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GAMES WITH SUMS AND DIGITS

By Carmen-Victorița Chirfot-Romania
Edited by Florică Anastase-Romania

Abstract: *In this paper are presented two short calculus formulas and a few applications.*

Introduction.

We know that: $\boldsymbol{a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2), a, b \in \mathbb{R}; \ (1)}$.

We want to find some a few sums in complex forms using two identities.

Replacing a and b with consecutive values, we have:

$$n^4 - (n-1)^4 = 1 \cdot (2n-1)((n-1)^2 + n^2), n \in \mathbb{N}^*$$

$$(n-1)^4 - (n-2)^4 = 1 \cdot (2n-3)((n-2)^2 + (n-1)^2)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$3^4 - 2^4 = 1 \cdot 5(2^2 + 3^2)$$

$$2^4 - 1^4 = 1 \cdot 3(1^2 + 2^2)$$

$$1^4 - 0^4 = 1 \cdot 1(0^2 + 1^2)$$

By adding these relationships, we get:

$$1(0^2 + 1^2) + 3(1^2 + 2^2) + 592^2 + 3^2) + \dots + (2n-3)((n-2)^2 + (n-1)^2)$$
$$+ (2n-1)((n-1)^2 + n^2) = n^4, n \in \mathbb{N}^*$$

Another sum who we can find using the relationship (1) is:

$$S = 1 \cdot 3(1^2 + 2^2) + 2 \cdot 6(2^2 + 4^2) + 3 \cdot 11(4^2 + 7^2) + 4 \cdot 18(7^2 + 11^2) + \dots +$$
$$+ n \cdot (n^2 + 2)\left(\left(\frac{n^2 - n + 2}{2}\right)^2 + \left(\frac{n^2 + n + 2}{2}\right)^2\right), n \in \mathbb{N}^*$$

For some terms, the relationship (1) has $a - b = k$ and $a + b = k^2, k = \overline{1, n}, n \in \mathbb{N}^*$. So,

$$1 \cdot 3(1^2 + 2^2) = 2^4 - 1^4$$

$$2 \cdot 6(2^2 + 4^2) = 4^4 - 2^4$$

$$3 \cdot 11(4^2 + 7^2) = 7^4 - 4^4$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$n\cdot(n^2+2)\left(\left(\frac{n^2-n+2}{2}\right)^2+\left(\frac{n^2+n+2}{2}\right)^2\right)=\left(\frac{n^2+n+2}{2}\right)^4-\left(\frac{n^2-n+2}{2}\right)^4$$

By adding, we get:

$$\boldsymbol{S=\left(\frac{n^2+n+2}{2}\right)^4-1;\quad (2)}$$

Let's using the identity: $\boldsymbol{a^3-b^3=(a-b)(a^2+ab+b^2), a,b\in\mathbb{R};\quad (3)}$. For example,

$$11^3-1^3=10(1^2+1\cdot 11+11^2)$$

$$111^3-11^3=100(11^2+11\cdot 111+111^2)$$

$$1111^3-111^3=1000(111^2+111\cdot 1111+1111^2)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\underbrace{\overline{11\dots 11}}_{2022-digits}{}^3-\underbrace{\overline{11\dots 11}}_{2021-digits}{}^3=\underbrace{\overline{100\dots 00}}_{2022-digits}\left(\underbrace{\overline{11\dots 11}}_{2021-digits}{}^2+\underbrace{\overline{11\dots 11}}_{2021-digits}\cdot\underbrace{\overline{11\dots 11}}_{2022-digits}+\underbrace{\overline{11\dots 11}}_{2022-digits}{}^2\right)$$

By adding, we get:

$$\boldsymbol{S=10(1^2+1\cdot 11+11^2)+10^2(11^2+11\cdot 111+111^2)+}$$

$$\boldsymbol{+10^3(111^2+111\cdot 1111+1111^2)+\dots+}$$

$$\boldsymbol{\dots+10^{2021}\left(\underbrace{\overline{11\dots 11}}_{2021-digits}{}^2+\underbrace{\overline{11\dots 11}}_{2021-digits}\cdot\underbrace{\overline{11\dots 11}}_{2022-digits}+\underbrace{\overline{11\dots 11}}_{2022-digits}{}^2\right)=}$$

$$\boldsymbol{=\underbrace{\overline{11\dots 11}}_{2022-digits}{}^3-1;\quad (4)}$$

Applications.

1) Find:

$$\boldsymbol{S=1(0^2+1^2)+3(1^2+2^2)+5(2^2+3^2)+\dots+2021(1010^2+1011^2)}$$

Solution.

$$1(0^2+1^2)+3(1^2+2^2)+5(2^2+3^2)+\dots+(2n-3)((n-2)^2+(n-1)^2)$$
$$+(2n-1)((n-1)^2+n^2)=n^4$$

The last term has $2n-1=2021\Rightarrow 2n=2022\Rightarrow n=1011$. So $S=1011^4$.

