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Crux Mathematicorum with Mathematical Mayhem

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Using Conway's triangle notation to solve algebraic problems

Daniel Sitaru

In this paper is presented a method for proving algebraic inequalities, based on a coordinate system for triangles invented by John H. Conway.

. . . .

1 Conway's Coordinates

Let $x, y, z \in \mathbb{R}$ be such that

$$x+y>0; y+z>0; z+x>0; xy+yz+zx>0; \tag{1}$$

and denote

$$a=\sqrt{x+y},\ b=\sqrt{y+z},\ c=\sqrt{z+x} \tag{2}$$

We will prove that a, b, c are edge lengths of a nondegenerate triangle ΔABC. We show that $a+b>c$; the other two cases of the triangle inequality follow by symmetry. Using (2), this becomes $\sqrt{x+y}+\sqrt{y+z}>\sqrt{z+x}$.

Squaring, we see (both sides being positive) that this is equivalent to

$$\begin{aligned} & x+y+y+z+2\sqrt{(x+y)(y+z)}>z+x \\ \Leftrightarrow\quad & y+\sqrt{(x+y)(y+z)}>0 \\ \Leftrightarrow\quad & y+\sqrt{xz+yz+xy+y^2}>y+|y|\geq 0 \end{aligned}$$

proving the claim. We now derive expressions for various values associated with this triangle.

The semiperimeter is clearly given by

$$s=\frac{1}{2}(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})\ . \tag{3}$$

The area is given by

$$F=\frac{1}{2}\sqrt{xy+yz+zx}. \tag{4}$$

Exercise. Prove this by plugging (3) into Heron's formula

$$F^2=s(s-a)(s-b)(s-c).$$

Let r, R be the inradius and circumradius of ΔABC. Using (3), we obtain:

$$r = \frac{F}{s} = \frac{\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}} \tag{5}$$

$$R = \frac{abc}{4F} = \frac{\sqrt{(x+y)(y+z)(z+x)}}{2\sqrt{xy+yz+zx}} \tag{6}$$

Let m_a, m_b, m_c be the medians of ΔABC; we have

$$m_a^2 = \frac{4x+y+z}{4}; \quad m_b^2 = \frac{x+4y+z}{4}; \quad m_c^2 = \frac{x+y+4z}{4}. \tag{7}$$

Exercise. Prove this using (2) and the familiar formula

$$m_a^2 = \frac{1}{2}(b^2+c^2) - \frac{1}{4}a^2.$$

Exercise. Let h_a, h_b, h_c be the altitudes of ΔABC. Show that

$$h_a = \sqrt{\frac{xy+yz+xz}{y+z}} = \sqrt{x+\frac{yz}{y+z}} \tag{8}$$

and similarly for h_b and h_c. (What geometric formula would you start with?)

To the problem solver, these formulae will have a familiar feel: they are very much like the cyclic sums that turn up in algebraic inequalities! Of course, triangle geometry is also a rich source of inequalities. If we were given a triangle inequality hidden in this way, could we unmask it? Here's a problem from a national Olympiad that seems to be built from the right parts. In particular, the given value for $xy+yz+zx$ and the existence of the square roots justify the use of Conway's coordinates - this could be a strong hint to the alert solver! If we translate it back into geometry, what do we find?

Example 1 (Turkey NMO - 2006). If x, y, z are positive numbers with $xy+yz+zx = 1$ then:

$$\frac{27}{4}(x+y)(y+z)(z+x) \geq (\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 6\sqrt{3}$$

Proof. Substituting (2), the first inequality to be proved becomes

$$\frac{27}{4}a^2b^2c^2 \geq (a+b+c)^2. \tag{9}$$

Using the familiar geometric formulae $R = abc/4F$ and $2s = a+b+c$, this becomes

$$27R^2F^2 \geq s^2. \tag{10}$$

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But (4) gives $F^2 = (xy + yz + zx)/4$ which is by hypothesis $1/4$. Plugging this in and taking square roots, (10) is equivalent to

$$\frac{3\sqrt{3}R}{2} \geq s,$$

one of a familiar pair of inequalities due to Mitrinović.

The second inequality to be proved becomes $4s^2 \geq 6\sqrt{3}$ or $s^2 \geq \frac{3}{2}\sqrt{3}$. Introducing $F = rs = 1/2$ (as we were given), this is equivalent to $s^2 \geq 3\sqrt{3}rs$, or

$$s \geq 3\sqrt{3}r,$$

the other Mitrinović inequality. □

Application 2 (IMO - DataBase - 2008). If x, y, z are three reals such that the numbers $y+z, z+x, x+y$ and $yz+zx+xy$ are all nonnegative, then:

$$\sum_{cyc} \sqrt{(z+x)(x+y)} \geq x+y+z+\sqrt{3(xy+yz+zx)}$$

Proof.
We will use Conway's substitutions:

$$a = \sqrt{y+z}; b = \sqrt{z+x}; c = \sqrt{x+y}$$

$$F = \frac{1}{2}\sqrt{xy+yz+zx}$$

The nequality can be written as:

$$\sum_{cyc} \sqrt{(z+x)(x+y)} \geq \frac{1}{2}\sum_{cyc}(\sqrt{x+y})^2 + \sqrt{3}\cdot\sqrt{xy+yz+zx}$$

$$\sum_{cyc} ab \geq \frac{1}{2}(a^2+b^2+c^2) + 2\sqrt{3}F$$

$$\frac{1}{2}(a^2+b^2+c^2) \geq a^2+b^2+c^2-(ab+bc+ca)+2\sqrt{3}F$$

$$a^2+b^2+c^2 \geq (a-b)^2+(b-c)^2+(c-a)^2+4\sqrt{3}F$$

which is the well known Hadwiger-Finsler's inequality in triangle. □

2 Problems

If $x, y, z \in \mathbb{R}$ are such that: $x+y>0; y+z>0; z+x>0; xy+yz+zx>0$ then:

1. $\sum_{cyc}(x+y)\sqrt{(x+z)(y+z)} \geq 4(xy+yz+zx)$

2. $(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 6\sqrt{3(xy+yz+zx)}$
3. $2(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})\sqrt{xy+yz+zx} \leq 3\sqrt{3(x+y)(y+z)(z+x)}$
4. $8(x+y+z)(xy+yz+zx) \leq 9(x+y)(y+z)(z+x)$
5. $x+y+z \geq \sqrt{3(xy+yz+zx)}$
6. $\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}} \geq 2\sqrt{\frac{3(xy+yz+zx)}{(x+y)(y+z)(z+x)}}$
7. $\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}} \leq \frac{\sqrt{3}(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})}{2\sqrt{xy+yz+zx}}$
8. $\frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}} \geq \frac{4(xy+yz+zx)}{(x+y)(y+z)(z+x)}$
9. $\frac{1}{\sqrt{(x+y)(x+z)}}+\frac{1}{\sqrt{(y+z)(y+x)}}+\frac{1}{\sqrt{(z+x)(z+y)}} \leq \frac{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2}{4(xy+yz+zx)}$
10. $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \geq \frac{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}{\sqrt{(x+y)(y+z)(z+x)}}$
11. $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \leq \frac{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2}{4(xy+yz+zx)}$
12. $\sqrt{\frac{x+y}{x+z}}+\sqrt{\frac{x+z}{x+y}} \leq \frac{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})\sqrt{(x+y)(y+z)(z+x)}}{2(xy+yz+zx)}$
13. $(x+y+z)(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 18(xy+yz+zx)$
14. $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)} \geq \frac{36(xy+yz+zx)}{(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2}$
15. $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)} \leq \frac{9(x+y)(y+z)(z+x)}{4(xy+yz+zx)}$
16. $\sqrt{(x+y)(x+z)}+\sqrt{(y+z)(y+x)}+\sqrt{(z+x)(z+y)} \geq 2\sqrt{3(xy+yz+zx)}$
17. $x^2+y^2+z^2 \geq xy+yz+zx$
18. $(x+y)(x+z)+(y+z)(y+x)+(z+x)(z+y) \geq 4(xy+yz+zx)$
19. $\frac{9\sqrt{(x+y)(y+z)(z+x)}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}} \geq 2\sqrt{3(xy+yz+zx)}$
20. $\sqrt{3}(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}) \leq 4\sqrt{\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}}+\frac{2\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$
21. $\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}} \geq \frac{9\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}}$
22. $\sqrt{x+\frac{y+z}{4}}+\sqrt{y+\frac{z+x}{4}}+\sqrt{z+\frac{x+y}{4}} \leq \frac{9\sqrt{(x+y)(y+z)(z+x)}}{4\sqrt{xy+yz+zx}}$
23. $\frac{3\sqrt{xy+yz+zx}}{\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}} \leq \sqrt[4]{\frac{3}{4}(xy+yz+zx)}$

3 Hints

With Conway's substitutions, the inequalities 1-23 are well known:

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1. $R \geq 2r$ (Euler)
2. $s \geq 3\sqrt{3}r$ (Mitrinović I)
3. $s \leq \frac{3\sqrt{3}}{2}R$ (Mitrinović II)
4. $a^2 + b^2 + c^2 \leq 9R^2$ (Leibniz)
5. $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$ (Ionescu-Weitzenbock)
6. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{\sqrt{3}}{R}$ (Leuenberger I)
7. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}$ (Leuenberger II)
8. $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}$ (Leuenberger III)
9. $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \leq \frac{1}{4r^2}$ (Leuenberger IV)
10. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{2Rr}$ (Steinig I)
11. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}$ (Steinig II)
12. $\frac{b}{c} + \frac{c}{b} \leq \frac{R}{r}$ (BĂndilĂ)
13. $a^2 + b^2 + c^2 \geq 36r^2$ (Neuberg)
14. $ab + bc + ca \geq 36r^2$ (Leuenberger V)
15. $ab + bc + ca \leq 9R^2$ (Leuenberger VI)
16. $ab + bc + ca \geq 4\sqrt{3}F$ (Gordon)
17. $a^4 + b^4 + c^4 \geq 16F^2$ (Goldner I)
18. $a^2b^2 + b^2c^2 + c^2a^2 \geq 16F^2$ (Goldner II)
19. $\frac{9abc}{a+b+c} \geq 4F\sqrt{3}$ (Curry)
20. $s\sqrt{3} \leq 4R + r$ (Doucet)
21. $m_a + m_b + m_c \geq 9r$ (Gotman I)
22. $m_a + m_b + m_c \leq \frac{9R}{2}$ (Gotman II)
23. $\frac{3}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} \leq \sqrt[4]{3F^2}$ (Makowski)

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