RMM - Calculus Marathon 2001 - 2100

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Proposed by

Daniel Sitaru – Romania, Asmat Qatea-Afghanistan

Togrul Ehmedov-Azerbaijan, Ankush Kumar Parcha-India

Le Thu-Vietnam, Naren Bhandari-Nepal, Florică Anastase-Romania

Srinivasa Raghava-AIRMC-India, Sakthi Vel-India

Kunihiko Chikaya-Japan, Ose Favour-Nigeria

Vasile Mircea Popa-Romania, Ngulmun George Baite-India

Hikmat Mammadov-Azerbaijan, Saboor Halimi-Afghanistan

Marian Ursărescu-Romania, Ty Halpen-USA

Bui Hong Suc-Vietnam, D.M.Bătinețu-Giurgiu-Romania

Mihaly Bencze-Romania, Syed Shahabudeen-India

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Solutions by

Daniel Sitaru – Romania, Pham Duc Nam-Vietnam

Togrul Ehmedov-Azerbaijan, Rana Ranino-Algerie, Amin Hajiyev-Azerbaijan

Bamidele Oluwatosin-Nigeria, Bamidele Benjamin-Nigeria

Ruxandra Daniela Tonilă-Romania, Mohamed Amine Ben Ajiba-Morocco

Tapas Das-India, Khaled Abd Imouti-Syria, Bui Hong Suc-Vietnam

Hikmat Mammadov-Azerbaijan, Ravi Prakash-India,Yen Tung Chung-Taiwan

Le Thu-Vietnam, Hikmat Mammadov-Azerbaijan

Ankush Kumar Parcha-India, Florentin Vișescu-Romania, Ngulmun George

Baite-Nigeria, Saboor Halimi-Afghanistan, Ose Favour-Nigeria

Samir Zaakouni-Morocco, Max Wong-Hong Kong, Asmat Qatea-Afghanistan

Toubal Fethi-Algerie, Soumava Chakraborty-India, Sakthi Vel-India

Syed Shahabudeen-India, Izumi Ainsworth-Japan, Samir Zaakouni-Morocco

Fao Ler-Iraq, Vuk Stojiljkovic-Serbie, Soumitra Mandal-India

Kartick Chandra Betal-India, Adrian Popa-Romania

2001. Find:

$$\Omega = \int_0^1 \int_0^1 \ln(x^2 + y^2 + \phi xy)\, dxdy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$\Omega = \int_0^1 \int_0^1 \ln(x^2 + y^2 + \phi xy)\, dxdy$$

$$= 2\int_0^1 \int_0^1 x \ln(x^2 + x^2y^2 + \phi x^2 y)\, dxdy = 2\int_0^1 \int_0^1 x \ln\left(x^2(1 + y^2 + \phi y)\right) dxdy$$

$$= 2\int_0^1 \int_0^1 x \ln x^2\, dxdy + 2\int_0^1 \int_0^1 x\ln(1 + y^2 + \phi y) dxdy$$

$$* \, 2\int_0^1 \int_0^1 x \ln x^2\, dxdy = 4\int_0^1 x \ln x\, dx \int_0^1 dy = -1$$

$$* \, 2\int_0^1 \int_0^1 x\ln(1 + y^2 + \phi y) dxdy = 2\int_0^1 xdx \int_0^1 \ln(1 + y^2 + \phi y) dy$$

$$= \int_0^1 \ln(1 + y^2 + \phi y) dy$$

$$Let: \begin{cases} u = \ln(1 + y^2 + \phi y) \\ dv = dy \end{cases} \Rightarrow \begin{cases} du = \dfrac{2y + \phi}{y^2 + \phi y + 1} \\ v = y \end{cases} \Rightarrow \int_0^1 \ln(1 + y^2 + \phi y)\, dy$$

$$= y \ln(1 + y^2 + \phi y)\Big|_0^1 - \int_0^1 \frac{2y^2 + \phi y}{y^2 + \phi y + 1} dy$$

$$= \ln(2 + \phi) + \int_0^1 \left(\frac{\phi y + 2}{y^2 + \phi y + 1} - 2\right) dy$$

$$= \ln(2 + \phi) - 2 + \int_0^1 \left(\frac{\phi(2y + \phi)}{2(y^2 + \phi y + 1)} - \frac{\phi^2 - 4}{2(y^2 + \phi y + 1)}\right) dy$$

$$= \ln(2 + \phi) - 2 + \frac{\phi}{2} \ln(y^2 + \phi y + 1)\Big|_0^1 + \sqrt{4 - \phi^2}\, \arctan\left(\frac{\phi + 2y}{\sqrt{4 - \phi^2}}\right)\Bigg|_0^1$$

$$= \ln(2 + \phi) - 2 + \frac{\phi}{2} \ln(2 + \phi)$$

$$+ \sqrt{4 - \phi^2}\left(\arctan\frac{\phi + 2}{\sqrt{4 - \phi^2}} - \arctan\frac{\phi}{\sqrt{4 - \phi^2}}\right)$$

$$=\left(1+\frac{\phi}{2}\right)ln(2+\phi)-2+\sqrt{4-\phi^2}\,arctan\frac{\frac{\phi+2}{\sqrt{4-\phi^2}}-\frac{\phi}{\sqrt{4-\phi^2}}}{1+\left(\frac{\phi+2}{\sqrt{4-\phi^2}}\right)\left(\frac{\phi}{\sqrt{4-\phi^2}}\right)}$$

$$=\left(1+\frac{\phi}{2}\right)ln(2+\phi)-2+\sqrt{3-\phi}\,arctan\sqrt{\frac{5-2\sqrt{5}}{5}}$$

$$=\left(\frac{2+\phi}{2}\right)ln(2+\phi)-2+\sqrt{3-\phi}.\frac{\pi}{10}$$

$$\Rightarrow \Omega=\left(\frac{2+\phi}{2}\right)ln(2+\phi)-2+\sqrt{3-\phi}\frac{\pi}{10}-1=\boxed{\left(\frac{2+\phi}{2}\right)ln(2+\phi)+\sqrt{3-\phi}.\frac{\pi}{10}-3}$$

$$*\ arctan\,x-arctan\,y=arctan\frac{x-y}{1+xy}$$

$$*\ Let:\varphi=\frac{\pi}{10}\Rightarrow 5\varphi=\frac{\pi}{2}\Rightarrow 3\varphi=\frac{\pi}{2}-2\varphi\Rightarrow cos\,3\,\varphi=sin\,2\,\varphi\Leftrightarrow 4\,cos^3\varphi-3\,cos\,\varphi$$

$$=2\,sin\,\varphi\,cos\,\varphi\Leftrightarrow -4\,sin^2\varphi+2\,sin\,\varphi+1=0$$

$$\Leftrightarrow\left[\begin{array}{l} sin\,\varphi=\frac{-1-\sqrt{5}}{4}(<0,reject)\\ sin\,\varphi=sin\frac{\pi}{10}=\frac{-1+\sqrt{5}}{4}\Rightarrow cos\frac{\pi}{10}=\frac{\sqrt{10+2\sqrt{5}}}{4}\Rightarrow\\ tan\frac{\pi}{10}=\frac{-1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}=\sqrt{\frac{\left(\sqrt{5}-1\right)^2}{10+2\sqrt{5}}}=\sqrt{\frac{3-\sqrt{5}}{5+\sqrt{5}}}=\\ \sqrt{\frac{(3-\sqrt{5})(\sqrt{5}-1)}{\sqrt{5}(\sqrt{5}+1)(\sqrt{5}-1)}}=\sqrt{\frac{4(\sqrt{5}-2)}{4\sqrt{5}}}=\sqrt{\frac{5-2\sqrt{5}}{5}}\Rightarrow arctan\sqrt{\frac{5-2\sqrt{5}}{5}}=\frac{\pi}{10}\end{array}\right.$$

2002. Prove that:

$$\Omega=\int_0^1\int_0^1\frac{arctan(xy)}{1+x^2y^2}dxdy=\frac{\pi}{4}G-\frac{7}{16}\zeta(3)$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^1\int_0^1\frac{arctan(xy)}{1+x^2y^2}dxdy=\frac{\pi}{4}G-\frac{7}{16}\zeta(3)\ =$$

$$=\int_0^1\left(\frac{arctan^2(xy)}{2x}\Bigg|_0^1\right)dx=\frac{1}{2}\int_0^1\frac{arctan^2\,x}{x}dx,\ let:x=tan\,t\Rightarrow dx=\frac{dt}{cos^2\,t}$$

$$\Rightarrow \int_0^1 \frac{\arctan^2 x}{x}dx = \int_0^{\frac{\pi}{4}} \frac{t^2}{\cos^2 t \tan t}dt = 2\int_0^{\frac{\pi}{4}} \frac{t^2}{\sin(2t)}dt$$

$$\begin{cases} u = t^2 \\ dv = \dfrac{dt}{\sin 2t} \end{cases} \Rightarrow \begin{cases} du = 2tdt \\ v = \dfrac{1}{2}(ln\sin t - ln\cos t) \end{cases} \Rightarrow 2\int_0^{\frac{\pi}{4}} \frac{t^2}{\sin(2t)}dt$$

$$= \underbrace{t^2(ln\sin t - ln\cos t)|_0^{\frac{\pi}{4}}}_{=0} - 2\int_0^{\frac{\pi}{4}} t\,(ln\sin t - ln\cos t)dt$$

$$= -2\int_0^{\frac{\pi}{4}} t\left(-\ln 2 - \sum_{k=1}^{+\infty}\frac{\cos 2kt}{k} + \ln 2 + \sum_{k=1}^{+\infty}(-1)^k\frac{\cos 2kt}{k}\right)dt$$

$$= 2\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{4}}\frac{t\cos 2kt}{k}dt - 2\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{4}}(-1)^k\frac{t\cos 2kt}{k}dt$$

$$* \; 2\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{4}}\frac{t\cos 2kt}{k}dt \underline{IBP} \sum_{k=1}^{+\infty}\frac{\pi k\sin\frac{\pi k}{2} + 2\cos\frac{\pi k}{2} - 2}{4k^3}$$

$$= \frac{\pi}{4}\sum_{k=1}^{+\infty}\frac{\sin\frac{\pi k}{2}}{k^2} + \frac{1}{2}\sum_{k=1}^{+\infty}\frac{\cos\frac{\pi k}{2}}{k^3} - \frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{k^3}$$

$$= \frac{\pi}{4}\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)^2} + \frac{1}{16}\sum_{k=1}^{+\infty}\frac{(-1)^j}{j^3} - \frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{k^3} = \frac{\pi}{4}G + \frac{1}{16}\left(-\frac{3}{4}\zeta(3)\right) - \frac{1}{2}\zeta(3)$$

$$= \frac{\pi}{4}G - \frac{35}{64}\zeta(3)$$

$$* \; 2\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{4}}(-1)^k\frac{t\cos 2kt}{k}dt \underline{IBP} \sum_{k=1}^{+\infty}\frac{(-1)^k}{4k^3}\left(\pi k\sin\frac{\pi k}{2} + 2\cos\frac{\pi k}{2} - 2\right)$$

$$= \frac{\pi}{4}\sum_{k=1}^{+\infty}\frac{(-1)^k\sin\frac{\pi k}{2}}{k^2} + \sum_{k=1}^{+\infty}\frac{(-1)^k\cos\frac{\pi k}{2}}{2k^3} - \sum_{k=1}^{+\infty}\frac{(-1)^k}{2k^3}$$

$$= \frac{\pi}{4}\sum_{j=0}^{+\infty}\frac{(-1)^{j+1}}{(2j+1)^2} + \frac{1}{16}\sum_{j=1}^{+\infty}\frac{(-1)^j}{j^3} - \sum_{k=1}^{+\infty}\frac{(-1)^k}{2k^3}$$

$$= -\frac{\pi}{4}\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)^2} + \frac{1}{16}\sum_{j=1}^{+\infty}\frac{(-1)^j}{j^3} - \sum_{k=1}^{+\infty}\frac{(-1)^k}{2k^3}$$

$$= -\frac{\pi}{4}G + \frac{1}{16}\left(-\frac{3}{4}\zeta(3)\right) + \frac{3}{8}\zeta(3) = \frac{21}{64}\zeta(3) - \frac{\pi}{4}G$$

$$\Rightarrow \Omega = \frac{1}{2}\left(\frac{\pi}{4}G - \frac{35}{64}\zeta(3) - \left(\frac{21}{64}\zeta(3) - \frac{\pi}{4}G\right)\right) = \boxed{\frac{\pi}{4}G - \frac{7}{16}\zeta(3)}$$

