n)\cdot\log\sqrt[n]{n}}{x_n}$.

Solution. From $(n+1)^k \geq \alpha \cdot n^k \Rightarrow \left(\frac{n+1}{n}\right)^k \geq \alpha \Rightarrow \left(1 + \frac{1}{n}\right)^k \geq \alpha \Rightarrow \log\left(1 + \frac{1}{n}\right)^\alpha \geq \log\alpha$

$$k \cdot \log\left(1 + \frac{1}{n}\right) \geq \log\alpha \Rightarrow k \geq \frac{\log\alpha}{\log\left(1 + \frac{1}{n}\right)}$$

Because $k(n) = \min\{k \in \mathbb{N} | (n+1)^k \geq \alpha \cdot n^k\} \Rightarrow k(n) = \left[\frac{\log\alpha}{\log\left(1+\frac{1}{n}\right)}\right]$ or

$k(n) = \left[\frac{\log a}{\log\left(1+\frac{1}{n}\right)}\right] + 1$. So, we have: $\frac{\log a}{\log\left(1+\frac{1}{n}\right)} \le k(n) \le \frac{\log a}{\log\left(1+\frac{1}{n}\right)} + 1 \Leftrightarrow$

$$\frac{\log a}{n \cdot \log\left(1+\frac{1}{n}\right)} \le \frac{k(n)}{n} \le \frac{\log a}{n \cdot \log\left(1+\frac{1}{n}\right)} + \frac{1}{n} \Leftrightarrow$$

$$\frac{\log a}{\log\left(1+\frac{1}{n}\right)^n} \le \frac{k(n)}{n} \le \frac{\log a}{\log\left(1+\frac{1}{n}\right)^n} + \frac{1}{n} \Rightarrow \lim_{n\to\infty} \frac{k(n)}{n} = \log a\,; (1)$$

Now, from $x_{n+1} = x_n + \frac{1}{e^{x_n}}, \forall n \in \mathbb{N} \Rightarrow x_{n+1} - x_n = \frac{1}{e^{x_n}} > 0, \forall n \in \mathbb{N} \Rightarrow (x_n)_{n\ge1} \nearrow$.

Suppose that exists $x \in \mathbb{R}$ such that $x = \lim_{n\to\infty} x_n \Rightarrow x - x = \frac{1}{e^x}\,(not\ possible!) \Rightarrow$

$\lim_{n\to\infty} x_n = +\infty; (2)$. From (1),(2) we have:

$$\Omega = \lim_{n\to\infty} \frac{k(n)\cdot \log\sqrt[n]{n}}{x_n} = \lim_{n\to\infty}\left(\frac{k(n)}{n}\cdot\frac{\log n}{x_n}\right) = \log \alpha \cdot \lim_{n\to\infty}\frac{\log n}{x_n} \stackrel{Stolz}{=}$$

$$= \log\alpha \cdot \lim_{n\to\infty} \frac{\log(n+1) - \log n}{x_{n+1} - x_n} = \log\alpha\cdot\lim_{n\to\infty}\frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{e^{x_n}}} =$$

$$= \log\alpha\cdot\lim_{n\to\infty}\frac{e^{x_n}}{n}\cdot\log\left(1+\frac{1}{n}\right)^n \stackrel{Stolz}{=} \log\alpha\cdot\lim_{n\to\infty}\frac{e^{x_{n+1}} - e^{x_n}}{n+1-n} =$$

$$= \log\alpha\cdot\lim_{n\to\infty} e^{x_n}\left(e^{x_{n+1}-x_n} - 1\right) = \log\alpha\cdot\lim_{n\to\infty}\frac{e^{\frac{1}{e^{x_n}}} - 1}{\frac{1}{e^{x_n}}} = \log\alpha.$$

Application 7. For all $n \in (1,\infty)$ denote $x(n)$ real solution of the equation

$$x(1+\log x) = n. \textbf{ Prove that: } \lim_{n\to\infty}\frac{x(n)}{n}\cdot\log n = 1$$

Solution. Let be $f:[1,\infty) \to \mathbb{R}, f(x) = x(1+\log x)$ continuous function and $f(1) = 1$,

$\lim_{x\to\infty} f(x) = \infty$, so $f([1,\infty)) = [1,\infty)$ which means that $f:[1,\infty) \to [1,\infty)$ is surjective.

Let's suppose that f –is not injective, then $\exists x, y \in [1,\infty), x < y$ such that $f(x) \ge f(y)$

$\Leftrightarrow x + x\cdot\log x \ge y + y\cdot\log y \Leftrightarrow x - y \ge y\cdot\log y - x\cdot\log x > x\cdot\log y - x\cdot\log x =$

$= x\cdot\log\left(\frac{y}{x}\right) > 0$, which proves that $x \ge y$ contradiction with $x < y$.

Because $f:[1,\infty) \to [1,\infty)$ is bijective, then $\forall n \in (1,\infty), \exists!\, x = x(n) \in [1,\infty)$ such that $f(x) = n \Leftrightarrow x(1+\log x) = n$. In conclusion, for all $n > 1$ equation $x(1+\log x) = n$ have only solution $x = x(n)$. In this conditions, we have:

$$\frac{x(n)\log n}{n}=\frac{x(n)}{x(n)\cdot(1+\log x(n))}\cdot\log n=\frac{1}{\frac{1}{\log n}+\frac{\log x(n)}{\log n}};(1)$$

Because $\lim_{n\to\infty}\frac{1}{\log n}=0$ to prove that $\lim_{n\to\infty}\frac{x(n)\cdot\log n}{n}=1$ it is suffices to prove that

$$\lim_{n\to\infty}\frac{\log x(n)}{\log n}=1$$

From $x(n)\cdot(1+\log x(n))=n$ we have $\log x(n)+\log(1+\log x(n))=\log n$.

$$\frac{\log x(n)}{\log n}=\frac{\log n-\log(1+\log x(n))}{\log n}=1-\frac{\log(1+\log x(n))}{\log n};(2)$$

Because $x(n)<n,\forall n\geq 1$ then $\log x(n)\leq\log n$ and hence,

$$0\leq\frac{\log(1+\log x(n))}{\log n}<\frac{\log(1+\log n)}{\log n},\forall n\geq 1$$

$$0\leq\lim_{n\to\infty}\frac{\log(1+\log x(n))}{\log n}\leq\lim_{n\to\infty}\frac{\log(1+\log n)}{\log n}=0;(3)$$

From (1),(2),(3) it follows that: $\lim_{n\to\infty}\frac{x(n)}{n}\cdot\log n=1$.

Application 8. For all $n\in(1,\infty)$ denote $x(n)$ solution of the equation

$x^k(1+\log x)=n,k\geq 1,k-fixed$. Prove that:

$$\lim_{n\to\infty}\frac{x^k(n)}{n}\cdot\log n=k.$$

Solution. For $n>1$, we have: $x^k(n)\cdot(1+\log x(n))=n\Rightarrow\frac{x^k(n)}{n}=\frac{1}{1+\log x(n)}$

$$\log n=k\cdot\log x(n)+\log(1+x(n))$$

$$\frac{x^k(n)}{n}\cdot\log n=\frac{k}{\frac{1}{\log x(n)}+1}+\frac{\log(1+\log x(n))}{1+\log x(n)};(1)$$

Now, using $\log(1+t)\leq t,\forall t\geq -1\Rightarrow 1+\log u\leq u,\forall u\geq 0$, we get:

$$x(n)\geq 1+\log x(n)\Rightarrow n=x^k(n)(1+\log x(n))\leq x^{k+1}(n)$$

$$x(n)\geq\sqrt[k+1]{n}\Rightarrow\lim_{n\to\infty}x(n)\geq\lim_{n\to\infty}\sqrt[k+1]{n}=+\infty\Rightarrow$$

$\lim_{n\to\infty}\log x(n)=+\infty$ and using $\lim_{n\to\infty}\frac{\log n}{n}=0$, we get:

$\lim_{n\to\infty}\frac{\log(1+x(n))}{1+\log x(n)}=1$ and from (1) we get: $\lim_{n\to\infty}\frac{x^k(n)}{n}\cdot\log n=k$.

Application 9. For all $n \in \mathbb{N}^*$ denote $x(n)$ solution of the equation

$$x^{n+2} - (n+2)x - (n+1) = 0. \text{ Find: } \lim_{n\to\infty} x(n)$$

Solution. Let be $f_n: [0,\infty) \to \mathbb{R}, f_n(x) = x^{n+2} - (n+2)x - (n+1)$. We have:

$$f_n(0) = -n - 1 < 0 \text{ and } \lim_{n\to\infty} f_n(x) = +\infty$$

$$f_n': [0,\infty) \to \mathbb{R}, f_n'(x) = (n+2)x^{n+1} - (n+2), f_n'(x) = 0 \Leftrightarrow x = 1$$

x	0		1		∞
$f_n'(x)$	$- - - - - - - - - - -$		0	$+ + + + + + + + +$	
$f_n(x)$	$-(n+1)$	$\searrow$	$-(2n+2)$	$\nearrow$	∞

How $f_n(1) = -(2n+2) < 0$ and $\lim_{n\to\infty} f_n(x) = +\infty$, then exists $x(n) = x_n \in (1,\infty)$ such that $f_n(x_n) = 0$. Because on $(1,\infty)$ function f_n is increasing, then f_n is injective which means that $x_n \in (1,\infty)$ is the unique solution of the equation $f_n(x) = 0$.

Observe that $f_n(2) = 2^{n+2} - 3n - 5 \geq 0$ and then $x_n \in (1,2]$; (1)

Now, from $f(x_n) = 0$ we have $x_n^{n+2} = (n+2)x_n + n + 1$. Thus,

$$x_n = \sqrt[n+2]{(n+2)x_n + n + 1}; (2)$$

From (1),(2) it follows that:

$$\sqrt[n+2]{2n+3} = \sqrt[n+2]{n+2+n+1} < x_n \leq \sqrt[n+2]{2(n+2)+n+1} = \sqrt[n+2]{3n+5}; 3)$$

From Cauchy-d'Alembert criterion, we have: $\lim_{n\to\infty} \sqrt[n+2]{2n+3} = \lim_{n\to\infty} \sqrt[n+2]{3n+5} = 1$.

Therefore, $\lim_{n\to\infty} x_n = 1$.

Application 10. For all $n \in \mathbb{N}^*, n \geq 3$ denote $x(n)$ solution of the equation

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^x = \frac{1}{2}. \text{ Prove that: } \lim_{n\to\infty} n(x(n) - 1) = 2.$$

Solution. $\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^x = \frac{1}{2}$; (1). Let be the function $f: \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^x - \frac{1}{2}$ continuous and decreasing. $f(1) = \frac{n(n+1)}{2n^2} - \frac{1}{2} = \frac{1}{2n} > 0$

$$f(2) = \frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{2} = \frac{-2n^2 + 3n + 2}{6n^2} < 0, \forall n \geq 3$$

So, equation (1) have unique solution $x(n) = x_n \in (1,2)$.

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{x_n} = \frac{1}{2}, \forall n \geq 3 \Leftrightarrow \frac{1}{n}\sum_{k=1}^{n}\left[\left(\frac{k}{n}\right)^{x_n} - \left(\frac{k}{n}\right)\right] = \frac{1}{2} - \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}, \forall n \geq 3$$

$$y_n \overset{not.}{=} \frac{1}{2} - \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}, \forall n \geq 3$$

Let be the function $g: (0, \infty) \to \mathbb{R}, g(x) = \left(\frac{k}{n}\right)^x$, then we have:

$y_n = \frac{1}{n}\sum_{k=1}^{n}[g(x_n) - g(1)]$; (2).From M.V.T. $\exists\, \xi_n \in (1, x_n)$ such that

$$g(x_n) - g(1) = (x_n - 1)g'(\xi_n) = (x_n - 1)\left(\frac{k}{n}\right)^{\xi_n}\log\left(\frac{k}{n}\right); (3)$$

But $\xi_n < x_n < 2$ and $0 < \frac{k}{n} \leq 1$, then $\left(\frac{k}{n}\right)^{\xi_n} \geq \left(\frac{k}{n}\right)^2$ and from $\log\left(\frac{k}{n}\right) \leq 0$ hence,

$\left(\frac{k}{n}\right)^{\xi_n}\log\left(\frac{k}{n}\right) \leq \left(\frac{k}{n}\right)^2\log\left(\frac{k}{n}\right)$. So, from (3) it follows that

$$g(x_n) - g(1) \leq (x_n - 1)\left(\frac{k}{n}\right)^2\log\left(\frac{k}{n}\right)$$

$$\frac{1}{n}\sum_{k=1}^{n}[g(x_n) - f(1)] \leq (x_n - 1)\cdot\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\log\left(\frac{k}{n}\right), z_n \overset{not.}{=} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\log\left(\frac{k}{n}\right)$$

From (2) we have: $y_n \leq (x_n - 1)z_n, \forall n \geq 3$

$z_n > 0, \forall n \geq 3$ then $0 < x_n - 1 \leq \frac{y_n}{z_n}, \forall n \geq 3$; (4)

Let $h_1: [0,1] \to \mathbb{R}, h_1(x) = \begin{cases} x^2\log x, & x \in (0,1] \\ 0, & x = 0 \end{cases}$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\log\left(\frac{k}{n}\right) = \int_0^1 h_1(x)\,dx = \lim_{\varepsilon\to 0_+}\left(\int_\varepsilon^1 x^2\log x\,dx\right) = -\frac{1}{9}$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[g(x_n) - g(1)] = 0, \text{then } \lim_{n\to\infty} x_n = 1.$$

Now, from Taylor, exists $\zeta_n \in (1, x_n)$ such that

$$g(x_n) - g(1) = (x_n - 1)g'(1) + \frac{(x_n - 1)^2}{2}g''(\zeta_n)$$

$$g(x_n) - g(1) - (x_n - 1)g'(1) = \frac{(x_n - 1)^2}{2}g''(\zeta_n)$$

$$\frac{g(x_n) - g(1)}{x_n - 1} - g'(1) = \frac{x_n - 1}{2}g''(\zeta_n)$$

How $g'(x) = \left(\frac{k}{n}\right)^x\log\left(\frac{k}{n}\right)$ and $g''(x) = \left(\frac{k}{n}\right)^x\log^2\left(\frac{k}{n}\right)$, we get:

$$0 \le \frac{f(x_n) - g(1)}{x_n - 1} - \frac{k}{n}\log\left(\frac{k}{n}\right) = \frac{x_n - 1}{2}\left(\frac{k}{n}\right)^{\zeta_n}\log^2\left(\frac{k}{n}\right) \le \frac{x_n - 1}{2}\cdot\frac{k}{n}\cdot\log^2\left(\frac{k}{n}\right)$$

Using (2) it follows that:

$$0 \le \frac{y_n}{x_n - 1} - \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\left(\frac{k}{n}\right) \le \frac{x_n - 1}{2}\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log^2\left(\frac{k}{n}\right); (5)$$

Let $h_2: [0,1] \to \mathbb{R}, h_2(x) = \begin{cases} x\log^2 x, x \in (0,1] \\ 0, \quad x = 0 \end{cases}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log^2\left(\frac{k}{n}\right) = \int_0^1 h_2(x)\,dx = \frac{x^2}{2}\log^2 x\Big|_0^1 - \int_0^1 x\log x\,dx = \frac{1}{4}$$

From (5) it follows that

$$\lim_{n\to\infty}\left(\frac{y_n}{x_n - 1} - \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\left(\frac{k}{n}\right)\right) = 0, \qquad \lim_{n\to\infty}\frac{y_n}{x_n - 1} = \int_0^1 x\log x\,dx = -\frac{1}{4}$$

$\lim_{n\to\infty}\frac{ny_n}{n(x_n-1)} = -\frac{1}{4}$. Therefore,

$$\lim_{n\to\infty} n(x_n - 1) = -4\lim_{n\to\infty} ny_n = -4\cdot\left(-\frac{1}{2}\right) = 2$$

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ABOUT THE SPEED OF CONVERGENCE OF THE SEQUENCE AND APPLICATIONS

By Tran Minh Vu,Tran Thi Thanh Minh-Vietnam

Abstract: *In this paper, we have used Cesaro-Stolz Theorem's in evaluating the convergence rate of the arrays relative to* n^α *and from there give new result for this article.*

1. Convergence rate of the sequence

Theorem 1.1. For $a, b, \alpha \in \mathbb{R}_+$ and $(u_n)_{n\in\mathbb{N}^*}$ be sequence of real numbers, such that

$$u_1 = a, u_{n+1} = u_n + \frac{b}{u_n^\alpha}, n \geq 1.$$

In these conditions,

$$\lim_{n\to\infty} \frac{u_n^{1+\alpha}}{n} = b(1+\alpha)$$

Proof. We have $u_{n+1} - u_n = \frac{b}{u_n^\alpha} > 0$. Hence, $(u_n)_{n\in\mathbb{N}^*}$ increase. Since equation below has not solution, $l = l + \frac{b}{l^\alpha} \Leftrightarrow \frac{b}{l^\alpha} = 0$, we have: $\lim_{n\to\infty} u_n = +\infty$. We have:

$$u_{n+1}^{\alpha+1} - u_n^{\alpha+1} = \left(u_n + \frac{b}{u_n^\alpha}\right)^{\alpha+1} - u_n^{\alpha+1} = \frac{\frac{\left(u_n+\frac{b}{u_n^\alpha}\right)^{\alpha+1} - u_n^{\alpha+1}}{u_n^{\alpha+1}}}{\frac{1}{u_n^{\alpha+1}}} = \frac{\left(1+\frac{b}{u_n^{\alpha+1}}\right)^{\alpha+1} - 1}{\frac{1}{u_n^{\alpha+1}}}$$

Let: $f(x) = (1+bx)^{\alpha+1}$ and $x_n = \frac{1}{u_n^{\alpha+1}}$, we have:

$$\lim_{n\to\infty} \frac{f(1+x_n) - f(0)}{x_n} = f'(0) = b(1+\alpha)$$

Thus, $\lim_{n\to\infty}(u_{n+1}^{\alpha+1} - u_n^{\alpha+1}) = b(1+\alpha)$, since Cesaro – Stolz theorem's

$$\lim_{n\to\infty} \frac{u_n^{1+\alpha}}{n} = b(1+\alpha)$$

Application 1 (TST Viet Nam 1993) Let $(u_n)_{n\in\mathbb{N}^*}$ be sequence of real numbers, such that

$$u_1 = 1 \text{ and } u_{n+1} = u_n + \frac{1}{\sqrt{u_n}}, n \geq 1.$$

Find all $\beta \in \mathbb{R}$ such that $\lim_{n\to\infty} \frac{u_n^\beta}{n} = a \neq 0$.

Solution. Application Theorem 1.1., since $\beta = 1 + \frac{1}{2} = \frac{3}{2}$, we have:

$$\lim_{n\to\infty} \frac{u_n^{\frac{3}{2}}}{n} = \frac{3}{2}.$$

Theorem 1.2. For $(b_i)_{i=1}^m, (\alpha_i)_{i=1}^n$ are $2m$ positive numbers. Let $(u_n)_{n=1}^\infty$ such that:

$$u_1 = a > 0 \text{ and } u_{n+1} = u_n + \sum_{i=1}^m \frac{b_i}{u_n^{\alpha_i}}, n \geq 1$$

Set $\alpha_l = \min\{\alpha_i\}, l \in \{1, 2, \dots, m\}$, we have:

$$\lim_{n\to\infty} \frac{u_n^{1+\alpha_l}}{n} = b_l(1 + \alpha_l)$$

Proof. Very easy we have $\lim_{n\to\infty} u_n = +\infty$

$$u_{n+1}^{1+\alpha_l} - u_n^{1+\alpha_l} = \left(u_n + \sum_{i=1}^m \frac{b_i}{u_n^{\alpha_i}}\right)^{1+\alpha_l} - u_n^{1+\alpha_l} = \frac{\frac{\left(u_n + \sum_{i=1}^m \frac{b_i}{u_n^{\alpha_i}}\right)^{1+\alpha_l} - u_n^{1+\alpha_l}}{u_n^{1+\alpha_l}}}{\frac{1}{u_n^{1+\alpha_l}}} =$$

$$= \frac{\left(1 + \sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}\right)^{1+\alpha_l} - 1}{\frac{1}{u_n^{1+\alpha_l}}} = \frac{\left(1 + \sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}\right)^{1+\alpha_l} - 1}{\sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}} \cdot \frac{\sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}}{\frac{1}{u_n^{1+\alpha_l}}} =$$

$$= \frac{\left(1 + \sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}\right)^{1+\alpha_l} - 1}{\sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}} \cdot \sum_{i=1}^m \frac{b_i}{u_n^{\alpha_i-\alpha_l}}$$

Set: $x_n = \sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}$, then $\lim_{n\to\infty} x_n = 0, f(x) = (1 + x)^{1+\alpha_l}$. Hence,

$$\lim_{n\to\infty} \frac{\left(1 + \sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}\right)^{1+\alpha_l} - 1}{\sum_{i=1}^m \frac{b_i}{u_n^{1+\alpha_i}}} = \lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n} = 1 + \alpha_l.$$

Because $\lim_{n\to\infty} \sum_{i=1}^m \frac{b_i}{u_n^{\alpha_i-\alpha_l}} = b_l$, thus, $\lim_{n\to\infty}\left(u_{n+1}^{1+\alpha_l} - u_n^{1+\alpha_l}\right) = b_l(1 + \alpha_l)$

Application Cesro-Stolz theorem's, proof complete.

Application 2. For $(u_n)_{n\geq 1}$ such that $u_1 = 2020, u_{n+!} = u_n + \frac{2}{u_n} + \frac{3}{u_n^2}, n \geq 1$.

Prove that: $\lim_{n\to\infty}\frac{u_n^2}{n} = 4$.

Solution. We have $\lim_{n\to\infty} u_n = +\infty$ and

$$u_{n+1}^2 - u_n^2 = \left(u_n + \frac{2}{u_n} + \frac{3}{u_n^2}\right)^2 - u_n^2 = \frac{\frac{\left(u_n+\frac{2}{u_n}+\frac{3}{u_n^2}\right)^2-u_n^2}{u_n^2}}{\frac{1}{u_n^2}} =$$

$$= \frac{\left(1+\frac{2}{u_n^2}+\frac{3}{u_n^3}\right)^2-1}{\frac{1}{u_n^2}} = \frac{\left(1+\frac{2}{u_n^2}+\frac{3}{u_n^3}\right)^2-1}{\frac{2}{u_n^2}+\frac{3}{u_n^3}}\cdot\frac{\frac{2}{u_n^2}+\frac{3}{u_n^3}}{\frac{1}{u_n^2}} =$$

$$= \frac{\left(1+\frac{2}{u_n^2}+\frac{3}{u_n^3}\right)^2-1}{\frac{2}{u_n^2}+\frac{3}{u_n^3}}\cdot\left(2+\frac{3}{u_n^2}\right)$$

Let: $x_n = \frac{2}{u_n^2} + \frac{3}{u_n^3}$, then $\lim_{n\to\infty} x_n = 0$. With function $f(x) = (1+x)^2$ differentiable on $\mathbb{R}$,

we have: $\lim_{n\to\infty}\frac{\left(1+\frac{2}{u_n^2}+\frac{3}{u_n^3}\right)^2-1}{\frac{2}{u_n^2}+\frac{3}{u_n^3}} = \lim_{n\to\infty}\frac{(1+x_n)^2-1}{x_n} = \lim_{n\to\infty}\frac{f(x_n)-f(0)}{x_n-0} = f'(0) = 2$

Since, $\lim_{n\to\infty}\left(2+\frac{3}{u_n^2}\right) = 2$, hence, $\lim_{n\to\infty}(u_{n+1}^2 - u_n^2) = 4$.

Application Cesaro-Stolz theorem's, proof complete.

Applicartion 3. Let $a > 0$ and $(u_n)_{n\geq 1}$ such that $u_1 = a, u_{n+1} = u_n + \frac{2}{\sqrt{u_n}} + \frac{5}{\sqrt[5]{u_n}}; n \geq 1$.

Prove that $v_n = \frac{u_n}{\sqrt[6]{n^5}}$ have a limit and find it.

Solution. Application Theorem 1.2. with $b_1 = 2; b_2 = 5$ and $\alpha_1 = \frac{1}{2}; \alpha_2 = \frac{1}{5}$.

We have $\lim_{n\to\infty}\frac{u_n^{\frac{6}{5}}}{n} = 5\cdot\frac{6}{5}$. Hence, $\lim_{n\to\infty} v_n = \lim_{n\to\infty}\frac{u_n}{n^{\frac{5}{6}}} = \sqrt[6]{6^5}$.

Application 4 (TST-Vung Tau-Viet Nam 2020). Let $(u_n)_{n\geq 1}$ be sequence of real numbers such that, $u_1 = 2, u_{n+1} = u_n + \frac{n}{u_n}, n \geq 1$.

Prove that $v_n = \frac{u_n}{n}$ have a limit and find it.

Solution. We have: $u_{n+1}^2 = u_n^2 + 2n + \frac{n^2}{u_n^2}$. So, $u_{n+1}^2 > u_n^2 + 2n$.

$$\begin{cases} u_{n+1}^2 > u_n^2 + 2n \\ u_n^2 > u_{n-1}^2 + 2(n-1) \\ \dots\dots\dots \\ u_2^2 > u_1^2 + 2\cdot 1 \\ u_1^2 = 4^2 \end{cases}$$

Adding the above inequalities and simplify, we have $u_{n+1}^2 > n^2 + n + 4$; (1.1)

Hence,

$$u_{n+1}^2 = u_n^2 + 2n + \frac{n^2}{u_n^2} < u_n^2 + 2n + \frac{n^2}{(n^2+n+4)^2} < u_n^2 + 2n + \frac{n^2}{u_n^2} < u_n^2 + 2n + \frac{1}{n}.$$

$$\begin{cases} u_{n+1}^2 < u_n^2 + 2n + \frac{1}{n} \\ u_n^2 < u_{n-1}^2 + 2(n-1) + \frac{1}{n-1} \\ \dots\dots\dots\dots\dots \\ u_2^2 < u_1^2 + 2\cdot 1 + \frac{1}{1} \\ u_1^2 = 4. \end{cases}$$

Add the above inequalities and simplify, we have:

$$u_{n+1}^2 < n^2 + n + 4 + \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right); (1.2)$$

Since (1.1) and (1.2), we have:

$$\frac{n^2+n+4}{(n+1)^2} < \frac{u_{n+1}^2}{(n+1)^2} < \frac{n^2+n+4}{(n+1)^2} + \frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{(n+1)^2}$$

Hence, according to the clamping principle $\lim_{n\to\infty} \frac{u_{n+1}^2}{(n+1)^2} = 1$ or $\lim_{n\to\infty} \frac{u_n}{n} = 1$.

2. Exercise.

Exercise 1. Let $(a_n)_{n=1}^{\infty}$ such that $a_0 = \frac{1}{2}; a_{n+1} = a_n - a_n^2, n \geq 1$.

Find $\lim_{n\to\infty}(na_n)$.

Exercise 2 (Romanian 2007). Let $(a_n)_{n=1}^{\infty}$ such that $a_0 \in (0,1); a_{n+1} = a_n(1 - a_n^2), n \geq 1$.

Find $\lim_{n\to\infty}(\sqrt{n}a_n)$.

Exercise 3. Let $(a_n)_{n\geq 1}$ such that $2a_{n+1} - 2a_n + a_n^2 = 0, n = 0,1,2,\dots$

1) Prove that the number sequence is decrease.

2) If $a_0 = 1$, then find $\lim_{n\to\infty} a_n$.

3) Find the condition for a limited sequence and find the limit.

Exercise 4. Let $(u_n)_{n=1}^{\infty}$ such that $u_1 = 1, u_{n+1} = \frac{\sqrt{u_n^2+2019u_n}+u_n}{2}, n \geq 1$.

a) Set $v_n = \sum_{k=1}^{n} \frac{1}{u_n^2}$. Find $\lim_{n\to\infty} v_n$.

b) Find $\lim_{n\to\infty} \frac{u_n}{n}$.

