THE EQUIVALENCE OF HADWIGER-FINSLER'S AND DOUCET'S TRIANGLE INEQUALITIES

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ABSTRACT. In this paper its presented a detailed proof for the equivalence of Hadwiger-Finsler's and Doucet's triangle inequalities.

1. NOTATIONS AND PRELIMINARIES:

For $x, y, z > 0$, denote: $p = x + y + z, q = xy + yz + zx, t = xyz$.
For any triangle ABC, denote: $a = BC, b = CA, c = AB-$sides, $s = \frac{a+b+c}{2}-$semiperimeter, $F-$area, $R-$circumradii, $r-$inradii.
We recall Voiculescu-Ravi's substitutions:

$$a = y + z, b = z + x, c = x + y; x, y, z > 0$$

It is clear that:

$$a + b > c, b + c > a, c + a > b$$

$$s = \frac{a + b + c}{2} = \frac{2(x + y + z)}{2} = x + y + z = p$$

$$s - a = x, s - b = y, s - c = z$$

By Heron's formula:

$$F = \sqrt{s(s - a)(s - b)(s - c)} = \sqrt{xyz(x + y + z)} = \sqrt{pt}$$

$$R = \frac{abc}{4F} = \frac{(x + y)(y + z)(z + x)}{4\sqrt{xyz(x + y + z)}} = \frac{pq - t}{4\sqrt{pt}}$$

$$r = \frac{F}{s} = \frac{\sqrt{xyz(x + y + z)}}{x + y + z} = \sqrt{\frac{xyz}{x + y + z}} = \sqrt{\frac{t}{p}}$$

Lemma 1.
In any triangle ABC the following relationship holds:

$$\boldsymbol{a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)}$$

Proof.

$$2(s^2 - 4Rr - r^2) = 2\left(p^2 - 4 \cdot \frac{pq - t}{4\sqrt{pt}} \cdot \sqrt{\frac{t}{p}} - \frac{t}{p}\right) =$$

$$= 2\left(p^2 - \frac{pq - t}{p} - \frac{t}{p}\right) = 2\left(p^2 - q + \frac{t}{p} - \frac{t}{p}\right) = 2(p^2 - q) =$$

$$= 2((x + y + z)^2 - (xy + yz + zx)) = 2(x^2 + y^2 + z^2 + xy + yz + zx) =$$

$$= x^2 + 2xy + y^2 + y^2 + 2yz + z^2 + z^2 + 2zx + x^2 =$$
$$= (x+y)^2 + (y+z)^2 + (z+x)^2 = a^2 + b^2 + c^2.$$

□

Lemma 2.
In any triangle ABC the following relationship holds:

$$\boldsymbol{ab + bc + ca = s^2 + 4Rr + r^2}$$

Proof.

$$s^2 + 4Rr + r^2 = p^2 + 4 \cdot \frac{pq - t}{4\sqrt{pt}} \cdot \sqrt{\frac{t}{p}} + \frac{t}{p} =$$
$$= p^2 + \frac{pq - t}{p} + \frac{t}{p} = p^2 + q = (x+y+z)^2 + xy + yz + zx =$$
$$= x^2 + xy + zx + zy + y^2 + yz + xy + xz + z^2 + zx + yz + yx =$$
$$= x(x+y) + z(x+y) + y(y+z) + x(y+z) + z(z+x) + y(z+x) =$$
$$= (x+y)(x+z) + (y+z)(y+x) + (z+x)(z+y) = ab + bc + ca.$$

□

Doucet's inequality:

$$\boldsymbol{s\sqrt{3} \leq 4R + r}$$

Proof. With Voiculescu-Ravi's substitutions:

$$(x+y+z)\sqrt{3} \leq 4 \cdot \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}} + \sqrt{\frac{xyz}{x+y+z}}$$
$$p\sqrt{3} \leq \frac{pq - t}{\sqrt{pt}} + \sqrt{\frac{t}{p}}$$
$$p\sqrt{3pt} \leq pq - t + t$$
$$p\sqrt{3pt} \leq pq \Leftrightarrow \sqrt{3pt} \leq q \Leftrightarrow 3pt \leq q^2$$
$$(xy + yz + zx)^2 \geq 3xyz(x+y+z)$$
$$x^2y^2 + y^2z^2 + z^2x^2 + 2xyz(x+y+z) \geq 3xyz(x+y+z)$$
$$x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x+y+z)$$
$$2x^2y^2 + 2y^2z^2 + 2z^2x^2 - 2x^2yz - 2xy^2z - 2xyz^2 \geq 0$$
$$x^2y^2 - 2xy \cdot yz + y^2z^2 + y^2z^2 - 2yz \cdot zx + z^2x^2 + z^2x^2 - 2zx \cdot xy + x^2y^2 \geq 0$$
$$(xy - yz)^2 + (yz - zx)^2 + (zx - xy)^2 \geq 0$$

□

2. Main result

The inequalities Hadwiger-Finsler and Doucet are equivalents.

Proof. We will write successively Hadwiger-Finsler's inequality using lemma 1 and lemma 2.

$$a^2+b^2+c^2 \geq 4\sqrt{3}F+(a-b)^2+(b-c)^2+(c-a)^2$$
$$a^2+b^2+c^2 \geq 4\sqrt{3}rs+2(a^2+b^2+c^2)-2(ab+bc+ca)$$
$$4\sqrt{3}rs+a^2+b^2+c^2 \leq 2(ab+bc+ca)$$
$$4\sqrt{3}rs+a^2+b^2+c^2 \leq 2(ab+bc+ca)$$
$$4\sqrt{3}rs+2s^2-8Rr-2r^2 \leq 2(s^2+4Rr+r^2)$$
$$4\sqrt{3}rs-8rr-2r^2 \leq 8Rr+2r^2$$
$$4\sqrt{3}rs \leq 16Rr+4r^2$$
$$s\sqrt{3} \leq 4R+r$$

which it's Doucet's inequality. □

References

[1] Romanian Mathematical Magazine-`www.ssmrmh.ro`