2003. Find:

$$\Omega = \int_0^1 \int_0^1 \ln(x^4 + y^4 + (x-y)^4)\,dxdy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$\Omega = \int_0^1 \int_0^1 \ln(x^4 + y^4 + (x-y)^4)\,dxdy$$

$$= 2\int_0^1 \int_0^1 x\ln(x^4 + x^4y^4 + (x-xy)^4)\,dxdy = 2\int_0^1 \int_0^1 x\ln\left(x^4(1 + y^4 + (1-y)^4)\right)dxdy$$

$$= 2\int_0^1 \int_0^1 x\ln x^4\,dxdy + 2\int_0^1 \int_0^1 x\ln(1 + y^4 + (1-y)^4)\,dxdy$$

$$*\ 2\int_0^1 \int_0^1 x\ln x^4\,dxdy = 8\int_0^1 x\ln x\,dx \int_0^1 dy = -2$$

$$*\ 2\int_0^1 \int_0^1 x\ln(1 + y^4 + (1-y)^4)\,dxdy = 2\int_0^1 xdx \int_0^1 \ln(1 + y^4 + (1-y)^4)\,dy$$

$$= \int_0^1 \ln(1 + y^4 + (1-y)^4)\,dy$$

$$\begin{cases} u = \ln(1 + y^4 + (1-y)^4) \\ dv = dy \end{cases} \Rightarrow \begin{cases} du = \dfrac{4y-2}{y^2 - y + 1} \\ v = y \end{cases}$$

$$\Rightarrow \int_0^1 \ln(1 + y^4 + (1-y)^4)\,dy = y\ln(1 + y^4 + (1-y)^4)\Big|_0^1 - \int_0^1 \frac{4y^2 - 2y}{y^2 - y + 1}dy$$

$$= \ln 2 - \int_0^1 \left(\frac{2y-1}{y^2 - y + 1} - \frac{3}{y^2 - y + 1} + 4\right)dy$$

$$= \ln 2 - 4 - \ln(y^2 - y + 1)\Big|_0^1 + 2\sqrt{3}\arctan\frac{2y-1}{\sqrt{3}}\Big|_0^1 = \ln 2 - 4 + 2\sqrt{3}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \ln 2 - 4 + \frac{2}{\sqrt{3}}\pi$$

$$\Rightarrow \Omega = \ln 2 - 4 + \frac{2}{\sqrt{3}}\pi - 2 = \boxed{\frac{2}{\sqrt{3}}\pi + \ln 2 - 6}$$

2004. Prove that:

$$\Omega = \int_0^1 \frac{\log(1-x^2)}{\sqrt{x}(\sqrt{x}+1)}dx = \frac{7}{2}\ln^2 2 - \frac{5}{4}\zeta(2)$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Pham Duc Nam-Vietnam

$$*\ Let: t = \sqrt{x} \Rightarrow dt = \frac{dx}{2\sqrt{x}} \Rightarrow \frac{dx}{\sqrt{x}} = 2dt \Rightarrow \Omega = 2\int_0^1 \frac{\ln(1-t^4)}{t+1}dt$$

$$= 2\int_0^1 \frac{\ln(1-t^2)}{t+1}dt + 2\int_0^1 \frac{\ln(1+t^2)}{t+1}dt = 2J + 2K$$

$$*\ K = \int_0^1 \frac{\ln(1+t^2)}{t+1}dt, let: K(a) = \int_0^1 \frac{\ln(1+at^2)}{t+1}dt \Rightarrow K(a) = \int_0^1 \frac{\partial}{\partial a}\left(\frac{\ln(1+at^2)}{t+1}\right)dt$$

$$= \int_0^1 \frac{t^2}{(t+1)(1+at^2)}dt = \frac{1}{a+1}\int_0^1 \left(\frac{t}{at^2+1} - \frac{1}{at^2+1} + \frac{1}{t+1}\right)dt$$

$$= \frac{\ln(at^2+1) + 2a(\ln t + 1) - 2\sqrt{a}\arctan(t\sqrt{a})}{2a^2+2a}\Bigg|_0^1$$

$$= \frac{\ln(a+1)}{2a(a+1)} + \frac{\ln 2}{a+1} - \frac{\arctan(\sqrt{a})}{\sqrt{a}(a+1)}, \text{integrating both sides from 0 to 1}$$

$$\Rightarrow K = K(1) = \int_0^1 \left(\frac{\ln(a+1)}{2a(a+1)} + \frac{\ln 2}{a+1} - \frac{\arctan(\sqrt{a})}{\sqrt{a}(a+1)}\right)da$$

$$= \ln^2 2 - \int_0^1 \frac{\arctan(\sqrt{a})}{\sqrt{a}(a+1)}da + \int_0^1 \frac{\ln(a+1)}{2a(a+1)}da$$

$$= \ln^2 2 - \left(\arctan\sqrt{a}\right)^2\Big|_0^1 + \frac{1}{2}\int_0^1 \ln(a+1)\left(\frac{1}{a} - \frac{1}{a+1}\right)da$$

$$= \ln^2 2 - \frac{\pi^2}{16} - \frac{1}{4}\ln^2(a+1)\Big|_0^1 - \frac{1}{2}Li_2(-a)\Big|_0^1 = \ln^2 2 - \frac{\pi^2}{16} - \frac{1}{4}\ln^2 2 + \frac{\pi^2}{24} = \frac{3}{4}\ln^2 2 - \frac{\pi^2}{48}$$

$$*\ J = \int_0^1 \frac{\ln(1-t^2)}{t+1}dt = \int_0^1 \frac{\ln(1-t)}{(t+1)}dt + \int_0^1 \frac{\ln(1+t)}{(t+1)}dt$$

$$= \int_0^1 \frac{\ln(1-t)}{(t+1)}dt + \frac{1}{2}\ln^2(1+t)\Big|_0^1 = \frac{1}{2}\ln^2 2 + \int_0^1 \frac{\ln(1-t)}{(t+1)}dt = \frac{1}{2}\ln^2 2 + I$$

$$*\, I=\int_0^1 \frac{ln(1-t)}{(t+1)}dt, let: I(k)=\int_0^1 \frac{ln(1-kt)}{(t+1)}dt \Rightarrow I'(k)=\int_0^1 \frac{\partial}{\partial k}\left(\frac{ln(1-kt)}{(t+1)}\right)dt$$

$$=-\int_0^1 \frac{t}{(1-kt)(t+1)}dt$$

$$=\frac{1}{k+1}\int_0^1\left(\frac{1}{t+1}-\frac{1}{1-kt}\right)dt=\frac{1}{k+1}\left(ln(1+t)+\frac{ln(1-kt)}{k}\right)\Big|_0^1$$

$$=\frac{ln\,2}{k+1}+\frac{ln(1-k)}{k(k+1)}, \text{integrating both sides from 0 to 1}$$

$$\Rightarrow I=I(1)=\int_0^1\left(\frac{ln\,2}{k+1}+\frac{ln(1-k)}{k(k+1)}\right)dk$$

$$=ln\,2\,ln(k+1)|_0^1+\int_0^1\frac{ln(1-k)}{k}dk-\int_0^1\frac{ln(1-k)}{k+1}dk \Rightarrow 2I$$

$$=ln\,2\,ln(k+1)|_0^1+\int_0^1\frac{ln(1-k)}{k}dk=ln^2 2-\frac{\pi^2}{6}$$

$$\Rightarrow I=\frac{1}{2}ln^2 2-\frac{\pi^2}{12}\Rightarrow J=\frac{1}{2}ln^2 2+\frac{1}{2}ln^2 2-\frac{\pi^2}{12}=ln^2 2-\frac{\pi^2}{12}$$

$$\Rightarrow \Omega=2J+2K=2\left(ln^2 2-\frac{\pi^2}{12}\right)+2\left(\frac{3}{4}ln^2 2-\frac{\pi^2}{48}\right)=\frac{7}{2}ln^2 2-\frac{5}{24}\pi^2=\frac{7}{2}ln^2 2-\frac{5}{4}\cdot\frac{\pi^2}{6}$$

$$=\boxed{\frac{7}{2}ln^2 2-\frac{5}{4}\zeta(2)}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^1\frac{\log(1-x^2)}{\sqrt{x}(1+\sqrt{x})}dx\overset{x=y^2}{=}2\int_0^1\frac{\log(1-y^4)}{1+y}dy=$$

$$=2\int_0^1\frac{\log(1-y)}{1+y}dy+2\int_0^1\frac{\log(1+y)}{1+y}dy+2\int_0^1\frac{\log(1+y^2)}{1+y}dy=$$

$$=2I_1+2I_2+2I_3$$

$$I_1=\int_0^1\frac{\log(1-y)}{1+y}dy=\int_0^1\frac{\log y}{2-y}dy=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}\int_0^1 y^k\log y\,dy=$$

$$=-Li_2\left(\frac{1}{2}\right)=\frac{1}{2}\log^2 2-\frac{\pi^2}{12}$$

$$I_2=\int_0^1 \frac{\log(1+y)}{1+y}dy=\frac{1}{2}\log^2 2$$

$$I_3=\int_0^1 \frac{\log(1+y^2)}{1+y}dy \overset{IBP}{=} \log^2 2-2\int_0^1 \frac{y\log(1+y)}{1+y^2}dy=\log^2 2-2I_{3a}$$

$$I_{3a}=\int_0^1 \frac{y\log(1+y)}{1+y^2}dy=\frac{\pi^2}{96}+\frac{\log^2 2}{8}\Rightarrow I_3=\frac{3}{4}\log^2 2-\frac{\pi^2}{48}$$

$$\Omega=2(I_1+I_2+I_3)=\frac{7}{2}\log^2 2-\frac{5\pi^2}{24}=\frac{7}{2}\log^2 2-\frac{5}{4}\zeta(2)$$

2005. Prove that:

$$\int_0^1 \frac{\cos(\ln x)-1}{\ln(x)(x+1)}dx=\frac{1}{2}\ln\left(\frac{2}{\pi}\cdot\frac{e^\pi-1}{e^\pi+1}\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1 \frac{\cos(\ln x)-1}{\ln(x)\,(1+x)}dx$$

$$\Omega \overset{x=e^{-t}}{=} \int_0^\infty \frac{1-\cos(t)}{t(1+e^{-t})}e^{-t}dt=-\sum_{n=1}^{\infty}(-1)^n\int_0^\infty \frac{1-\cos(t)}{t}e^{-nt}dt$$

$$\text{Take}: I(y)=\int_0^\infty \frac{1-\cos(yt)}{t}e^{-nt}dt \quad \text{we have}: I(0)=0 \quad and \quad \Omega=-\sum_{n=1}^{\infty}(-1)^n I(1)$$

$$\frac{dI(y)}{dy}=\int_0^\infty \sin(yt)\,e^{-nt}=\frac{y}{n^2+y^2} \quad \therefore \quad I(1)=\int_0^1 \frac{y}{n^2+y^2}dy$$

$$\Omega=-\sum_{n=1}^{\infty}(-1)^n\int_0^1 \frac{y}{n^2+y^2}dy$$

$$\text{Known}: \sum_{n=-\infty}^{\infty}\frac{(-1)^n y}{y^2+n^2}=\frac{\pi}{\sinh(\pi y)} \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{(-1)^n y}{y^2+n^2}=\frac{\pi}{2\sinh(\pi y)}-\frac{1}{2y}$$

$$\Omega=\frac{1}{2}\int_0^1\left(\frac{1}{y}-\frac{\pi}{\sinh(\pi y)}\right)dy=\frac{1}{2}\left[\ln y-\ln\tanh\left(\frac{\pi y}{2}\right)\right]_0^1=-\frac{1}{2}\ln\tanh\left(\frac{\pi}{2}\right)+\frac{1}{2}\lim_{y\to 0}\ln\left(\frac{\tanh\left(\frac{\pi y}{2}\right)}{y}\right)$$

$$\Omega=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-\frac{1}{2}\ln\left(\frac{e^\pi-1}{e^\pi+1}\right)=\frac{1}{2}\ln\left(\frac{\pi}{2}\frac{e^\pi+1}{e^\pi-1}\right)$$

$$\int_0^1 \frac{\cos(\ln x)-1}{\ln(x)\,(1+x)}dx=-\frac{1}{2}\ln\left(\frac{2}{\pi}\cdot\frac{e^\pi-1}{e^\pi+1}\right)$$