Exercise 5. Let $(u_n)_{n=1}^{\infty}$ such that $u_1 = 1, u_{n+1} = u_n + \frac{1}{2u_n}, n \geq 1$. Prove that:

a) $n \leq u_n^2 < n + \sqrt[3]{n}$.

b) $\lim_{n\to\infty} (u_n - n) = 0$.

Exercise 6 (TST-Vinh-Viet Nam 2020). Let $(u_n)_{n=1}^{\infty}$ such that $u_1 = 1, u_{n+!} = u_n + \frac{n^2}{u_n^2}, n \geq 1$.

Prove that $\lim_{n\to\infty} \frac{u_n}{n} = 1$.

ABOUT NESBITT –IONESCU INEQUALITY

By D.M.Bătinețu-Giurgiu, Mihaly Bencze,Daniel Sitaru-Romania

If $a, b, c \in (0, \infty)$, then:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}; \quad (N.I.)$$

Generalized: If $a, b, c, t, u \in \mathbb{R}_+^*$, then:

$$\frac{a}{tb+uc} + \frac{b}{tc+ua} + \frac{c}{ta+ub} \geq \frac{3}{t+u}; \quad (1)$$

Let be $n \in \mathbb{N}^* - \{1\}$ and $x_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}, X_v = \sum_{k=1}^{n} x_k^v, \forall v \in \mathbb{R}_+^*; \quad (2)$

Theorem.

If $n \in \mathbb{N}^* - \{1\}, a \in [0, \infty); b, c, d, m, t \in \mathbb{R}_+^*, \forall k = \overline{1,n}$ and $x_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}$,

$$X_s = \sum_{k=1}^{n} x_k^s, \forall s \in \mathbb{R}_+^*, c \cdot X_t > d \cdot \max_{1\leq k\leq n} x_k^t, \text{then holds:}$$

$$\sum_{k=1}^{n} \frac{a \cdot X_m + b \cdot x_k^m}{c \cdot X_t - d \cdot x_k^t} \geq \frac{(an+b)n}{cn-d} \cdot \frac{X_m}{X_t}; \quad (*)$$

Proof. WLOG, suppose $x_1 \geq x_2 \geq \cdots \geq x_n$ and then:

$$\frac{1}{c\cdot X_t-d\cdot x_1^t}\ge\frac{1}{c\cdot X_t-dx_2^t}\ge\cdots\ge\frac{1}{c\cdot X_t-d\cdot x_n^t}$$

Applying Chebyshev's inequality for:

$$a\cdot X_m+b\cdot x_1^m\ge a\cdot X_m+b\cdot x_2^m\ge\cdots\ge a\cdot X_m+b\cdot x_n^m;\ \ (3)$$

$$\frac{1}{c\cdot X_t-d\cdot x_1^t}\ge\frac{1}{c\cdot X_t-dx_2^t}\ge\cdots\ge\frac{1}{c\cdot X_t-d\cdot x_n^t};\ \ (4)$$

We get:

$$\sum_{k=1}^{n}\frac{a\cdot X_m+b\cdot x_k^m}{c\cdot X_t-d\cdot x_k^t}\ge\frac{1}{n}\left(\sum_{k=1}^{n}(a\cdot X_m+b\cdot x_k^m)\right)\cdot\sum_{k=1}^{n}\frac{1}{c\cdot X_t-d\cdot x_k^t}=$$

$$=\frac{1}{n}\left(a\cdot n\cdot X_m+b\cdot\sum_{k=1}^{n}x_k^m\right)\cdot\sum_{k=1}^{n}\frac{1}{c\cdot X_t-d\cdot x_k^t}=$$

$$=\frac{1}{n}(a\cdot n\cdot X_m+b\cdot X_m)\cdot\sum_{k=1}^{n}\frac{1}{c\cdot X_t-d\cdot x_k^t}\overset{Bergstrom}{\ge}$$

$$\ge\frac{a\cdot n+b}{n}\cdot X_m\cdot\frac{n^2}{\sum_{k=1}^{n}(c\cdot X_t-d\cdot x_k^t)}=$$

$$=(a\cdot n+b)\cdot\frac{n}{c\cdot n\cdot X_t-d\cdot X_t}\cdot X_m=\frac{(a\cdot n+b)n}{c\cdot n-d}\cdot\frac{X_m}{X_n}$$

If $m=t$, then inequality $(*)$ becomes:

$$\sum_{k=1}^{n}\frac{a\cdot X_m+b\cdot x_k^t}{c\cdot X_t-d\cdot x_k^t}\ge\frac{(a\cdot n+b)n}{c\cdot n-d};\ \ (**)$$

If $a=0, b=c=d=1$, then we get:

$$\sum_{k=1}^{n}\frac{x_k^m}{X_m-x_k^m}\ge\frac{n}{n-1};\ \ (***)$$

If $m=1$, then we get:

$$\sum_{k=1}^{n}\frac{x_k}{X-x_k}\ge\frac{n}{n-1},\text{ where } X=X_1=\sum_{k=1}^{n}x_k;\ \ (N.I.)$$

For $n=3$, we have:

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\ge\frac{3}{2};\forall x,y,z\in\mathbb{R}_+^*$$

If $n = 3$ and $a = 0, b = c = d = 1$, then $(*)$ becomes as:

$$\frac{x_a^m}{x_2^t + x_3^t} + \frac{x_2^m}{x_3^t + x_1^t} + \frac{x_3^m}{x_1^t + x_2^t} \geq \frac{3(x_1^m + x_2^m + x_3^m)}{2(x_1^t + x_2^t + x_3^t)}; \quad (****)$$

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

A SIMPLE PROOF FOR MAVLO'S INEQUALITY

By Daniel Sitaru-Romania

If $\boldsymbol{a}, \boldsymbol{b} > \mathbf{0}; \boldsymbol{n} \in \mathbb{N}^*$ then:

$$\left(\frac{\boldsymbol{a}+\boldsymbol{b}}{\mathbf{2}}\right)^{\boldsymbol{n}} - \left(\sqrt{\boldsymbol{ab}}\right)^{\boldsymbol{n}} \geq \frac{\left(\sqrt{\boldsymbol{a}^{\boldsymbol{n}}} - \sqrt{\boldsymbol{b}^{\boldsymbol{n}}}\right)^2}{\mathbf{2}^{\boldsymbol{n}}}$$

Proof. For $n = 1$:

$$\frac{a+b}{2} - \sqrt{ab} \geq \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2}$$

$$a + b - 2\sqrt{ab} \geq \left(\sqrt{a} - \sqrt{b}\right)^2 \Leftrightarrow \left(\sqrt{a} - \sqrt{b}\right)^2 \geq \left(\sqrt{a} - \sqrt{b}\right)^2$$

Suppose $n \geq 2$. Denote $a = x^2; b = y^2$. Inequality can be written:

$$\left(\frac{x^2 + y^2}{2}\right)^n - \left(\sqrt{x^2y^2}\right)^n \geq \frac{\left(\sqrt{x^{2n}} - \sqrt{y^{2n}}\right)^2}{2^n}$$

$$\frac{(x^2 + y^2)^n}{2^n} - x^n y^n \geq \frac{(x^n - y^n)^2}{2^n}$$

$$(x^2 + y^2)^n - 2^n \cdot x^n y^n \geq x^{2n} + y^{2n} - 2x^n y^n$$

$$(x^2 + y^2)^n - x^{2n} - y^{2n}. = 2^n \cdot x^n y^n - 2 \cdot x^n y^n$$

$$\sum_{k=1}^{n-1} \binom{n}{k} x^{2n-2k} y^{2k} \geq x^n y^n (2^n - 2)$$

$$2\sum_{k=1}^{n-1} \binom{n}{k} x^{2n-2k} y^{2k} \geq 2x^n y^n \cdot \sum_{k=1}^{n} \binom{n}{k}$$

$$\sum_{k=1}^{n-1} \binom{n}{k} x^{2n-2k} y^{2k} + \sum_{k=1}^{n-1} \binom{n}{k} y^{2n-2k} x^{2k} - 2\sum_{k=1}^{n-1} \binom{n}{k} x^n y^n \geq 0$$

$$\sum_{k=1}^{n-1} \binom{n}{k} x^n y^{2k} \left(x^{n-2k} - y^{n-2k}\right) + \sum_{k=1}^{n-1} \binom{n}{k} x^{2k} y^n \left(y^{n-2k} - x^{n-2k}\right) \geq 0$$

$$\sum_{k=1}^{n-1}\binom{n}{k}\left(x^n y^{2k}-x^{2k}y^n\right)\left(x^{n-2k}-y^{n-2k}\right)\geq 0$$

$$\sum_{k=1}^{n-1}\binom{n}{k}x^{2k}y^{2k}\left(x^{n-2k}-y^{n-2k}\right)^2\geq 0$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A SIMPLE PROOF FOR ABI-KHUZAM'S INEQUALITY

By Daniel Sitaru – Romania

Abstract: In this paper is presented an elementary, detailed proof for the famous Abi-Khuzam's inequality.

Lemma 1: If $x,y,z,A,B,C\in\mathbb{R}; A+B+C=\pi$ then:

$$x^2+y^2+z^2\geq 2(yz\cos A+zx\cos B+xy\cos C)\quad (1)$$

Proof: $0\leq\left(z-(x\cos B+y\cos A)\right)^2+(x\sin B-y\sin A)^2=$

$$=z^2-2z(x\cos B+y\cos A)+(x\cos B+y\cos A)^2+$$

$$+x^2\sin^2 B+y^2\sin^2 A-2xy\sin A\sin B=$$

$$=z^2-2xz\cos B-2zy\cos A+x^2(\sin^2 B+\cos^2 B)+$$

$$+y^2(\cos^2 A+\sin^2 A)+2xy(\cos A\cos B-\sin A\sin B)=$$

$$=x^2+y^2+z^2-2yz\cos A-2zx\cos B+2xy\cos(A+B)=$$

$$=x^2+y^2+z^2-2yz\cos A-2zx\cos B+2xy\cos(\pi-C)=$$

$$=x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C$$

$$0\leq x^2+y^2+z^2-2yz\cos A-2zx\cos B-2xy\cos C$$

$$x^2+y^2+z^2\geq 2(xy\cos C+yz\cos A+zx\cos B)$$

Lemma 2: If $x,y,z,A,B,C\in\mathbb{R}; x,y,z>0, A+B+C=\pi$ then:

$$x\cos A+y\cos B+z\cos C\leq\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)\quad (2)$$

Proof. Replace in (1):

$$x\to\sqrt{\frac{yz}{x}};\ y\to\sqrt{\frac{zx}{y}};\ z\to\sqrt{\frac{xy}{z}}$$

$$\left(\sqrt{\frac{yz}{x}}\right)^2+\left(\sqrt{\frac{zx}{y}}\right)^2+\left(\sqrt{\frac{xy}{z}}\right)^2\ge$$

$$\ge 2\left(\sqrt{\frac{zx}{y}}\sqrt{\frac{xy}{z}}\cos A+\sqrt{\frac{xy}{z}}\cdot\sqrt{\frac{yz}{x}}\cos B+\sqrt{\frac{yz}{x}}\cdot\sqrt{\frac{zx}{y}}\cos C\right)$$

$$\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)\ge x\cos A+y\cos B+z\cos C$$

$$x\cos A+y\cos B+z\cos C\le\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)$$

Theorem (ABI-KHUZAM'S INEQUALITY)

If $x,y,z,t>0$; $A,B,C,D\in\mathbb{R}$; $A+B+C+D=\pi$ then:

$$x\cos A+y\cos B+z\cos C+t\cos D\le\sqrt{\frac{(xy+zt)(xz+yt)(xt+yz)}{xyzt}}\quad\textbf{(3)}$$

Proof. Denote: $p=\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{z}{t}+\frac{t}{z}\right)$; $q=\frac{xy+zt}{2}$

By (2): $x\cos A+y\cos B+\sqrt{\frac{q}{p}}\cos(C+D)\le\frac{1}{2}\left(\frac{xy}{\sqrt{\frac{q}{p}}}+\sqrt{\frac{q}{p}}\left(\frac{x}{y}+\frac{y}{x}\right)\right)$ (4)

$$z\cos C+t\cos D+\sqrt{\frac{q}{p}}\cos(A+B)\le\frac{1}{2}\left(\frac{zt}{\sqrt{\frac{q}{p}}}+\sqrt{\frac{q}{p}}\left(\frac{z}{t}+\frac{t}{z}\right)\right)\quad(5)$$

$$\cos(A+B)+\cos(C+D)=\cos(A+B)+\cos(\pi-(A+B))=\cos(A+B)-\cos(A+B)=0$$

By adding (4); (5):

$$x\cos A+y\cos B+z\cos C+t\cos D+\sqrt{\frac{q}{p}}(\cos(A+B)+\cos(C+D))\le$$

$$\le\frac{1}{2}\left(\frac{xy+zt}{\sqrt{\frac{q}{p}}}+\sqrt{\frac{q}{p}}\left(\frac{x}{y}+\frac{y}{x}+\frac{z}{t}+\frac{t}{z}\right)\right)$$

$$x\cos A+y\cos B+z\cos C+t\cos D\le\frac{xy+zt}{2}\cdot\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}\cdot\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{z}{t}+\frac{t}{z}\right)$$

$$x\cos A+y\cos B+z\cos C+t\cos D\le q\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}\cdot p=\sqrt{pq}+\sqrt{pq}=2\sqrt{pq}$$

$$x\cos A+y\cos B+z\cos C+t\cos D\le 2\sqrt{\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{z}{t}+\frac{t}{z}\right)\frac{xy+zt}{2}}=$$

$$=\sqrt{4\cdot\frac{x^2tz+y^2tz+z^2xy+t^2xy}{2xyzt}\cdot\frac{xy+zt}{2}}=\sqrt{\frac{xz(xt+yz)+yt(xt+yz)}{xyzt}\cdot(xy+zt)}=$$

$$=\sqrt{\frac{(xy+zt)(xz+yt)(xt+yz)}{xyzt}}$$

Corollary 1: If $A,B,C,D\in\mathbb{R}; A+B+C+D=\pi$ then:

$$\cos A+\cos B+\cos C+\cos D\le 2\sqrt{2}\quad (6)$$

Proof. We take in (3): $x=y=z=t\neq 0$.

Corollary 2: If $A,B,C\in\mathbb{R}; A+B+C=\frac{\pi}{2}$ then:

$$\cos A+\cos B+\cos C\le 2\sqrt{2}$$

Proof. We take in (6): $D=\frac{\pi}{2}\Rightarrow A+B+C=\pi-\frac{\pi}{2}\Rightarrow A+B+C=\frac{\pi}{2}; \cos D=0$

Corollary 3: If $x,y,z,t>0$ then:

$$xyzt(x+y+z+t)^2\le 2(xy+zt)(xz+yt)(xt+yz)$$

Proof. We take in (3): $A=B=C=D=\frac{\pi}{4}$

$$\Rightarrow \cos A=\cos B=\cos C=\cos D=\frac{1}{\sqrt{2}}; A+B+C+D=\pi$$

$$\frac{1}{\sqrt{2}}(x+y+z+t)\le\sqrt{\frac{(xy+zt)(xz+yt)(xt+yz)}{xyzt}}$$

By squaring:

$$\frac{(x+y+z+t)^2}{2}\le\frac{(xy+zt)(xz+yt)(xt+yz)}{xyzt}$$

$$xyzt(x+y+z+t)^2\le 2(xy+zt)(xz+yt)(xt+yz)$$

Equality holds for $x=y=z=t$.

Reference: [1] Romanian Mathematical Magazine – www.ssmrmh.ro

POWER MEANS INEQUALITY AND APPLICATIONS

By Daniel Sitaru – Romania

Abstract. In this paper are presented power means concepts, a few connections and applications.

Proposition 1: If $a, b > 0, a, b$ – fixed, $x \geq y > 0$ then:

$$\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} \geq \left(\frac{a^y + b^y}{2}\right)^{\frac{1}{y}}$$

Proof. Let be $f: \mathbb{R} \to \mathbb{R}$:

$$f(a,b,x) = \begin{cases} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}}; x \neq 0 \\ \sqrt{ab}; x = 0 \end{cases}$$

$$\lim_{\substack{x\to 0\\x>0}} f(a,b,x) = \lim_{\substack{x\to 0\\x>0}} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} = e^{\lim_{\substack{x\to 0\\x>0}}\frac{1}{x}\log\left(\frac{a^x+b^x}{2}\right)} = e^{\lim_{\substack{x\to 0\\x>0}}\frac{a^x\log a + b^x \log b}{2}\cdot\frac{2}{a^x+b^x}} =$$

$$= e^{\frac{\log a + \log b}{1+1}} = e^{\log\sqrt{ab}} = \sqrt{ab} = f(a,b,0)$$

f continuous

$$f'(a,b,x) = \frac{1}{x}\left(\frac{a^x + b^x}{2}\right)' \cdot \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}-1} - \frac{1}{x^2}\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} \cdot \log\left(\frac{a^x + b^x}{2}\right)$$

$$f'(a,b,x) = \frac{1}{x} \cdot \frac{a^x \log a + b^x \log b}{2} \cdot \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}-1} - \frac{1}{x^2}\left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} \cdot \log\left(\frac{a^x + b^x}{2}\right)$$

$$x^2 f'(a,b,x) = \frac{x(a^x \log a + b^x \log b)}{2} \cdot \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}-1} - \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} \cdot \log\left(\frac{a^x + b^x}{2}\right)$$

$$x^2 f'(a,b,x) = \frac{1}{2}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}-1} \left(a^x \log a^x + b^x \log b^x - (a^x + b^x)\log\left(\frac{a^x+b^x}{2}\right)\right) \quad (1)$$

Define $g: (0,\infty) \to \mathbb{R}; g(x) = x\log x$

$$g'(x) = \log x + 1; g''(x) = \frac{1}{x} > 0; g\text{ – convexe}$$

By Jensen's inequality:

$$g(u) + g(v) \geq 2g\left(\frac{u+v}{2}\right); u, v > 0$$

For $u = a^x; v = b^x$

$$g(a^x)+g(b^x)\geq 2g\left(\frac{a^x+b^x}{2}\right)$$

$$a^x\log a^x+b^x\log b^x\geq 2\cdot\frac{a^x+b^x}{2}\cdot\log\left(\frac{a^x+b^x}{2}\right)$$

$$a^x\log a^x+b^x\log b^x-(a^x+b^x)\log\left(\frac{a^x+b^x}{2}\right)\geq 0 \quad (2)$$

By (1); (2): $x^2f'(a,b,x)\geq 0\Rightarrow f$ increasing

$$x\geq y>0; f \text{ increasing}\Rightarrow f(a,b,x)\geq f(a,b,y)$$

$$\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}\geq\left(\frac{a^y+b^y}{2}\right)^{\frac{1}{y}}$$

Corollary 1:

f increasing and $2>1>0>-1\Rightarrow f(a,b,2)\geq f(a,b,1)\geq f(a,b,0)\geq f(a,b,-1)$

$$\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}\geq\left(\frac{a^1+b^1}{2}\right)^{\frac{1}{1}}\geq\sqrt{ab}\geq\left(\frac{a^{-1}+b^{-1}}{2}\right)^{\frac{1}{-1}}$$

$$\sqrt{\frac{a^2+b^2}{2}}\geq\frac{a+b}{2}\geq\sqrt{ab}\geq\frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Corollary 2: If $n\in\mathbb{N}; n\geq 1;\ n>n-1>n-2>\cdots>3>2>1>0$

f increasing, then: $f(a,b,n)\geq f(a,b,n-1)\geq\cdots\geq f(a,b,1)\geq f(a,b,0)$

$$\left(\frac{a^n+b^n}{2}\right)^{\frac{1}{n}}\geq\left(\frac{a^{n-1}+b^{n-1}}{2}\right)^{\frac{1}{n-1}}\geq\cdots\geq\left(\frac{a^1+b^1}{2}\right)^{\frac{1}{1}}\geq\sqrt{ab}$$

$$\sqrt[n]{\frac{a^n+b^n}{2}}\geq\sqrt[n-1]{\frac{a^{n-1}+b^{n-1}}{2}}\geq\cdots\geq\frac{a+b}{2}\geq\sqrt{ab}$$

Corollary 3: If $n\in\mathbb{N}; n\geq 1; f$ increasing; $0<\frac{1}{n}<\frac{1}{n-1}<\frac{1}{n-2}<\cdots<\frac{1}{3}<\frac{1}{2}<1$

$$f(a,b,0)\leq f\left(a,b,\frac{1}{n}\right)\leq f\left(a,b,\frac{1}{n-1}\right)\leq\cdots\leq f\left(a,b,\frac{1}{3}\right)\leq f\left(a,b,\frac{1}{2}\right)\leq f(a,b,1)$$

$$\sqrt{ab}\leq\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^n\leq\left(\frac{a^{\frac{1}{n-1}}+b^{\frac{1}{n-1}}}{2}\right)^{n-1}\leq\cdots\leq\left(\frac{a^{\frac{1}{3}}+b^{\frac{1}{3}}}{2}\right)^3\leq\left(\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}\right)^2\leq\frac{a+b}{2}$$

$$\sqrt{ab}\leq\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n\leq\left(\frac{\sqrt[n-1]{a}+\sqrt[n-1]{b}}{2}\right)^{n-1}\leq\cdots\leq\left(\frac{\sqrt[3]{a}+\sqrt[3]{b}}{2}\right)^3\leq\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\leq\frac{a+b}{2}$$

Observation: In corollaries 1,2,3 equality holds for $a = b$.

Proposition 2: If $a, b, c > 0$; a, b, c – fixed; $x \geq y > 0$ then:

$$\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \geq \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{y}}$$

Proof. Let be $f: \mathbb{R} \to \mathbb{R}; f(a, b, x) = \begin{cases} \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}; x \neq 0 \\ \sqrt[3]{abc}; x = 0 \end{cases}$

$$\lim_{\substack{x\to 0\\ x>0}} f(a, b, c, x) = \lim_{\substack{x\to 0\\ x>0}} \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} = e^{\lim_{\substack{x\to 0\\ x>0}} \frac{1}{x}\log\left(\frac{a^x+b^x+c^x}{3}\right)} =$$

$$= e^{\lim_{\substack{x\to 0\\ x>0}} \frac{a^x \log a + b^x \log b + c^x \log c}{3} \cdot \frac{3}{a^x+b^x+c^x}} = e^{\frac{\log a + \log b + \log c}{1+1+1}} = e^{\log \sqrt[3]{abc}} = f(a, b, c, 0)$$

f continuous, $f'(a, b, c, x) = \frac{1}{x}\left(\frac{a^x+b^x+c^x}{2}\right)' \cdot \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}-1} -$

$$-\frac{1}{x^2}\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \cdot \log\left(\frac{a^x + b^x + c^x}{3}\right)$$

$$f'(a, b, c, x) = \frac{1}{x} \cdot \frac{a^x \log a + b^x \log b + c^x \log c}{3} \cdot \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}-1} -$$

$$-\frac{1}{x^2}\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \cdot \log\left(\frac{a^x + b^x + c^x}{3}\right)$$

$$x^2 f'(a, b, c, x) = \frac{x(a^x \log a + b^x \log b + c^x \log c)}{3} \cdot \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}-1} -$$

$$-\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \cdot \log\left(\frac{a^x + b^x + c^x}{3}\right)$$

$x^2 f'(a, b, c, x) = \frac{1}{3}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}-1} \left(a^x \log a + b^x \log b + c^x \log c - (a^x + b^x + c^x) \log\left(\frac{a^x+b^x+c^x}{3}\right)\right)$
(3)

Define $g: (0, \infty) \to \mathbb{R}; g(x) = x \log x, g'(x) = \log x + 1; g''(x) = \frac{1}{x} > 0; g$ – convexe

By Jensen's inequality:

$$g(u) + g(v) + g(w) \geq 3g\left(\frac{u + v + w}{3}\right); u, v, w > 0$$

For $u = a^x; v = b^x; w = c^x, g(a^x) + g(b^x) + g(c^x) \geq 3g\left(\frac{a^x+b^x+c^x}{3}\right)$

$$a^x \log a^x + b^x \log b^x + c^x \log c^x \geq 3 \cdot \frac{a^x + b^x + c^x}{3} \log\left(\frac{a^x + b^x + c^x}{3}\right)$$

$$a^x \log a^x + b^x \log b^x + c^x \log c^x - (a^x + b^x + c^x) \log\left(\frac{a^x+b^x+c^x}{3}\right) \geq 0 \quad (4)$$

By (3); (4): $x^2 f'(a,b,c,x) \geq 0 \Rightarrow f$ increasing

$$x \geq y > 0; f \text{ increasing} \Rightarrow f(a,b,x) \geq f(a,b,y)$$

$$\left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} \geq \left(\frac{a^y + b^y + c^y}{3}\right)^{\frac{1}{y}}$$

Corollary 4: $\boldsymbol{f}$ **increasing and** $\boldsymbol{2 > 1 > 0 > -1}$

$$\boldsymbol{f(a,b,c,2) \geq f(a,b,c,1) \geq f(a,b,c,0) \geq f(a,b,c,-1)}$$

$$\boldsymbol{\left(\frac{a^2 + b^2 + c^2}{3}\right)^{\frac{1}{2}} \geq \left(\frac{a^1 + b^1 + c^1}{3}\right)^{\frac{1}{1}} \geq \sqrt[3]{abc} \geq \left(\frac{a^{-1} + b^{-1} + c^{-1}}{3}\right)^{\frac{1}{-1}}}$$

$$\boldsymbol{\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}}$$

Corollary 5: If $\boldsymbol{n \in \mathbb{N}; n \geq 1;}$ $\boldsymbol{f}$ **increasing and:** $\boldsymbol{n > n-1 > n-2 > \cdots > 3 > 2 > 1 > 0}$

$$\boldsymbol{f(a,b,c,n) \geq f(a,b,c,n-1) \geq \cdots \geq f(a,b,c,1) \geq f(a,b,c,0)}$$

$$\boldsymbol{\left(\frac{a^n + b^n + c^n}{3}\right)^{\frac{1}{n}} \geq \left(\frac{a^{n-1} + b^{n-1} + c^{n-1}}{3}\right)^{\frac{1}{n-1}} \geq \cdots \geq \left(\frac{a^1 + b^1 + c^1}{3}\right)^{\frac{1}{1}} \geq \sqrt[3]{abc}}$$

$$\boldsymbol{\sqrt[n]{\frac{a^n + b^n + c^n}{3}} \geq \sqrt[n-1]{\frac{a^{n-1} + b^{n-1} + c^{n-1}}{3}} \geq \cdots \geq \sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3} \geq \sqrt[3]{abc}}$$

Corollary 6: If $\boldsymbol{n \in \mathbb{N}; n > 1;}$ $\boldsymbol{f}$ **increasing;** $\boldsymbol{0 < \frac{1}{n} < \frac{1}{n-1} < \frac{1}{n-2} < \cdots < \frac{1}{3} < \frac{1}{2} < 1}$

$$\boldsymbol{f(a,b,c,0) \leq f\left(a,b,c,\frac{1}{n}\right) \leq f\left(a,b,c,\frac{1}{n-1}\right) \leq \cdots}$$

$$\boldsymbol{\ldots \leq f\left(a,b,c,\frac{1}{3}\right) \leq f\left(a,b,c,\frac{1}{2}\right) \leq f(a,b,c,1)}$$

$$\boldsymbol{\sqrt[3]{abc} \leq \left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}} + c^{\frac{1}{n}}}{3}\right)^n \leq \left(\frac{a^{\frac{1}{n-1}} + b^{\frac{1}{n-1}} + c^{\frac{1}{n-1}}}{3}\right)^{n-1} \leq \cdots}$$

$$\dots \le \left(\frac{a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}}{3}\right)^3 \le \left(\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}}{3}\right)^3 \le \frac{a + b + c}{3}$$

$$\sqrt[3]{abc} \le \left(\frac{\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c}}{3}\right)^n \le \left(\frac{\sqrt[n-1]{a} + \sqrt[n-1]{b} + \sqrt[n-1]{c}}{3}\right)^{n-1} \le \cdots$$

$$\dots \le \left(\frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{3}\right)^3 \le \left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^2 \le \frac{a + b + c}{3}$$

Observation: In corollaries 4,5,6 equality holds for $a = b = c$.