2) Prove that the sum

$S=1\cdot 3(1^2+2^2)+2\cdot 6(2^2+4^2)+3\cdot 11(4^2+7^2)+4\cdot 18(7^2+11^2)+\dots+10\cdot 102(46^2+56^2)$ is multiply of 11.

Solution.

From (2): $S=\left(\frac{n^2+n+2}{2}\right)^4-1=\left(\frac{100+10+2}{2}\right)^4-1=56^4-1=55\cdot 57(1+56^2)$.

3) Find:

$$S=10\cdot 12(1^2+11^2)+10^2\cdot 122(11^2+111^2)+10^3\cdot 1222(111^2+1111^2)+\cdots$$
$$+10^{2021}\cdot \underbrace{\overline{122\dots 22}}_{2022-digits}\left(\underbrace{\overline{11\dots 11}^2}_{2021-digits}+\underbrace{\overline{11\dots 11}^2}_{2022-digits}\right)$$

Solution.

$$11^4-1^4=10\cdot 12(1^2+11^2)$$
$$111^4-11^4=100\cdot 122(11^2+111^2)$$
$$1111^4-111^4=1000\cdot 1222(111^2+1111^2)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\underbrace{\overline{11\dots 11}^4}_{2022-digits}-\underbrace{\overline{11\dots 11}^4}_{2021-digits}=\underbrace{\overline{100\dots 00}}_{2022-digits}\cdot\underbrace{\overline{122\dots 22}}_{2022-digits}(\underbrace{\overline{11\dots 11}^2}_{2021-digits}+\underbrace{\overline{11\dots 11}^2}_{2022-diggits}$$

So, we have:

$$S=10\cdot 12(1^2+11^2)+10^2\cdot 122(11^2+111^2)+10^3\cdot 1222(111^2+1111^2)+\cdots$$
$$+10^{2021}\cdot \underbrace{\overline{122\dots 22}}_{2022-digits}\left(\underbrace{\overline{11\dots 11}^2}_{2021-digits}+\underbrace{\overline{11\dots 11}^2}_{2022-digits}\right)=\underbrace{\overline{11\dots 11}^4}_{2022-digits}-1$$

4) Prove that the sum

$S=11\cdot 13(1^2+12^2)+111\cdot 135(12^2+123^2)+1111\cdot 1357(123^2+1234^2)+11111\cdot 13579(1234^2+12345^2)$ cannot be a perfect square.

Solution.

$$12^4-1^4=11\cdot 13(1^2+12^2)$$
$$123^4-12^4=111\cdot 135(12^2+123^2)$$
$$1234^4-123^4=1111\cdot 1357(123^2+1234^2)$$
$$12345^4-1234^4=11111\cdot 13579(1234^2+12345^2)$$

Therefore, $S=12345^4-1$.

5) Prove that:

$$\mathbf{1(1^2+3^2)+2(3^2+5^2)+3(5^2+7^2)+\cdots+1009(20172+2019^2)=\frac{2019^4-1}{8}}$$

Solution. We have:

$$2\cdot 4(1^2+3^2)+2\cdot 8(3^2+5^2)+2\cdot 12(5^2+7^2)+\cdots+2\cdot 4036(2017^2+2019^2)$$
$$=2019^4-1$$
$$3^4-1=2\cdot 4(1^2+3^2), 5^4-3^4=2\cdot 8(3^2+5^2)$$
$$7^4-5^4=2\cdot 12(5^2+7^2),\dots,2019^4-2017^4=2\cdot 4036(2017^2+2019^2)$$

Proposed problems.

P 1) Find:

$$\mathbf{S=11(1^2+1\cdot 12+12^2)+111(12^2+12\cdot 123+123^2)}$$
$$\mathbf{+1111(123^2+123\cdot 1234+1234^2)+\cdots+}$$
$$\mathbf{+\underbrace{\overline{11\dots 11}}_{9-digits}(\overline{12\dots 78}^2+\overline{12\dots 78}\cdot\overline{12\dots 89}+\overline{12\dots 89}^2)}$$

Indications. Using the relationship (3), we have:

$$12^3-1^3=11(1^2+1\cdot 12+12^2)$$
$$123^3-12^3=111(12^2+12\cdot 123+123^2)$$
$$1234^3-123^3=1111(123^2+123\cdot 1234+1234^2)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\overline{12\dots 89}^3-\overline{12\dots 78}^3=\underbrace{\overline{11\dots 11}}_{9-digits}(\overline{12\dots 78}^2+\overline{12\dots 78}\cdot\overline{12\dots 89}+\overline{12\dots 89}^2)$$

Therefore, $S=\overline{12\dots 89}^3-1$.