2006. Find a closed form:

$$\Omega=\int_0^1\frac{x\,ln(1-x^2)}{x^2+1}dx$$

Proposed by Le Thu-Vietnam

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^1\frac{x\,ln(1-x^2)}{x^2+1}dx,\ let:\begin{cases}t=\dfrac{1-x^2}{1+x^2}\\ \Rightarrow x^2=\dfrac{1-t}{1+t}\end{cases}\Rightarrow dt=-\frac{4x}{(x^2+1)^2}dx$$

$$\Rightarrow\frac{xdx}{x^2+1}=-\frac{dt}{2(t+1)}\Rightarrow\Omega=\frac{1}{2}\int_0^1\frac{ln\left(1-\frac{1-t}{1+t}\right)}{t+1}dt=\frac{1}{2}\int_0^1\frac{ln\left(\frac{2t}{1+t}\right)}{t+1}dt$$

$$=\frac{1}{2}\int_0^1\frac{ln(2)}{t+1}dt+\frac{1}{2}\int_0^1\frac{ln(t)}{t+1}dt-\frac{1}{2}\int_0^1\frac{ln(t+1)}{t+1}dt$$

$$=\frac{1}{2}ln\,2\,ln(t+1)|_0^1+\frac{1}{2}\left(-\frac{\pi^2}{12}\right)-\frac{1}{4}ln^2(t+1)\Big|_0^1=$$

$$=\frac{1}{2}ln^2 2-\frac{\pi^2}{24}-\frac{1}{4}ln^2 2=\frac{1}{4}ln^2 2-\frac{\pi^2}{24}$$

$$*\ Known:\zeta(-1)=-\frac{1}{12}\Rightarrow\Omega=\boxed{\frac{ln^2 2+2\pi^2\zeta(-1)}{4}}$$

2007.Prove that:

$$I=\int_0^{\pi}\prod_{k=-1}^{3}cos^2(2^kx)\,dx=\frac{\pi}{32}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Pham Duc Nam-Vietnam

$$I=\int_0^{\pi}\prod_{k=-1}^{3}cos^2(2^kx)\,dx=\frac{\pi}{32}$$

$$*\ f(x)=\prod_{k=-1}^{3}cos(2^kx)\Rightarrow sin\frac{x}{2}f(x)=sin\frac{x}{2}\prod_{k=-1}^{3}cos(2^kx)=\frac{1}{32}sin(16x)\Rightarrow\prod_{k=-1}^{3}cos^2(2^kx)$$

$$=\frac{1}{32^2}\cdot\frac{sin^2(16x)}{sin^2\frac{x}{2}}$$

$$*\ Apply: \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \Rightarrow I = \frac{1}{32^2}\int_0^\pi \frac{\sin^2(16x)}{\sin^2\frac{x}{2}}dx = \frac{1}{32^2}\int_0^\pi \frac{\sin^2(16x)}{\cos^2\frac{x}{2}}dx$$

$$\Rightarrow 2I = \frac{4}{32^2}\int_0^\pi \frac{\sin^2(16x)}{\sin^2 x}dx$$

$$*\ Define: I(n) = \int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}dx, n \in \mathbb{N} \Rightarrow I(n+1) = \int_0^\pi \frac{\sin^2\big((n+1)x\big)}{\sin^2 x}dx$$

$$\Rightarrow I(n+1) - I(n) = \int_0^\pi \frac{\sin\big((2n+1)x\big)\sin x}{\sin^2 x}dx = \int_0^\pi \frac{\sin\big((2n+1)x\big)}{\sin x}dx$$

$$= K(n)$$

$$\Rightarrow K(n+1) = \int_0^\pi \frac{\sin\big((2n+3)x\big)}{\sin x}dx \Rightarrow K(n+1) - K(n)$$

$$= \int_0^\pi \frac{2\cos\frac{(2n+3)x+(2n+1)x}{2}\sin\frac{(2n+3)x-(2n+1)x}{2}}{\sin x}dx$$

$$= 2\int_0^\pi \cos\big((2n+2)x\big)\,dx = \frac{\sin\big((2n+1)x\big)}{n+1}\Bigg|_0^\pi = 0$$

$$\Rightarrow K(n+1) = K(n) = K(n-1) = \ldots = K(0) = \int_0^\pi \frac{\sin x}{\sin x}dx = \pi$$

$$\Rightarrow I(n+1) - I(n) = \pi \Rightarrow I(n+1) = I(n) + \pi = I(n-1) + 2\pi = \ldots = I(1) + n\pi$$

$$= \int_0^\pi \frac{\sin^2 x}{\sin^2 x}dx + n\pi = \pi + n\pi = (n+1)\pi$$

$$\Rightarrow I(n) = n\pi \Rightarrow \int_0^\pi \frac{\sin^2(16x)}{\sin^2 x}dx = 16\pi \Rightarrow 2I = \frac{4}{32^2}.16\pi = \frac{\pi}{16} \Rightarrow I = \frac{\pi}{32}$$

Solution 2 by Bamidele Oluwatosin-Nigeria

$$I = \int_0^\pi \cos^2\left(\frac{x}{2}\right)\cos^2 x\cos^2(2x)\cos^2(4x)\cos^2(8x)\,dx =$$

$$= \int_0^\pi \cos^2\left(\frac{\pi-x}{2}\right)\cos^2(\pi-x)\cos^2\big(2(\pi-x)\big)\cos^2\big(4(\pi-x)\big)\cos^2\big(8(\pi-x)\big)\,dx$$

$$\Rightarrow 2I = \int_0^\pi \left(\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right)\right)\cos^2 x\cos^2(2x)\cos^2(4x)\cos^2(8x)\,dx =$$

$$= \int_0^\pi \cos^2 x\cos^2(2x)\cos^2(4x)\cos^2(8x)\,dx = \int_0^\pi \left(\prod_{n=0}^{3}\cos(2^n x)\right)^2 dx\,;$$

$$\text{Ola} = \prod_{n=0}^{3}\cos(2^n x), \mathit{from}:\ \sin(2x) = 2\sin x\cos x \Rightarrow \text{Ola} = \frac{\sin(16x)}{2^4\sin x}$$

$$I=\frac{1}{512}\int_0^{\pi}\frac{\sin^2(16x)}{\sin^2 x}dx=\left[-\frac{\cot x\sin^2(16x)}{512}\right]_0^{\pi}+\frac{1}{32}\int_0^{\pi}\frac{\sin(32x)\cos x}{\sin x}dx=$$

$$=\frac{1}{16}\int_0^{\frac{\pi}{2}}\frac{\sin(32x)\cos x}{\sin x}dx\overset{King-Rulle}{=}-\frac{1}{16}\int_0^{\frac{\pi}{2}}\frac{\sin(32x)\sin x}{\cos x}dx=$$

$$=\frac{1}{16}\int_0^{\frac{\pi}{2}}\frac{\sin(32x)\left(\cos^2 x-\sin^2 x\right)}{2\sin x\cos x}dx=\frac{1}{16}\int_0^{\frac{\pi}{2}}\frac{\sin(16x)\cos x}{\sin x}dx=$$

$$=\frac{1}{16}\int_0^{\frac{\pi}{2}}\frac{\sin(8x)\cos x}{\sin x}dx=\cdots=\frac{1}{16}\int_0^{\frac{\pi}{2}}\frac{\sin(2x)\cos x}{\sin x}dx$$

$$=\frac{1}{16}\int_0^{\frac{\pi}{2}}2\cos^2 x\,dx=\frac{\pi}{32}$$

2008.Find:

$$\Omega=\int_0^1\int_0^1 ln\left((x^2+y^2-2xya_1)(x^2+y^2+2xya_2)(x^2+y^2-2xya_3)\right)dxdy$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^1\int_0^1 ln\left((x^2+y^2-2xya_1)(x^2+y^2+2xya_2)(x^2+y^2-2xya_3)\right)dxdy$$

$$=\int_0^1\int_0^1 ln(x^2+y^2-2xya_1)\,dxdy+\int_0^1\int_0^1 ln(x^2+y^2+2xya_2)\,dxdy$$

$$+\int_0^1\int_0^1 ln(x^2+y^2-2xy2xya_3)\,dxdy=\Omega_1+\Omega_2+\Omega_3$$

$$*\,\Omega_1=\int_0^1\int_0^1 ln(x^2+y^2-2xya_1)\,dxdy=2\int_0^1\int_0^1 x\,ln(x^2+x^2y^2-2x^2ya_1)\,dxdy$$

$$=2\int_0^1\int_0^1 x\ln x^2\,dxdy+2\int_0^1\int_0^1 x\,ln(1+y^2-2ya_1)\,dxdy$$

$$=-1+\int_0^1 ln(y^2-2ya_1+1)\,dy,\begin{cases}u=ln(y^2-2ya_1+1)\\dv=dy\end{cases}\Rightarrow\begin{cases}du=\dfrac{2y-2a_1}{y^2-2ya_1+1}\\v=y\end{cases}$$

$$\Rightarrow \Omega_1 = -1 + y\,ln(y^2 - 2ya_1 + 1)\Big|_0^1 - \int_0^1 \frac{2y^2 - 2a_1 y}{y^2 - 2ya_1 + 1}dy$$

$$= -1 + ln(2 - 2a_1) - \int_0^1 \left(\frac{a_1(2y - 2a_1)}{y^2 - 2ya_1 + 1} - \frac{2 - 2{a_1}^2}{y^2 - 2ya_1 + 1} + 2\right)dy$$

$$= -1 + ln(2 - 2a_1) - 2y|_0^1 - a_1 ln(y^2 - 2ya_1 + 1)\Big|_0^1 + 2\sqrt{1 - {a_1}^2}\,arctan\frac{y - a_1}{\sqrt{1 - {a_1}^2}}\Bigg|_0^1$$

$$= -3 + (1 - a_1)\,ln(2 - 2a_1)$$

$$+ 2\sqrt{1 - {a_1}^2}\left(arctan\frac{1 - a_1}{\sqrt{1 - {a_1}^2}} - arctan\frac{-a_1}{\sqrt{1 - {a_1}^2}}\right)$$

$$= -3 + (1 - a_1)\,ln(2 - 2a_1) + 2\sqrt{1 - {a_1}^2}\,arctan\sqrt{\frac{1 + a_1}{1 - a_1}}$$

$$= -3 + 2\sin^2\frac{\pi}{14}\,ln\left(4\sin^2\frac{\pi}{14}\right) + \frac{6\pi}{7}\sin\frac{\pi}{7}$$

$$= -3 + 4\sin^2\frac{\pi}{14}\,ln\left(2\sin\frac{\pi}{14}\right) + \frac{6\pi}{7}\sin\frac{\pi}{7}$$

$$\Rightarrow \Omega_3 = -3 + (1 - a_3)\,ln(2 - 2a_3) + 2\sqrt{1 - {a_3}^2}\,arctan\sqrt{\frac{1 + a_3}{1 - a_3}}$$

$$= -3 + 2\sin^2\frac{3\pi}{14}\,ln\left(4\sin^2\frac{3\pi}{14}\right) + \frac{4\pi}{7}\sin\frac{3\pi}{7}$$

$$= -3 + 4\sin^2\frac{3\pi}{14}\,ln\left(2\sin\frac{3\pi}{14}\right) + \frac{4\pi}{7}\sin\frac{3\pi}{7}$$

$$and\ \Omega_2 = -3 + (1 + a_2)\,ln(2 + 2a_2) + 2\sqrt{1 - {a_2}^2}\,arctan\sqrt{\frac{1 - a_2}{1 + a_2}}$$

$$= -3 + 2\cos^2\frac{\pi}{7}\,ln\left(4\cos^2\frac{\pi}{7}\right) + \frac{2\pi}{7}\sin\frac{2\pi}{7}$$

$$= -3 + 4\cos^2\frac{\pi}{7}\,ln\left(2\cos\frac{\pi}{7}\right) + \frac{2\pi}{7}\sin\frac{2\pi}{7}$$

$$\Rightarrow \Omega = -9 + 5\,ln\,2 + 4\sin^2\frac{\pi}{14}\,lnsin\frac{\pi}{14} + 4\sin^2\frac{3\pi}{14}\,lnsin\frac{3\pi}{14} + 4\cos^2\frac{\pi}{7}\,lncos\frac{\pi}{7}$$

$$+ \underbrace{\left(\frac{6\pi}{7}\sin\frac{\pi}{7} + \frac{4\pi}{7}\sin\frac{3\pi}{7} + \frac{2\pi}{7}\sin\frac{2\pi}{7}\right)}_{=\frac{\pi}{2}csc\frac{\pi}{7}}$$

$$= -9 + 5\,ln\,2 + \frac{\pi}{2}csc\frac{\pi}{7} + 4\sin^2\frac{\pi}{14}\,lnsin\frac{\pi}{14} + 4\sin^2\frac{3\pi}{14}\,lnsin\frac{3\pi}{14} + 4\cos^2\frac{\pi}{7}\,lncos\frac{\pi}{7}$$