Reference: [1] Romanian Mathematical Magazine – www.ssmrmh.ro

PROBLEMS FOR JUNIORS

J.1099 Solve for real numbers:

$$\frac{(2x^4 + x^3 - x^2 - 2)^2 + 4x - 4}{x^3 - x^2 + x - 4} + \frac{1}{2x^2} = 0$$

Proposed by Carlos Paiva-Brazil

J.1100 If $a, b, c \in (0, \infty), \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + (a + b + c)^2 \le 12$, then:

$$ab + bc + ca + \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le 6$$

Proposed by Dan Radu Seclăman – Romania

J.1101 Solve in $\mathbb{R}$:

i) $x^3 + x^2 + x = (x + 1)(y + 2)\sqrt{(x + 1)(y + 1)}$

ii) $\sqrt{y + 1} + 2 = \left(x - 1 - \frac{3}{4x}\right)\sqrt{x + 1}$

Proposed by Carlos Paiva-Brazil

J.1102 Solve for real positive numbers:

$$3t^2 + t - \sqrt{16 - 16t + 4t^3 - t^4} + \sqrt{8 + 4t - 2t^2 - t^3} - \sqrt{8 - 12t + 6t^2 - t^3} + $$
$$+\sqrt{4 - t^2} = 2$$

Proposed by Samir Cabiyev-Azerbaijan

J.1103 If $a, b, c \in (0, \infty), \frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} \le 1$ then:

$$0 < \min(a, b, c) \le \frac{1}{2}$$

Proposed by Dan Radu Seclăman – Romania

J.1104 Prove that if n is a perfect number, then:

$$\forall d \le n, d \nmid n, (d,n) \ne 1$$

We have:

$$\sum d = \frac{(n-1)(n-2) - n \cdot \phi(n)}{2}$$

where $\phi(n)$ is Euler's totient function.

Proposed by Amrit Awasthi-India

J.1105 Solve in $\mathbb{C}$:

i) $(x+y)^2 = 5 + xy$

ii) $9x^3 - 5x + 2xy^2 = 26y^3 + 5y - 2x^2y$

Proposed by Carlos Paiva-Brazil

J.1106 If $a,b,c \in [0,\infty), a+b+c = 3$ then find:

$$\Omega = \max(2(a^3+b^3+c^3) + 15(ab+bc+ca) + 6abc)$$

Proposed by Dan Radu Seclăman – Romania

J.1107 Solve for real numbers:

$$\begin{cases} a,b,c \in (0,\infty) \\ \dfrac{a}{a+1} + \dfrac{b}{b+1} + \dfrac{c}{c+1} \le 1 \\ \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \le 6 \end{cases}$$

Proposed by Dan Radu Seclăman – Romania

J.1108 If $a,b,c \in [0,\infty), a+b+c = 3$ then:

$$(1-a)(1-b)(1-c) + 2 \ge 2abc$$

Proposed by Dan Radu Seclăman – Romania

J.1109 Solve the equation:

$$\frac{(\sqrt[3]{x}+2)(\sqrt[3]{x}-1)(\sqrt[3]{x}-3)(\sqrt[3]{x}-6)}{(\sqrt[3]{x}+4)(\sqrt[3]{x}+1)(\sqrt[3]{x}-5)(\sqrt[3]{x}-8)} = 3$$

Proposed by Asmat Qatea-Afghanistan

J.1110 Find $z_1, z_2, z_3 \in \mathbb{C}, Rez_1, Rez_2, Rez_3 < 0$ such that exists $a,b,c > 0$

$$|a-z_1|^2 + |b-z_2|^2 + |c-z_3|^2 \le 2(a|z_1| + b|z_2| + c|z_3|)$$

Proposed by Dan Radu Seclăman – Romania

J.1111 Solve

$$6\sqrt[3]{46-\sqrt{x+5}} + 6\sqrt{8-\sqrt[4]{x+381}} - 123 + 2\sqrt{x+5} + 3\sqrt[4]{x+381} = 0$$

Proposed by Lazaros Zachariadis-Greece

J.1112 Solve in $\mathbb{R}$:

i) $\sqrt[3]{4x-4} - \left(\frac{x^2+y^2+4}{3}\right) = \frac{2(xy+y)}{3}$ ii) $x^2y - xy^2 + 1 = 0$

Proposed by Carlos Paiva-Brazil

J.1113 Let ABC be a triangle with the sides a,b,c and the area F_1, XYZ another triangle with the sides x,y,z and the area F_2 and $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$,
$f(x) = (a^2+x^2)(b^2+x^2)(c^2+x^2)$, then:

$$f(x) + f(y) + f(z) \ge 144\sqrt{3} \cdot F_1 \cdot F_2^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1114 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and a,b,c are the lengths sides of ABC triangle with the area F, then:

$$\left(\left(\frac{x+y}{z}\right)^2a^8+1\right)\left(\left(\frac{y+z}{x}\right)^2b^8+1\right)\left(\left(\frac{z+x}{y}\right)^2c^8+1\right)\ge 768F^4$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1115 Let $A_1B_1C_1, A_2B_2C_2$ be triangles of area F_1 respectively F_2, then:

$$(a_1^4(-a_2^2+b_2^2+c_2^2)^2+1)(b_1^4(a_2^2-b_2^2+c_2^2)^2+1)(c_1^4(a_2^2+b_2^2-c_2^2)^2+1)\ge 192F_1^2F_2^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1116 In any ABC triangle having the area F, the following inequality holds:

$$\frac{a}{(b+c)^3h_a^2}+\frac{b}{(c+a)^3h_b^2}+\frac{c}{(a+b)^3h_c^2}\ge\frac{3}{32F^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

J.1117 If $t,u,v,x,y,z>0$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{(t+u)(x+y)}{vz}ab+\frac{(u+v)(y+z)}{tx}bc+\frac{(v+t)(z+x)}{uy}ca\ge 16\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1118 If $x,y,z,u,v,w>0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{(x+y)(u+v)}{zw}a^2+\frac{(y+z)(v+w)}{xu}b^2+\frac{(z+x)(w+u)}{yv}\ge 16\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1119 If $x,y,z>0$ and ABC is a triangle with the area F, then:

$$\frac{xh_a+yh_b}{z}c^2+\frac{yh_b+zh_c}{x}a^2+\frac{zh_c+xh_a}{y}b^2\ge 8\sqrt[4]{27}\cdot F\cdot\sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1120 If $m\in\mathbb{R}_+=[0,\infty)$; $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{y+z}{x}\left(b+c-\sqrt{bc}\right)^{m+1}+\frac{z+x}{y}\left(c+a-\sqrt{ca}\right)^{m+1}+\frac{x+y}{z}\left(a+b-\sqrt{ab}\right)^{m+1}\ge$$

$$\ge 2^{m+2}\left(\sqrt[4]{27}\right)^{m+1}\cdot\left(\sqrt{F}\right)^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1121 If $x,y,z>0$ then in any ABC triangle with the area F the following inequality holds:

$$(m_b+m_c)(h_b+h_c)a^4+(m_c+m_a)(h_c+h_a)b^4+(m_a+m_b)(h_a+h_b)c^4\ge 16\sqrt{3}F^3$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1122 If $x,y,z>0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{xa}{y+z}+\frac{yb}{z+x}+\frac{zc}{x+y}\geq\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1123 If $a,b,c,d,m\in\mathbb{R}_+^*=(0,\infty)$ and $a\cdot b\cdot c\cdot d=1$, then:

$$\frac{a^m\cdot b^{3m+1}}{a^{4m+1}+b+c+d}+\frac{b^m\cdot 3^{m+1}}{b^{4m+1}+c+d+a}+\frac{c^m\cdot d^{3m+1}}{c^{4m+1}+d+a+b}+\frac{d^m\cdot a^{3m+1}}{d^{4m+1}+a+b+c}\geq 1,$$
$$\forall m>0$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1124 If $m\geq 0$ and $x,y,z,t,u,v>0$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{(t+u)(x+y)}{vz}a^{m+1}+\frac{(u+v)(y+z)}{tx}b^{m+1}+\frac{(v+t)(z+x)}{uy}c^{m+1}\geq$$
$$\geq 2^{m+3}\cdot 3^{\frac{3-m}{4}}\cdot F^{\frac{m+1}{2}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1125 Let be $x,y,z>0$, then in ABC triangle with the area F, the following inequality holds:

$$\frac{x+y}{z}(h_a+h_b)c^3+\frac{y+z}{x}(h_b+h_c)a^3+\frac{z+x}{y}(h_c+h_a)b^4\geq 32\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1126 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle the following inequality holds:

$$\frac{x}{\sqrt{yz}}\cdot\frac{a^2}{h_b^2}+\frac{y}{\sqrt{zx}}\cdot\frac{b^2}{h_c^2}+\frac{z}{\sqrt{xy}}\cdot\frac{c^2}{h_a^2}\geq 4$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1127 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{x}{\sqrt{yz}}\cdot a^2+\frac{y}{\sqrt{zx}}\cdot b^2+\frac{z}{\sqrt{xy}}\cdot c^2\geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1128 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle with the area F, the following inequality holds:

$$\frac{y+z}{x\cdot h_bh_c}+\frac{z+x}{y\cdot h_ch_a}+\frac{x+y}{z\cdot h_ah_b}\geq\frac{2\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1129 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{x}{\sqrt{yz}}\cdot a^2b^2+\frac{y}{\sqrt{zx}}\cdot b^2c^2+\frac{z}{\sqrt{xy}}\cdot c^2a^2\geq 16F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1130 If $t,u,v\in\mathbb{R}_+=[0,\infty),x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then:

$$\frac{vx+y+(u+1)z}{x+ty}+\frac{(v+1)x+ty+z}{y+uz}+\frac{x+(t+1)y+uz}{z+vx}\geq 6$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1131 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in ABC triangle with the area F the following inequality holds:

$$\frac{x\cdot m_a^2}{y+z}+\frac{y\cdot m_b^2}{z+x}+\frac{z\cdot m_c^2}{x+y}\geq 2\left(\frac{F}{R}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1132 If $x,y,z>0$ then in ABC triangle with the area F the following inequality holds:

$$\frac{y+z}{x\cdot h_a}+\frac{z+x}{y\cdot h_b}+\frac{x+y}{z\cdot h_c}\geq\frac{2\cdot\sqrt[4]{27}}{\sqrt{F}}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1133 Let ABC be a triangle with the area F and the points $M\in(BC),N\in(CA),P\in(AB)$. If the cevians AM,BN,CP are concurrent, then: $\frac{MB}{MC\cdot h_a}+\frac{NC}{NA\cdot h_b}+\frac{PA}{PB\cdot h_c}\geq\sqrt{\frac{3\sqrt{3}}{F}}$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1134 If $a,b,c,d\in\mathbb{R}_+^*=(0,\infty)$, then:

$$(a^4+d^2)(b^4+d^2)(c^4+d^2)\geq\frac{3}{4}(ab+bc+ca)^2d^4$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru - Romania

J.1135 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in ABC triangle with the area F the following inequality holds:

$$\frac{x\cdot m_a}{y+z}+\frac{y\cdot m_b}{z+x}+\frac{z\cdot m_c}{x+y}\geq\sqrt{3}\cdot\frac{F}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.1136 Let ABC be a triangle:

$$x^2a^2+y^2b^2+z^2c^2\geq\frac{4\sqrt{3}}{3}(xy+yz+zx)F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.1137 If $m,n,x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and $\sqrt{x}+\sqrt{y}+\sqrt{z}=a$, then:

$$\frac{x^2}{m\sqrt{y}+n\sqrt{z}}+\frac{y^2}{m\sqrt{z}+n\sqrt{x}}+\frac{z^2}{m\sqrt{x}+n\sqrt{y}}\ge\frac{a^3}{9(m+n)}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1138 If $m,n,x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then:

$$\sum_{cyc}\frac{(mx+ny)(mx+nz)}{yz}\ge 12\cdot m\cdot n$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1139 If $a,b,c>0$, then:

$$\sqrt{(a+b)^2+(b+c)^2+(c+a)^2}+\frac{3abc}{ab+bc+ca}\ge 3\cdot\sqrt[3]{abc}$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

J.1140 If ABC is a triangle with the area F and the semiperimeter s, then:

$$\frac{4s}{3}+\frac{3a^2b^2c^2}{ab+bc+ca}\ge 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

J.1141 If $a,b\ge 0$ such that $a+b=2$ then: $(2+a^4)(2+b^4)\ge(2+a^3)(2+b^3)$

Proposed by Marin Chirciu – Romania

J.1142 If $a_1,a_2,\dots,a_n>0$ such that $a_1+a_2+\dots+a_n\le n$ then:

$$\frac{1}{a_1^3}+\frac{1}{a_2^3}+\dots+\frac{1}{a_n^3}\ge n$$

Proposed by Marin Chirciu – Romania

J.1143 In ΔABC the following relationship holds:

$$\frac{16r^2}{3R}(4R+r)^2\le\sum a^3\cot\frac{A}{2}\le\frac{4R}{3}(4R+r)^2$$

Proposed by Marin Chirciu – Romania

J.1144 If $a,b,x>0$, then:

$$\frac{a^2b^2}{(a+b)^4}+\frac{b^2c^2}{(b+c)^4}+\frac{c^2a^2}{(c+a)^4}+\frac{(a+b)(b+c)(c+a)}{32abc}\ge\frac{7}{16}$$

Proposed by Marin Chirciu – Romania

J.1145 If $a,b\ge 0$ such that $a+b=2$ then:

$(2+a^5)(2+b^5)\ge(2+a^4)(2+b^4)\ge(2+a^3)(2+b^3)\ge(2+a^2)(2+b^2)\ge$
$(2+a)(2+b)$

Proposed by Marin Chirciu – Romania

J.1146 If $x,y,z>0$ such that $\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=3$ and $\lambda\ge 0,\mu\ge 0$ then:

$$\lambda\sum x^3+\mu\left(\sum x^2+\sum\frac{1}{x^2}\right)\ge 3(\lambda+2\mu)$$

Proposed by Marin Chirciu – Romania

J.1147 In ΔABC the following relationship holds:

$$\frac{3r}{4R^2}\le\sum\frac{h_a}{bc}\sin^2\frac{A}{2}\le\frac{1}{4R}\left(1-\frac{r}{2R}\right)$$

Proposed by Marin Chirciu – Romania

J.1148 In ΔABC the following relationship holds:

$$\frac{18r^2}{R^2} \le \sum m_a m_b \left(\frac{1}{b^2} + \frac{1}{c^2}\right) \le \frac{9R^2}{8r^2}$$

Proposed by Marin Chirciu – Romania

J.1149 If $a, b, c > 0$ and $n \in \mathbb{N}^*$ then:

$$\sum \frac{a^{2n}}{b^{2n}} \sum \frac{a^{2n-1}}{b^{2n-1}} \ge \left(\sum \frac{a^n}{b^n}\right)^2$$

Proposed by Marin Chirciu – Romania

J.1150 If $a, b, c > 0$ such that $a + b + c = 3$ and $\lambda \ge 0$ then:

$$\frac{a^3 + \lambda b^3}{ab} + \frac{b^3 + \lambda c^3}{bc} + \frac{c^3 + \lambda a^3}{ca} \ge 3(\lambda + 1)$$

Proposed by Marin Chirciu – Romania

J.1151 $a, b \in (1, \infty), a + b = 10$. Solve for real numbers:

$$\log_a(10^x - b) = \lg\left(b + (a^x + b)^{\lg a}\right)$$

Proposed by Marin Chirciu – Romania

J.1152 In ΔABC the following relationship holds:

$$rp \le \sum (p - a)^2 \tan \frac{A}{2} \le \frac{R}{2} p$$

Proposed by Marin Chirciu – Romania

J.1153 If $x, y, z, t > 0$ then: $\frac{x}{y} + \frac{y}{z} + \frac{z}{t} + \frac{t}{x} \ge 2 + \frac{x}{t+x} + \frac{y}{x+y} + \frac{z}{y+z} + \frac{t}{z+t}$

Proposed by Marin Chirciu – Romania

J.1154 In ΔABC the following relationship holds:

$$\frac{1}{4Rr^2} \le \sum \frac{1}{a^3} \cot \frac{A}{2} \le \frac{R^3}{64r^6}$$

Proposed by Marin Chirciu – Romania

J.1155 If $a, b, c, d > 0$ such that $a + b + c + d = 4$ then:

$$\frac{a^2}{a + 2b^2} + \frac{b^2}{b + 2c^2} + \frac{c^2}{c + 2d^2} + \frac{d^2}{d + 2a^2} \ge \frac{4}{3}$$

Proposed by Marin Chirciu – Romania

J.1156 If $a, b > 0$ fixed then solve for real numbers:

$$\left(\frac{a}{b}\right)^{\log_x a^3 b} = \frac{x}{a^2 b^2}$$

Proposed by Marin Chirciu – Romania

J.1157 Let $a > 0$, fixed. Solve for real numbers:

$$2x\sqrt{2x-1} = (ax-a+1)^2(ax-a+2) + (a-2)(x-1)$$

Proposed by Marin Chirciu – Romania

J.1158 Find $\overline{abc}$ such that $\overline{abcb} = \overline{ca}^2$.

Proposed by Ștefan Marica-Romania

J.1159 Find $\overline{ab}$ such that $\overline{ab}^2 = \overline{(a+1)(b-1)}^2 - \overline{(a-1)(b+1)}^2$.

Proposed by Ștefan Marica-Romania

J.1160 Find $\overline{ab}, \overline{ab_1}$ and $\overline{ab_2}$ such that $\overline{ab}^2 - a = \overline{ab_1}^2 \cdot \overline{ab_2}^2$, where $\overline{ab_1}$ and $\overline{ab_2}$ are prime numbers.

Proposed by Ștefan Marica-Romania

J.1161 In $\Delta ABC, AH$ –altitude, $5AH = 12BH, 9AH = 12CH$ and $2P_{\Delta ABC} = A_{\Delta ABC}$.
Find area of ΔABC .

Proposed by Ștefan Marica-Romania

J.1162 Find $\overline{abc}$ such that $\overline{ab}^2 - b^2 = c!$, where $c! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot c$.

Proposed by Ștefan Marica-Romania

J.1163 For n –natural number solve the equation

$$(1^2 + 2^2 + \cdots + n^2)^2 + \overline{nn}^2 = \frac{n(n+1)(2n+1)}{3} \cdot \overline{nn}^2.$$

Proposed by Ștefan Marica-Romania

J.1164 If $a, b, c, x, y, z > 0$ then:

$$\sqrt{b\{x\} + c\left\{\frac{1}{y}\right\}} + \sqrt{c\{y\} + a\left\{\frac{1}{z}\right\}} + \sqrt{a\{z\} + b\left\{\frac{1}{x}\right\}} < 3\sqrt{\frac{a+b+c}{2}}$$

Proposed by Ionuț Florin Voinea – Romania

J.1165 Solve the system of equations:

$$\begin{cases} x^4 + 2x + 2 = x^2y^2 \\ \left(x + \sqrt{5-y^2}\right)^2 + x^2\sqrt{2y-1} = -y^2 - 2y + 5 \end{cases}$$

Proposed by Minh Nhat Nguyen – Vietnam

J.1166 Solve for natural numbers:

$$\begin{cases} xy + zy + xz = 11 \\ \dfrac{x+y}{z} + \dfrac{x+z}{y} + \dfrac{y+z}{x} = 8 \\ \dfrac{x}{yz} + \dfrac{y}{xz} + \dfrac{z}{xy} = \dfrac{2x+y+z}{x+y} \end{cases}$$

Proposed by Mokhtar Khassani-Algerie

J.1167 If $x, y \geq 0$ then:

$$x^3y^3(x+y)^3 \leq (x^2+y^2)(x^3+y^3)(x^4+y^4)$$

Proposed by Daniel Sitaru – Romania

J.1168 If $x, y, z > 0, xyz = 1$ then:

$$\sum_{cyc} \frac{z(x+y)^3}{(\sqrt{x}+\sqrt{y})(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[6]{x}+\sqrt[6]{y})} \geq 3$$

Proposed by Daniel Sitaru – Romania

J.1169 If $a, b, c, d > 0, abcd = 1$ then:

$$\frac{a^2b^2}{a^3b^3+cd} + \frac{c^2d^2}{c^3d^3+ab} \geq \frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

Proposed by Daniel Sitaru – Romania

J.1170 In ΔABC the following relationship holds:

$$s^5 \geq (s-a)^5 + (s-b)^5 + (s-c)^5 + 2160\sqrt{3}r^5$$

Proposed by Daniel Sitaru – Romania

J.1171 If $a, b > 0$ then:

$$\frac{2(a^2+b^2)+3(a+b)^2+20ab}{4} + \frac{28a^2b^2}{(a+b)^2} \leq \left(\sqrt{\frac{a^2+b^2}{2}} + \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{2} + \frac{2ab}{a+b}\right)^2$$

Proposed by Daniel Sitaru – Romania

J.1172 If $x, y, z > 0, x^3 \cdot y + y^3 \cdot z + z^3 \cdot x = \sqrt[3]{3}$ then:

$$(x^3+y^3+z^3)^4 \geq (x^4+y^4+z^4)^3 + 6$$

Proposed by Daniel Sitaru – Romania

J.1173 Solve for real numbers:

$$\sqrt[7]{x+3} + \sqrt[7]{6-x} = \sqrt[7]{9}$$

Proposed by Daniel Sitaru – Romania

J.1174 If in $\Delta ABC, a^2+b^2 = 2c^2$ then:

$$2am_a + m_c^2 \cdot \sqrt{\frac{ab}{m_bm_c}} \leq \frac{\sqrt{3}}{2}(a^2+b^2+c^2)$$

Proposed by Daniel Sitaru – Romania

J.1175 In ΔABC the following relationship holds:

$$\frac{m_a^2}{h_a^2} \geq 1 + \frac{\left(4 - \frac{2r}{R}\right)\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 (b^2 - c^2)^2}{(a + b + c)^4}$$

Proposed by Bogdan Fuștei – Romania

J.1176 In ΔABC the following relationship holds:

$$a\sqrt{3}(m_a - h_a) \geq |(m_b - m_c)(b - c)|$$

Proposed by Bogdan Fuștei – Romania

J.1177 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{|b - c|}{m_b + m_c} \geq \frac{2}{3s} \sum_{cyc} |m_a - m_b|$$

Proposed by Bogdan Fuștei – Romania

J.1178 In ΔABC the following relationship holds:

$$(1 - \cos A)(1 - \cos B)(1 - \cos C)\left(\frac{R}{r}\right)^3 = \frac{r_a r_b r_c}{h_a h_b h_c}$$

Proposed by Bogdan Fuștei – Romania

J.1179 In ΔABC the following relationship holds:

$$(m_a + m_b + m_c)^2 \geq 3\sqrt{3}S\left(\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}\right)$$

Proposed by Bogdan Fuștei – Romania

J.1180 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{m_a}{w_a} + \sqrt{\frac{m_a}{r_a}} + \sqrt{\frac{h_a}{h_b}} + \sqrt{\frac{h_b}{h_c}}\right) \leq \sqrt{\frac{2R}{r}} \sum_{cyc} \frac{b + c}{a}$$

Proposed by Bogdan Fuștei – Romania

J.1181 In ΔABC the following relationship holds:

$$\frac{R}{2r} \geq \sqrt{1 + \frac{\left(4 - \frac{2r}{R}\right)\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)^2 (b^2 - c^2)^2}{(a + b + c)^4}}$$

Proposed by Bogdan Fuștei – Romania

J.1182 In ΔABC the following relationship holds:

$$8(1 - \cos A)(1 - \cos B)(1 - \cos C) \leq \frac{r_a r_b r_c}{m_a m_b m_c}$$

Proposed by Bogdan Fuștei – Romania

J.1183 In acute ΔABC the following relationship holds:

$$\frac{\sin^5 A}{\sin^3 B}+\frac{\sin^5 B}{\sin^3 C}+\frac{\sin^5 C}{\sin^3 A}\geq\left(1+\frac{r}{R}\right)^2$$

Proposed by Marian Ursărescu – Romania

J.1184 In ΔABC the following relationship holds:

$$\frac{w_b+w_c}{h_a^2}+\frac{w_c+w_a}{h_b^2}+\frac{w_a+w_a}{h_c^2}\geq\frac{2}{r}$$

Proposed by Marian Ursărescu – Romania

J.1185 In ΔABC the following relationship holds:

$$\frac{\cos^5 A}{\cos^3 B}+\frac{\cos^5 B}{\cos^3 C}+\frac{\cos^5 C}{\cos^3 A}\geq 1-\left(\frac{r}{R}\right)^2$$

Proposed by Marian Ursărescu – Romania

J.1186 If $x,y,z>0, xyz=1, n\in(0,2]$ then:

$$\sum_{cyc}\frac{(xy+z)(xz+y)}{(x+yz)\big(1+n(xy+z)(xz+y)\big)}\leq\frac{2}{n}$$

Proposed by Florică Anastase – Romania

J.1187 If $a,b,c,m,n>0$ then:

$$\sum_{cyc}\frac{8a}{ma^2+nbc}\leq(m+n)\left(\frac{1}{m^2}+\frac{1}{n^2}\right)\left(\sum_{cyc}\frac{a}{bc}\right)$$

Proposed by Florică Anastase – Romania

J.1188 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_b+r_c}{a}\cdot\sum_{cyc}\left(\frac{a^2\sin^3 A}{b}+\frac{a^2\cos^3 A}{c}\right)\geq\frac{27r}{2}$$

Proposed by Florică Anastase – Romania

J.1189 In $\Delta ABC, I$ – incenter, R_a,R_b,R_c – circumradii of $\Delta IAB,\Delta IBC,\Delta IBC$.
Prove that:

$$\frac{a^2\cdot R_b^3R_c^3}{R_a}+\frac{b^2\cdot R_c^3R_a^3}{R_b}+\frac{c^2\cdot R_a^3R_b^3}{R_c}\geq\frac{16R^3F}{3}$$

Proposed by Florică Anastase – Romania

J.1190 In any scalene ΔABC holds:

$$\frac{(2s+a)bc}{(a-b)(a-c)}+\frac{(2s+b)ca}{(b-a)(b-c)}+\frac{(2s+c)ab}{(c-a)(c-b)}>6\sqrt{3}r$$

Proposed by Daniel Sitaru – Romania

J.1191 In ΔABC let R_A – let the radii of circle tangent simultaneous to AB, AC and external tangent to circumcircle of ΔABC. R_B, R_C – are defined similar.