P 2) Find:

$$\mathbf{S=2\cdot 10(1^2+1\cdot 21+21^2)+3\cdot 10^2(21^2+21\cdot 321+321^2)+}$$
$$\mathbf{+4\cdot 10^3(321^2+321\cdot 4321+4321^2)+\cdots+}$$
$$\mathbf{+10^8(\overline{87\dots 21}^2+\overline{87\dots 21}\cdot\overline{98\dots 21}+\overline{98\dots 21}^2)}$$

Indications. By the relationship (3), we have:

$$21^3-1^3=20(1^2+1\cdot 21+21^2)$$
$$321^3-21^3=300(21^2+21\cdot 321+321^2)$$
$$4321^3-321^3=4000(321^2+321\cdot 4321+4321^2)$$

… …

$$\overline{98\dots21}^3-\overline{87\dots21}^3=\underbrace{\overline{10\dots00}}_{9-digits}\left(\overline{87\dots21}^2+\overline{87\dots21}\cdot\overline{98\dots21}+\overline{98\dots21}^2\right)$$

Therefore, $S=\overline{98\dots21}^3-1$.

P 3) Determine the real number a such that

$$\boldsymbol{a\cdot10(1^2+1\cdot11+11^2)+a\cdot10^2(11^2+11\cdot111+111^2)+}$$
$$\boldsymbol{+a\cdot10^3(111^2+111\cdot1111+1111^2)+\dots+}$$
$$\boldsymbol{\dots+a\cdot10^{2021}\left(\underbrace{\overline{11\dots1}1^2}_{2021-digits}+\underbrace{\overline{11\dots11}}_{2021-digits}\cdot\underbrace{\overline{11\dots11}}_{2022-digits}+\underbrace{\overline{11\dots11}^2}_{2022-digits}\right)=}$$
$$\boldsymbol{=\underbrace{\overline{11\dots11}^3}_{2022-digits}-1}$$

Indications.

$$a[10(1^2+1\cdot11+11^2)+10^2(11^2+11\cdot111+111^2)+$$
$$+10^3(111^2+111\cdot1111+1111^2)+\dots+$$
$$\dots+10^{2021}\left(\underbrace{\overline{11\dots11}^2}_{2021-digits}+\underbrace{\overline{11\dots11}}_{2021-digits}\cdot\underbrace{\overline{11\dots11}}_{2022-digits}+\underbrace{\overline{11\dots11}^2}_{2022-digits}\right)]=$$
$$=\underbrace{\overline{11\dots11}^3}_{2022-digits}-1$$

From (4): $S=10(1^2+1\cdot11+11^2)+10^2(11^2+11\cdot111+111^2)+$

$$+10^3(111^2+111\cdot1111+1111^2)+\dots+$$
$$\dots+10^{2021}\left(\underbrace{\overline{11\dots11}^2}_{2021-digits}+\underbrace{\overline{11\dots11}}_{2021-digits}\cdot\underbrace{\overline{11\dots11}}_{2022-digits}+\underbrace{\overline{11\dots11}^2}_{2022-digits}\right)=$$
$$=\underbrace{\overline{11\dots11}^3}_{2022-digits}-1$$

$$a\left(\underbrace{\overline{11\dots11}^3}_{2022-digits}-1\right)\dots+10^{2021}\left(\underbrace{\overline{11\dots11}^2}_{2021-digits}+\underbrace{\overline{11\dots11}}_{2021-digits}\cdot\underbrace{\overline{11\dots11}}_{2022-digits}+\underbrace{\overline{11\dots11}^2}_{2022-digits}\right)=$$
$$=\underbrace{\overline{11\dots11}^3}_{2022-digits}-1$$

But from (3):

$$\underbrace{\overline{11\ldots11}^3}_{2022-digits} - \underbrace{\overline{11\ldots11}^3}_{2021-digits} = \underbrace{\overline{100\ldots00}}_{2022-digits}\left(\underbrace{\overline{11\ldots11}^2}_{2021-digits} + \underbrace{\overline{11\ldots11}}_{2021-digits} \cdot \underbrace{\overline{11\ldots11}}_{2022-digits} + \underbrace{\overline{11\ldots11}^2}_{2022-digits}\right)$$

Hence,

$$a\left(\underbrace{\overline{11\ldots11}^3}_{2022-digits} - 1\right) = \underbrace{\overline{11\ldots11}^3}_{2022-digits} - 1$$

So, $a = 1$.

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