$$*\ By\ Gauss(1813): If: gcd(a,q)=1, 1\le a<q\ then:\ \psi\left(\frac{a}{q}\right)$$

$$=-\gamma-ln(2q)-\frac{\pi}{2}cot\frac{\pi a}{q}+2\sum_{k=1}^{q/2}cos\frac{2\pi ak}{q}lnsin\frac{\pi k}{q}$$

$$\Rightarrow\frac{1}{2}\psi\left(\frac{4}{7}\right)=\frac{1}{2}\left(-\gamma-ln\,14-\frac{\pi}{2}cot\frac{4\pi}{7}+2\sum_{k=1}^{3}cos\frac{8\pi k}{7}lnsin\frac{\pi k}{7}\right)$$

$$=-\frac{\gamma}{2}-\frac{ln\,14}{2}-\frac{\pi}{4}cot\frac{4\pi}{7}+\sum_{k=1}^{3}cos\frac{8\pi k}{7}lnsin\frac{\pi k}{7}$$

$$and: -\frac{1}{2}\psi\left(\frac{15}{14}\right)=-\frac{1}{2}\psi\left(1+\frac{1}{14}\right)=-\frac{1}{2}\left(\psi\left(\frac{1}{14}\right)+14\right)$$

$$=-\frac{1}{2}\left(14-\gamma-ln\,28-\frac{\pi}{2}cot\frac{\pi}{14}+2\sum_{k=1}^{7}cos\frac{\pi k}{7}lnsin\frac{\pi k}{14}\right)$$

$$=-7+\frac{\gamma}{2}+\frac{ln\,28}{2}+\frac{\pi}{4}cot\frac{\pi}{14}-\sum_{k=1}^{7}cos\frac{\pi k}{7}lnsin\frac{\pi k}{14}$$

$$\Rightarrow\frac{1}{2}\psi\left(\frac{4}{7}\right)-\frac{1}{2}\psi\left(\frac{15}{14}\right)-ln\,2-2$$

$$=-9-\frac{ln\,2}{2}+\frac{\pi}{2}csc\frac{\pi}{7}+\sum_{k=1}^{3}cos\frac{8\pi k}{7}lnsin\frac{\pi k}{7}-\sum_{k=1}^{7}cos\frac{\pi k}{7}lnsin\frac{\pi k}{14}$$

$$=-9-\frac{ln\,2}{2}+\frac{\pi}{2}csc\frac{\pi}{7}-cos\frac{\pi}{7}lnsin\frac{\pi}{7}+sin\frac{3\pi}{14}lnsin\frac{2\pi}{7}-sin\frac{\pi}{14}lncos\frac{\pi}{14}$$

$$-\left(sin\frac{\pi}{14}\left(lnsin\frac{3\pi}{14}-lncos\frac{3\pi}{14}\right)+sin\frac{3\pi}{14}\left(lnsin\frac{\pi}{7}-lncos\frac{\pi}{7}\right)\right.$$

$$\left.+cos\frac{\pi}{7}\left(lnsin\frac{\pi}{14}-lncos\frac{\pi}{14}\right)\right)$$

$$=-9-\frac{ln\,2}{2}+\frac{\pi}{2}csc\frac{\pi}{7}-cos\frac{\pi}{7}\left(lnsin\frac{\pi}{7}+lnsin\frac{\pi}{14}-lncos\frac{\pi}{14}\right)$$

$$+sin\frac{3\pi}{14}\left(lnsin\frac{2\pi}{7}-lnsin\frac{\pi}{7}+lncos\frac{\pi}{7}\right)$$

$$-sin\frac{\pi}{14}\left(lncos\frac{\pi}{14}+lnsin\frac{3\pi}{14}-lncos\frac{3\pi}{14}\right)$$

$$= -9 - \frac{\ln 2}{2} + \frac{\pi}{2}\csc\frac{\pi}{7} - \cos\frac{\pi}{7}\ln\left(2\sin^2\frac{\pi}{14}\right) + \sin\frac{3\pi}{14}\ln\left(2\cos^2\frac{\pi}{7}\right) - \sin\frac{\pi}{14}\ln\left(2\sin^2\frac{3\pi}{14}\right)$$

$$= -9 - \frac{\ln 2}{2} + \frac{\pi}{2}\csc\frac{\pi}{7} - \ln 2\cos\frac{\pi}{7} - 2\cos\frac{\pi}{7}\ln\left(\sin\frac{\pi}{14}\right) + \ln 2\sin\frac{3\pi}{14}$$

$$+ 2\sin\frac{3\pi}{14}\ln\left(\cos\frac{\pi}{7}\right) - \ln 2\sin\frac{\pi}{14} - 2\sin\frac{\pi}{14}\ln\left(\sin\frac{3\pi}{14}\right)$$

$$= -9 - \frac{\ln 2}{2} + \frac{\pi}{2}\csc\frac{\pi}{7} + 4\sin^2\frac{\pi}{14}\ln\sin\frac{\pi}{14} - 2\ln\sin\frac{\pi}{14} + 4\sin^2\frac{3\pi}{14}\ln\sin\frac{3\pi}{14} - 2\ln\sin\frac{3\pi}{14}$$

$$+ 4\cos^2\frac{\pi}{7}\ln\cos\frac{\pi}{7} - 2\ln\cos\frac{\pi}{7} - \ln 2\cos\frac{\pi}{7} - \ln 2\sin\frac{\pi}{14} + \ln 2\sin\frac{3\pi}{14}$$

$$= -9 - \frac{\ln 2}{2} + \frac{\pi}{2}\csc\frac{\pi}{7} + 4\sin^2\frac{\pi}{14}\ln\sin\frac{\pi}{14} + 4\sin^2\frac{3\pi}{14}\ln\sin\frac{3\pi}{14} + 4\cos^2\frac{\pi}{7}\ln\cos\frac{\pi}{7}$$

$$- 2\ln\left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\cos\frac{\pi}{7}\right) - \ln 2\left(\cos\frac{\pi}{7} + \sin\frac{\pi}{14} - \sin\frac{3\pi}{14}\right)$$

$$= -9 - \frac{\ln 2}{2} + \frac{\pi}{2}\csc\frac{\pi}{7} + 4\sin^2\frac{\pi}{14}\ln\sin\frac{\pi}{14} + 4\sin^2\frac{3\pi}{14}\ln\sin\frac{3\pi}{14} + 4\cos^2\frac{\pi}{7}\ln\cos\frac{\pi}{7} + 6\ln 2 - \frac{\ln 2}{2}$$

$$= -9 + 5\ln 2 + \frac{\pi}{2}\csc\frac{\pi}{7} + 4\sin^2\frac{\pi}{14}\ln\sin\frac{\pi}{14} + 4\sin^2\frac{3\pi}{14}\ln\sin\frac{3\pi}{14} + 4\cos^2\frac{\pi}{7}\ln\cos\frac{\pi}{7} = \Omega$$

2009. Prove that:

$$\int_0^{\frac{\pi}{4}} \frac{\log\left(\log\left(\tan\left(\frac{\pi}{4}+x\right)\right)\right)\log\left(\tan\left(\frac{\pi}{4}+x\right)\right)}{\tan 2x}dx = \frac{\pi^2}{48}\log\left(\frac{2e^3\pi^3}{A^{36}}\right)$$

where A is Glaisher-Kinkelin constant.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\log\left(\log\left(\tan\left(\frac{\pi}{4}+x\right)\right)\right)\log\left(\tan\left(\frac{\pi}{4}+x\right)\right)}{\tan 2x}dx \overset{x\to\frac{\pi}{4}-x}{=}$$

$$= -\int_0^{\frac{\pi}{4}} \log(-\log(\tan x))\log(\tan x)\tan(2x)\,dx =$$

$$= -2\int_0^{\frac{\pi}{4}} \frac{\log(-\log(\tan x))\log(\tan x)\tan x}{1-\tan^2 x}dx \overset{\tan x = e^{-t}}{=}$$

$$= 2\int_0^{\infty} \frac{t\log t\, e^{-2t}}{1-e^{-4t}}dt = 2\sum_{n=0}^{\infty}\int_0^{\infty} t\log t\, e^{-(4n+2)t}dt = 2\sum_{n=0}^{\infty} L\{t\log t\}(4n+2)$$

$$L\{t\log t\}(p) = -\frac{d}{dp}L(\log t\}(p) = \frac{d}{dp}\left(\frac{\gamma+\log p}{p}\right) = \frac{1-\gamma-\log p}{p^2}$$

$$\Omega = 2\sum_{n=0}^{\infty}\frac{1-\gamma-\log(4n+2)}{(4n+2)^2} = \frac{1-\gamma-\log 2}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{\log(2n+1)}{(2n+1)^2} =$$

$$= \frac{\pi^2}{16}(1-\gamma-\log 2) - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\log(2n+1)}{(2n+1)^2}$$

Now,

$$\sum_{n=1}^{\infty}\frac{(-1)^n\log n}{n^2} = \sum_{n=1}^{\infty}\frac{\log(2n)}{(2n)^2} - \sum_{n=1}^{\infty}\frac{\log(2n+1)}{(2n+1)^2}$$

$$\sum_{n=1}^{\infty}\frac{\log(2n+1)}{(2n+1)^2} = \frac{\pi^2\log 2}{24} + \frac{1}{4}\sum_{n=2}^{\infty}\frac{\log n}{n^2} - \sum_{n=2}^{\infty}\frac{(-1)^n\log n}{n^2}$$

$$\sum_{n=2}^{\infty}\frac{\log n}{n^2} = -\lim_{s\to 2}\frac{d}{ds}\sum_{n=0}^{\infty}\frac{1}{n^s} = -\zeta'(2)$$

$$\sum_{n=2}^{\infty}\frac{(-1)^n\log n}{n^2} = \lim_{s\to 2}\frac{d}{ds}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^s} = \eta'(2)$$

$$\because \eta'(s) = 2^{1-s}\log(2)\zeta(s) + (12^{1-s})\zeta'(s)$$

$$\eta'(2) = \frac{\pi^2}{12}\log(2) + \frac{1}{2}\zeta'(2)$$

$$\sum_{n=1}^{\infty}\frac{\log(2n+1)}{(2n+1)^2} = -\frac{3}{4}\zeta'(2) - \frac{\pi^2}{24}\log(2)$$

$$\Omega = \frac{\pi^2}{16}\left(1-\gamma-\frac{2}{3}\log(2) + \frac{6}{\pi^2}\zeta'(2)\right)$$

$$\frac{6}{\pi^2}\zeta'(2) = \gamma + \log(2) + \log(\pi) - 12\log(A)$$

Hence:

$$\Omega=\frac{\pi^2}{16}\left(1+\frac{1}{3}\log(2)+\log(\pi)-12\log(A)\right)$$

$$=\frac{\pi^2}{48}(3+\log(2)+3\log(\pi)-36\log(A))$$

$$\int_0^{\frac{\pi}{4}}\frac{\log\left(\log\left(\tan\left(\frac{\pi}{4}+x\right)\right)\right)\log\left(\tan\left(\frac{\pi}{4}+x\right)\right)}{\tan 2x}dx=\frac{\pi^2}{48}\log\left(\frac{2e^3\pi^3}{A^{36}}\right)$$

2010. Prove that:

$$\int_0^{\frac{\pi}{2}}\frac{\log(\cos x)}{\sin x}\log\left(tan\left(\frac{x}{2}\right)\right)dx=\frac{7}{16}\zeta(3)$$

where $\zeta(s),\Re(s)>1$ is Euler-Riemann zeta function.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Bamidele Benjamin-Nigeria