Prove that: $R_AR_B + R_BR_C + R_CR_A \ge 48r^2$

Proposed by Daniel Sitaru – Romania

J.1192 In ΔABC holds:

$$\sqrt{2}a\cos\frac{B}{2}\cos\frac{C}{2} = s \Leftrightarrow 2m_a = a$$

Proposed by Daniel Sitaru – Romania

J.1193 In ΔABC the following relationship holds:

$$\left(2\cos\frac{A}{2}\cos\frac{C}{2} + 3\sin\frac{B}{2}\right)^2 = 24\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} \Leftrightarrow 2b = a + c$$

Proposed by Daniel Sitaru – Romania

J.1194 If $a, b, c > 0$ then:

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1 \Rightarrow a + b + c \ge 6$$

Proposed by Daniel Sitaru – Romania

J.1195 In ΔABC the following relationship holds:

$$\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc} \le \frac{3R}{4r}$$

Proposed by Marian Ursărescu – Romania

J.1196 In ΔABC the following relationship holds:

$$\frac{\cot\frac{A}{2}}{a^2} + \frac{\cot\frac{B}{2}}{b^2} + \frac{\cot\frac{C}{2}}{c^2} \ge \frac{9}{4F}$$

Proposed by Marian Ursărescu – Romania

J.1197 If $x, y, z > 0$ then prove:

$$\sum_{cyc} x(y^2 + yz + z^2) \ge \sqrt{3(xy + yz + zx)^3}$$

Proposed by Bogdan Fuștei – Romania

J.1198 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a^2 - 2m_bm_c}{\sqrt{5(b^2 + c^2) + 2a^2}} \ge 0$$

Proposed by Bogdan Fuștei – Romania

J.1199 In ΔABC, n_a – Nagel's cevian, the following relationship holds:

$$3\sqrt{3}\sum_{cyc} a\sin\frac{A}{2} \ge m_a + m_b + m_c + 2(w_a + w_b + w_c)$$

Proposed by Bogdan Fuștei – Romania

J.1200 If $a_i, b_i > 0, i \in \overline{1,n}$ then:

$$\left(\sum_{i=1}^{n}(a_i + b_i)\right)\left(\sum_{i=1}^{n}\frac{a_i b_i}{a_i + b_i}\right) \ge \left(\sum_{i=1}^{n}\sqrt{a_i b_i}\right)^2$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.1201 If $a, b, c > 0$ then prove:

$$\frac{ab}{\sqrt{a+b}} + \frac{bc}{\sqrt{b+c}} + \frac{ca}{\sqrt{c+a}} > 2\sqrt{abc}$$

Proposed by Olimjon Jalilov-Uzbekistan

J.1202 Let $f: \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \ge m, \forall x \ne y$. Find all functions f if $m \in \mathbb{N}$ and $m \in \mathbb{R}^*$.

Proposed by Surjeet Singhania-India

J.1203 If $f: \mathbb{R}^* \to \mathbb{R}$ such that $x(2x+1)f(x) + f\left(\frac{1}{x}\right) = x + 1, \forall x \in \mathbb{R}^*$. Find:

$$\Omega = \sum_{k=1}^{2050} f(k)$$

Proposed by Mohammad Hamed Nasery-Afghanistan

J.1204 If $a, b > 0$ then:

$$\sqrt{\frac{a^2 + b^2}{2}} + \frac{4ab}{a+b} \ge 3\sqrt{ab}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.1205 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x^2 y) + f\left(y^2 f(y)\right) = f\left(xf(x)\right), \forall x, y \in \mathbb{R}$.

Proposed by Mokhtar Khassani-Algerie

J.1206 If $k \in \mathbb{N} - \{0\}, k = \overline{a_m a_{m-1} \dots a_1 a_0}$ denote $p(k) = a_m . a_{m-1} \cdot \dots \cdot a_1 \cdot a_0$.
Find $n \in \mathbb{N} - \{0\}$ such that: $p\left(p(n)\right) = n^2 - 29n + 8$.

Proposed by Ionuț Florin Voinea-Romania

J.1207 In ΔABC the following relationship holds:

$$\frac{2F}{r} < \sum_{cyc}\left(m_a + \frac{a^2}{4m_a}\right) \le 6R$$

Proposed by Rajeev Rastogi-India

J.1208 If $a, b, c > 0$ then:

$$\frac{(a+b)^4}{13abc + 3c^3} + \frac{(b+c)^4}{13abc + 3a^3} + \frac{(c+a)^4}{13abc + 3b^3} \ge 3\sqrt[3]{abc}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

J.1209 Solve for real numbers:

$$3+\sin(2x)=4\sin\left(x+\frac{\pi}{4}\right)$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

J.1210 In ΔABC the following relationship holds:

$$\frac{m_a h_a}{s-a}+\frac{m_b h_b}{s-b}+\frac{m_c s_c}{s-c}\ge\frac{6F}{R}$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1211 In ΔABC the following relationship holds:

$$2\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\ge\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}+3$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1212 If $x,y,z>0$ then:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge 9\sqrt{\frac{x^2+y^2+z^2}{xy+yz+zx}}$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1213 In ΔABC the following relationship holds:

$$\cos(A-B)+\cos(B-C)+\cos(C-A)\le\frac{3}{2}+\frac{3r}{R}$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1214 In ΔABC the following relationship holds:

$$\frac{1}{4}(a+b+c)^2\sum_{cyc}(a-b)^2+16F^2\ge abc(a+b+c)$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1215 In ΔABC the following relationship holds:

$$4+\sum\frac{a^2}{r_b r_c}\ge 8\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1216 In ΔABC the following relationship holds:

$$\sum w_a\left(\frac{b}{c}+\frac{c}{b}\right)\ge 2(m_a+m_b+m_c).$$

Proposed by Adil Abdullayev-Azerbaijan

J.1217 In ΔABC the following relationship holds:

$$3\left(\sum r_a^2\right)\left(\sum \frac{1}{r_a^2}\right) \le \frac{4R^3}{r^3} - 5.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1218 Solve:

$$\frac{2\sqrt{x}}{\sqrt{x}+2} + \frac{16\sqrt[3]{x}}{\sqrt[3]{x}+16} = \frac{\left(\sqrt[3]{x}+2\right)\left(\sqrt{x}+16\right)}{\sqrt{x}+\sqrt[3]{x}+18}$$

Proposed by Jalil Hajimir-Canada

J.1219 If $x, y > 0, m \ge 0$ and $x \cdot \min\{h_a, h_b, h_c\} > yr$ then in ABC triangle the following inequality holds:

$$\frac{a}{(xh_a - yr)^m} + \frac{b}{(xh_b - yr)^m} + \frac{c}{(xh_c - yr)^m} \ge \frac{6\sqrt{3}}{r^{m-1}(3x-y)^m}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

J.1220 If ABC is a triangle with the area F and M an interior point in the triangle and $x = MA, y = MB, z = MC$, then:

$$(x^2 + y^2)h_a h_b + (y^2 + z^2)h_b h_c + (z^2 + x^2)h_c h_a \ge 8F$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

J.1221 If $x, y, z > 0$ and $A_1B_1C_1; A_2B_2C_2$ are two triangles with the area F respectively F_2, then:

$$\frac{x+y}{z}a_1b_2 + \frac{y+z}{x}b_1c_2 + \frac{z+x}{y}c_1a_2 \ge 8\sqrt{3}\sqrt{F_1F_2}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.664 $f[a, a+1] \to \mathbb{R}, f$ – continuous, $a \ge 0$ – fixed, $n \in \mathbb{N}, n \ge 2$. Prove that exists

$c_1, c_2, \dots, c_{n-1} \in (a, a+1)$ – different in pairs such that:

$$\left|\int_a^{a+1} f(x)dx - \frac{f(c_1)+f(c_2)+\dots+f(c_{n-1})}{n}\right| \le \int_a^{a+\frac{1}{2}} |f(x)|\,dx$$

Proposed by Dan Radu Seclăman – Romania

S.665

$$x_1 = 4, x_{n+1} = \frac{(1-2n^2)x_n - 4n^2}{(2+x_n)n^2+1}, n \ge 1$$

Find:

$$\Omega = \lim_{n\to\infty}\left((3+x_n)^{\sum_{k=1}^n \frac{k^2}{n}}\right)^{\sum_{k=1}^n\left(1-\frac{k}{n+k}\right)}$$

Proposed by Ruxandra Daniela Tonilă-Romania

S.666 Solve for integers:

$$\frac{2x^2+x}{x^2+x+1} + \frac{18x^2+16x+30}{x^2+x+2} + \frac{84x^2+81x+240}{x^2+x+3} \dots +$$
$$+\frac{a_n\cdot x^2 + b_n\cdot x + c_n}{x^2+x+n} = \frac{6n^5+15n^4+10n^3-n}{30}, n \in \mathbb{N}^*$$

and find:

$$\Omega = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)^{\frac{c_n}{n^2}}$$

Proposed by Costel Florea – Romania

S.667 $(x_n)_{n\in\mathbb{N}} > 0, x_n(x_{n-1}+x_{n+1}) < 2x_{n-1}\cdot x_{n+1}, (\forall)n \ge 1$.Prove that: $x_0 \ge x_1$

Proposed by Dan Radu Seclăman – Romania

S.668 $x^{3n+4} + x^2 + 1 = P(x)\cdot Q(x), n \in \mathbb{N}^*$, with degree $(P) <$ degree (Q).

$A(n)$ = number of terms to $Q(x)$; $B(n) = b_1 - b_2 + b_3 - b_4 + \dots + b_{n-1} - b_n + a_0$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{P(1) - Q(1)\cdot\sum_{k=1}^n B(k)}{\sum_{k=1}^n A(k)}\right)^n$$

Proposed by Costel Florea – Romania

S.669 Prove inequality for $a, b \in \left(0, \frac{\pi}{2}\right)$

$$e^{\sin\left(\frac{b-a}{4}\right)\cos\left(\frac{b+3a}{4}\right)}\cos a + e^{\sin\left(\frac{a-b}{4}\right)\cos\left(\frac{a+3b}{4}\right)}\cos b \ge \frac{2}{e^2}\cos\frac{a+b}{2}$$

Proposed by Olimjon Jalilov – Uzbekistan

S.670 $x_1 = 1, x_2 = \frac{3}{2}, x_{n+2} = \frac{(n+2)!\cdot(5x_n - 2x_{n+1}) + 5n + 13}{3(n+2)!}$. Find:

$$\Omega = \lim_{n\to\infty} 2^n x_n \left(\left(\tan^{-1}\frac{1}{n^2} + \tan^{-1}\frac{1}{(n+1)^2}\right)\sum_{k=2}^{n} k(k-1)\binom{n}{k}\right)^{-1}$$

Proposed by Ruxandra Daniela Tonilă-Romania

S.671 $2x^{3n+2} - x^{3n-1} + x + 1 = P(x)\cdot Q(x), n \in \mathbb{N}^*, degree(P) < degree(Q)$

$Q(x) = a_1x^{b_1} + a_2x^{b_2} + \cdots + a_nx^{b_n}$, with $b_1 > b_2 > \cdots > b_n$

$$A(n) = a_1 - a_2 + a_3 - a_4 + \cdots + a_{n-1} - a_n + a_0$$

$$B(n) = b_1 - b_2 + b_3 - b_4 + \cdots + b_{n-1} - b_n + a_0$$

$C(n)$ = number of terms to $Q(x)$.Solve for natural numbers:

$$\left|\frac{2\big(C(n) - B(n)\big) - 3}{15\cdot P(1)(A(n) - 16)}\cdot\prod_{k=3}^{n}\frac{8k^3 - 12k^2 - 26k + 15}{8k^3 + 12k^2 - 26k - 15}\right| = Q(1)$$

Proposed by Costel Florea – Romania

S.672 Find without softs:

$$\Omega = \int_0^1 \frac{\sqrt{x}}{x^3 + 4x\sqrt{x} + 8}dx$$

Proposed by Mustapha Issah-Ghana

S.673 If ABC is a triangle with the area F and the points $M \in (BC), N \in (CA), P \in (AB)$ then: $(AM + BN)c^3 + (BN + CP)a^3 + (CP + AM)b^3 \geq 16\sqrt{3}F^2$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.674 If $x, y, z, t > 0$, then in ABC triangle with the area F the following inequality holds:

$$\frac{t^4 + x^2}{y + z}\cdot a^4 + \frac{t^4 + y^2}{z + x}\cdot b^4 + \frac{t^4 + z^2}{x + y}\cdot c^4 \geq 16t^2F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.675 If $n \in \mathbb{N}^* - \{1\}$ and $x_k \in \mathbb{R}^* = (-\infty, 0)\cup(0,\infty)$ and $X_n = \sum_{k=1}^{n} x_k^2$, then:

$$\sum_{k=1}^{n}\left(x_k^2 + \frac{1}{x_k^2}\right)^{m+1} \geq \frac{1}{n^m}\left(\frac{X_n^2 + n^2}{n}\right)^{m+1}, \forall m \geq 0$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.676 If $m, n \geq 0, t, u, v, x, y, z > 0$ then in any ABC triangle with the area F the following inequality holds:

$$\sum_{cyc}\left(\frac{t + u}{v}(ab)^{m+1} + \left(\frac{xc^2}{y + z}\right)^{m+1}\right)^{n+1} \geq$$

$$\geq 2^{mn+m+n+1}\left(\sqrt{3}\right)^{1-mn-m-n}\cdot\left(2^{m+2}+1\right)^{n+1}F^{(m+1)(n+1)}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.677 If ABC is a non right triangle, then:

$$\left(\frac{\sin A}{\cos^2 B}+\frac{\sin B}{\cos^2 C}+\frac{\sin C}{\cos^2 A}\right)\left(\frac{1}{(\sin A+\sin B)^2}+\frac{1}{(\sin B+\sin C)^2}+\frac{1}{(\sin C+\sin A)^2}\right)\geq\frac{27}{8}\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.678 Let $n\in\mathbb{N}, n\geq 3$ and $x_k\in\mathbb{R}_+^*=(0,\infty), \forall k=\overline{1,n}$, then:

$$n\cdot\sum_{k=1}^{n}\frac{1}{x_k x_{k+1}}\geq 4\left(\sum_{k=1}^{n}\frac{1}{x_k+x_{k+1}}\right)^2$$

where $x_{n+1}=x_1$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.679 Let $x,y>0$ and ABC triangle with the area F, then there are two triangles MNP and UVW with the sides m,n,p, respectively u,v,w such that: $mu+nv+pw\geq 4xy\sqrt{3}F$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.680 If $x,y,z\in(0,1)$, then in any ABC triangle the following inequality holds:

$$\frac{a}{(xy+xz)(1-x)h_a}+\frac{b}{(yz+yx)(1-y)h_b}+\frac{c}{(zx+zy)(1-z)h_c}\geq\frac{27\cdot\sqrt{3}}{4}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.681 If $m,n\geq 0, m+n>0$, then in any triangle with the area F the following inequality holds:

$$\sum_{cyc}\frac{a^2-ab+b^2}{b^2+ab+a^2}\left(mb^2+nc^2\right)^2\geq\frac{16(m+n)^2}{3}\cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.682 If $m,n\in\mathbb{R}_+^*=(0,\infty)$ and ABC is a triangle having the area F, then:

$$\sum_{cyc}\left(\sqrt[3]{\frac{a^{3m}+b^{3m}}{2}}+\frac{a^{2n+2}\cdot b^{2n+2}}{a^m+b^m}\right)\geq 2^{2m+3}\left(\sqrt{3}\right)^{1-m}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.683 If $x,y,z\in(0,1)$ and ABC is a triangle with the area F, then:

$$\frac{a^2}{(xy+xz)(1-x)}+\frac{b^2}{(yz+yx)(1-y)}+\frac{c^2}{(zx+zy)(1-z)}\geq\frac{27\sqrt{3}}{2}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.684 Let $n \in \mathbb{N}, n \geq 3, A_1A_2 \ldots A_n$ be a convex polygon with the sides $A_kA_{k+1} = a_k$, $A_{n+1} = A_1, k = \overline{1,n}$ and the area F and $x_k \in \left(0, \frac{\pi}{2}\right), \forall k = \overline{1,n}$, then:

$$\sum_{k=1}^{n} \frac{a_k^2}{\sin x_k \cdot \cos^2 x_k} \geq 6\sqrt{3} \cdot F \cdot \tan\frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.685 If $m, n > 0$ and $x, y, z \in (0,1)$, then:

$$\frac{1}{(my + nz)^3(1 - x^2)} + \frac{1}{(mz + nx)(1 - y^2)} + \frac{1}{(mx + ny)(1 - z^2)} \geq$$

$$\geq \frac{81\sqrt{3}}{2(m + n)^2(x + y + z)^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.686 Let $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC be a triangle with the area F, then:

$$\frac{x\sqrt{(a^4 + 1)(b^4 + 1)}}{y + z} + \frac{y\sqrt{(b^4 + 1)(c^4 + 1)}}{z + x} + \frac{z\sqrt{(c^4 + 1)(a^4 + 1)}}{x + y} \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.687 If $a, b, c, d \in [1, \infty)$ and m is their arithmetic means, then:

$$(a^a + b^a + c^a + d^a)(a^b + b^b + c^b + d^b)(a^c + b^c + c^c + d^c)(a^d + b^d + c^d + d^d) \geq 256m^{4m}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.688 Let be $m \in \mathbb{R}_+ = [0, \infty)$ and $a, b, c, d \in \mathbb{R}_+^* = (0, \infty)$, then:

$$(a^{2m+2} + d^2)(b^{2m+2} + d^2)(c^{2m+2} + d^2) \geq \frac{3^{2-m}}{4} d^4(ab + bc + ca)^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.689 If $x \in \left(0, \frac{\pi}{2}\right)$, then in ABC triangle with the area F the following inequality holds:

$$bc\left(\frac{\sin x}{x}\right)^3 + ca\left(\frac{\sin x}{x}\right)^2 + ab\left(\frac{\sin x}{x}\right) + 3\sqrt[3]{(abc)^2} \cdot \left(\frac{\tan x}{x}\right) \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.690 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ABC triangle with the area F, the following inequality holds:

$$\frac{y + z}{x} \cdot \frac{b^7 + c^7}{b^5 + c^5} + \frac{z + x}{y} \cdot \frac{c^7 + a^7}{c^5 + a^5} + \frac{x + y}{z} \cdot \frac{a^7 + b^7}{a^5 + b^5} \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.691 In any ABC triangle with the area F the following inequality holds:

$$a^2\tan\frac{B}{2}+b^2\tan\frac{C}{2}+c^2\tan\frac{A}{2}>2\sqrt{AB+BC+CA}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.692 If $x,y,z>0,t\geq 0$, then in any ABC triangle with the area F the following inequality holds: $(x\cdot m_a)^{t+1}+(y\cdot m_b)^{t+1}+(z\cdot m_c)^{t+1}\geq\frac{2^{t+1}}{3^t}(xy+yz+zx)^{\frac{t+1}{2}}\left(\frac{F}{R}\right)^{t+1}$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.693 If $n\in\mathbb{N},n\geq 2,x_k\in\mathbb{R}_+^*=(0,\infty),\forall k=\overline{1,n}$ and $X_n=\sum_{k=1}^n x_k$, then:

$$\sum_{k=1}^n\frac{x_k^{m+1}}{\left(x_k+(n+1)\cdot(X_n-x_k)\right)^m}\geq\frac{X_n}{n^2},\forall m\in\mathbb{R}_+=[0,\infty)$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.694 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ABC triangle the following inequality holds:

$$\frac{x}{\sqrt{yz}}\cdot\frac{a}{h_b}+\frac{y}{\sqrt{zx}}\cdot\frac{b}{h_c}+\frac{z}{\sqrt{zy}}\cdot\frac{c}{h_a}\geq 2$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.695 Let be $a,b,c,d>0$ such that $a\cdot b^3c^3d^3=1$, then:

$$\frac{ab^3c^7}{ab^{10}+d+c}+\frac{ac^3d^7}{ac^{10}+b+d}+\frac{ad^3b^7}{ad^{10}+c+b}\geq 1$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.696 Let be $x,y,z>0$ and $t\geq 0$, then in ABC triangle with the area F and the other usual notations the following inequality holds:

$$\frac{y+z+2t}{x+t}\cdot a^4+\frac{z+x+2t}{y+t}\cdot b^4+\frac{x+y+2t}{z+t}\cdot c^4\geq 32F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.697 If $a,b,c,d\in\mathbb{R}_+^*=(0,\infty)$ and $\frac{y+z}{x}a+\frac{z+x}{y}b+\frac{x+y}{z}c\geq d,\forall x,y,z\in\mathbb{R}_+^*$, then:

$$\frac{y+z}{x}\cdot a^{m+1}+\frac{z+x}{y}\cdot b^{m+1}+\frac{x+y}{z}\cdot c^{m+1}\geq\frac{d^{m+1}}{6^m},\forall m\in\mathbb{R}_+=[0,\infty)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.698 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle the following inequality holds:

$$\frac{x}{\sqrt{yz}}\cdot\frac{a}{h_a}+\frac{y}{\sqrt{zx}}\cdot\frac{b}{h_b}+\frac{z}{\sqrt{xy}}\cdot\frac{c}{h_c}\geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.699 If $x,y>0$, then in any ABC triangle with the semiperimeter s the following inequality holds:

$$\frac{x\cdot b+yc}{r_br_c}+\frac{xc+ya}{r_c\cdot r_a}+\frac{xa+yb}{r_a\cdot r_b}\ge\frac{6(x+y)}{s}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.700 Let be $n\in\mathbb{N},n\ge 3$ and $A_1A_2\dots A_n$ a convexe polygon with the area F having the length sides $A_kA_{k+1}=a_k,k=\overline{1,n},A_{n+1}=A_1$, then:

$$\sum_{k=1}^{n}\sqrt{(a_k^4+1)(a_{k+1}^4+1)}\ge 8\cdot F\cdot\tan\frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.701 In any ABC triangle the following inequality holds:

$$\left(a\left(\frac{b}{a}\right)^{\frac{m_b}{h_b}}+b\left(\frac{a}{b}\right)^{\frac{m_b}{h_b}}\right)\cdot\left(b\left(\frac{c}{b}\right)^{\frac{m_c}{h_c}}+c\left(\frac{b}{c}\right)^{\frac{m_c}{h_c}}\right)\cdot\left(c\left(\frac{a}{c}\right)^{\frac{m_a}{h_a}}+a\left(\frac{c}{a}\right)^{\frac{m_a}{h_a}}\right)\ge 8abc$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.702 If $t,u,v,x,y,z>0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{(t+u)(x+y)}{vz}\cdot a^2b^2+\frac{(u+v)(y+z)}{tx}\cdot b^2c^2+\frac{(v+t)(z+x)}{uy}\cdot c^2a^2\ge 64F^2$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.703 If $x,y,z>0$ and ABC is a triangle with the area F and the points $M\in(BC)$, $N\in(CA),P\in(AB)$, then:

$$\frac{x\cdot AM+y\cdot BN}{z}\cdot c^3+\frac{y\cdot BN+z\cdot CP}{x}\cdot a^3+\frac{z\cdot CP+x\cdot AM}{y}\cdot b^3\ge 16\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.704 If $x,y>0$ then in any ABC triangle the following inequality holds:

$$\frac{(xb+yc)^4}{w_b\cdot w_c}+\frac{(xc+ya)^4}{w_c\cdot w_a}+\frac{(xa+yb)^4}{w_a\cdot w_b}\ge 48\cdot(x+y)^4\cdot r^2$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.705 In $\Delta ABC, I$ – incenter, R_a,R_b,R_c – circumradii of $\Delta BIC,\Delta CIA,\Delta AIB$. Prove that:

$$\frac{1}{4}\left(5-\frac{2r}{R}\right)\le\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2\le\frac{1}{4}\left(2+\frac{R}{r}\right)$$

Proposed by Marin Chirciu – Romania

S.706 If $a, b, c > 0$ such that $abc = 1$ and $\lambda \geq 0, n \in \mathbb{N}, n \geq 2$ then:

$$\frac{b^n + \lambda c^n}{a} + \frac{c^n + \lambda a^n}{b} + \frac{a^n + \lambda b^n}{c} \geq \lambda(a + b + c) + 3$$

Proposed by Marin Chirciu – Romania

S.707 In ΔABC the following relationship holds:

$$32r^3(4R + r)^2 \leq \sum a^5 \cot\frac{A}{2} \leq \frac{2R^4}{r}(4R + r)^2$$

Proposed by Marin Chirciu – Romania

S.708 In ΔABC the following relationship holds:

$$3\sum a^4 \tan\frac{A}{2} \geq \sum a^4 \cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.709 In ΔABC the following relationship holds:

$$\frac{3}{2Rp} \leq \sum \frac{1}{a^2}\tan\frac{A}{2} \leq \frac{3}{4rp}$$

Proposed by Marin Chirciu – Romania

S.710 In non-right ΔABC the following relationship holds:

$$\frac{(4R + r)^2}{6Rp} \leq \sum \frac{bc}{a^2(\tan A + \cot A)} \leq \frac{R(4R + r)^2}{24r^2 p}$$

Proposed by Marin Chirciu – Romania

S.711 In ΔABC the following relationship holds:

$$\sum \frac{(m_b^{n+1} + m_c^{n+1})^2}{m_b^n + m_c^n} \leq \frac{27}{2}R^2, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

S.712 If $x, y, z > 0$ then: $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{2y}{x+y} + \frac{2z}{y+z} + \frac{2z}{z+x}$

Proposed by Marin Chirciu – Romania

S.713 If $a, b, c > 0$ such that $a + b + c = 1$ and $\lambda \geq 0, n \in \mathbb{N}, n \geq 2$ then:

$$\sum \frac{a^n}{1 + b(c + \lambda)} \geq \frac{27}{3^n(10 + 3\lambda)}$$

Proposed by Marin Chirciu – Romania

S.714 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{bc}\sin^2\frac{A}{2} \leq \sum \frac{r_a}{bc}\sin^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.715 If $x_1, x_2, \dots, x_n > 0$ then:

$$\frac{x_1}{2x_2}+\frac{x_2}{2x_3}+\cdots+\frac{x_n}{2x_1}+\frac{2^n x_1 x_2 \dots x_n}{(x_1+x_2)(x_2+x_3)\dots(x_n+x_1)} \geq 1+\frac{1}{2}n$$

Proposed by Marin Chirciu – Romania

S.716 In ΔABC the following relationship holds:

$$\frac{r}{p}\cdot 16r^2(4R+r)^2 \leq \sum b^2c^2\tan\frac{A}{2} \leq \frac{r}{p}\cdot 4R^2(4R+r)^2$$

Proposed by Marin Chirciu – Romania

S.717 In ΔABC the following relationship holds:

$$9\sqrt{3}r^{\frac{3}{2}} \leq m_a\sqrt{w_a}+m_b\sqrt{w_b}+m_c\sqrt{w_c} \leq \frac{9\sqrt{6}}{4}R^{\frac{3}{2}}$$

Proposed by Marin Chirciu – Romania

S.718 In ΔABC the following relationship holds: $\sum\frac{h_a}{w_a} \geq 3\left(\frac{2r}{R}\right)^{\frac{2}{3}}$

Proposed by Marin Chirciu – Romania

S.719 In ΔABC the following relationship holds:

$$(4R+r)^2\cdot\frac{48R^2r^3}{p} \leq \sum b^3c^3\tan\frac{A}{2} \leq (4R+r)^2\cdot\frac{6R^5}{p}$$

Proposed by Marin Chirciu – Romania

S.720 I_a, I_b, I_c – excenters in ΔABC. Prove that:

$$\frac{2}{S}\left(2-\frac{r}{R}\right) \leq \frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]} \leq \frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Proposed by Marin Chirciu – Romania

S.721 In ΔABC the following relationship holds:

$$3\sum(p-a)^3\tan\frac{A}{2} \leq \sum(p-a)^3\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.722 In acute ΔABC the following relationship holds:

$$\sum\cos A\left(\frac{\cos B}{\cos C}\right)^n \geq \left(\frac{3}{2}\right)^n\left(\frac{R}{R+r}\right)^{n-1}, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

S.723 In ΔABC the following relationship holds:

$$16R^2rp^3 \leq \sum b^3c^3\cot\frac{A}{2} \leq \frac{R^6p^3}{r^3}$$

Proposed by Marin Chirciu – Romania

S.724 If $x_1, x_2, \dots, x_n > 0$ then:

$$\frac{x_1}{x_2}+\frac{x_2}{x_3}+\cdots+\frac{x_n}{x_1}\geq\frac{n}{2}+\frac{x_1}{x_n+x_1}+\frac{x_2}{x_1+x_2}+\cdots+\frac{x_n}{x_{n-1}+x_n}$$

Proposed by Marin Chirciu – Romania

S.725 In ΔABC the following relationship holds:

$$24Rr\leq\frac{a^4}{h_bh_c}+\frac{b^4}{h_ch_a}+\frac{c^4}{h_ah_b}\leq 4R^2\left(\frac{2R}{r}-1\right)$$

Proposed by Marin Chirciu – Romania

S.726 Find:

$$\Omega=\lim_{n\to\infty}\frac{1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\cdots+\frac{1}{\sqrt[3]{n}}}{\sqrt[3]{n^2}}$$

Proposed by Vasile Mircea Popa – Romania

S.727 Solve in $\mathbb{R}$:

$$\left(2-\frac{2e+\pi-\frac{2\pi^2}{x}+\frac{e^2+1}{x}}{x+e-\frac{\pi^2}{x}+\frac{1}{x}}\right)^9=1+\left(1-\frac{\frac{\pi+2e}{x}-\frac{2\pi^2-e^2-1}{x^2}}{1+\frac{e}{x}-\frac{\pi^2-1}{x^2}}\right)^9$$

Proposed by Orlando Irahola Ortega-Bolivia

S.728 Solve in $\mathbb{R}$

$$\sqrt{x^2-x}=\frac{6x^6-18x^5+20x^4-10x^3+2x^2}{x^6-9x^5+18x^4-21x^3+15x^2-6x+1}$$

Proposed by Orlando Irahola Ortega-Bolivia

S.729 The polygone $A_1A_2A_3A_4A_5A_6$ is tangent to a circle with O – center. If $\sphericalangle A_1\equiv\sphericalangle A_3\equiv\sphericalangle A_5, \sphericalangle A_2\equiv\sphericalangle A_4\equiv\sphericalangle A_6$ then find:

$$\Omega=\sum_{i=1}^{6}\overline{OA_l}$$

Proposed by Ionuț Florin Voinea – Romania

S.730 Find:

$$\Omega=\lim_{x\to 0}\left(\frac{\sin(\sin 2x-\sin x)-\sin(\tan 2x-\tan x)}{x(\sin(\cos^{-1}x)-1)}\right)$$

Proposed by Qusay Yousef-Algerie

S.731 Evaluate: $\sin\left(\frac{\pi}{13}\right)\sin\left(\frac{2\pi}{13}\right)\sin\left(\frac{3\pi}{13}\right)\sin\left(\frac{4\pi}{13}\right)\sin\left(\frac{5\pi}{13}\right)\sin\left(\frac{6\pi}{13}\right)$

Proposed by Rajesh Darbi-India

S.732 Find:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x + x^3 + \tan^3 x + x^2}{1 + x^2} dx$$

Proposed by Rajesh Darbi-India

S.733 Prove without softs:

$$\int_0^{\sqrt{2}} \frac{dx}{1 + x^2 + \cos^{100} x} < 1$$

Proposed by Rajesh Darbi-India

S.734 Let a, b, c be non-negative real numbers such that no two of them is equal to zero.