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\log(\cos x)}{\sin x}\log\left(tan\left(\frac{x}{2}\right)\right)dx\overset{u=-\log\left(tan\left(\frac{x}{2}\right)\right)}{=}-\int_0^{\infty}u\cdot\log\left(\frac{1-e^{-2u}}{1+e^{-2u}}\right)du=$$

$$=-\int_0^{\infty}u\cdot\log(1-e^{-2u})\,du+\frac{1}{2}\int_0^{\infty}u\cdot\log(1+e^{-2u})\,du=$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}e^{-2ku}u\,du-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\infty}e^{-2ku}u\,du=$$

$$=\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^3}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}=\frac{1}{4}\zeta(3)-\frac{1}{4}Li_3(-1)=\frac{7}{16}\zeta(3)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\log(\cos x)}{\sin x}\log\left(tan\left(\frac{x}{2}\right)\right)dx\overset{\tan\left(\frac{x}{2}\right)=y}{=}\int_0^1\frac{\log y\log\left(\frac{1-y^2}{1+y^2}\right)}{y}dy=$$

$$=\int_0^1\frac{\log y\log(1-y^2)}{y}dy-\int_0^1\frac{\log y\log(1+y^2)}{y}dy\overset{y^2=z}{=}$$

$$=\frac{1}{4}\int_0^1\frac{\log z\log(1-z)}{z}dz-\frac{1}{4}\int_0^1\frac{\log z\,log(1+z)}{z}dz=\frac{1}{4}I_1-\frac{1}{4}I_2$$

$$I_1=\int_0^1\frac{\log z\log(1-z)}{z}dz=\zeta(3),\qquad I_2=\int_0^1\frac{\log z\log(1+z)}{z}dz=-\frac{3}{4}\zeta(3)$$

$$\Omega=\frac{1}{4}\zeta(3)+\frac{3}{16}\zeta(3)=\frac{7}{16}\zeta(3)$$

2011.Prove:

$$\Omega=\int_0^\pi\frac{a\sin x+b\cos x}{\frac{1}{2}(a+b)\sin^2 x+\sqrt{ab}\cos^2 x}dx$$

$$=\frac{4a}{a-b}\sqrt{1+\frac{2\sqrt{ab}}{a+b}}\tanh^{-1}\frac{a-b}{\sqrt{(a+b)\left(2\sqrt{ab}+a+b\right)}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^\pi\frac{a\sin x+b\cos x}{\frac{1}{2}(a+b)\sin^2 x+\sqrt{ab}\cos^2 x}dx$$

$$=\frac{4a}{a-b}\sqrt{1+\frac{2\sqrt{ab}}{a+b}}\tanh^{-1}\frac{a-b}{\sqrt{(a+b)\left(2\sqrt{ab}+a+b\right)}}$$

$$*\ \frac{1}{2}(a+b)-\sqrt{ab}=\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{\sqrt{2^2}},\qquad *\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

$$\Rightarrow\Omega=\int_0^\pi\frac{a\sin x+b\cos x}{\frac{1}{2}(a+b)\sin^2 x+\sqrt{ab}\cos^2 x}dx=\int_0^\pi\frac{a\sin x-b\cos x}{\frac{1}{2}(a+b)\sin^2 x+\sqrt{ab}\cos^2 x}dx$$

$$\Rightarrow 2\Omega=\int_0^\pi\frac{2a\sin x}{\frac{1}{2}(a+b)\sin^2 x+\sqrt{ab}\cos^2 x}dx\Rightarrow\Omega$$

$$=-\int_0^\pi\frac{a}{\frac{1}{2}(a+b)(1-\cos^2 x)+\sqrt{ab}\cos^2 x}d(\cos x)$$

$$=-\int_0^\pi\frac{a}{\frac{1}{2}(a+b)-\left(\frac{1}{2}(a+b)-\sqrt{ab}\right)\cos^2 x}d(\cos x)$$

$$=-\frac{a}{\left(\frac{1}{2}(a+b)-\sqrt{ab}\right)}\int_0^\pi\frac{d(\cos x)}{\sqrt{\left(\frac{\frac{1}{2}(a+b)}{\left(\frac{1}{2}(a+b)-\sqrt{ab}\right)}\right)^2}-\cos^2 x}$$

$$=-\frac{a}{\left(\frac{1}{2}(a+b)-\sqrt{ab}\right)}\cdot\sqrt{\frac{\left(\frac{1}{2}(a+b)-\sqrt{ab}\right)}{\frac{1}{2}(a+b)}}\tanh^{-1}\left(\cos x\sqrt{\frac{\left(\frac{1}{2}(a+b)-\sqrt{ab}\right)}{\frac{1}{2}(a+b)}}\right)\Bigg|_0^{\pi}$$

$$=-\frac{2a}{\sqrt{a+b}\sqrt{(a+b)-2\sqrt{ab}}}\left(\tanh^{-1}\frac{-(\sqrt{a}-\sqrt{b})}{\sqrt{a+b}}-\tanh^{-1}\frac{(\sqrt{a}-\sqrt{b})}{\sqrt{a+b}}\right)$$

$$=\frac{4a}{\sqrt{a+b}\sqrt{(a+b)-2\sqrt{ab}}}\tanh^{-1}\frac{(\sqrt{a}-\sqrt{b})}{\sqrt{a+b}}$$

$$=\frac{4a}{a-b}\cdot\sqrt{\frac{(\sqrt{a}-\sqrt{b})^2(\sqrt{a}+\sqrt{b})^2}{(a+b)(\sqrt{a}-\sqrt{b})^2}}\tanh^{-1}\frac{a-b}{\sqrt{(a+b)(\sqrt{a}+\sqrt{b})^2}}$$

$$=\frac{4a}{a-b}\cdot\sqrt{\frac{a+b+2\sqrt{ab}}{a+b}}\tanh^{-1}\frac{a-b}{\sqrt{(a+b)(2\sqrt{ab}+a+b)}}$$

$$=\frac{4a}{a-b}\cdot\sqrt{1+\frac{2\sqrt{ab}}{a+b}}\tanh^{-1}\frac{a-b}{\sqrt{(a+b)(2\sqrt{ab}+a+b)}}$$

2012. Find:

$$\Omega=\int_0^1\int_0^{+\infty}\frac{(x-y)(x+y)}{(1+x)(1+x^2)(x^2+y^2)}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^1\int_0^{+\infty}\frac{(x-y)(x+y)}{(1+x)(1+x^2)(x^2+y^2)}dxdy$$

$$=\int_0^1\int_0^{+\infty}\frac{x^2-y^2}{(1+x)(1+x^2)(x^2+y^2)}dxdy=\int_0^{+\infty}\frac{dx}{(1+x)(1+x^2)}\int_0^1\frac{x^2-y^2}{(x^2+y^2)}dy$$

$$=\int_0^{+\infty}\frac{2x\arctan\frac{1}{x}-1}{(1+x)(1+x^2)}dx=\int_0^{+\infty}\frac{2x\left(\frac{\pi}{2}-\arctan x\right)-1}{(1+x)(1+x^2)}dx=\int_0^{+\infty}\frac{\pi x-2x\arctan x-1}{(1+x)(1+x^2)}dx$$

$$=\int_0^{+\infty}\frac{\pi x}{(1+x)(1+x^2)}dx-\int_0^{+\infty}\frac{2x\arctan x}{(1+x)(1+x^2)}dx-\int_0^{+\infty}\frac{1}{(1+x)(1+x^2)}dx$$

$$*\int_0^{+\infty}\frac{\pi x}{(1+x)(1+x^2)}dx=\lim_{b\to+\infty}\int_0^b\pi\left(\frac{x+1}{2(x^2+1)}-\frac{1}{2(x+1)}\right)dx$$

$$=\pi\lim_{b\to+\infty}\left(\frac{1}{4}\ln(x^2+1)-\frac{1}{2}\ln(x+1)+\frac{1}{2}\arctan x\right)\Bigg|_0^b=\frac{\pi^2}{4}$$

$$*\int_0^{+\infty}\frac{1}{(1+x)(1+x^2)}dx=\lim_{b\to+\infty}\int_0^b\left(\frac{1-x}{2(x^2+1)}+\frac{1}{2(x+1)}\right)dx$$

$$=\lim_{b\to+\infty}\left(-\frac{1}{4}ln(x^2+1)+\frac{1}{2}ln(x+1)+\frac{1}{2}arctan\,x\right)\Big|_0^b=\frac{\pi}{4}$$

$$*\int_0^{+\infty}\frac{x\,arctan\,x}{(1+x)(1+x^2)}dx\Rightarrow, let: x=tan\,t\Rightarrow dx=(1+tan^2 t)dt$$

$$\Rightarrow\int_0^{+\infty}\frac{x\,arctan\,x}{(1+x)(1+x^2)}dx=\int_0^{\frac{\pi}{2}}\frac{t\,tan\,t\,(1+tan^2 t)dt}{(1+tan\,t)(1+tan^2 t)}=\int_0^{\frac{\pi}{2}}\frac{t\,tan\,t\,dt}{(1+tan\,t)},\begin{cases}u=t\\ dv=\dfrac{tan\,t}{1+tan\,t}dt\end{cases}$$

$$\Rightarrow\begin{cases}du=dt\\ v=\dfrac{1}{2}(t-ln(sin\,t+cos\,t))\end{cases}$$

$$\Rightarrow\int_0^{+\infty}\frac{x\,arctan\,x}{(1+x)(1+x^2)}dx=\frac{1}{2}t(t-ln(sin\,t+cos\,t))\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}(t-ln(sin\,t+cos\,t))dt$$

$$=\frac{\pi^2}{16}+\frac{1}{2}\int_0^{\frac{\pi}{2}}ln(sin\,t+cos\,t)\,dt$$

$$*\int_0^{\frac{\pi}{2}}ln(sin\,t+cos\,t)\,dt=\frac{\pi}{4}ln\,2+\int_0^{\frac{\pi}{2}}ln\left(sin\left(t+\frac{\pi}{4}\right)\right)dt$$

$$=\frac{\pi}{4}ln\,2+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}ln(sin\,t)\,dt=\frac{\pi}{4}ln\,2+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(-ln\,2-\sum_{k=1}^{+\infty}\frac{cos(2kt)}{k}\right)dt$$

$$=-\frac{\pi}{4}ln\,2-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\sum_{k=1}^{+\infty}\frac{cos(2kt)}{k}dt$$

$$=-\frac{\pi}{4}ln\,2$$

$$-\sum_{k=1}^{+\infty}\frac{sin\frac{3\pi k}{2}}{2k^2}+\sum_{k=1}^{+\infty}\frac{sin\frac{\pi k}{2}}{2k^2}=-\frac{\pi}{4}ln\,2+\frac{1}{2}\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)^2}+\frac{1}{2}\sum_{j=0}^{+\infty}\frac{(-1)^j}{(2j+1)^2}$$

$$=-\frac{\pi}{4}ln\,2+G\Rightarrow\frac{1}{2}\int_0^{\frac{\pi}{2}}ln(sin\,t+cos\,t)\,dt=-\frac{\pi}{8}ln\,2+\frac{G}{2}$$

$$\Rightarrow\int_0^{+\infty}\frac{x\,arctan\,x}{(1+x)(1+x^2)}dx=\frac{\pi^2}{16}-\frac{\pi}{8}ln\,2+\frac{G}{2}$$

$$\Rightarrow\Omega=\frac{\pi^2}{4}-2\left(\frac{\pi^2}{16}-\frac{\pi}{8}ln\,2+\frac{G}{2}\right)-\frac{\pi}{4}=\frac{\pi^2}{4}-\left(\frac{\pi^2}{8}-\frac{\pi}{4}ln\,2+G\right)-\frac{\pi}{4}$$

$$=\frac{\pi^2}{8}-\frac{8G}{8}+\frac{\pi}{4}ln\,2-\frac{\pi}{4}ln\,e=\frac{\pi^2-8G+2\pi\,ln\frac{2}{e}}{8}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\infty}\frac{1}{(1+x)(1+x^2)}dx\int_0^1\frac{x^2-y^2}{x^2+y^2}dx=$$

$$= \int_0^\infty \frac{\pi x - 1 - 2x \cdot \tan^{-1} x}{(1+x)(1+x^2)} dx = \pi \underbrace{\int_0^\infty \frac{x}{(1+x)(1+x^2)} dx}_{I_1} -$$

$$- \underbrace{\int_0^\infty \frac{dx}{(1+x)(1+x^2)}}_{I_2} - 2 \underbrace{\int_0^\infty \frac{x \tan^{-1} x}{(1+x)(1+x^2)} dx}_{I_3}$$

$$I_1 = \int_0^\infty \frac{x}{(1+x)(1+x^2)} dx = \frac{\pi}{4}, \qquad I_2 = \int_0^\infty \frac{dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$$

$$I_3 = \int_0^\infty \frac{x \tan^{-1} x}{(1+x)(1+x^2)} dx = \frac{1}{2} G - \frac{\pi}{8} \log 2 + \frac{\pi^2}{16}$$

$$I = \pi I_1 - I_2 - 2I_3 = \frac{\pi^2}{8} - \frac{\pi}{4} - G + \frac{\pi}{4} \log 2 = \frac{\pi^2 - 8G + 2\pi \log\left(\frac{2}{e}\right)}{8}$$