Prove that if $a + b + c = 2$ and $a \geq b > c \geq 0$ then:

$$\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} + \frac{\sqrt{ab + bc + ca}}{\sqrt{2 - ab - bc - ca}} \geq 3 - 2c$$

Proposed by Minh Nhat Nguyen – Vietnam

S.735 $a, b \in (0,1] \Rightarrow \left(8 + ab - 2(a + b)\right) \cdot a^{1-\frac{4}{5}} \cdot b^{b-\frac{4}{5}} \geq 5$

Proposed by Pavlos Trifon-Greece

S.736 Find x: $x^x + x^{(9e)^x} - \pi^{\left(x^{\left(e^x - 9x\right)}\right)} + x^{(2020x)^x} = \frac{9}{e^{-\pi x}}$

Proposed by Arslan Ahmed-Yemen

S.737 Solve for natural numbers: $(m!)^{n^2} + (n!)^{m^2} = 80 + m^m n^{m!}$

Proposed by Mokhtar Khassani-Algerie

S.738 If $x, y, z > 0, x + y + z = 1$ then:

$$(xm_a + ym_b + zm_c)^2 + (xm_b + ym_c + zm_a)^2 + (xm_c + ym_a + zm_b)^2 \leq \frac{27R^2}{4}$$

Proposed by Hikmat Mammadov-Azerbaijan

S.739 If $0 < a \leq b$ then:

$$\left(\int_a^b e^{-13x^2} dx\right)\left(\int_a^b e^{-8x^2} dx\right) \geq \left(\int_a^b e^{-10x^2} dx\right)\left(\int_a^b e^{-11x^2} dx\right)$$

Proposed by Daniel Sitaru – Romania

S.740 In ΔABC the following relationship holds:

$$\left(\frac{4}{(b + c)^2} + \frac{9}{(c + a)^2} + \frac{1}{(a + b)^2}\right)\left(\frac{9}{(b + c)^2} + \frac{1}{(c + a)^2} + \frac{4}{(a + b)^2}\right) > 49 \sum_{cyc} \frac{1}{(a + b)^2 (b + c)^2}$$

Proposed by Daniel Sitaru – Romania

S.741 If in $\Delta ABC, m(\sphericalangle B) = 2m(\sphericalangle A), m(\sphericalangle C) = 4m(\sphericalangle A)$ then:

$$h_a^2 + h_b^2 + h_c^2 > 7\sqrt{21}R^2$$

Proposed by Daniel Sitaru – Romania

S.742 Find:

$$\Omega = \lim_{n\to\infty}\left(n\left(\log 2 - \sum_{i=1}^{n}\frac{(n+i)^4}{3+(n+i)^5+\cot^{-1}(n+i)}\right)\right)$$

Proposed by Daniel Sitaru – Romania

S.743 In acute ΔABC holds:

$$\sum_{cyc}\sqrt{(\sin A\cos A + \sin B\cos B)\sin 2C} \le \sqrt{6(1+\cos A\cos B\cos C)}$$

Proposed by Daniel Sitaru – Romania

S.744 In $\Delta ABC, m_a = b, m_b = a, cm_a = am_c$. Find:

$$\Omega = \frac{w_a w_b w_c}{n_a n_b n_c} + \frac{g_a g_b g_c}{h_a h_b h_c} + \frac{m_a m_b m_c}{s_a s_b s_c}$$

Proposed by Daniel Sitaru – Romania

S.745 Solve for real numbers:

$$\sum_{k=0}^{10}\binom{20}{2k}(x+k-1)(x+k-2)\cdot\ldots\cdot(x+k-19) = 0$$

Proposed by Daniel Sitaru – Romania

S.746 In ΔABC the following relationship holds:

$$(m_a)^{m_a}\cdot(m_b)^{m_b}\cdot(m_c)^{m_c} \ge (r_a r_b r_c)^{3r}$$

Proposed by Daniel Sitaru – Romania

S.747

$$G = \left\{0, \frac{1}{2021}, \frac{2}{2021}, \ldots, \frac{2020}{2021}\right\}, x * y = x + y - [x+y], [*] - GIF$$

Prove that: $(G,*) \cong (\mathbb{Z}_{2021}, +)$

Proposed by Daniel Sitaru – Romania

S.748 Solve for real numbers:

$$\frac{1}{1+|\sin x|} + \frac{1}{1+|\cos y|} = 1 + \frac{1}{1+|\sin x + \cos y|}$$

Proposed by Daniel Sitaru – Romania

S.749 Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{(2n)!\cdot\left(2\sum_{k=0}^{n}\frac{1}{(n-k)!\cdot(n+k)!}-\frac{4^n}{(2n)!}\right)}$$

Proposed by Daniel Sitaru – Romania

S.750 In acute ΔABC the following relationship holds:

$$(\tan A)^{3\tan A}\cdot(\tan B)^{3\tan B}\cdot(\tan C)^{3\tan C}\geq(\tan A\cdot\tan B\cdot\tan C)^{\tan A\tan B\tan C}$$

Proposed by Daniel Sitaru – Romania

S.751 Solve for real numbers:

$$\frac{(307-x)\sqrt[5]{x-64}-(x-63)\sqrt[5]{307-x}}{\sqrt[5]{307-x}-\sqrt[5]{x-63}}=120$$

Proposed by Daniel Sitaru – Romania

S.752 $ABCD$ – cyclic quadrilateral, R – circumradii.If $AB=a, BC=b, CD=c, DA=2R$ then: $R^3\geq abc$. When equality holds?

Proposed by Daniel Sitaru – Romania

S.753 Find $x,y,y\geq 1$ such that:

$$\begin{cases} x^3+y^2+2z^2=4 \\ 729\cdot\prod_{cyc}(\log(xy)\cdot\log z)=8\cdot\log^6(xyz) \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.754 In ΔABC: p: "$bc\sqrt{4\cos^2 B+4\cos^2 C+1}=3\sqrt{3}R^2, 3a=\pi$"

q: "$a=\mu(A), b=\mu(B), c=\mu(C)$". Prove that: $p\Leftrightarrow q$

Proposed by Daniel Sitaru – Romania

S.755 In ΔABC the following relationship holds:

$$\frac{\sum_{cyc}\sin^2\frac{A}{5}\cdot\sum_{cyc}\sin^2\frac{A}{7}\cdot\sum_{cyc}\sin^2\frac{A}{9}}{\left(1-\cos\frac{2\pi}{15}\right)\left(1-\cos\frac{2\pi}{21}\right)\left(1-\cos\frac{2\pi}{27}\right)}\geq\frac{27}{8}$$

Proposed by Daniel Sitaru – Romania

S.756 If $0<a\leq b<\frac{\pi}{2}$ then:

$$\sin b-\sin a\leq\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Proposed by Daniel Sitaru – Romania

S.757 If $a,b>0, x\in\mathbb{R}$ then:

$$(1+a\sin^2 x+b\cos^2 x)^{a\sin^2 x+b\cos^2 x}\le(1+a)^{a\sin^2 x}\cdot(1+b)^{b\cos^2 x}$$

Proposed by Daniel Sitaru – Romania

S.758 If a,b,c,d – sides, e,f – diagonals, R – circumradii in a cyclic quadrilateral then:

$$R\ge\frac{2\sqrt{abcd}}{e+f}$$

Proposed by Daniel Sitaru – Romania

S.759 ΔMNP – the intouch triangle of ΔABC, Γ – Gergonne's point. Prove that:

$$\frac{3}{r^2 s}\left(\frac{\Gamma M}{\Gamma A}+\frac{\Gamma N}{\Gamma B}+\frac{\Gamma P}{\Gamma C}\right)\le\sum_{cyc}\frac{1}{a}\cdot\sum_{cyc}\frac{1}{(s-a)^2}$$

Proposed by Daniel Sitaru – Romania

S.760 F – area, R – circumradii, r – inradii, s – semiperimeter in a bicentric octagon.

Prove that:

$$\frac{r^2}{R\cos\frac{\pi}{8}}\le\frac{F}{s}\le\frac{R^2\cos^2\frac{\pi}{8}}{r}$$

Proposed by Daniel Sitaru – Romania

S.761 O – circumcenter, I – incenter, R – circumradii in a bicentric quadrilateral $ABCD$. If $3\sin A\sin B=1$ then find:

$$\Omega=\frac{R}{OI}$$

Proposed by Daniel Sitaru – Romania

S.762 Find:

$$\Omega=\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{n}\cdot 3^{\frac{k}{n}}\right)\left(1+\frac{1}{n}\cdot 5^{\frac{k}{n}}\right)\left(1+\frac{1}{n}\cdot 7^{\frac{k}{n}}\right)$$

Proposed by Daniel Sitaru – Romania

S.763 If $0<a\le b$ then:

$$\left(\int_0^{\frac{3a+b}{4}} t^5 e^{t^2}\,dt\right)\cdot\left(\int_0^{\frac{a+3b}{4}} t^4 e^{t^2}\,dt\right)\le\left(\int_0^{\frac{a+3b}{4}} t^5 e^{t^2}\,dt\right)\cdot\left(\int_0^{\frac{3a+b}{4}} t^4 e^{t^2}\,dt\right)$$

Proposed by Daniel Sitaru – Romania

S.764 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \dfrac{x^x\cdot y^y\cdot z^z\cdot\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)}{\sqrt{x^{y+z}\cdot y^{z+x}\cdot z^{x+y}}}=1 \\ x+y+z=1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.765 Solve for real numbers:

$$\begin{cases} \cos^2 x\cdot\cos^2 y\cdot\cos^2 z=\dfrac{1}{8} \\ \prod_{cyc}(\cos^2 x-\cos^2 x\cdot\cos^2 y+\cos^2 y)=\dfrac{8}{27} \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.766 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\frac{n_a}{h_a}\le\sum_{cyc}\frac{\sqrt{4R^2+(n_a-g_a)^2+2(n_ag_a-w_a^2)}}{2r}$$

Proposed by Bogdan Fuștei – Romania

S.767 In ΔABC the following relationship holds:

$$2\sqrt{3}\cdot\sum_{cyc}\frac{h_a}{n_a+g_a+\sqrt{2r_br_c}}\ge\sum_{cyc}\cos(A-B)$$

Proposed by Bogdan Fuștei – Romania

S.768 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$2\sum_{cyc}\frac{h_a}{s-n_a}=\frac{s}{r}+\sum_{cyc}\frac{n_a}{r_a}$$

Proposed by Bogdan Fuștei – Romania

S.769 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\prod_{cyc}(s+n_a)\left(\cot\frac{B}{2}+\cot\frac{C}{2}-\frac{2n_a}{h_a}\right)\le 64\cdot\sqrt[4]{\prod_{cyc}m_an_ag_aw_a}$$

Proposed by Bogdan Fuștei – Romania

S.770 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sqrt{\frac{R}{2r}}\sum_{cyc}(n_a+g_a)\ge 2\sum_{cyc}r_a$$

Proposed by Bogdan Fuștei – Romania

S.771 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\frac{n_a g_a + n_b g_b + n_c g_c}{h_a h_b + h_b h_c + h_c h_a} \geq \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c}\right)^2$$

Proposed by Bogdan Fuștei – Romania

S.772 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$2\sum_{cyc} \frac{r_a}{s + n_a} + \sum_{cyc} \frac{n_a}{h_a} = \frac{s}{r}$$

Proposed by Bogdan Fuștei – Romania

S.773 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{n_a + n_b + n_c}{3r} + \frac{2}{3} \cdot \sum_{cyc} \frac{2r_a + h_a}{n_a + s} \geq \sqrt{\left(4 - \frac{2r}{R}\right)\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{c}{b} + \frac{b}{a} + \frac{a}{c}\right)}$$

Proposed by Bogdan Fuștei – Romania

S.774 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{n_a n_b + n_b n_c + n_c n_a}{h_a h_b + h_b h_c + h_c h_a} \geq \left(\frac{64a^2b^2c^2}{(4a^2 - (b-c)^2)(4b^2 - (c-a)^2)(4c^2 - (a-b)^2)}\right)^2$$

Proposed by Bogdan Fuștei – Romania

S.775 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc} \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a}\right) \geq 4\sum_{cyc} \frac{m_a}{b + c}$$

Proposed by Bogdan Fuștei – Romania

S.776 $A \in M_2(\mathbb{R}), Tr\ A + \det A = 0$. Prove that:

$$\det(A^2 + 3A + 3I_2) + \det(A^2 - 3A + 3I_2) \geq 30 \det A$$

Proposed by Marian Ursărescu – Romania

S.777 $z_1, z_2, z_3 \in \mathbb{C}^*$, different in pairs, $|z_1| = |z_2| = |z_3|, A(z_1), B(z_2), C(z_3)$.Prove that:

$$\sum_{cyc} \left|\frac{2z_1 - z_2 - z_3}{z_2 - z_3}\right|^2 = 9 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

S.778 $A, B \in M_3(\mathbb{C}), 2021AB = I_3 + 2020BA$. Find:

$$\Omega = Tr((AB - BA)^3)$$

Proposed by Marian Ursărescu – Romania

S.779 In ΔABC the following relationship holds:

$$\frac{m_b + m_c}{g_a^2} + \frac{m_c + m_a}{g_b^2} + \frac{m_a + m_b}{g_c^2} \geq \frac{4}{R}$$

Proposed by Marian Ursărescu – Romania

S.780 Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n}(2k-1)\binom{n}{k-1}\binom{n}{k}}$$

Proposed by Marian Ursărescu – Romania

S.781 $z_1, z_2, z_3 \in \mathbb{C}^*$ - different in pairs, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$

$$\prod_{cyc}\left|(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|\right| = \left(\sum_{cyc}|z_1 - z_2|\right)^3 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

S.782 In acute $\Delta ABC, n \in \mathbb{N}, n \geq 2$ the following relationship holds:

$$\sum_{cyc}\left(1 - \sqrt[n]{\sin A}\right) \geq \sum_{cyc}\frac{1 - \sin A \sin B}{2n + 1 - \sin A \sin B}$$

Proposed by Florică Anastase – Romania

S.783 $(a_n)_{n\geq1}, (b_n)_{n\geq1}; a_n = \int_1^n \left[\frac{n^2}{x}\right] dx, b_1 > 1, b_{n+1} = 1 + \log(b_n), [*] - GIF$

Find:

$$\Omega = \lim_{n\to\infty}\frac{a_n \cdot \log \sqrt[n]{b_n}}{\log n}$$

Proposed by Florică Anastase – Romania

S.784 $\Omega(a) = \int_0^a \log(1 + x) \cdot \tan^{-1}(\sqrt{x})\, dx, a > 0$. Prove that:

$$\Omega(a) + \Omega(b) + \Omega(c) < (a + b + c)\left(a + b + c + \frac{1}{2}\right)$$

Proposed by Florică Anastase – Romania

S.785 Let $(a_n)_{n\geq1}$ – be sequence of real numbers with $a_1 = 1$ and $[(a_n - a_{n-1})(n + 1)!\, n - a_n a_{n-1}](n + 1) = n^2 a_n a_{n-1}, n \geq 1$. Find:

$$\lim_{n\to\infty}\frac{\left(\sqrt[n+1]{\frac{a_{n+1}}{n+2}}\right)^a - \left(\sqrt[n]{\frac{a_n}{n+1}}\right)^a}{\left(\sqrt[n]{\frac{a_n}{n+1}}\right)^{n-1}}$$

Proposed by Florică Anastase – Romania

S.786 If $a, b, c > 1$, then:

$$\sum_{cyc}\log_{a+b}(1 + b^{b+1})(1 + c^{c+1}) \geq 6(a + b)^{c-b}(b + c)^{a-c}(c + a)^{b-a}$$

Proposed by Florică Anastase – Romania

S.787 If $a,b,c\in(1,2), f:(2,3)\to\mathbb{R}_+$ continuous with $f'(x)<0$ and $f''(x)<0, \forall x\in(2,3)$ then prove:

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right)\ge 2\cdot\sqrt[4]{\prod_{cyc} f(a+1)\cdot\sum_{cyc} f(a+1)}$$

Proposed by Florică Anastase – Romania

S.788 If $f:[0,1]\to(0,\infty), f$ – continuous then:

$$\log\left(8\int_0^1\left(f(x)\int_x^1\left(f(y)\int_x^y f(z)dz\right)dy\right)dx\right)\ge 3\int_0^1\log(f(x))\,dx$$

Proposed by Daniel Sitaru – Romania

S.789 Find without any software:

$$\Omega=\int\frac{(x^2+6x+15)\sin x}{x^4+12x^3+54x^2+108x+81}dx$$

Proposed by Daniel Sitaru – Romania

S.790 In any ΔABC the following relationship holds:

$$R\sum_{cyc}(b\sin 3C-c\sin 3B)\ge 12\sqrt{3}r^2\sum_{cyc}\sin(B-C)$$

Proposed by Daniel Sitaru – Romania

S.791 In $\Delta ABC, I_a, I_b, I_c$ – excenters, the following relationship holds:

$$\frac{(AI_a)^{n+1}}{b^n}+\frac{(BI_b)^{n+1}}{c^n}+\frac{(CI_c)^{n+1}}{a^n}\ge 2^{n+1}\cdot\sqrt{3^{n+4}}\cdot\frac{r^{n+1}}{R^n}, n\in\mathbb{N}$$

Proposed by Daniel Sitaru – Romania

S.792 Find without any software: $\Omega=\int\frac{1}{14+(x+4)^4+(x+6)^4}dx$

Proposed by Daniel Sitaru – Romania

S.793 If F_n, L_n, P_n – Fibonacci, Lucas, Pell numbers then in ΔABC holds:

$$6\sqrt{3}r\cdot F_n+2s\cdot L_n+3\sqrt{3}R\cdot P_n\ge\sqrt{3}(F_n+L_n+P_n)(4r+R)$$

Proposed by Daniel Sitaru – Romania

S.794 If $0<a\le b<\frac{\pi}{2}$ then:

$$\frac{(\cos a-\cos b)\left(\sqrt{1+b^2}+\sqrt{1+a^2}\right)}{b+a}\ge b-a$$

Proposed by Daniel Sitaru – Romania

S.795 $x_1 = \frac{1}{2}, 2x_{n+1}^2 + \sqrt{1 - x_n^2} = 1, n \geq 1$. Find:

$$\Omega = \lim_{n\to\infty} (2^n \cdot x_n)$$

Proposed by Daniel Sitaru – Romania

S.796 If $0 < a \leq b$ then:

$$\int_a^b \left(\cos^7 x - \cos^7\left(x + \frac{\pi}{3}\right) + \cos^7\left(x + \frac{2\pi}{3}\right)\right) dx \leq \frac{21}{32}\sin\frac{3(b-a)}{2}$$

Proposed by Daniel Sitaru – Romania

S.797 Find:

$$\Omega = \underbrace{\min}_{x\in\mathbb{R}}\left(\sqrt{x^2 - 8x + 64} + \sqrt{x^2 - 6\sqrt{3}x + 36}\right)$$

Proposed by Daniel Sitaru – Romania

S.798 In ΔABC the following relationship holds:

$$a^a \cdot b^b \cdot c^c \cdot \left(6\sqrt{3}r\right)^{6\sqrt{3}r} \leq (a^2 + b^2 + c^2)^{2s}$$

Proposed by Daniel Sitaru – Romania

S.799

$$A(2,1010), B\left(x, \frac{2020}{x}\right), C\left(y, \frac{2020}{y}\right)$$

Find $x, y \in \mathbb{R}$ such that $H(11,2020)$ is the orthocenter of ΔABC.

Proposed by Daniel Sitaru – Romania

S.800 If $a, b, c > 0$ then:

$$\left(\frac{a}{a+b+c}\right)^{\frac{a}{b+c}} \cdot \left(\frac{b}{a+b+c}\right)^{\frac{b}{c+a}} \cdot \left(\frac{c}{a+b+c}\right)^{\frac{c}{a+b}} \geq \sqrt{\frac{abc}{(a+b+c)^3}}$$

Proposed by Daniel Sitaru – Romania

S.801 In acute ΔABC the following relationship holds:

$$\prod_{cyc}(1 + \tan A)^{\tan A} \geq \left(1 + \sqrt[3]{\sum_{cyc}\tan A}\right)^{3\cdot\sqrt[3]{\sum_{cyc}\tan A}}$$

Proposed by Daniel Sitaru – Romania

S.802 If $0 \leq x < 1$ then: $x^1 + 2^{2^1} + x^{2^2} + x^{2^3} + \cdots \geq \ln\left(\frac{1}{1-x}\right)$

Proposed by Asmat Qatea-Afganistan

S.803 Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\frac{n^7\cdot k^3 - k^7\cdot n^3}{n^{11}(\ln(n)-\ln(k))}\right)$$

Proposed by Asmat Qatea-Afganistan

S.804 Prove that:

$$\lim_{n\to\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+n}\right)\left(4^n\sqrt{n-1}\right) = \sqrt{\pi}$$

Proposed by Asmat Qatea-Afganistan

S.805 If $x \geq 1$ then:

$$x!\cdot e^{(\gamma+1)(x-1)} \geq x^x$$

Proposed by Asmat Qatea-Afganistan

S.806 If $x \geq 1$ then:

$$\int_1^x t\sqrt[t]{t+1}\,dt \leq \gamma(x-1) + \ln(x!) + \frac{x^2-1}{2}$$

Proposed by Asmat Qatea-Afganistan

S.807 If $n \in \mathbb{N}$ and $(0 \leq x \leq 1)$ then:

$$x^{2n} + (2n-1)x^n \leq (2n-1)x^{n+1} + x$$

Proposed by Asmat Qatea-Afganistan

S.808 Find:

$$\Omega = \left(\sum_{k=1}^{99}\cos\left(\sqrt{3}+\frac{2\pi k}{3\cdot 99}\right)\right)\left(\sum_{k=1}^{99}\cos\left(\sqrt{5}+\frac{2\pi k}{99}\right)\right)$$

Proposed by Asmat Qatea-Afganistan

S.809 If $n \in \mathbb{N}$ and $[*]$ denotes greatest integer function then prove that:

$$\cos^n(x) = \frac{1}{2^{n-1}}\left(\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{k}\cos((n-2k)x)\right) - \frac{1}{2^n}\binom{n}{\left[\frac{n}{2}\right]}\cos^2\left(\frac{n\pi}{2}\right)$$

Proposed by Asmat Qatea-Afganistan

S.810 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{2+\sum_{k=1}^{n}\frac{2k-1}{2^k}}{\sum_{k=1}^{n}\frac{2k+1}{2^k}}\right)^{2^n}$$

Proposed by Costel Florea – Romania

S.811 Find:

$$\Omega = \lim_{x\to\infty}\left((1+x^3)\int_a^{a+x}\frac{dt}{(t^2-(2a-3x)t+a^2-3ax+ax^2)^2}\right)^{x^3}$$

Proposed by Costel Florea – Romania

S.812

$$P(x) = n^3x^{n+1}-(n^3+3n^2-3n+1)x^2+(6n-6)x^{n-1}+(6n-12)x^{n-2}+\cdots+$$
$$+18x^3+12x^2+1$$

If $P(x) = (ax+b)\cdot\sum_{k=0}^{n}u_kx^k$, then find:

$$\Omega = \lim_{n\to\infty}\left(\frac{u_{n-200}-u_{200}}{u_{n-100}-u_{100}}\right)^{\frac{\phi n}{100(a-b)}}$$

ϕ – golden ratio.

Proposed by Costel Florea – Romania

S.813 $\omega(n) = \int_0^{\tan^{-1}n}\frac{2\sin^2 x+\sin x+7\sin x\cos x+2\cos x+6\cos^2 x}{6\sin^2 x+3\cos^2 x+3\sin x+11\sin x\cos x+\cos x}dx$

Find:

$$\Omega(n) = \lim_{n\to\infty}\omega(n)$$

Proposed by Costel Florea – Romania

S.814 $n^2x^{n+1}-(x^2+2n-1)x^n+2x^{n-1}+2x^{n-2}+\cdots+2x^2+2x+1 = (ax+b)P(x)$

$$n(n+1)x^n-2nx^{n-1}-(2n-2)x^{n-2}-(2n-4)x^{n-3}-\cdots-6x^2-4x-2$$
$$= (cx+d)Q(x)$$

$a,b,c,d\in\mathbb{R}$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(a-b)Q(1)}{(c-d)P(1)}\right)^{2\pi n}$$

Proposed by Costel Florea – Romania

S.815 $a_1 = 1, a_{n+1} = 4a_n+n^2+1, n\ge 1$. Find:

$$\Omega = \lim_{n\to\infty}\frac{a_n}{4^n}$$

Proposed by Costel Florea – Romania

S.816 If

$$\Omega(n) = \sum_{k=1}^{n}\int_0^k\frac{x^{2k+1}-x^{2k-1}+1}{x^{2k+3}-(1-k^2)x^{2k+1}-k^2x^{2k-1}+x^2+k^2}dx$$

Find:

$$\Omega = \lim_{n\to\infty}\frac{\Omega(n)}{n}$$

Proposed by Costel Florea – Romania

S.817 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{2^n\cdot\sum_{k=1}^{n}(3k-2)2^{3k-2}}{\sum_{k=1}^{n}(4k-3)2^{4k-3}}\right)$$

Proposed by Costel Florea – Romania

S.818

$$\Omega(n) = \sum_{k=1}^{n}\int_0^n \frac{x^{2k+3}-x^{2k+2}+1}{(x+1)(x^{2k+4}-x^{2k+3}+\cdots+1)-x(x^{2k+1}-x-1)}dx$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{9\Omega(n)}{4\pi(n-1)}\right)^n$$

Proposed by Costel Florea – Romania

S.819

$$\Omega_1(n) = \int_0^1 \frac{x^{n+4}-(n^3+3n^2-3n+1)x^n+(6n-6)x^{n+1}+(6n-12)x^{n-2}+\cdots+12x^2+6x+1}{(x-1)^2}dx$$

$$\Omega_2(n) = \int_0^1 \frac{(n+1)x^{n+2}-(n+2)x^{n+1}-x^2+2x}{(x-1)^2}dx$$

$$\Omega_3(n) = \int_0^1 \frac{n^2x^{n+1}-(x^2+2x-1)x^n+2x^{n-1}+2x^{n-2}+\cdots+2x+1}{(x-1)^2}dx$$

Find: $\Omega = \lim_{n\to\infty}\left(\frac{3\Omega_1(n)}{2\Omega_2(n)\cdot\Omega_3(n)}\right)^{2n\phi}$, ϕ – golden ratio.

Proposed by Costel Florea – Romania

S.820 $u_n = \log\sqrt[3]{\frac{n^4+6n^3+13n^2+12n+4}{n^4+14n^3+73n^2+168n+144}}$, $S_n = u_1+u_2+\cdots+u_n$.Find:

$$\Omega = \lim_{n\to\infty} n\left(\frac{3}{2}S_n-4\log n-\log 72\right)$$

Proposed by Costel Florea – Romania

S.821

$$E(x,n) = \sum_{k=0}^{n}\frac{1}{x^3+3(k+4)x^2+(3k^2+24k+47)x+k^3+12k^2+47k+60}$$

Solve for natural numbers: $E(0,n) = \frac{5}{132}$ and find:

$$\Omega = \lim_{n\to\infty}\left[(2n^2+19n+41)\left(\frac{1}{24}-E(0,n)\right)\right]^{2\phi n}$$

ϕ – Golden ratio.

Proposed by Costel Florea – Romania

S.822 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \int_{2^{n-n^2}}^{3^{n+n^2}} \frac{dx}{\sqrt[n]{x^{n+3}} + 5\sqrt[n+1]{x^{n+2+\frac{1}{n}}} + 4\sqrt[n+2]{x^{\frac{n^3+3n^2-4}{n^2+2n}}}}$$

Proposed by Costel Florea – Romania

S.823 Find without software: $\Omega = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{3\cos^2 x+1}{\sin^5 x} dx$

Proposed by Costel Florea – Romania

S.824 Find:

$$\Omega = \lim_{n\to\infty} \left(2^{7n} \sin\left(\frac{7}{2} \int_{2^{8n}}^{2^{16n}} \frac{dx}{\sqrt[8]{x}\left(1 + 16x\sqrt{x\sqrt{x}}\right)} \right) \right)$$

Proposed by Costel Florea – Romania

S.825 Find without softs:

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\tan^2 x}{x^2 + \sec^2 x - x(2\tan x + 3) + 1 + 3\tan x} dx$$

Proposed by Costel Florea – Romania

S.826 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{n^2} \lim_{x\to 0} \frac{1 - \cos x \cdot \cos 2\sqrt{2}x \cdot \ldots \cdot \cos n\sqrt{n}x}{1 - \sqrt{\cos x} \cdot \sqrt[4]{\cos 2x} \cdot \ldots \cdot \sqrt[2n]{\cos nx}} \right)^n$$

Proposed by Costel Florea – Romania

S.827 Find:

$$\Omega = \lim_{n\to\infty} \left(\int_{-n}^{n} \frac{2^{2^{x+1}+x}}{2^{2^{x+2}-2^{2^{x+1}+1}+49}} dx \right)$$

Proposed by Costel Florea – Romania

S.828 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{n^2} \lim_{x\to 0} \frac{1 - \sqrt[3]{\cos x} \cdot \sqrt[5]{\cos 3x} \cdot \ldots \cdot \sqrt[2n+1]{\cos(2n-1)x}}{x^2} \right)$$

Proposed by Costel Florea – Romania

S.829 Solve for real numbers:

$$\begin{cases} x+y=10 \\ \begin{vmatrix} \frac{1}{10} & \frac{1}{x+2^y} & \frac{1}{x+8} \\ \frac{1}{2^x+y} & \frac{1}{2^x+2^y} & \frac{1}{2^x+8} \\ \frac{1}{4+y} & \frac{1}{4+2^y} & \frac{1}{12} \end{vmatrix} = 0 \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.830 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \sum_{cyc} x^{2021}\left(\frac{y}{z}+\frac{z}{x}\right) = \sum_{cyc} x^{2020}(y+z) \\ 3^x+4^y=5^z \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.831 Prove without any software:

$$e(e+2) < \frac{(e+1)(e+2)\log\left(1+\frac{1}{e+1}\right)}{\log\left(1+\frac{1}{e}\right)} < (e+1)^2$$

Proposed by Daniel Sitaru – Romania

S.832 a,b,c,d,e,f – sides, r – inradii in a bicentric quadrilateral. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{e}+\frac{e^2}{f}+\frac{f^2}{a} \geq 4\sqrt{3}r$$

Proposed by Daniel Sitaru – Romania

S.833 If $x,y \in \mathbb{C}$ then: $|x|+|y|+|3x+2y| \leq |4x+3y|+2|x+y|+|y-x|$

Proposed by Daniel Sitaru – Romania

S.834 In ΔABC the following relationship holds:

$$\frac{a^6b^2+b^6a^2}{c}+\frac{b^6c^2+c^6b^2}{a}+\frac{c^6a^2+a^6c^2}{b} \geq 256r^4s^3$$

Proposed by Daniel Sitaru – Romania

S.835 In ΔABC the following relationship holds:

$$\frac{2}{R} \leq \frac{1}{s_a}+\frac{1}{s_b}+\frac{1}{s_c} \leq \frac{R}{2r^2}$$

Proposed by Marian Ursărescu – Romania

S.836 In ΔABC the following relationship holds:

$$\frac{s_a}{m_a^2+m_bm_c}+\frac{s_b}{m_b^2+m_cm_a}+\frac{s_c}{m_c^2+m_am_b} \leq \frac{1}{2r}$$

Proposed by Marian Ursărescu – Romania

S.837 $z_1, z_2, z_3 \in \mathbb{C}$, different in pairs, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc} \frac{(z_2 - z_3)^2}{z_2^2 - 6z_2z_3 + z_3^2} = \frac{9}{7} \Rightarrow \Delta ABC\ right$$

Proposed by Marian Ursărescu – Romania

S.838 $z_1, z_2, z_3 \in \mathbb{C}$, different in pairs, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc} \frac{(z_2 + z_3)^2}{-z_2^2 + 6z_2z_3 - z_3^2} = \frac{3}{7} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

S.839 If $a, b, c, d \in (0,1) \vee a, b, c, d \in (1, \infty)$ then:

$$\log_{bc^2d^3}(a^3b^2c) + \log_{cd^2a^3}(b^3c^2d) + \log_{da^2b^3}(c^3d^2a) + \log_{ab^2c^3}(d^3a^2b) \geq 4$$

Proposed by Marian Ursărescu – Romania

S.840 Solve for real numbers:

$$\log_9 x \cdot \log_2(7 - x) = 1$$

Proposed by Marian Ursărescu – Romania

S.841 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\frac{s}{r} + \sum_{cyc} \frac{n_a}{r_a} \geq 8 \cdot \sum_{cyc} \frac{h_a - 2r}{g_a}$$

Proposed by Bogdan Fuștei – Romania

S.842 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc} \sqrt{\frac{g_a}{r_a}} \geq 2\sqrt{2} \cdot \sum_{cyc} \sqrt{\frac{h_a - 2r}{n_a + s}}$$

Proposed by Bogdan Fuștei – Romania

S.843 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$2\sqrt{3} \cdot \sum_{cyc} \frac{r_a}{n_a + s} \geq \sum_{cyc} \frac{m_a + w_b + w_c - n_a\sqrt{3}}{h_a}$$

Proposed by Bogdan Fuștei – Romania

S.844 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \geq \frac{\sum (n_a + h_a)}{s}$$

Proposed by Bogdan Fuștei – Romania

S.845 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{b+c}{n_a+h_a}\geq\frac{2(h_a+h_b+h_c)}{s}$$

Proposed by Bogdan Fuștei – Romania

S.846 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds: $\sum_{cyc}\frac{n_a}{h_a}\cdot\cos\frac{A}{2}\geq\frac{s}{2R}$

Proposed by Bogdan Fuștei – Romania

S.847 In $\Delta ABC, n_a$ – Nagel's cevian, $x, y, z > 0$, the following relationship holds:

$$\frac{3}{4}\cdot\frac{a^2x+b^2y+c^2z}{r\sqrt{xy+yz+zx}}\geq n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

Proposed by Bogdan Fuștei – Romania

S.848 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{m_an_a}{h_a}\geq\sqrt{\frac{1}{8r^2}\sum_{cyc}m_a^2(b^2+c^2-a^2)+\frac{3}{2}s^2}$$

Proposed by Bogdan Fuștei – Romania

S.849 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\frac{h_a}{g_a+s-a}\leq\frac{\sum(3g_a+n_a)}{6r}+\frac{2}{3r}\sum_{cyc}\frac{m_ar_aw_a}{(n_a+s)(r_b+r_c)}$$

Proposed by Bogdan Fuștei – Romania

S.850 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{m_aw_a}{n_ar_a}}\geq 2\sqrt{2}\cdot\sum_{cyc}\sqrt{\frac{h_a-2r}{n_a+s}}$$

Proposed by Bogdan Fuștei – Romania

S.851 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\sqrt{2m_a(m_b+m_c)}\leq\sum_{cyc}(n_a+g_a)$$

Proposed by Bogdan Fuștei – Romania

S.852 In $\Delta ABc, n_a$ – Nagel's cevian, the following relationship holds:

$$2\sqrt{3}\cdot\sum_{cyc}\frac{h_a}{n_a+s}\geq\sum_{cyc}\frac{m_a+w_b+w_c-n_a\sqrt{3}}{r_a}$$

Proposed by Bogdan Fuștei – Romania

S.853 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{m_aw_a}{(2m_a-g_a)r_a}}\ge 2\sqrt{2}\cdot\sum_{cyc}\sqrt{\frac{h_a-2r}{n_a+s}}$$

Proposed by Bogdan Fuștei – Romania

S.854 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{s}{r}\ge\frac{1}{4\sqrt{2}}\sum_{cyc}\frac{2n_a+n_b+n_c}{r_a}+\frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\right)$$

Proposed by Bogdan Fuștei – Romania

S.855 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{1}{4}\cdot\sum_{cyc}\sqrt{\frac{n_a+s}{n_a}}\ge\sum_{cyc}\sqrt{\frac{r_b+r_c}{r}}$$

Proposed by Bogdan Fuștei – Romania

S.856 Find:

$$\Omega=\lim_{k\to\infty}\int_{\sqrt{2}}^{\frac{\pi}{2}}\frac{\log(x+k)+\log(x-k)-2\log x}{k^2}$$

Proposed by Abdul Mukhtar-Nigeria

S.857 For $\forall i\in\mathbb{N}, a_i>0, \lambda>0$ prove that:

$$\sum_{cyc}\frac{a_1}{a_2+a_3+\cdots+a_n+\lambda a_1}\le\frac{n}{\lambda+(n-1)}$$

Proposed by Amrit Awasthi-India

S.858 Find without any software:

$$\Omega=\int\frac{x^5+x^8+x^9}{x^{10}+x^9+x^7+2x^6+x^4+x^3+1}dx$$

Proposed by Pranesh Pyara Shrestha-Nepal

S.859 Find all numbers $\alpha\ge 0$ such that $\tan(\alpha x)\ge\cot(\alpha x), \forall x\in\left(0,\frac{\pi}{2}\right)$.

Proposed by Nguyen Van Canh-Vietnam

S.860 Solve for real numbers: $2\sqrt{ex}-ex-e=2\left(\sqrt{e^x}-e^{\sqrt{x}}\right)$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

S.861 If $x,y,z>0, x\sqrt[3]{x}+y\sqrt[3]{y}+z\sqrt[3]{z}=3$ then:

$$\sum_{cyc}\frac{1}{xy}\cdot\left(\sum_{cyc}\left(\frac{1}{\sqrt[3]{x}+\sqrt[3]{y}}\right)^3\right)^{-1}\ge\frac{9}{8}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

S.862 If $x, y > 0$ then:

$$\ln x^2 \cdot \ln(xy)^2 \geq \ln\left(x^{\sqrt{3}} \cdot y^{\sqrt{3}+2}\right) \cdot \ln\left(x^{\sqrt{3}} \cdot y^{\sqrt{3}-2}\right)$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

S.863 If $x, y, z > 0$ then:

$$\frac{x}{\frac{y+z}{2} + \sqrt{2(y^2+z^2)}} + \frac{y}{\frac{x+z}{2} + \sqrt{2(z^2+x^2)}} + \frac{z}{\frac{x+y}{2} + \sqrt{2(x^2+y^2)}} \geq 1$$

Proposed by Rahim Shahbazov-Azerbaijan

S.864 If $a, b, c, d > 0, a + b + c + d = 4$ then:

$$\frac{1}{a^3+a^2+a+1} + \frac{1}{b^3+b^2+b+1} + \frac{1}{c^3+c^2+c+1} + \frac{1}{d^3+d^2+d+1} \geq 1$$

Proposed by Rahim Shahbazov-Azerbaijan

S.865 If $x, y, z > 0$ then:

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.866 If $a, b, c, d, e > 0, abcde = 1$ then:

$$\frac{a^2+1}{a^9+4} + \frac{b^2+1}{b^9+4} + \frac{c^2+1}{c^9+4} + \frac{d^2+1}{d^9+4} + \frac{e^2+1}{e^9+4} \leq 2$$

Proposed by Rahim Shahbazov-Azerbaijan

S.867 If $x, y, z > 0, xyz = 1$ then:

$$\frac{x^8+1}{x^{15}+2} + \frac{y^8+1}{y^{15}+2} + \frac{z^8+1}{z^{15}+2} \leq 2$$

Proposed by Rahim Shahbazov-Azerbaijan

S.868 If $x, y > 0$ then:

$$\frac{x^3+y^3}{x^2+y^2} \geq \sqrt[5]{\frac{x^5+y^5}{2}}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.869 In $\Delta ABC, \alpha \geq 2$, the following relationship holds:

$$\left(\frac{R}{2r}\right)^{\alpha} \sum w_a^2 \geq \sum m_a^2$$

Proposed by Nguyen Van Canh-Vietnam

S.870 In $\Delta ABC, \alpha \geq 3$, the following relationship holds:

$$\left(\frac{R}{2r}\right)^{\alpha}\sum h_a^2 \geq \sum m_a^2$$

Proposed by Nguyen Van Canh-Vietnam

S.871 In $\Delta ABC, \alpha \geq 2$, the following relationship holds:

$$\left(\frac{R}{2r}\right)^{\alpha}\sum g_a^2 \geq \sum m_a^2$$

Proposed by Nguyen Van Canh-Vietnam

S.872 Let $\alpha > \beta > 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$\alpha f(xf(x) + f(y)) = \beta f(f(x) + yf(y)) + x^{\alpha}y^{\beta}f(xy), \qquad \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

S.873 Let $\alpha, \beta > 0$. Find all continous functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$\beta f(\alpha x) = \alpha f(\beta x) - (\alpha + \beta)x^{\alpha+\beta}, \qquad \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

S.874 H –orthocenter in acute $\Delta ABC, r_1, r_2, r_3$ –inradii in $\Delta BHC, \Delta CHA, \Delta AHB$. Prove that:

$$r_1 + r_2 + r_3 \leq \left(2 - \sqrt{3}\right)s.$$

Proposed by Adil Abdullayev-Azerbaijan

S.875 In $\Delta ABC, H$ –orthocenter, I –incenter, N_a –Nagel's point, G –centroid, S_p –Spieker point, the following relationship holds: $[HIN_a] = 6[HGS_p]$.

Proposed by Adil Abdullayev-Azerbaijan

S.876 In ΔABC the following relationship holds:

$$2 + \sum\left(\frac{r_a^2}{bc} + \frac{bc}{r_a^2}\right) \geq \frac{8(m_a^2 + m_b^2 + m_c^2)}{m_a m_b + m_b m_c + m_c m_a}.$$

Proposed by Adil Abdullayev-Azerbaijan

S.877 In ΔABC the following relationship holds:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{w_a w_b w_c}{8r_a r_b r_c} \geq \frac{13}{8}.$$

Proposed by Adil Abdullayev-Azerbaijan

S.878 In ΔABC the following relationship holds:

$$1 + \sum \sin\frac{A}{2} \leq \sqrt{6 + \frac{m_a m_b + m_b m_c + m_c m_a}{4(m_a^2 + m_b^2 + m_c^2)}}.$$

Proposed by Adil Abdullayev-Azerbaijan

S.879 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{r_a}{h_a}}+\sqrt[3]{\frac{r_b}{h_b}}+\sqrt[3]{\frac{r_c}{h_c}}\le\frac{3m_am_bm_c}{h_ah_bh_c}.$$

Proposed by Adil Abdullayev-Azerbaijan

S.880 In ΔABC holds: $\frac{27}{2}\left(\frac{r}{R}\right)^2\le(1+\cos A)(1+\cos B)(1+\cos C)\le\frac{27}{8}$.

Proposed by Adil Abdullayev-Azerbaijan

S.881 In ΔABC the following relationship holds:

$$\frac{m_am_bm_c}{r_ar_br_c}\le\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{abc(a+b)(b+c)(c+a)}.$$

Proposed by Adil Abdullayev-Azerbaijan

S.882 Prove that:

$$\sum_{k=0}^{n}\binom{n}{k}^4\ge\frac{2^{4n}}{(n+1)^3},\forall n\in\mathbb{N}$$

Proposed by Jalil Hajimir-Canada

S.883 In ΔABC find:

$$\Omega=\min\left(\left(\sum_{cyc}\tan A\right)\left(\sum_{cyc}\frac{1}{\mu(A)}\right)\right)$$

Proposed by Jalil Hajimir-Canada

S.884 Prove that:

$$\sum_{k=2}^{n}\left[x+\frac{1}{k}\right]\log_n\left[x+\frac{1}{k}\right]\ge[nx](\log_n[nx]-1),n\in\mathbb{N}-\{1\},x\in\mathbb{R}_+$$

$[*]$ –the greatest integer part of $*$.

Proposed by Jalil Hajimir-Canada

S.885 Let $x_1,x_2,\dots,x_7$ be the roots of the equation:

$$10x^7+20x^6-573x^5-1146x^4+8951x^3+17902x^2-24738x-49476=0$$

Find: $\sum_{k=1}^{7}\{x_i\}$, where $\{*\}$ –is fractional part of $*$.

Proposed by Jalil Hajimir-Canada

S.886 If $p_1,p_2,\dots,p_n$ are prime numbers.

Prove that $N=\sqrt[p_1]{p_2}+\sqrt[p_2]{p_3}+\dots+\sqrt[p_n]{p_1}$ is an irrational number.

Proposed by Jalil Hajimir-Canada

S.887 Let x, y be positive real numbers, prove that:

$$\frac{x}{3}+\frac{2y}{3}\le\sqrt{\log\left(\frac{e^{x^2}}{3}+\frac{2e^{y^2}}{3}\right)}$$

Proposed by Jalil Hajimir-Canada

S.888 Find without softs:

$$\Omega=\int_0^{2\pi}\frac{x\sin(\cos x)}{x^2+1}dx$$

Proposed by Jalil Hajimir-Canada

S.889 Find without softs:

$$\int_0^{2\pi}\cos^2\left(\frac{\pi}{4}+4e^{i\theta}\right)d\theta$$

Proposed by Jalil Hajimir-Canada

S.890 Solve for real numbers: $\left|2021^{[\tan x]}-2021^{1-[\tan x]}\right|=2029$

$[*]$ –is the greatest integer part of $*$. ***Proposed by Jalil Hajimir-Canada***

S.891 Let $x, y, z \in [1,\infty); u, v, w > 0$ and m is the arithmetic means of the numbers x, y, z. If ABC is a triangle with the area F, then:

$$\frac{(x^x+y^x+z^x)(u+v)a^2}{w}+\frac{(x^y+y^y+z^z)(v+w)b^2}{u}+\frac{(x^z+y^z+z^z)(w+u)c^2}{v}\ge$$
$$\ge 24\sqrt{3}\cdot m^m\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.892 If $x, y, z \in [1,\infty), t \ge 0$ and m is an arithmetic mean of the numbers x, y, z and u, v, w, then in any ABC triangle with the area F the following inequality holds:

$$\frac{(x^x+y^x+z^x)(u+v)}{w}(ab)^{t+1}+\frac{(x^y+y^y+z^y)(v+w)}{u}(bc)^{t+1}+$$
$$+\frac{(x^z+y^z+z^z)(w+u)}{v}(ca)^{t+1}\ge 2^{2t+3}\left(\sqrt{3}\right)^{3-t}F^{t+1}\cdot m^m$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.893 Let be $m, n \in \mathbb{R}_+ = [0,\infty), m+n=2$ and M an interior point in ΔABC with the area F and $x = MA, y = MB, z = MC$, then:

$$\frac{a^m x^2}{h_a^n}+\frac{b^m y^2}{h_b^n}+\frac{c^m z^2}{h_c^n}\ge\frac{1}{3\cdot 2^{n-4}F^{n-2}}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.894 If $m,x,y,z>0$, then in any ΔABC with the semiperimeter s the following inequality holds: $\frac{(x+y)b^mc^m}{z(s-a)^{2m}}+\frac{(y+z)c^ma^m}{x(s-b)^{2m}}+\frac{(z+x)a^mb^m}{y(s-c)^{2m}}\geq 3\cdot 2^{2m+1}$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.895 If $x,y,z\in[1,\infty)$ and $x+y+z=3m$ then in any ABC triangle with the semiperimeter s the following inequality holds:

$$\frac{(x^2+y^x+z^x)bc}{(s-a)^2}+\frac{(x^y+y^y+z^h)ca}{(s-b)^2}+\frac{(x^z+y^z+z^z)ab}{(s-c)^2}\geq 36m^m$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.896 If $x,y,z\in[1,\infty)$; $x+y+z=3m$ and ABC is a triangle with the area F and the points $M\in(BC),N\in(CA),P\in(AB)$ such that the cevians AM,BN,CP are concurrent, then:

$$(x^x+y^x+z^x)\frac{MB}{NA}bc+(x^y+y^z+z^y)\frac{NC}{PB}ca+(x^z+y^z+z^z)\frac{PA}{MC}ab\geq 12m^m\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.897 If $x,y,z,m\in[1,\infty),x+y+z=3m$ and ABC is a triangle with the area F and the points $M\in(BC),N\in(CA),P\in(AB)$ such that the cevians AM,BN,CP are concurrent, then:

$$(x^x+y^x+z^x)\frac{MB}{NA}\cdot b+(x^y+y^y+z^y)\frac{NC}{PB}+(x^z+y^z+z^z)\frac{PA}{MC}a\geq$$
$$\geq 6m^m\sqrt[4]{27}\sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.898 Let $m\geq 0,u,v>0$ and M an interior point in ΔABC with the area F and $x=MA$, $y=MB,z=MC$, then:

$$\sum_{cyc}\left(\frac{x}{a}\left(u\frac{y}{b}+v\frac{z}{c}\right)\right)^{m+1}\geq\frac{(u+v)^{m+1}}{3^m}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.899 If $u,v>0,M$ is an interior point in ΔABC and $x=MA,y=MB,z=MC$ then:

$$\sum_{cyc}\left(\frac{x}{a}\left(u\cdot\frac{u}{b}+v\cdot\frac{z}{c}\right)\right)^4\geq\frac{(u+v)^4}{27}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.900 If $m\geq 0,x,y,z>0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{x}{h_a^2}+\frac{y}{h_b^2}+\frac{z}{h_c^2}\right)^{2m+2}\cdot\left(\frac{1}{(x+y)^{2m+2}}+\frac{1}{(y+z)^{2m+2}}+\frac{1}{(z+x)^{2m+2}}\right)\geq\frac{3^{m+2}}{4^{m+1}F^{2m+2}}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.901 Let $t>0$ and M an interior point in ΔABC with the area F and x,y,z are the distances from M to the apices A,B,C respectively u,v,w the distances from M to the sides BC,CA,AB respectively. If $X=x+y+z, U=u+v+w$, then:

$$\frac{X+tu}{v+w}a^2+\frac{X+tv}{w+u}b^2+\frac{X+tw}{u+v}c^2\geq 2(6+t)\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.902 If $m\in\mathbb{R}_+=[0,\infty); x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ΔABC the following inequality holds:

$$\frac{y+z}{x}\left(\frac{b^7+c^7}{b^5+c^5}\right)^{m+1}+\frac{z+x}{y}\left(\frac{c^7+a^7}{c^5+a^5}\right)^{m+1}+\frac{z+x}{y}\left(\frac{a^7+b^7}{a^5+b^5}\right)^{m+1}\geq 2^{2m+3}\left(\sqrt{3}\right)^{1-m}F^{m+1}$$

where F is the area of ΔABC.