2013. Find:

$$I = \int_0^1 (x^2 - x + 1)^{\frac{7}{2}} \cos\left(7 \arctan \frac{x\sqrt{3}}{2-x}\right) dx$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$I = \int_0^1 (x^2 - x + 1)^{\frac{7}{2}} \cos\left(7 \arctan \frac{x\sqrt{3}}{2-x}\right) dx$$

$$* If: \frac{t}{7} = \arctan \frac{x\sqrt{3}}{2-x} \Rightarrow \tan \frac{t}{7} = \frac{x\sqrt{3}}{2-x} \Rightarrow 1 + \tan^2 \frac{t}{7} = \frac{1}{\cos^2 \frac{t}{7}} = 1 + \left(\frac{x\sqrt{3}}{2-x}\right)^2$$

$$= \frac{4(x^2 - x + 1)}{(x-2)^2} \Rightarrow \cos^2 \frac{t}{7} = \frac{(x-2)^2}{4(x^2 - x + 1)} \Rightarrow \cos \frac{t}{7}$$

$$= \frac{2 - x}{2(x^2 - x + 1)^{\frac{1}{2}}} (x \in (0,1))$$

$$* \text{By De Moivre's formula and Binomial Theorem} \Rightarrow \cos(t) = \cos\left(7.\frac{t}{7}\right)$$

$$= \cos^7 \frac{t}{7} - 21 \sin^2 \frac{t}{7} \cos^5 \frac{t}{7} + 35 \sin^4 \frac{t}{7} \cos^3 \frac{t}{7} - 7 \sin^6 \frac{t}{7} \cos \frac{t}{7}$$

$$=\left(\frac{2-x}{2(x^2-x+1)^{\frac{1}{2}}}\right)^7-21\left(1-\frac{(x-2)^2}{4(x^2-x+1)}\right)\left(\frac{2-x}{2(x^2-x+1)^{\frac{1}{2}}}\right)^5$$

$$+35\left(1-\frac{(x-2)^2}{4(x^2-x+1)}\right)^2\left(\frac{2-x}{2(x^2-x+1)^{\frac{1}{2}}}\right)^3$$

$$-7\left(1-\frac{(x-2)^2}{4(x^2-x+1)}\right)^3\frac{2-x}{2(x^2-x+1)^{\frac{1}{2}}}$$

$$=\frac{(2-x)(x^6-12x^5-3x^4+29x^3-12x^2-3x+1)}{2(x^2-x+1)^{\frac{7}{2}}}$$

$$\Rightarrow I=\int_0^1(x^2-x+1)^{\frac{7}{2}}\cos\left(7\arctan\frac{x\sqrt{3}}{2-x}\right)dx$$

$$=\int_0^1\frac{(x^2-x+1)^{\frac{7}{2}}(2-x)(x^6-12x^5-3x^4+29x^3-12x^2-3x+1)}{2(x^2-x+1)^{\frac{7}{2}}}dx$$

$$=\int_0^1\frac{(2-x)(x^6-12x^5-3x^4+29x^3-12x^2-3x+1)}{2}dx$$

$$=\frac{1}{2}\int_0^1(-x^7+14x^6-21x^5-35x^4+70x^3-21x^2-7x+2)dx$$

$$=\frac{1}{2}\left(-\frac{x^8}{8}+2x^7-\frac{7}{2}x^6-7x^5+\frac{35}{2}x^4-7x^3-\frac{7}{2}x^2+2x\right)\Bigg|_0^1$$

$$=\frac{1}{2}\left(-\frac{1}{8}+2-\frac{7}{2}-7+\frac{35}{2}-7-\frac{7}{2}+2\right)=\frac{1}{2}\left(\frac{3}{8}\right)=\frac{3}{16}$$

2014. Find:

$$I=\int_0^{\frac{\pi}{2}}\ln(a^2\sin^2 x+b^2\cos^2 x+c^2\tan^2 x)\,dx$$

Proposed by Sakthi Vel-India

Solution by Pham Duc Nam-Vietnam

$*$ *Apply some trigonometry formulas*:

$$\cos^2 x=\frac{1}{1+\tan^2 x},\sin^2 x=1-\cos^2 x=1-\frac{1}{1+\tan^2 x}=\frac{\tan^2 x}{1+\tan^2 x}\Rightarrow I$$

$$=\int_0^{\frac{\pi}{2}}\ln\left(a^2\frac{\tan^2 x}{1+\tan^2 x}+b^2\frac{1}{1+\tan^2 x}+c^2\tan^2 x\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\ln\left(\frac{a^2\tan^2 x+b^2+c^2\tan^2 x\,(1+\tan^2 x)}{1+\tan^2 x}\right)dx$$

$$=\int_0^{\frac{\pi}{2}} ln\left(a^2\tan^2 x+b^2+c^2\tan^2 x\,(1+\tan^2 x)\right)dx-\int_0^{\frac{\pi}{2}} ln(1+\tan^2 x)\,dx$$

$$=\int_0^{\frac{\pi}{2}} ln\left(a^2\tan^2 x+b^2+c^2\tan^2 x\,(1+\tan^2 x)\right)dx+2\int_0^{\frac{\pi}{2}} ln\cos x\,dx$$

$$=\int_0^{\frac{\pi}{2}} ln\left(a^2\tan^2 x+b^2+c^2\tan^2 x\,(1+\tan^2 x)\right)dx-\pi\ln 2=K(a,b,c)-\pi\ln 2$$

$$*\,K(a,b,c)=\int_0^{\frac{\pi}{2}} ln\left(a^2\tan^2 x+b^2+c^2\tan^2 x\,(1+\tan^2 x)\right)dx, let: t=\tan x\Rightarrow x$$

$$=\arctan t\Rightarrow dx=\frac{dt}{t^2+1}$$

$$\Rightarrow K(a,b,c)=\int_0^{+\infty}\frac{ln\left(a^2t^2+b^2+c^2t^2(1+t^2)\right)}{t^2+1}dt\Rightarrow\frac{\partial}{\partial c}K(a,b,c)$$

$$=\int_0^{+\infty}\frac{\partial}{\partial c}\left(\frac{ln\left(a^2t^2+b^2+c^2t^2(1+t^2)\right)}{t^2+1}\right)dt$$

$$=\int_0^{+\infty}\frac{2ct^2}{(a^2+c^2)t^2+b^2+c^2t^4}dt\xrightarrow{t=\sqrt{\frac{b}{c}}k}\frac{2}{\sqrt{bc}}\int_0^{+\infty}\frac{k^2}{k^4+k^2\frac{(a^2+c^2)}{bc}+1}dk$$

$$\xrightarrow{k=\frac{1}{k}}\frac{2}{\sqrt{bc}}\int_0^{+\infty}\frac{1}{k^4+k^2\frac{(a^2+c^2)}{bc}+1}dk\Rightarrow\int_0^{+\infty}\frac{2ct^2}{(a^2+c^2)t^2+b^2+c^2t^4}dt$$

$$=\frac{1}{\sqrt{bc}}\int_0^{+\infty}\frac{k^2+1}{k^4+k^2\frac{(a^2+c^2)}{bc}+1}dk$$

$$=\frac{1}{\sqrt{bc}}\int_0^{+\infty}\frac{d\left(k-\frac{1}{k}\right)}{\left(k-\frac{1}{k}\right)^2+\frac{a^2+c^2+2bc}{bc}}dk\xrightarrow{u=k-\frac{1}{k}}\frac{1}{\sqrt{bc}}\cdot\sqrt{\frac{bc}{a^2+c^2+2bc}}\arctan\left(u\sqrt{\frac{bc}{a^2+c^2+2bc}}\right)\Bigg|_{-\infty}^{+\infty}$$

$$=\frac{\pi}{\sqrt{a^2+c^2+2bc}}\Rightarrow\frac{\partial}{\partial c}K(a,b,c)=\frac{\pi}{\sqrt{a^2+c^2+2bc}}$$

$$*\,K(a,b,0)=K(a,b)=\int_0^{+\infty}\frac{ln(a^2t^2+b^2)}{t^2+1}dt\Rightarrow\frac{\partial}{\partial a}K(a,b)$$

$$=\int_0^{+\infty}\frac{\partial}{\partial a}\left(\frac{ln(a^2t^2+b^2)}{t^2+1}\right)dt=\int_0^{+\infty}\frac{2at^2}{(t^2+1)(a^2t^2+b^2)}dt$$

$$=2a\int_0^{+\infty}\left(\frac{1}{(a^2-b^2)t^2+(a^2-b^2)}-\frac{b^2}{a^2t^2(a^2-b^2)+(a^2-b^2)}\right)dt$$

$$=\frac{2a\arctan t-2b\arctan\frac{at}{b}}{a^2-b^2}\Bigg|_0^{+\infty}=\frac{\pi}{a+b}\Rightarrow K(a,b)=\pi\ln(a+b)+C,let:a=0$$

$$\Rightarrow\pi\ln b=\pi\ln(b)+C\Rightarrow C=0\Rightarrow K(a,b)=\pi\ln(a+b)$$

$$\Rightarrow K(a,b,c)=\int\frac{\pi}{\sqrt{a^2+c^2+2bc}}dc=\pi\ln\left(b+c+\sqrt{a^2+c^2+2bc}\right)+C,let:c=0$$

$$\Rightarrow\pi\ln(a+b)=\pi\ln(a+b)+C\Rightarrow C=0\Rightarrow K(a,b,c)$$

$$=\pi\ln\left(b+c+\sqrt{a^2+c^2+2bc}\right)$$

$$\Rightarrow I=\pi\ln\left(b+c+\sqrt{a^2+c^2+2bc}\right)-\pi\ln 2=\pi\ln\frac{b+c+\sqrt{a^2+c^2+2bc}}{2}$$

2015. Let a,b,c be real numbers. Prove that:

$$\frac{(\pi-1)a^2+(\pi-9)b^2+(\pi-4)c^2-6b+\pi-1}{a^2+b^2+c^2+1}\le\int_0^1\frac{6x+5(\pi^2-1)\sqrt{x}}{4\sqrt{x+(\pi^2-1)\sqrt{x}}}dx$$

Proposed by Kunihiko Chikaya-Tokyo-Japan

Solution by Ruxandra Daniela Tonilă-Romania

$$\frac{(\pi-1)a^2+(\pi-9)b^2+(\pi-4)c^2-6b+\pi-1}{a^2+b^2+c^2+1}=$$

$$=\frac{(a^2+b^2+c^2+1)\pi-a^2-9b^2-6b-4c^1-1}{a^2+b^2+c^2+1}=$$

$$=\pi-\frac{a^2+9b^2+6b+4c^2+1}{a^2+b^2+c^2+1}\le\pi,\forall a,b,c\in\mathbb{R};\quad(1)$$

$$\int_0^1\frac{6x+5(\pi^2-1)\sqrt{x}}{4\sqrt{x+(\pi^2-1)\sqrt{x}}}dx=\int_0^1\left(\frac{4\left(x+(\pi^2-1)\sqrt{x}\right)}{4\sqrt{x+(\pi^2-1)\sqrt{x}}}+\frac{2x+(\pi^2-1)\sqrt{x}}{4\sqrt{x+(\pi^2-1)\sqrt{x}}}\right)dx=$$

$$=\int_0^1\left(\sqrt{x+(\pi^2-1)\sqrt{x}}+\frac{x}{2}\left(1+\frac{\pi^2-1}{2\sqrt{x}}\right)\left(x+(\pi^2-1)\sqrt{x}\right)^{-\frac{1}{2}}\right)dx=$$

$$=\int_0^1\left(\sqrt{x+(\pi^2-1)\sqrt{x}}+x\left(\sqrt{x+(\pi^2-1)\sqrt{x}}\right)'\right)dx=$$

$$=\int_0^1\left[x\cdot\sqrt{x+(\pi^2-1)\sqrt{x}}\right]'dx=x\cdot\sqrt{x+(\pi^2-1)\sqrt{x}}\Bigg|_0^1=\pi;\quad(2)$$

From (1) and (2), it follows that:

$$\frac{(\pi-1)a^2+(\pi-9)b^2+(\pi-4)c^2-6b+\pi-1}{a^2+b^2+c^2+1}\le\int_0^1\frac{6x+5(\pi^2-1)\sqrt{x}}{4\sqrt{x+(\pi^2-1)\sqrt{x}}}dx$$

2016. ***If*** $0 < a \leq b < \frac{\pi}{2}$ ***then :***

$$\int_a^b \frac{(e^x - e^{\sin x})(\tan x - x)}{e^{\tan x} - e^x} dx \leq (b-a)\left(1 + \sin\frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that:\ \sin x < x < \tan x,\ \ \forall x \in \left(0, \frac{\pi}{2}\right)\ \ and\ \ e^t - 1 \geq t,\ \ \forall t \in \mathbb{R}.$$

$$Then:\ e^{\tan x - x} - 1 \geq \tan x - x\ \ or\ \ \frac{\tan x - x}{e^{\tan x} - e^x} \leq e^{-x},\ \ \forall x \in \left(0, \frac{\pi}{2}\right).$$

$$Thus,\ \ \frac{(e^x - e^{\sin x})(\tan x - x)}{e^{\tan x} - e^x} \leq (e^x - e^{\sin x})e^{-x} = 1 - e^{\sin x - x} \leq 1,\ \ \forall x \in \left(0, \frac{\pi}{2}\right).$$

$$Therefore,\ \ \int_a^b \frac{(e^x - e^{\sin x})(\tan x - x)}{e^{\tan x} - e^x} dx \leq \int_a^b dx = b - a \leq (b-a)\left(1 + \sin\frac{a+b}{2}\right).$$

$$Equality\ holds\ for\ a = b.$$

Solution 2 by Tapas Das-India

$$We\ know\ that: e^t = 1 + t + \frac{t^2}{2!} + \cdots$$

$$e^x(1 + \tan x - x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)(\tan x - x) + \cdots =$$

$$= (\tan x - x) + x(\tan x - x) + \frac{x^2}{2!}(\tan x - x) + \cdots; \qquad (1)$$

$$e^{\tan x} - e^x = \left(1 + \tan x + \frac{\tan^2 x}{2!} + \frac{\tan^3 x}{3!} + \cdots\right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) =$$

$$= (\tan x - x) + \frac{\tan^2 x - x^2}{2!} + \frac{\tan^3 x - x^3}{3!} + \cdots =$$

$$= (\tan x - x) + \frac{(\tan x - x)(\tan x + x)}{2!} + \frac{(\tan x - x)(\tan^2 x + x\tan x + x^2)}{3!}$$

$$+ \cdots; \quad (3)$$

$$From\ (1)\ and\ (2):\ e^x(\tan x - x) \leq e^{\tan x} - e^x$$

$$\frac{\tan x - x}{e^{\tan x} - e^x} \le \frac{1}{e^x}$$

$$\frac{e^x - e^{\sin x}}{e^{\tan x} - e^x}(\tan x - x) \le \frac{e^x - e^{\sin x}}{e^x} \quad since: \frac{\tan x - x}{e^{\tan x} - e^x} \le \frac{1}{e^x}$$

$$We\ know\ that: \sin x < x; (\forall) x \in \left(0, \frac{\pi}{2}\right), \qquad e^{\sin x} < e^x; e^x - e^{\sin x} < e^x$$

$$\frac{e^x - e^{\sin x}}{e^{\tan x} - e^x}(\tan x - x) \le \frac{e^x - e^{\sin x}}{e^x} \le \frac{e^x}{e^x} = 1$$

$$Therefore, \quad \int_a^b \frac{(e^x - e^{\sin x})(\tan x - x)}{e^{\tan x} - e^x} dx \le \int_a^b dx = b - a \le (b - a)\left(1 + \sin\frac{a+b}{2}\right).$$

$$Equality\ holds\ for\ a = b.$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$f(x) = \frac{1}{\frac{e^{\tan x} - e^x}{\tan x - x}}; x \in \left(0, \frac{\pi}{2}\right)\ and\ since\ \sin x < x; (\forall) x \in \left(0, \frac{\pi}{2}\right)$$

$$f(x) = \frac{1}{e^x \cdot \frac{e^{\tan x - x} - 1}{\tan x - x}} = \frac{e^{-x}}{\frac{e^{\tan x - x} - 1}{\tan x - x}}$$

$$\lim_{x \to 0} f(x) = 1; \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{y \to \infty} \frac{e^y - 1}{y} = 0$$

$$f'(x) < 0; (\forall) x \in \left(0, \frac{\pi}{2}\right) \Rightarrow f - decreasing, so\ f(x) \le 1; (\forall) x \in \left(0, \frac{\pi}{2}\right)$$

$$e^x - e^{\sin x} = e^x(1 - e^{\sin x - x})$$

$$e^x(1 - e^{\sin x - x}) f(x) \le 1 - e^{\sin x - x}; (\sin x - x < 0), so\ 1 - e^{\sin x - x} \le 1$$

$$Therefore, \quad \int_a^b \frac{(e^x - e^{\sin x})(\tan x - x)}{e^{\tan x} - e^x} dx \le \int_a^b dx = b - a \le (b - a)\left(1 + \sin\frac{a+b}{2}\right).$$

$$Equality\ holds\ for\ a = b.$$

2017. If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{(1 - \sin^5 x)^7}{(1 - \sin^3 x)^8} dx \ge b - a$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

$$We\ know\ that: 0 < \sin x < 1; (\forall)x \in \left(0, \frac{\pi}{2}\right), then\ \sin^3 x > \sin^5 x\ and$$

$$1 - \sin^2 x < 1 - \sin^5 x, also\ we\ have: \sin x < x; (\forall)x \in \left(0, \frac{\pi}{2}\right)$$

$$Hence: \frac{(1 - \sin^5 x)^7}{(1 - \sin^3 x)^8} > \frac{(1 - \sin^5 x)^7}{(1 - \sin^5 x)^8} = \frac{1}{1 - \sin^5 x} \geq 1$$

$$Therefore, \int_a^b \frac{(1 - \sin^5 x)^7}{(1 - \sin^3 x)^8} dx \geq \int_a^b dx = b - a$$

Equality holds for $a = b$.

2018. $B(x, y) = \int_0^1 t^x (1 - t)^y\, dt, x, y > 0$

Prove that:

$$B(x, y) \cdot B(y, z) \cdot B(z, x) \geq B(x, x) \cdot B(y, y) \cdot B(z, z)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \text{ and } \Gamma(\alpha) = \int_0^\infty S^{\alpha - 1} e^{-s}\, ds$$

$$\Gamma(\lambda x + (1 - \lambda)y) = \int_0^\infty S^{\lambda x + (1 - \lambda)y - 1} e^{-S}\, ds =$$

$$= \int_0^\infty S^{\lambda(x-1) + (1-\lambda)(y-1)} e^{-\lambda s} e^{-(1-\lambda)s}\, ds =$$

$$= \int_0^\infty (S^{x-1} e^{-S})^\lambda (S^{y-1} e^{-S})^{1-\lambda}\, ds \overset{Holder}{\leq}$$

$$\leq \left(\int_0^\infty \left((S^{x-1} e^{-S})^\lambda\right)^{\frac{1}{\lambda}} ds\right)^\lambda \left(\int_0^\infty \left((S^{y-1} e^{-S})^{1-\lambda}\right)^{\frac{1}{1-\lambda}} ds\right)^{1-\lambda} =$$

$$= \left(\int_0^\infty S^{x-1} e^{-S}\, ds\right)^\lambda \left(\int_0^\infty S^{y-1} e^{-S}\, ds\right)^{1-\lambda} = \Gamma(x)^\lambda \Gamma(y)^{1-\lambda}$$

$\Gamma(\cdot)$ —is log-convex, then:

$$\Gamma\left(\frac{1}{2} \cdot 2x + \frac{1}{2} \cdot 2y\right) \leq \Gamma(2x)^{\frac{1}{2}} \Gamma(y)^{\frac{1}{2}}$$

$$\Gamma(x + y) \leq \sqrt{\Gamma(2x)\Gamma(2y)}$$

Hence:

$$\frac{1}{\Gamma(x+y)} \geq \frac{1}{\sqrt{\Gamma(2x)\Gamma(2y)}} \Rightarrow \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \geq \frac{\Gamma(x)\Gamma(y)}{\sqrt{\Gamma(2x)\Gamma(2y)}}$$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \geq \sqrt{\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+x)}} \cdot \sqrt{\frac{\Gamma(x)\Gamma(y)}{\Gamma(y+y)}}$$

$$B(x,y) \geq \sqrt{B(x,x)} \cdot \sqrt{B(y,y)}$$

$$B(y,z) \geq \sqrt{B(y,y)} \cdot \sqrt{B(z,z)}$$

$$B(z,x) \geq \sqrt{B(z,z)} \cdot \sqrt{B(x,x)}$$

Therefore,

$$B(x,y) \cdot B(y,z) \cdot B(z,x) \geq B(x,x) \cdot B(y,y) \cdot B(z,z)$$

2019. $If\ 0 < a \leq b\ then\ prove:$

$$\frac{\sqrt{3}}{2}(b^2-a^2)(b-a)^2 \leq \int_a^b \int_a^b \int_a^b \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz \leq \frac{\sqrt{3}}{4}(b^2-a^2)^2 ln\left(\frac{b}{a}\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$Lemma:\ If\ x,y,z > 0\ then:$

$$x+y+z \leq \sqrt{3(x^2+y^2+z^2)} \leq \frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}.$$

$Proof:\ By\ CBS\ inequality, we\ have: \sqrt{3(x^2+y^2+z^2)} \geq x+y+z.$

$Using\ the\ well\ known\ inequality\ (u+v+w)^2 \geq 3(uv+vw+wu), we\ have:$

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y} = \frac{(xy)^2+(yz)^2+(zx)^2}{xyz} \geq \frac{\sqrt{3[(xy)^2.(yz)^2+(yz)^2.(zx)^2+(zx)^2.(xy)^2]}}{xyz}$$
$$= \sqrt{3(x^2+y^2+z^2)}.$$

$So\ the\ proof\ of\ the\ lemma\ is\ done. Using\ this\ lemma\ we\ have:$

$$\int_a^b \int_a^b \int_a^b \frac{x+y+z}{\sqrt{3}}\,dx\,dy\,dz \leq \int_a^b \int_a^b \int_a^b \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$$
$$\leq \int_a^b \int_a^b \int_a^b \frac{1}{\sqrt{3}}\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)dx\,dy\,dz,$$

$$with:\ \int_a^b\int_a^b\int_a^b\frac{x+y+z}{\sqrt{3}}dx\,dy\,dz=\frac{3}{\sqrt{3}}\int_a^b xdx\int_a^b dy\int_a^b dz=\frac{\sqrt{3}}{2}(b^2-a^2)(b-a)^2,$$

$$and:$$

$$\int_a^b\int_a^b\int_a^b\frac{1}{\sqrt{3}}\left(\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\right)dx\,dy\,dz=\frac{3}{\sqrt{3}}\int_a^b xdx\int_a^b ydy\int_a^b\frac{dz}{z}=\frac{\sqrt{3}}{4}(b^2-a^2)^2 ln\left(\frac{b}{a}\right).$$

$$Therefore,$$

$$\frac{\sqrt{3}}{2}(b^2-a^2)(b-a)^2\le\int_a^b\int_a^b\int_a^b\sqrt{x^2+y^2+z^2}dx\,dy\,dz\le\frac{\sqrt{3}}{4}(b^2-a^2)^2 ln\left(\frac{b}{a}\right).$$

2020. Prove that:

$$S=\sum_{k=1}^{+\infty}\left(\frac{1}{(12k)!}+\frac{1}{(12k-6)!}\right)=\frac{cosh(1)}{3}+\frac{2}{3}cosh\left(\frac{1}{2}\right)cos\left(\frac{\sqrt{3}}{2}\right)-1$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$S=\sum_{k=1}^{+\infty}\left(\frac{1}{(12k)!}+\frac{1}{(12k-6)!}\right)=\frac{cosh(1)}{3}+\frac{2}{3}cosh\left(\frac{1}{2}\right)cos\left(\frac{\sqrt{3}}{2}\right)-1$$

$$*\ \mathcal{L}(t^n)=\frac{n!}{s^{n+1}}$$

$$PROVE: \mathcal{L}(t^n)=\int_0^{+\infty}t^ne^{-st}dt,\begin{cases}u=t^n\\dv=e^{-st}dt\end{cases}\Rightarrow\begin{cases}du=nt^{n-1}\\v=-\dfrac{e^{-st}}{s}\end{cases}\Rightarrow\mathcal{L}(t^n)$$

$$=\underbrace{-\frac{t^ne^{-st}}{s}\Big|_0^{+\infty}}_{=0}+\frac{n}{s}\int_0^{+\infty}t^{n-1}e^{-st}dt=\frac{n}{s}\mathcal{L}(t^{n-1})=\frac{n(n-1)}{s^2}\mathcal{L}(t^{n-2})=\ldots=\frac{n!}{s^{n+1}}$$

$$*\ Consider: S(t)=\sum_{k=1}^{+\infty}\left(\frac{t^{12k}}{(12k)!}+\frac{t^{12k-6}}{(12k-6)!}\right)\Rightarrow S=S(1)$$

$$*\ TakingLaplaceTransform: \mathcal{L}(S(t))=\sum_{k=1}^{+\infty}\left(\mathcal{L}\left(\frac{t^{12k}}{(12k)!}\right)+\mathcal{L}\left(\frac{t^{12k-6}}{(12k-6)!}\right)\right)$$

$$=\sum_{k=1}^{+\infty}\left(\frac{1}{s^{12k+1}}+\frac{1}{s^{12k-5}}\right)=\frac{1}{s(s^6-1)}$$

$$=\frac{2s-1}{6(s^2-s+1)}+\frac{2s+1}{6(s^2+s+1)}+\frac{1}{6}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)-\frac{1}{s}$$

$$*TakingInverseLaplaceTransform: S(t)$$

$$= \mathcal{L}^{-1}\left(\frac{2s-1}{6(s^2-s+1)}+\frac{2s+1}{6(s^2+s+1)}+\frac{1}{6}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)-\frac{1}{s}\right)$$

$$= \frac{1}{6}\underbrace{(e^{-t}+e^{t})}_{=2\cosh t}-1+\frac{1}{6}\mathcal{L}^{-1}\left(\frac{2\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{2\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= \frac{1}{3}\cosh t-1+\frac{1}{6}\left(2e^{\frac{1}{2}t}\cos\frac{t\sqrt{3}}{2}+2e^{-\frac{1}{2}t}\cos\frac{t\sqrt{3}}{2}\right)$$

$$= \frac{1}{3}\cosh t-1+\frac{1}{3}\cos\frac{t\sqrt{3}}{2}\underbrace{\left(e^{\frac{1}{2}t}+e^{-\frac{1}{2}t}\right)}_{=2\cosh\frac{t}{2}}$$

$$= \frac{1}{3}\cosh t-1+\frac{2}{3}\cos\frac{t\sqrt{3}}{2}\cosh\frac{t}{2}. Let: t=1 \Rightarrow S = \boxed{\frac{\cosh(1)}{3}+\frac{2}{3}\cosh\left(\frac{1}{2}\right)\cos\left(\frac{\sqrt{3}}{2}\right)-1}$$