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.903 If ABC and XYZ are two triangles with the area F, respectively S, then:

$$\frac{xa}{h_a}+\frac{yb}{h_b}+\frac{zc}{h_c}\geq 4\sqrt[4]{3}\sqrt{S}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.904 Let M be an interior point in ABC triangle and $x=MA, y=MB, z=MC$ and u,v,w the distances from M to the sides BC,CA,AB respectively, then:

$$(xu+yv+zw)\left(\frac{1}{(u+v)^2}+\frac{1}{(v+w)^2}+\frac{1}{(w+u)^2}\right)\geq\frac{9}{2}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.905 Let ABC be a triangle with the area F and M an interior point in the triangle. If x,y,z are the distances of point M respectively to the apices A,B,C and u,v,w the distances from M to the sides BC,CA,AB, then: $\frac{x^2a^3}{u}+\frac{y^2b^3}{v}+\frac{z^2c^3}{w}\geq 8(xy+yz+zx)F$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.906 If $x,y>0$ and ABC is a triangle with the area F, then:

$$(ax^2+by^2)\sqrt{(a+c)(b+c)}+(bx^2+cy^2)\sqrt{(b+a)(c+a)}+$$
$$+(cx^2+ay^2)\sqrt{(c+b)(a+b)}\geq 4\sqrt{3}(x+y)^2F$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.907 If $x,y,z>0, n\in\mathbb{N}-\{1\}$ and $A_kB_kC_k, k=\overline{1,n}$ are triangles with the area F_k the following inequality holds:

$$\frac{x+y}{z}a_1a_2\dots a_n+\frac{y+z}{x}b_1b_2\dots b_n+\frac{z+x}{y}c_1c_2\dots c_n\ge 8\sqrt{3}\sqrt{F_1\cdot F_2\dots F_n}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.323 Evaluate,

$$\int_{[0,1]^n}\prod_{1\le i\le n}\sqrt{x_i(1-\ln(x_i))}\,dx_1dx_2\dots dx_n$$

Where,

$$\int_{[0,1]^n} denotes \int_0^1\int_0^1\int_0^1\dots\int_0^1 \quad (n-times)$$

Proposed by Akerele Olofin – Nigeria

U.324 Find in a closed form:

$$\int_0^1\frac{\arctan\left(\frac{x}{\sqrt{3}}\right)}{x}\left(\sqrt[3]{\frac{1-x}{1+x}}+\sqrt[3]{\frac{1+x}{1-x}}\right)dx$$

Proposed by Sujeethan Balendran– SriLanka

U.325 Show that:

$$\int_0^{\frac{\pi}{2}}x\cot x\log^3(\cos x)\,dx=\frac{3\pi}{8}\left\{4Li_4\left(\frac{1}{2}\right)+\frac{3}{2}\zeta(3)\log(2)-\frac{\pi^4}{45}-\frac{\log^4(2)}{6}-\frac{1}{3}\pi^2\log^2(2)\right\}$$

Proposed by Sujeethan Balendran– SriLanka

U.326

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\cos\left(\frac{\pi n}{4}\right)=\frac{\sqrt{2}}{16}\left(3\pi+4\ln\left(1+\sqrt{2}\right)-\ln(4)\right)$$

Proposed by Asmat Qatea-Afghanistan

U.327 Prove that:

$$\int_0^{\infty}\frac{x^n}{\cosh(x^m)}dx=\frac{\Gamma(p)}{m2^{2p-1}}\left[\zeta\left(p,\frac{1}{4}\right)-\zeta\left(p,\frac{3}{4}\right)\right]$$

Where $p=\frac{n+1}{m}, n\geq 0$ and $m>0, \Gamma(p)$ = Euler's Gamma function and $\zeta(s,q)$ = Hurwit's zeta function

Proposed by Lunjapao Baite – India

U.328 Find without any software:

$$\Omega=\int_0^1\int_x^{\sqrt{2-x^2}}\frac{x}{\sqrt{x^2+y^2}}dxdy$$

Proposed by Durmuș Qgmen-Turkiye

U.329 For $n>0$, prove that:

$$\int_0^{\infty}\int_0^{\infty}\frac{x\cos(xt)}{\sinh\left(\frac{\pi x}{n}\right)}dtdx=\frac{\pi n}{4}$$

Proposed by Lunjapao Baite – India

U.330 Prove that:

$$\prod_{n=1}^{\infty}\frac{2n+(-1)^{\frac{n^2+n}{2}}}{2n+\cos\left(\frac{n\pi}{2}\right)}=\frac{\sqrt{4-2\sqrt{2}}}{2}$$

Proposed by Asmat Qatea-Afghanistan

U.331 Prove that: $\int_0^1\frac{\ln\ln(x)}{1-x+x^2}dx=\frac{2\pi}{\sqrt{3}}\ln\Gamma\left(\frac{5}{6}\right)-\frac{\pi}{3\sqrt{3}}\ln(2\pi)$

Proposed by Lunjapao Baite – India

U.332

$$\int_0^1\frac{x^2(\arctan(x^2))^2}{(1+x^4)\left(\sqrt{1-x^2}\right)}dx+$$

$$+\int_0^1\frac{\arctan(x^2)\log\frac{(1+x^4)}{(1-x^2)^2}-x^2\log^2\frac{1-x^2}{\sqrt{1+x^4}}}{(1+x^4)\sqrt{(1-x^2)}}dx=-\frac{2^{\frac{3}{4}}\pi\sin\frac{\pi}{8}}{12}\left(\pi^2+12\ln^2(2)\right)$$

Proposed by Sujeethan Balendran– SriLanka

U.333

$$\Omega(n)=\int_0^1\left(\tan^{-1}x+\frac{nx^n}{n^2x^2+1}\right)dx+\int_0^1\left(2x\cdot\tan^{-1}(2x)+\frac{(n-1)x^{n-1}}{(n-1)^2x^2+1}\right)dx+\cdots$$
$$+\int_0^1\left(nx^{n-1}\cdot\tan^{-1}(nx)+\frac{x}{x^2+1}\right)dx$$

Find:

$$\lim_{n\to\infty}\frac{\Omega(n)}{n}$$

Proposed by Costel Florea – Romania

U.334

$$\int_0^{\frac{\pi}{6}}\frac{(1-\sin^4(x))}{(1+\sin^4(x))\sqrt{(1+\sin^2(x))}}dx$$

$$\frac{1}{4}\log\left(\frac{23+4\sqrt{15}}{17}\right)+\frac{1}{2}\operatorname{arccot}\left(2\sqrt{\frac{3}{5}}\right)$$

Proposed by Sujeethan Balendran– SriLanka

U.335 Find a closed form:

$$\Omega=\int_0^1\frac{x\log x}{1-x+x^2-x^3}dx$$

Proposed by Abdul Mukhtar-Nigeria

U.336 Prove that:

$$\sum_{n=0}^{\infty}\frac{{}_2F_1(2n;n;n+1;-1)}{{}_1F_1(1;n+1;2)-{}_1F_1(1;n+1;-2)}G_{3,5}^{1,2}\left(1\,\middle|\,\begin{matrix}n,n+\frac{1}{2},n+1\\ n+\frac{1}{2},n,n+1,\frac{n}{2},\frac{n+1}{2}\end{matrix}\right)=$$
$$=\frac{1}{2\pi}\left(\sqrt{\frac{e}{2}}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{\pi}}\right)$$

Where

${}_1F_1(a;b;z)\to$ Confluent hypergeometric function, $G_{p,q}^{m,n}\left(a\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)\to$ Meijer G – function

${}_2F_1(a,b;c;z)\to$ Gauss hypergeometric function, $\operatorname{erf}(x)\to$ Error function

Proposed by Izumi Ainsworth-Peru

U.337 Prove that:

$$\sum_{k=0}^{\infty}\sum_{p=1}^{3}\frac{(9i)^{-2k}}{k!}\left(\frac{p}{3}\right)^{4-p}G_{1,3}^{3,1}\left(G^{-2}\,\middle|\,\begin{matrix}\frac{4p-6k-2031}{4}\\ \frac{4p-6k+2011}{4},0,\frac{1}{2}\end{matrix}\right)=\frac{2021\sqrt[3]{\pi^2}}{3^{-2021}}$$

Where $G_{p,q}^{m,n}\left(z\middle|\begin{matrix}a_p\\b_q\end{matrix}\right)$ → Meijer G – function, I → Imaginary number

Proposed by Izumi Ainsworth-Peru

U.338 Show that:

$$\sum_{k=1}^{n}\cot^4\left(\frac{k\pi}{2n+1}\right)=\frac{1}{15}\sum_{k=1}^{n}\cot^2\left(\frac{k\pi}{2n+1}\right)P(n)$$

also show that:

$$\sum_{n=1}^{\infty}\left(\left(\sum_{k=1}^{n}\cot^4\left(\frac{k\pi}{2n+1}\right)\right)^{-1}-\frac{1}{P(n)}\right)=7\gamma-24\ln 2-$$

$$\left(\frac{7}{2}+\sqrt{\frac{200}{17}}\right)\psi(6-\sqrt{34})+\left(\sqrt{\frac{200}{17}}-\frac{7}{2}\right)\psi(6+\sqrt{34})$$

Proposed by Naren Bhandari-Nepal

U.339 Find a closed form:

$$\Omega=\int_0^{\infty}\frac{x\ln(1+x)}{x^4+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.340 Find:

$$\Omega=\int_0^{\infty}\frac{\sqrt{x}\arctan(x)}{x^2+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.341 Prove that:

$$\Psi_1\left(\frac{1}{8}\right)-\Psi_1\left(\frac{3}{8}\right)-\Psi_1\left(\frac{5}{8}\right)+\Psi_1\left(\frac{7}{8}\right)=4\pi^2\sqrt{2}$$

where $\Psi_1(x)$ is the trigamma function.

Proposed by Vasile Mircea Popa – Romania

U.342 Prove:

$$\psi=\int_{-1}^{1}\frac{\ln(1-x)\ln(1+x)}{1+2021^{\tan(2021x)}}dx=(\ln 2)^2+2-\zeta(2)-\ln 4$$

Proposed by Hussain Reza Zadah-Afghanistan

U.343 Suppose $f:\mathbb{R}\to\mathbb{R}$ is a three times differentiable function satisfying:

$$f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}, f'\left(\frac{\pi}{2}\right)=-1, f''\left(\frac{\pi}{2}\right)=\frac{1}{2}$$

and for all $x\in\left[0,\frac{\pi}{2}\right]$, $xf'''(x)+f''(x)-xf'(x)\geq\sin x$, $2xf''(x)+f'(x)\geq\cos x$

Prove that for all $x\in\left[0,\frac{\pi}{2}\right]$, $\int_0^{\frac{\pi}{2}}f(x)\cos x\,dx\leq\pi$

Proposed by Olimjon Jalilov – Uzbekistan

U.344 Let f be a twice differentiable function such that: $xf''(x)+f'(x)\geq f^2(x)$
for all $x\in(0,1), x\in\mathbb{R}$. Prove that:

$$\int_0^1\left(x^3f'(x)+6\right)f(x)dx\geq 1$$

Proposed by Olimjon Jalilov – Uzbekistan

U.345 If

$$\int_0^\infty e^{-t}J_0(t)dt=\frac{x}{y}$$

then find the value of $[x+y]$. Where $[.]$ is the greatest integer function an J_0 is the Bessel function.

Proposed by Tobi Josua-Nigeria

U.346 Evaluate:

$$\int\frac{\ln\left(\varphi\sqrt{x}-7\right)}{x^{\sqrt{x}}\ln\left(\varphi\sqrt{x}+7\right)-1}dx$$

φ: Golden ratio.

Proposed by Arslan Ahmed-Yemen

U.347 Prove that:

$$\Omega=\lim_{n\to\infty}\left(1-\log n+\sum_{k=2}^{n}\frac{2^k}{k^2(k+1)\log 2\cdot\log 3\cdot\ldots\cdot\log n}\right)<\gamma$$

Proposed by Daniel Sitaru – Romania

U.348 If $n\in\mathbb{N}, n\geq 1, K(n)-K$ function, then:

$$K(n)\cdot\left(\sum_{k=1}^{n}\sum_{i=1}^{k}\binom{k}{i}\right)^n\geq n!\cdot K(n+1)$$

Proposed by Daniel Sitaru – Romania

U.349 If $a,b,c>0, a+b+c=3, F_n$ – Fibonacci numbers, L_n – Lucas numbers, P_n – Pell numbers, then:

$$\frac{a^2(P_n-F_n)(P_n-L_n)}{F_nL_n}+\frac{b^2(F_n-L_n)(F_n-P_n)}{L_nP_n}+\frac{c^2(L_n-P_n)(L_n-F_n)}{P_nF_n}\geq 9$$

Proposed by Daniel Sitaru – Romania

U.350

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(k^2+n^2-1)(-1)^{k+n}}{(k+n)!}\binom{2n-k}{n}\binom{2k-n}{k}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.351 Prove that:

$$\int_0^{\frac{\pi}{2}}\sin(x)\sin^{-1}(\cos(\tan^{-1}(\sin(x))))\,dx=\pi\left(1-\frac{1}{\sqrt{2}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.352 Prove that:

$$\int_{-\infty}^{\infty}\frac{e^{\pi x}}{e^{4\pi x}\phi+e^{2\pi x}+1}dx=\frac{1}{2}\sqrt{\frac{1}{19}\left(1-2\sqrt{5}+\sqrt{2(17\sqrt{5}+1)}\right)}$$

where ϕ – Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.353 For $n>0$, let $U(n)=\int_0^{\infty}(1-x\sin(x))\log(e^{-nx}+1)\,dx$

then show that

$$\int_1^{\infty}\frac{U(n)}{n^2}dn=\frac{\pi^2}{24}+\frac{1}{2}+\log(2)-\log(\pi)-\frac{\pi}{2\sinh(\pi)}+\log\left(\tanh\left(\frac{\pi}{2}\right)\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.354 For $m,n>0$, we have

$$\int_{-\infty}^{\infty}x\tan^{-1}\left(\frac{m}{x}\right)e^{-nx^2}\,dx=\frac{\pi}{2n}-\frac{\pi e^{m^2n}\operatorname{erfc}(m\sqrt{n})}{2n}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.355 Prove that:

$$\int_0^{\infty}\frac{e^{-\pi x}}{(\sinh(\pi x)+\phi)(\cosh(\pi x)+\phi)}dx=\frac{\log(T)}{\pi\sqrt{3\sqrt{5}+5}}$$

Where

$$T=\left(\frac{1}{2}\left(\sqrt{5}-\sqrt{2(\sqrt{5}+1)}+1\right)\right)^{\sqrt{\sqrt{5}+5}}\left(\sqrt{5}+\sqrt{2\sqrt{5}+5}+1\right)^{\sqrt{\sqrt{5}+1}}$$

Where ϕ is Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.356 Let, for any complex number y

$$\int_{-\infty}^{\infty} \frac{e^{-\pi(x^2+xy)}}{(\tanh(\pi x)+1)^2}\,dx = \psi(y)\int_{-\infty}^{\infty} \frac{e^{-\pi(x^2+xy)}}{(\coth(\pi x)+1)^2}\,dx$$

then prove that

$$\int_{-\infty}^{\infty} (\psi(y)-1)\,dy = \frac{4(\pi - \sec^{-1}(e^{\pi}))}{\pi\sqrt{e^{2\pi}-1}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.357 Let $f(n)$ is the real root of the equation $x^3 - x = n$ then show that

$$\int_0^1 f(n)\,dn = \frac{4-9p}{4p-12}$$

where $p = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1\ldots}}}}}}}}$

Proposed by Srinivasa Raghava-AIRMC-India

U.358 Prove that

$$\int_{-\infty}^{\infty}\int_0^{\infty} \frac{x\left(\sin(\pi x\sqrt{z})+\cos(\pi x\sqrt{z})\right)}{\left(\cosh\left(\frac{2\pi x}{\sqrt{z}}\right)+1\right)}\frac{dzdx}{z} = \frac{1}{\pi^2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.359 F_n – Fibonacci number and φ – Golden Ratio. Let the recurrence relation

$y(n-1) + y(n+1) = F_n,\ y(0) = \frac{1}{\varphi}, y(1) = \varphi$ then show that

$$\sum_{m=0}^{\infty} \frac{y(m)}{\varphi^{2m}} = 5\sum_{m=0}^{\infty} \frac{(-1)^m y(m)}{\varphi^{2m-2}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.360 Let the 4×4 Matrix

$$M(t) = \begin{pmatrix} t & -t & 0 & it \\ -t & 0 & it & t \\ 0 & it & t & -t \\ it & t & -t & 0 \end{pmatrix}$$

Evaluate the limit: $\lim_{n\to 0}\int_0^{\infty} Tr\left[e^{M(t)}\right]e^{i\pi nt}dt$

Proposed by Srinivasa Raghava-AIRMC-India

U.361 Prove the integral

$$\int_0^{\frac{\pi}{3}} \left(\frac{\tan^2(x)}{\cos^3\left(\frac{x}{2}\right)} - \frac{16\sin(x)}{5-4\cos(x)} + \frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1} \right) dx = 4$$

Proposed by Srinivasa Raghava-AIRMC-India

U.362 Evaluate the integral in a closed – form

$$\int_0^{\frac{\pi}{3}} \left(\frac{\tan^2(x)}{\cos^3\left(\frac{x}{2}\right)} + \frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1} + \frac{16\sin(x)}{4\cos(x)+1} \right) dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.363 If we have, for $\mathcal{R}(y) > 0$, $\int_{-\infty}^{\infty} e^{-(x^2y+xy^2+xy)}\,(x^2y + xy^2 + xy)dx = 0$

then find the value of: $y + 2y^2 + y^3$

Proposed by Srinivasa Raghava-AIRMC-India

U.364 Prove the integral relation

$$\int_0^1\int_0^1 Li_3(\max(x,y))Li_3(\min(x,y))\,dydx = \left(1-\zeta(2)+\zeta(3)\right)^2$$

Proposed by Srinivasa Raghava-AIRMC-India

U.365 Prove the summation

$$\sum_{n=2}^{\infty} \frac{(-1)^n H_n H_{n+1}}{n^3-n}$$

$$= \frac{19\zeta(3)}{16} - \frac{5}{2} + \frac{1}{6}\log(2)\,(2\log(2)-3)^2 + \frac{1}{48}\pi^2(11-8\log(2))$$

Proposed by Srinivasa Raghava-AIRMC-India

U.366 Prove the inequality

For any $y \geq 1$, we have

$$0 \leq \int_0^{\infty} \frac{\sin^3(\pi x)}{e^{\frac{\pi x(y^2+1)}{y}}}dx \leq \frac{6}{65\pi}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.367 Prove the integral relation

$$\int_0^\infty\left(\int_0^1 \mathcal{L}_x[e^{-\varphi x}J_z(x)](y)dy\right)dz = \log\left(\frac{\log\left(\frac{1}{2}\left(\sqrt{5}+\sqrt{6(\sqrt{5}+3)}+3\right)\right)}{\log\left(\frac{1}{2}\left(\sqrt{5}+\sqrt{2(\sqrt{5}+5)}+1\right)\right)}\right)$$

φ – Golden Ratio, $J_n(x)$ – Bessel function, $\mathcal{L}_x[f](y)$ – Laplace Transform

Proposed by Srinivasa Raghava-AIRMC-India

U.368 Prove that the flow of the vector field

$$\frac{\partial}{\partial\phi} \text{ on } S^2 \text{ is } \varphi_t(x) = xe^{tE_z}$$

$$\text{where, } E_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.369

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{2020}\frac{\Gamma(n+\sin^2 k)}{e^{\psi(kn+1)}\Gamma(n-\cos^2 k)}$$

Proposed by Asmat Qatea-Afganistan

U.370 If $n\in\mathbb{N}$ then prove that:

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+3)(k+5)\dots(k+2n+1)} = \frac{1}{2^n\cdot n!}\left|\sum_{k=0}^{n}\binom{n}{k}(-1)^k H_{2k+1}\right|$$

H_n – Harmonic Number

Proposed by Asmat Qatea-Afganistan

U.371 Find a closed form:

$$\int_1^2\left(\frac{x}{2}+\sqrt{\frac{x^2}{4}+\frac{1}{x}}\right)^{99}+\left(\frac{x}{2}-\sqrt{\frac{x^3+4}{4x}}\right)^{99}dx$$

Proposed by Asmat Qatea-Afganistan

U.372 Prove:

$$\prod_{k=1}^{\infty}\left(1-\frac{5}{4k^2}+\frac{5}{16k^4}-\frac{1}{32k^5}\right) = \left(\prod_{k=1}^{5}\left(-\cos\left(\frac{(6k+1)\pi}{15}\right)\right)!\right)^{-1}$$

Proposed by Asmat Qatea-Afganistan

U.373 Prove:

$$\int_a^b \frac{dx}{x(x+2)(x+4)(x+6)\dots(x+2n)} = \frac{1}{2^n \cdot n!}\left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k \ln\left(\frac{b+2k}{a+2k}\right)\right)$$

Proposed by Asmat Qatea-Afganistan

U.374 If, for $n \geq 1$

$$\int_0^\infty \frac{1}{e^{\pi\sqrt{n}x} + \sqrt{n}} \frac{dx}{\sqrt{x}} = f(n) \int_0^\infty \frac{\sqrt{x}}{e^{\pi\sqrt{n}x} + \sqrt{n}} dx$$

then show that: $f(n) = O(\sqrt{n})$

Proposed by Srinivasa Raghava-AIRMC-India

U.375 Find the value of α, if

$$\int_0^1 \int_0^1 \frac{(x+y)\sin^{-1}\left(\sqrt{1-x}\sqrt{y}\right)}{\sqrt{1-y}\sqrt{xy-y+1}} dydx$$

$$+a \int_0^1 \int_0^1 \frac{(xy)\sin^{-1}\left(\sqrt{1-x}\sqrt{y}\right)}{\sqrt{1-y}\sqrt{xy-y+1}} dydx = 0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.376 Prove the integral relation

$$\int_0^1 \int_0^1 \frac{\sin^{-1}\left(\sqrt{1-x}\sqrt{y}\right)\cos^{-1}\left(\sqrt{1-x}\sqrt{y}\right)}{\sqrt{1-y}\sqrt{xy-y+1}} dydx = 8\log(2) - \frac{7\zeta(3)}{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.377 If we define the function f

$$f(x,y) = \frac{\left(\sqrt{x}+\sqrt{y}\right)\sqrt{xy}}{\sqrt{x\sqrt{y} + y\sqrt{x}}}$$

then establish the inequality

$$\int_0^1 \int_0^1 f\left(\frac{x+y}{2}, \sqrt{xy}\right) dydx < \frac{\pi}{4}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.378 If we define

$$\sum_{n=0}^{\infty} \frac{\varphi^{3n+1} + \varphi^{3n-1}(-x)^n}{\varphi^{4n}} = f(x) \sum_{n=0}^{\infty} \frac{\varphi^{3n-1} + \varphi^{3n+1}(-x)^n}{\varphi^{4n}}$$

then prove that

$$\int_{\frac{1}{\varphi}}^{\infty} \frac{f(x)}{x^2} dx = \frac{3\varphi}{2} - \frac{1}{4}(\varphi - 3)\log(2\varphi + 3)$$

$\varphi -$ Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.379 Evaluate the expression in a closed-form:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{7x^2+6}+\frac{1}{7x^2+1}\right)\left(\frac{1}{7x^2+5}+\frac{1}{7x^2+2}\right)\left(\frac{1}{7x^2+4}+\frac{1}{7x^2+3}\right)dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.380 If we define the function

$$\psi(y) = \frac{\mathcal{F}_x[(e^{-\pi x}\sin(e^{-\pi x}))^2](y)}{\mathcal{F}_x[e^{-\pi x}\sin(e^{-\pi x})](y)}$$

then prove the integral relation

$$\int_0^{\pi} \sqrt{y}\,\psi(y)\psi(-y)dy = \int_0^{\pi} \frac{\psi(y)\psi(-y)}{\sqrt{y}} dy$$

$\mathcal{F}_x[f](y)$ – Fourier Transform

Proposed by Srinivasa Raghava-AIRMC-India

U.381 For $n \geq 2$, we have

$$\int_0^{\infty} \frac{\sin(2x)\sinh\left(\frac{x}{2}\right)}{(e^x - 1)} \frac{dx}{x^{\frac{1}{n}}}$$

$$= 2^{-\frac{1}{n}} 17^{\frac{1}{2}\left(\frac{1}{n}-1\right)} \Gamma\left(\frac{n-1}{n}\right) \sin\left(\frac{(n-1)\tan^{-1}(4)}{n}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.382 If we have the Sum

$$\sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{\left(x+\frac{1}{x}\right)^{m+\frac{1}{2}}\left(x-\frac{1}{x}\right)^{m-\frac{1}{2}}} = \frac{1+\sqrt{5}}{2}$$

then find the value of x

Proposed by Srinivasa Raghava-AIRMC-India

U.383

$$\int_0^{\infty} \frac{(2n\log(x) + \pi x)^2}{\log^2(x) + \frac{\pi^2}{4}} \frac{dx}{(x^2+1)^2} = \pi(n^2 - (n^2 - 1)\log(2))$$

Proposed by Srinivasa Raghava-AIRMC-India

U.384 If we have the sum

$$A(n) = \sum_{m=1}^{3n} \frac{1+2+3+4+\cdots+m}{\cos\left(\frac{\pi m}{3}\right)}$$

then show that: $\sum_{n=1}^{\infty} A(n)x^n = \frac{7}{8(x-1)} - \frac{23}{8(x+1)} - \frac{3}{4(x+1)^2} + \frac{9}{2(x+1)^3}$

Proposed by Srinivasa Raghava-AIRMC-India

U.385 Prove via Complex – Analysis

$$\sum_{n=1}^{m} (-1)^{1+2+3+4+\cdots+n}(1+2+3+4+\cdots+n)$$
$$= \frac{1}{2}m(m+2)\cos\left(\frac{\pi m}{2}\right) - \frac{1}{2}(m+1)\sin\left(\frac{\pi m}{2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.386 Solve for x

$$\sum_{n=0}^{\infty} (-1)^n \binom{3n}{n} \left(\frac{1-x}{1+x}\right)^n = \varphi$$

$\varphi -$ Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.387 If we have the integral $\psi(z) = \int_0^1 \mathcal{L}_x\left[e^{-\varphi x} J_z(x)\right](y)dy$ and if

$$\int_0^{\infty} \frac{\psi(z)}{\sqrt{x}} dz = 2\left(\sqrt{\pi \log(A)} - \sqrt{\pi \log(B)}\right)$$

then find the value of $AB - (A+B)$, $\varphi -$ Golden Ratio, $J_n(x)$ – Bessel function, $\mathcal{L}_x[f](y)$ – Laplace Transform .

Proposed by Srinivasa Raghava-AIRMC-India

U.388 If we define the integral, for $y \geq 1$

$$\eta(y) = \int_0^{\infty} \frac{\cos(\pi x\sqrt{y})}{\cosh\left(\frac{2\pi x}{\sqrt{y}}\right) + 1} dx$$

then show that

$$\int_0^{\infty} \eta(y)^2 e^{-\pi y} dy = \left(1 - \frac{\zeta(4)}{\zeta(3)}\right) \int_0^{\infty} \eta(y)^2 dy$$

Proposed by Srinivasa Raghava-AIRMC-India

U.389 Prove that:

$$\int_0^1 \frac{\log(x)\left(\tan^{-1}(x) + \cot^{-1}(x)\right)^2}{(x^2+1)^2} dx$$

$$+\frac{4C+\pi}{2+\pi}\int_0^1 \frac{(\tan^{-1}(x)+\cot^{-1}(x))^2}{(x^2+1)^2}dx = 0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.390 Let for $n \geq 0$

$$\phi(n) = \int_{-\infty}^{\infty} \frac{e^{i\pi nx}\sin(\pi x)}{x^2+i}dx$$

and if

$$\int_0^{\infty} \phi(n)\, e^{-i\pi n} dn = \alpha \int_0^{\infty} \phi(n)\sin(\pi n)\, dn$$

then show that

$$\alpha^4 + 2\alpha^2 - 4\alpha + 2 = 0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.391 For any complex numbers x, y

If $\frac{x}{y} + \frac{y}{x} = 1$

then prove that

$$2019 - \left(\frac{y}{x}\right)^{2020} - \left(\frac{x}{y}\right)^{2020} = 2020$$

Proposed by Srinivasa Raghava-AIRMC-India

U.392 For any real number $n \geq 1$, we have:

$$\int_{-\infty}^{\infty} \frac{\sin\left(n\left(x-\frac{1}{x}\right)\right)}{x+\frac{1}{x}}dx = 2\int_{-\infty}^{\infty} \frac{\cos\left(n\left(x-\frac{1}{x}\right)\right)}{\left(x+\frac{1}{x}\right)^2}dx = \frac{\pi}{e^{2n}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.393 If $\varphi(x,t) = Li_2(tx^2) - Li_2(t^2x^2) - t^2\big(Li_2(t^2x^2) - Li_2(tx^2) - 2\log(1+$

$t2Li2t-$

$$-2t\log(1+t^2)\,Li_2(t) - 2t^3 Li_2(t)(\log(1+t^2)+1) + 2t^4 Li_2(t)(\log(1+t^2)+1).$$

Find:

$$\Phi(x) = \int_0^1 \frac{\varphi(x,t)}{t^3-t^2+t-1}dt$$

Proposed by Abdul Hafeez Ayinde-Nigeria

U.394 Find a closed form:

$$\Omega = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\,(2n+1)(3n+2)}$$

Proposed by Ajentunmobi Abdulqoyyum-Nigeria

U.395 Prove that:

$$\sum_{n=1}^{\infty} \frac{1}{n64^n}\binom{4n}{2n}\binom{2n}{n} = 6\log 2 - \sqrt{2}\pi + \frac{1}{2\sqrt{2}\pi}\left(\psi_1\left(\frac{5}{8}\right) + \psi_2\left(\frac{7}{8}\right)\right)$$

Where $\psi_1(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$ is trigamma function.

Proposed by Naren Bhandari-Nepal

U.396 Find a closed form:

$$\Omega = \int_0^{\infty} \frac{x\tan^{-1} x}{x^4 - x^2 + 1}\,dx$$

Proposed by Vasile Mircea Popa-Romania

U.397 For all $m, n \in \mathbb{N}$ prove that $\frac{H_n}{n} + \frac{H_m}{m} \le \frac{m+2}{m+1} + \frac{\psi}{nm}$, where H_n is n^{th} harmonic number and

$$\psi = \sum_{k=1}^{n} \frac{\zeta_n(k+1)}{k+1};\ \zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}$$

Proposed by Amrit Awasthi-India

U.398 For $b > a > 0, n \in \mathbb{N}$ find a closed form:

$$\Omega(a,b,n) = \int_0^a x(b-x)^{-1}\left(\frac{a-x}{x}\right)^n \log\left(\frac{a-x}{x}\right)dx$$

Proposed by Ghazaly Abiodun-Nigeria

U.399 Find a closed form:

$$\Omega = \int_0^{\infty} \frac{\sin(\pi x)}{x(1-x^2)}\,dx$$

Proposed by Abdul Mukhtar-Nigeria

U.400 Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \frac{1}{n}\int_0^1 (1+x^2)^{-n} \cdot \tan^{-1} x\,dx$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

U.401 Prove that:

$$1-\frac{23}{3}\left(\frac{3}{10}\right)^3+\frac{43}{3}\left(\frac{3\cdot 13}{10\cdot 10}\right)^3\frac{1}{8}-\frac{63}{3}\left(\frac{3\cdot 13\cdot 23}{10\cdot 10\cdot 10}\right)^3\frac{1}{216}+\cdots=\frac{5}{3\pi}\varphi$$

Where $\varphi=\frac{1+\sqrt{5}}{2}$ –golden ratio.

Proposed by Ngulmun George Baite-India

U.402

$$\int_0^1\int_0^1\cdots\int_0^1\frac{\sqrt{\ln\left(\frac{1}{x_1}\right)+\ln\left(\frac{1}{x_2}\right)+\ln\left(\frac{1}{x_3}\right)+\cdots+\ln\left(\frac{1}{x_n}\right)}\,dx_1dx_2\ldots dx_n}{1+\prod_{k=1}^n x_k}=$$

$$=\frac{n}{4^n}\binom{2n}{n}\eta\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right),\forall n\in\mathbb{N}$$

Proposed by Kaushik Mahanta – Assam – India

U.403 Prove that: $\int_0^1\sqrt[n]{\left(\frac{x}{1-x}\right)^x}\sin\left(\frac{\pi x}{n}\right)dx=\frac{\sqrt[n]{e\pi^n}}{2n},\forall n>1$

Proposed by Surjeet Singhania, Kaushik Mahanta – India

U.404 Prove that: $\sum_n^\infty\frac{\tan^{-1}n}{n^2}<\phi$, where ϕ is Golden ratio.

Proposed by Kaushik Mahanta – Assam – India

U.405 Prove that:

$$G=\int_0^1\frac{\cot^{-1}(x)-\tan^{-1}(x)}{1-x^2}dx$$

where $G=\sum_{n=1}^\infty\frac{(-1)^{n-1}}{(2n-1)^2}$, Catalan's constant

Proposed by Surjeet Singhania, Kaushik Mahanta – India

U.406 Prove that:

$$\int_0^\theta\log(\cos x)\,dx=-\pi\log\left(\frac{G\left(\frac{1}{2}+\frac{\theta}{\pi}\right)}{G\left(\frac{3}{2}+\frac{\theta}{\pi}\right)}\right)-\left(\frac{\pi}{2}+\theta\right)\log\left(\frac{\pi}{\cos\theta}\right)-\theta\ln 2$$

Where $G(z)$ is the Barnes $G-$function.

Proposed by Kaushik Mahanta – Assam – India

U.407 Prove that:

$$\int_0^1\int_0^1\int_0^1\cdots\int_0^1\frac{dx_1dx_2dx_3\ldots dx_n}{\sqrt{x_1x_2x_3\ldots x_n(1-x_1)(1-x_2)\ldots(1-x_n)(1+x_1x_2x_3\ldots x_n)}}$$

$$=\sqrt{\pi^{n+3}}\,{}_{n+1}F_n\left(\underbrace{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}}_{n+1\ times};\overbrace{1,1,1,\ldots,1}^{n\ times};-1\right)$$

Proposed by Kaushik Mahanta – Assam – India

U.408 $\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{\cos(x_1)\cos(x_2)\cos(x_3)\ldots\cos(x_{n+1})dx_1dx_2\ldots dx_{n+1}}{x_1+x_2+x_3+\cdots+x_{n+1}}=$

$$= \frac{1}{4}\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma(n)} = \frac{1}{2n\binom{n-1}{\frac{n}{2}}}, \forall n \in \mathbb{N}$$

Proposed by Kaushik Mahanta – Assam – India

U.409 Prove that:

$$\int_0^{\frac{\pi}{16}} \log(\cos x)\, dx = 2\pi \log\left(\frac{G\left(\frac{7}{16}\right)}{G\left(\frac{25}{16}\right)}\right) + \frac{9\pi}{8}\log\left(\frac{2\pi}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)$$

where $G(z)$ is the Barnes G – function.

Proposed by Kaushik Mahanta – Assam – India

U.410 If $0 < a, b < 1$ then:

$$\int_{-a}^{a}\int_{-b}^{b}\frac{e^{x^2+y^2-2}}{(a^x+1)(b^x+1)}\,dxdy < 1$$

Proposed by Jalil Hajimir-Canada

U.411 Evaluate this integral:

$$\Omega = \int_0^\infty \frac{3x^{10}+x^8-4x^6+9x^4-5x^2+1}{3x^{14}+x^{12}-10x^{10}+3x^8-42x^6+26x^4-8x^2+1}\,dx$$

Proposed by Simon Peter-Madagascar

U.412 Solve this differential equation:

$$a\frac{\partial L(\alpha)}{\partial a} + b\frac{\partial L(\alpha)}{\partial b} = L(\alpha)$$

where:

$$L(\alpha) = \int_0^a \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\,dt$$

Proposed by Simon Peter-Madagascar

U.413 Evaluate:

$$\int_0^1 \log(2-x)\log(x)\log(2+x)\,dx$$

Proposed by Simon Peter-Madagascar

U.414 Show that:

$$\Phi = \int_0^1 \sqrt{\frac{1-x^2}{1+x^2}}\,dx = \frac{\sqrt{\pi}}{4}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - 4\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)$$

where: Γ – Gamma function

Proposed by Simon Peter-Madagascar

U.415 Evaluate:

$$\Omega = \int_{-\infty}^{\infty} \frac{1}{(1+x^{2n})^2} dx, n \in \mathbb{Z}$$

Proposed by Simon Peter-Madagascar

U.416 Prove that:

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_k^{(3)}}{2n+1} = \frac{\pi^2}{6} G + 4\beta(4) - \frac{3\pi}{2}\zeta(3)$$

where: $\beta(.)$: Beta Dirichlet function, H_k: Harmonic number, G: Catalan's constant

$\beta(4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4}$ and $H_k^3 = \sum_{k=1}^{\infty} \frac{1}{k^3}$

Proposed by Simon Peter-Madagascar

U.417 Evaluate:

$$\Phi = \int_0^1 \log\left[\log\left({}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{2}{3},\frac{4}{3};x\right)\right)\right] dx$$

where:

${}_3F_2(.)$: hypergeometric function

Proposed by Simon Peter-Madagascar

U.418

$$\Omega = \int_0^{2\pi}\int_0^{2\pi} \frac{\sinh(\eta)}{(\cosh(\eta)-\cos(\theta))^2} \cdot \sqrt{1 - c\cdot\sinh^2(\eta)\sin(\phi)}\, d\theta d\phi$$

where η and c are the parameters such that $\sinh^2(\eta) = 2$ and $c \cdot \sinh^2(\eta) < 1$

Proposed by Simon Peter-Madagascar

U.419 Prove that:

$$A = \int_{-\infty}^{\infty} \frac{\ln(t+1)}{t^2+1} dt = \frac{\pi}{2}\left(\ln(2) + \frac{\pi}{2} i\right)$$

Proposed by Simon Peter-Madagascar

U.420 If $0 < b < a$, show that:

$$I = \frac{1}{(2\pi i)^2} \int_{a-i\infty}^{a+i\infty} \int_{b-i\infty}^{b+i\infty} \frac{dvdu}{Ai(u)Ai(v)(u-v)} = \frac{1}{2}$$

Note: $Ai(z) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{t^3}{3} + zt\right) dt$ satisfies $Ai''(z) = zAi(z)$

Proposed by Simon Peter-Madagascar

U.421 Calculate the following integral for a fixed positive integers $d, n_0, \dots, n_d$

$$\int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} \dots \int_0^{1-x_1-\dots-x_d} (1-x_1-x_2-\dots-x_d)^{n_0} x_1^{n_1} x_2^{n_2} \dots x_d^{n_d} (x_1+x_2+\dots x_d) \cdot$$
$$\cdot (1-x_1)(1-x_2)\dots(1-x_d) dx_d dx_{d-1} \dots dx_1$$

Proposed by Simon Peter-Madagascar

U.422 Evaluate:

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{\ln[\sin(x)]\ln[\cos(x)]}{\tan(x)} dx = \frac{\zeta(3)}{8}$$

where: ζ: Zeta function

Proposed by Simon Peter-Madagascar

U.423 Generalized summation:

Prove that:

$$\sum_{x=1}^{\infty}\frac{\sin^n(x)}{x^n}=\frac{1}{2}\cdot\left(\frac{\pi}{2^{(n-1)}\cdot(n-1)!}\left(\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\cdot{}^nC_k\cdot(n-2k)^{(n-1)}\right)-1\right)$$

where [.] is greatest integer function.

Proposed by Amrit Awasthi-India

U.424 Prove that:

$$\forall\, n,m\in\mathbb{N}$$

$$\int_0^{\pi}\frac{\sin^{2m}(nx)}{x}dx\geq\frac{(2m)!}{2^{2m}(m!)^2}H_n$$

where H_n is nth Harmonic number.

Proposed by Amrit Awasthi-India

U.425 If for some x and y we have $\pi\cdot{}^xC_y\cdot{}^yC_x=\frac{2}{x-y}$

Then, find the value of: $\Omega=x-y$ ***Proposed by Amrit Awasthi-India***

U.426 Find z if:

$$\frac{-\ln(2-2\cos(1))}{2z}+\frac{i(\pi-1)}{2z}={}_2F_1(1,1;2;z)$$

where ${}_2F_1(a,b;c;z)$ is Gaussian hypergeometric function and $i=\sqrt{-i}$

Proposed by Amrit Awasthi-India

U.427 If: $x_k=\frac{1}{k}$ and $H_n=\sum_{k=1}^{n}x_k$ then find:

$$\Omega=\lim_{n\to\infty}e^{H_n}\prod_{k=1}^{\infty}\frac{1}{1+\pi x_k}$$

Proposed by Amrit Awasthi-India

U.428 If for $n\geq k$ and $n,k,a>0, n,k\in\mathbb{N}$;

$$\xi_a(k;n)=\sum_{r=k}^{n}\sqrt[r]{a}=\sqrt[k]{a}+\sqrt[k+1]{a}+\cdots+\cdots+\sqrt[n]{a}$$

and also $\zeta_n(s)=\sum_{k=1}^{n}\frac{1}{k^s}=\frac{1}{1^s}+\frac{1}{2^s}+\cdots+\cdots+\frac{1}{n^s}$ then, prove that:

$$2\left(\sqrt{n+1}-1\right)+\xi_e(2,n+1)<\zeta_n(0)+\zeta_n\left(\frac{1}{2}\right)+\zeta_n(1)<\xi_e(1,n)+2\sqrt{n}$$

where e is Euler's number that is $e=\sum_{n=0}^{\infty}\frac{1}{n!}$

Proposed by Amrit Awasthi-India

U.429 Find S:

$$S=\sum_{k=0}^{\infty}\frac{(-1)^k(5k^4+80k^3+465k^2+1160k+1044)}{(2+k)(3+k)(4+k)(5+k)(6+k)k!}$$

And also prove that: $I-J=S$ where

$$I=\int_0^1\ln\left(\frac{1}{t}\right)\left(1+\ln\left(\frac{1}{t}\right)+\ln^2\left(\frac{1}{t}\right)+\ln^3\left(\frac{1}{t}\right)+\ln^4\left(\frac{1}{t}\right)\right)dt$$

$$J = \int_1^\infty e^{-t}\, t(1 + t + t^2 + t^3 + t^4)dt$$

Proposed by Amrit Awasthi-India

U.430 Prove that:

$$\int_0^\infty \ln(\sqrt{2}x)\left(\frac{\pi x\sinh(2\pi x) - \cosh(2\pi x) + \pi x\sin(2\pi x) + \cos(2\pi x)}{x^3(\cosh(2\pi x) - \cos(2\pi x))}\right)dx = -\pi\zeta'(2)$$

Proposed by Amrit Awasthi-India

U.431 Prove that:

$$\forall n \in \mathbb{N}$$

$$\int_0^\pi \frac{\sin^2(nx)}{x}dx \geq \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \frac{H_n}{2}$$

Proposed by Amrit Awasthi-India

U.432 Prove that:

$$\forall m, n \in \mathbb{N}$$

$$\int_0^\pi \left|\frac{\sin^{2m+1}(nx)}{x}\right|dx \geq \frac{2^{2m+1}(m!)^2}{\pi(2m+1)!}H_n$$

where H_n is nth Harmonic number.

Proposed by Amrit Awasthi-India

U.433 If we have: $\xi(x,s) = \sum_{n=1}^\infty \frac{1}{x^s+n^s}$ then without the use of software prove that:

$$\xi(1,3) < \ln\left(\frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{\pi}\right) < \zeta(3)$$

Proposed by Amrit Awasthi-India

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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PROBLEMS FOR JUNIORS

JP.406 If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > 0; a + b + c = 3$ **then:** $(\boldsymbol{a^3+2})(\boldsymbol{b^3+2})(\boldsymbol{c^3+2}) \geq \boldsymbol{27}$

Proposed by Daniel Sitaru-Romania

JP.407 In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^3 \geq \frac{(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3)}{27a^2b^2c^2}$$

Proposed by Alex Szoros-Romania

JP.408 In ΔABC the following relationship holds:

$$\left(\frac{R}{r}\right)^2 + 4 \geq \frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{r_a r_b r_c} \geq \frac{3R+2r}{r} \geq 2\left(\frac{a}{b}+\frac{b}{a}\right)+4$$

Proposed by Alex Szoros-Romania

JP.409 If $a,b,c>1$ and $0 \leq \lambda \leq 1$ then

$$\frac{\log_b a}{\lambda + \log_a b + \log_a c} + \frac{\log_c b}{\lambda + \log_b a + \log_b c} + \frac{\log_a c}{\lambda + \log_c a + \log_c b} \geq \frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

JP.410 If $x,y,z>0$ and $n \in \mathbb{N}, n \geq 2$ then:

$$\sum_{cyc} \sqrt[n]{x^{2n-1}(y+z)} \geq \left(1+\frac{1}{2^n}\right)(xy+yz+zx)$$

Proposed by Marin Chirciu-Romania

JP.411 In ΔABC the following relationship holds:

$$\frac{ab(a+b)}{\sqrt{2(a^2+b^2)}} + \frac{bc(b+c)}{\sqrt{2(b^2+c^2)}} + \frac{ca(c+a)}{\sqrt{2(c^2+a^2)}} \geq 4\sqrt{3}F$$

Proposed by Marian Ursărescu-Romania

JP.412 In $\Delta ABC, I$ –incenter, the following relationship holds:

$$AI^6 + BI^6 + CI^6 \leq 64[(R^2 - Rr + r^2)^3 - 24r^6]$$

Proposed by Marian Ursărescu-Romania

JP.413 If $(a_n)_{n\geq 1}$ be increasing sequence with $a_i > 0, \forall i = \overline{1,n}$ and $k \in \mathbb{N}, k \geq 2$ solve for real numbers:

$$\sqrt[k]{\frac{a_1^x+1}{a_2^x+1}} + \sqrt[k]{\frac{a_2^x+1}{a_3^x+1}} + \sqrt[k]{\frac{a_{n-1}^x+1}{a_n^x+1}} = n-1+\sqrt[k]{\frac{a_1^x+1}{a_n^x+1}}$$

Proposed by Florică Anastase-Romania

JP.414 Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ x^4 + 1 = 5(y^2 + z^2) \\ x + y + z = 2\sqrt[3]{xyz} + \dfrac{3xyz}{xy + yz + zx} \end{cases}$$

Proposed by Daniel Sitaru-Romania

JP.415 In any quadrilateral with the sides' lengths a, b, c, d

$$\frac{1}{a(b+c+d-a)} + \frac{1}{b(a+c+d-b)} + \frac{1}{c(a+b+d-c)} + \frac{1}{d(a+b+c-d)} \geq \frac{32}{(a+b+c+d)^2}$$

Proposed by Florentin Vișescu – Romania

JP.416 Solve in $\mathbb{R}_+$ the equation:

$$\sqrt{3n-1+\sqrt{8n^2-4n}} \cdot \sqrt{7n-5+\sqrt{48n^2-68n+24}} \cdot$$

$$\cdot \sqrt{5n-3+\sqrt{24n^2-28n+8}} \cdot \sqrt{5n-3+\sqrt{16n^2-12n}} = (10n-6)^2$$

Proposed by George – Florin Șerban – Romania

JP.417 Prove that in any ΔABC the following inequality holds:

$$\sum \sin^3\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \geq \frac{3}{16}\sum \cos A$$

Proposed by Gheorghe Alexe and George Florin Șerban – Romania

JP.418 Let be $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0$,

$$\left(\sum_{k=1}^{n} x_k\right)^2 > 2\prod_{k=1}^{n} x_k, \sum_{k=1}^{n} x_k < \sum_{k=1}^{n} y_k$$

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) < \prod_{k=1}^{n} x_k + \prod_{k=1}^{n} y_k$$

Prove that:

$$\left(\sum_{k=1}^{n} x_k\right) \cdot \left(\prod_{k=1}^{n} y_k\right) > \left(\sum_{k=1}^{n} y_k\right) \cdot \left(\prod_{k=1}^{n} x_k\right)$$

Proposed by George Florin Șerban – Romania

JP.419 Find all $a,b\in\mathbb{Z}$ such that

$$\sqrt[3]{1+\sqrt{2019-ab}}+\sqrt[3]{1-\sqrt{2019-ab}}\in\mathbb{Z}$$

Proposed by Pedro Pantoja-Natal-Brazil

JP.420 Let a,b,c be positive real numbers such that $a^2+b^2+c^2=3$.

Prove that:

$$\frac{a^3+b^3+c^2+1}{b^3(c^2+1)}+\frac{b^3+c^3+a^2+1}{c^3(a^2+1)}+\frac{c^3+a^3+b^2+1}{a^3(b^2+1)}\geq 6$$

Proposed by Pedro Pantoja-Natal-Brazil

PROBLEMS FOR SENIORS

SP.406 Let a,b,c,d be positive real numbers. Find the maximum value of the expression:

$$\frac{\sqrt[4]{\frac{abc}{ab+ac+bc}}+\sqrt[4]{\frac{abd}{ab+ad+bd}}+\sqrt[4]{\frac{acd}{ac+ad+cd}}+\sqrt[4]{\frac{bcd}{bc+bd+cd}}}{\sqrt[4]{a^4+b^4+c^4+d^4}}$$

Proposed by Kunihiko Chikaya-Japan

SP.407 If $X,Y,Z\in M_{11}(\mathbb{C});X^3=Y^5=Z^7=I_{11};XY=YX;YZ=ZY;$

$$ZX=XZ;\Omega=2XYZ+X^2(Y+Z)+Y^2(Z+X)+Z^2(X+Y)$$

then $\det(\Omega)\neq 0$.

Proposed by Daniel Sitaru-Romania

SP.408Let ABC be an equilateral triangle such that $|z_A|=|z_B|=|z_C|$. Find $z\in\mathbb{C}$ such that

$$\begin{cases}|z-z_A|\leq|z_B+z_C|\\|z-z_B|\leq|z_C+z_A|\\|z-z_C|\leq|z_A+z_B|\end{cases}$$

Proposed by Ionuț Florin Voinea-Romania

SP.409 Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that

$$f(x-y)=f(x)-f(y)-xy(x-y),\forall x,y\in\mathbb{Q}$$

Proposed by Ionuț Florin Voinea-Romania

SP.410 Let $z_1,z_2,z_3\in\mathbb{C}^*$ different in pairs such that $|z_1|=|z_2|=|z_3|=1$,

$$A(z_1), B(z_2), C(z_3). \sum_{cyc} |z_1 - z_2 - z_3|^4 = 243 \Rightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

SP.411 Let $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that
$|z_1| = |z_2| = |z_3|, A(z_1), B(z_2), C(z_3)$.

$$\sum_{cyc} \frac{1}{8z_1z_2z_3 - (z_1^2 + z_2z_3)(z_2 + z_3)} = \frac{3}{10z_1z_2z_2} \Rightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

SP.412 Let $A \in M_n(\mathbb{R})$ such that $A^{2021} = I_n + A + A^2 + \cdots + A^{2019}$. Prove that:

$$\det(A^3 + I_n) \geq 0$$

Proposed by Marian Ursărescu-Romania

SP.413 Let $\alpha > 1$ fixed. For $\forall n \in \mathbb{N}^*$ denote $k(n) = \min\{k \in \mathbb{N} \mid (n+1)^k \geq \alpha \cdot n^k\}$ and $(x_n)_{n\geq1}, x_{n+1} = x_n + \frac{1}{e^{x_n}}$. Find: $\Omega = \lim_{n\to\infty} \frac{k(n)\cdot\log\sqrt[n]{n}}{x_n}$

Proposed by Florică Anastase-Romania

SP.414 Solve for real numbers:

$$\begin{cases} x^4 = \sqrt{y^4 + 8} - \sqrt{y^4 + 3} \\ y^4 = \sqrt{z^4 + 8} - \sqrt{z^4 + 3} \\ z^4 = \sqrt{t^4 + 8} - \sqrt{t^4 + 3} \\ t^4 = \sqrt{x^4 + 8} - \sqrt{x^4 + 3} \end{cases}$$

Proposed by Daniel Sitaru-Romania

SP.415 Solve for real numbers: $\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = 1$

Proposed by Daniel Sitaru-Romania

SP.416 If $-3 < x, y, z < 3, x + y + z = 0$ then:

$$\left|\frac{xyz}{9 + xy + yz + zx}\right| < 3$$

Proposed by Daniel Sitaru-Romania

SP.417 Let $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ be sequences of real numbers such that

$$x_n = \sum_{k=3}^{n} \tan\left(\frac{\pi}{k}\right) - \pi\log n, \; y_n = \sum_{k=1}^{n} 2^{k-1} \cdot \left[\frac{k^2}{k+1}\right], [*] - \text{GIF}.$$

$$\text{Find: } \Omega = \lim_{n\to\infty} \frac{2^n \cdot x_n}{y_n}$$

Proposed by Florică Anastase-Romania

SP.418 Solve for real numbers:

$$\begin{cases} \sin^3 x + \cos^3 y + z^3 + 3z = 3z^2 + 2 \\ \sin^2 x + \cos^2 y + z^2 = 2z + 2 \\ \sin x + \cos y + z = 2 \end{cases}$$

Proposed by Daniel Sitaru-Romania

SP.419 If $a, b, c \in \mathbb{R}; \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$, then solve for real numbers:

$$\sin x \cdot \sin y \cdot \sin z = \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}$$

Proposed by Daniel Sitaru-Romania

SP.420 If $x, y, z \in \mathbb{R}, 32(x^5 + y^5 + z^5) = 3$, then:

$$\sum_{cyc} (2x^6 + x^4 + x^3 + x^2) + \frac{51}{32} \geq 2(x + y + z)$$

Proposed by Daniel Sitaru-Romania

UNDERGRADUATE PROBLEMS

UP.406 If $0 < a \leq b$ then:

$$\left(\int_a^b \frac{x^2+1}{x^3+1} dx\right)\left(\int_a^b \frac{\sqrt{x}}{x^3+1} dx\right) \leq \frac{(b-a)^2}{\sqrt{a}(1+a^2)}$$

Proposed by Daniel Sitaru-Romania

UP.407 If $0 < a \leq b$ then:

$$2\int_a^b \int_a^b \sqrt{x^2 + xy + y^2}\, dxdy \geq \sqrt{3}(b+a)(b-a)^2$$

Proposed by Daniel Sitaru – Romania

UP.408 If $f, g: [a, b] \to (0, \infty); 0 < a \leq b; f, g$ – continuous, then:

$$6\int_a^b \frac{f(x)g(x)}{f(x)+g(x)} dx \leq \int_a^b \big(f(x) + g(x)\big) dx + \int_a^b \sqrt{f(x)g(x)}\, dx$$

Proposed by Daniel Sitaru – Romania

UP.409 If $a, b, c, d \in \left(0, \frac{4\pi}{\pi^2-4}\right)$ then:

$$\int_a^b \frac{\tan^{-1} x}{x} dx + \int_c^d \frac{\tan^{-1} x}{x} dx > \frac{\pi}{2} \cdot \log\left(\frac{4\pi\sqrt{bd}}{(2a+\pi)(2c+\pi)}\right)$$

Proposed by Daniel Sitaru-Romania

UP.410 Let $(x_n)_{n\geq 1}$ be sequence of real numbers such that $x_n = \sum_{k=1}^{n} \sin\frac{\pi}{k} - \pi \log n$

$$\text{Find: } \Omega = \lim_{n\to\infty} x_n \cdot \sum_{k=1}^{n} \frac{1}{n + \sqrt[3]{(k+1)^2(k^2+1)^2}}$$

Proposed by Florică Anastase-Romania

UP.411 Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of real numbers such that

$$x_n = \sum_{k=1}^{n} \sin\frac{1}{k} + \log\left(\sin\frac{1}{n}\right), y_n = \sum_{k=1}^{n^2+n} \left[\sqrt{k} + \frac{1}{2}\right], [*] - \text{GIF}.$$

$$\text{Find: } \Omega = \lim_{n\to\infty} \frac{x_n}{y_n}$$

Proposed by Florică Anastase-Romania

UP.412 Find:

$$\Omega = \lim_{n\to\infty} \frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}} + \frac{1}{\sqrt{1-\frac{1}{2^2}}} + \cdots + \frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{\left[\sqrt[3]{1}\right] + \left[\sqrt[3]{2}\right] + \left[\sqrt[3]{3}\right] + \cdots + \left[\sqrt[3]{n^3-1}\right]}, [*] - \text{GIF}, \alpha \in \mathbb{R}$$

Proposed by Florică Anastase-Romania

UP.413 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\tan^{-1} k}{k} \cdot \tan^{-1}(n-k+1)$$

Proposed by Daniel Sitaru-Romania

UP.414 If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \int_a^b \frac{z \cdot \min\left(x, \frac{1}{y}, y + \frac{1}{x}\right) dxdydz}{z^2+1} \leq \frac{\sqrt{2}}{2}(b-a)^2 \log\left(\frac{b^2+1}{a^2+1}\right)$$

Proposed by Daniel Sitaru-Romania

UP.415 Let ABC denote a triangle and H its orthocenter. Let point M be the middle of the segment AH. Prove that: (a) angle BMC is acute. (b) area $\Delta BMC = \frac{1}{8}\cdot AH^2\cdot \tan\widehat{BMC}$.

Proposed by George Apostolopoulos-Messolonghi-Greece

UP.416 Let ABC denote a triangle with circumradius R. Let D, E, F be chosen on sides BC, CA, AB, respectively, so that AD, BE and CF bisect the angles of ABC. Prove:

$R \geq 2R'$, where R' denotes the circumradius of triangle DEF.

Proposed by George Apostolopoulos-Messolonghi-Greece

UP.417 Find:

$$\Omega(\mathrm{a}) = \lim_{x\to\infty}\left((x+a)\sqrt[x+1]{\Gamma(x+2)}\sin\frac{1}{x+a} - x\sqrt[x]{\Gamma(x+1)}\sin\frac{1}{x}\right); a > 0$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP.418 In ΔABC the following relationship holds:

$$\frac{3}{2}\cdot\sqrt[6]{\frac{4F}{R^2}} \leq \sum_{cyc}\sqrt{\frac{r_a}{b+c}} \leq \frac{1}{2}\left(1+\frac{4R}{r}\right)\sqrt{\frac{Rr}{2F}}$$

Proposed by Marin Chirciu-Romania

UP.419 If $n \in \mathbb{N}; n \geq 3$ then:

$$n^{\frac{1}{n}+\frac{1}{n^2}+\cdots+\frac{1}{n^n}} > (n+1)^{\sqrt[n]{(n+1)^{n+1}}}$$

Proposed by Daniel Sitaru-Romania

UP.420 If $x \geq 0$ then:

$$\frac{3\cosh(4x)+5\cosh(3x)}{\cosh x\,(3+5e^{-x})(3+5e^{x})} \geq \frac{\operatorname{sech}^5 x}{3+5\operatorname{sech} x}$$

Proposed by Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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