2021. If $x, y > 0$ then:

$$\Gamma(x+5)\cdot\Gamma(y+7)\leq 11!\cdot\Gamma(x+y)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

The inequality holds if $0 < x < 7, 0 < y < 5$ or $x > 7, y > 5$.

If $x = 5$ or $y = 5$ holds equality.

If $0 < x < 7$ or $y > 5$ or $x > 7, 0 < y < 5$ the inequality is reversed.

Let be the functions: $\alpha(u), \mu(u), c(u), u \in [0,1], \alpha(u) \geq 0$

$\mu(u), c(u)$ have opposite monotonicity

If μ is increasing and c is decreasing: $\begin{cases} u > v \Rightarrow \mu(u)-\mu(v)\geq 0; c(u)-c(v)\leq 0 \\ u < v \Rightarrow \mu(u)-\mu(v)\leq 0; c(u)-c(v)\geq 0 \end{cases}$

If μ is decreasing and c is increasing: $\begin{cases} u > v \Rightarrow \mu(u)-\mu(v)\leq 0; c(u)-c(v)\geq 0 \\ u < v \Rightarrow \mu(u)-\mu(v)\geq 0; c(u)-c(v)\leq 0 \end{cases}$

$(\forall)u, v \in [0,1]$ and $\big(\mu(u)-\mu(v)\big)\big(c(u)-c(v)\big)\leq 0$

Since $\alpha(u)\geq 0$ then we have:

$$\alpha(u)\alpha(v)\big(\mu(u)-\mu(v)\big)\big(c(u)-c(v)\big)\leq 0; (\forall)u, v \in [0,1]$$

$$\int_0^1 \alpha(v)\,dv\int_0^1 \alpha(u)\mu(u)c(u)\,du-\int_0^1 \alpha(u)\mu(u)du\int_0^1 \alpha(v)c(v)\,dv-$$

$$-\int_0^1 \alpha(u)c(u)du\int_0^1 \alpha(v)\mu(v)dv+\int_0^1 \alpha(u)du\int_0^1 \alpha(v)\mu(v)c(v)dv \le 0$$

$$\int_0^1 \alpha(u)\,du\int_0^1 \alpha(u)\mu(u)c(u)\,du \le \int_0^1 \alpha(u)\mu(u)\,du\int_0^1 \alpha(u)c(u)\,du$$

$$(I)\text{ Set } \alpha(u)=u^{x-1}\underbrace{(1-y)^{y-1}}_{\ge 0};\mu(u)=u^{7-x};(\text{increasing})$$

$$c(u)=(1-u)^{5-y};(\text{decreasing}),0<x<7;0<y<5$$

$\mu(u),c(u)$ –have opposite monotonicity.

$$\int_0^1 u^{x-1}(1-u)^{y-1}du\int_0^1 u^{7-1}(1-u)^{(y-1)}u^{7-x}(1-u)^{5-y}\,du=$$

$$=\beta(x,y)\beta(7,5)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}$$

$$\int_0^1 u^{x-1}(1-u)^{y-1}u^{7-x}\,dx\int_0^1 u^{x-1}(1-u)^{y-1}(1-u)^{5-y}\,du=$$

$$=\beta(7,y)\beta(x,5)=\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)}\cdot\frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)}$$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}\le\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)}\cdot\frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)}$$

$$\Rightarrow \Gamma(x+y)\Gamma(12)\ge\Gamma(7+y)\Gamma(5+x)$$

$$(II)\text{ Set } \alpha(u)=u^{7-1}(1-u)^{5-1}\ge 0;\ \mu(u)=u^{x-7};(\text{increasing}),c(u)=(1-u)^{y-5},$$

$$x>7,y>5$$

$\mu(u),c(u)$ –have opposite monotonicity

$$\int_0^1 u^{7-1}(1-u)^{5-1}du\int_0^1 u^{7-1}(1-u)^{5-1}u^{5-1}(1-u)^{y-5}\,du=$$

$$=\beta(7,5)\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}$$

$$\int_0^1 u^{7-1}(1-u)^{5-1}u^{x-7}\,dx\int_0^1 u^{7-1}(1-u)(1-u)^{y-5}\,du=$$

$$=\beta(7,y)\beta(x,5)=\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)}\cdot\frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)}$$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}\le\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)}\cdot\frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)}$$

$$\Rightarrow \Gamma(x+y)\Gamma(12) \geq \Gamma(7+y)\Gamma(5+x)$$

(III) **Set** $\alpha(u) = u^{x-1}(1-u)^{5-1} \geq 0;\ \mu(u) = u^{7-x};$ **(increasing)**, $c(u) = (1-u)^{y-5}$,

$$0x < 7, y > 5$$

$\mu(u), c(u)$ **–have opposite monotonicity**

$$\int_0^1 u^{x-1}(1-u)^{5-1}du \int_0^1 u^{x-1}(1-u)^{y-5}u^{7-x}(1-u)^{y-5}\,du =$$

$$= \beta(x,5)\beta(7,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+5)} \cdot \frac{\Gamma(7)\Gamma(5)}{\Gamma(7+y)}$$

$$\int_0^1 u^{x-1}(1-u)^{5-1}u^{7-x}\,dx \int_0^1 u^{x-1}(1-u)^{5-1}(1-u)^{y-5}\,du =$$

$$= \beta(7,5)\beta(x,y) = \frac{\Gamma(7)\Gamma(5)}{\Gamma(12)} \cdot \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)} \cdot \frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)} \leq \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \cdot \frac{\Gamma(7)\Gamma(5)}{\Gamma(12)} \Rightarrow \Gamma(x+y)\Gamma(12) \leq \Gamma(7+y)\Gamma(5+x)$$

(IV) **Set** $\alpha(u) = u^{7-1}(1-u)^{y-1} \geq 0;\ \mu(u) = u^{x-7};$ **(increasing)**, $c(u) = (1-u)^{5-y}$,

$x > 7, 0 < y < 5,$ $\mu(u), c(u)$ **–have opposite monotonicity**

$$\int_0^1 u^{7-1}(1-u)^{y-1}du \int_0^1 u^{7-1}(1-u)^{y-1}u^{x-7}(1-u)^{5-y}\,du =$$

$$= \beta(7,y)\beta(x,5) = \frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)} \cdot \frac{\Gamma(5)\Gamma(x)}{\Gamma(5+x)}$$

$$\int_0^1 u^{7-1}(1-u)^{y-1}u^{x-7}\,dx \int_0^1 u^{7-1}(1-u)^{y-1}(1-u)^{5-y}\,du =$$

$$= \beta(x,y)\beta(7,5) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \cdot \frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}$$

$$\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)} \cdot \frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)} \leq \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \cdot \frac{\Gamma(7+5)}{\Gamma(12)} \Rightarrow \Gamma(x+y)\Gamma(12) \leq \Gamma(7+y)\Gamma(5+x)$$

2022. Find:

$$\Omega = \prod_{n=1}^{\infty} \frac{49n^2+42n+9}{49n^2+42n-40}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$Let\ a_n=\frac{49n^2+42n+9}{49n^2+42n-40}=\frac{(7n+3)^2}{(7n-4)(7n+10)}$$

$$\log a_n=2\log(7n+3)-\log(7n-4)-\log(7n-10)=$$

$$=\log(7n+3)-\log(7n-4)-[\log(7n+10)-\log(7n+3)]$$

$$Let\ P_n=\prod_{k=1}^{n}a_k\,,then\log P_n=\sum_{k=1}^{n}\log a_k=$$

$$=\sum_{k=1}^{n}[\log(7k+3)-\log(7k-4)]-\sum_{k=1}^{n}[\log(7k+10)-\log(7k+3)]=$$

$$=\log(7n+3)-\log 3-\log(7n+10)+\log 10=$$

$$=\log\left(\frac{7n+3}{7n+10}\right)+\log\left(\frac{10}{3}\right),hence\ P_n=\frac{10}{3}\cdot\frac{7n+3}{7n+10}$$

$$\lim_{n\to\infty}P_n=\frac{10}{3}\cdot\lim_{n\to\infty}\frac{1+\frac{3}{7n}}{1+\frac{10}{7n}}=\frac{10}{3}$$

Solution 2 by Le Thu-Vietnam

$$Write:\frac{49n^2+42n+9}{49n^2+42n-40}=\frac{49\left(n+\frac{3}{7}\right)^2}{49\left(n-\frac{4}{7}\right)\left(n+\frac{10}{7}\right)}=\frac{\left(n+\frac{3}{7}\right)\left(n+\frac{3}{7}\right)}{\left(n-\frac{4}{7}\right)\left(n+\frac{10}{7}\right)}\ we\ get:$$

$$\Omega=\prod_{n=1}^{\infty}\frac{\left(n+\frac{3}{7}\right)\left(n+\frac{3}{7}\right)}{\left(n-\frac{4}{7}\right)\left(n+\frac{10}{7}\right)}\overset{n=k+1}{=}\prod_{k=0}^{\infty}\frac{k+\frac{10}{7}}{k+\frac{3}{7}}\cdot\prod_{n=1}^{\infty}\frac{n+\frac{3}{7}}{n+\frac{10}{7}}=\frac{10}{3}$$

Solution 3 by Adrian Popa-Romania

$$\Omega=\prod_{n=1}^{\infty}\frac{49n^2+42n+9}{49n^2+42n-40}=\prod_{n=1}^{\infty}\frac{(7n+3)^2}{(7n-4)(7n+10)}=$$

$$=\prod_{n=1}^{\infty}\frac{7n+3}{7n-4}\cdot\prod_{n=1}^{\infty}\frac{7n+3}{7n+10}=P_1\cdot P_2$$

$$P_1=\lim_{n\to\infty}\prod_{k=1}^{n}\frac{7k+3}{7k-4}=\frac{10}{3}\cdot\frac{17}{10}\cdot\ldots\cdot\frac{7n+3}{7n-4}=\frac{7n+3}{3}$$

$$P_2=\lim_{n\to\infty}\prod_{k=1}^{n}\frac{7k+3}{7k+10}=\frac{10}{17}\cdot\frac{17}{24}\cdot\ldots\cdot\frac{7n+3}{7n+10}=\frac{10}{7n+10}$$

$$Therefore,\Omega=\lim_{n\to\infty}\frac{10(7n+3)}{3(7n+10)}=\frac{10}{3}.$$

Solution 4 by Ankush Kumar Parcha-India

$$\Omega=\prod_{n=1}^{\infty}\frac{49n^2+42n+9}{49n^2+42n-40}=\prod_{n=1}^{\infty}\frac{(7n+3)^2}{(7n+3)^2-7^2}=\prod_{n=1}^{\infty}\frac{(7n+3)^2}{(7n-4)(7n+10)}=$$

$$=\prod_{n=1}^{\infty}\frac{7n+3}{7n-4}\cdot\prod_{n=1}^{\infty}\frac{7n+3}{7n+10}=\frac{1}{3}\cdot\frac{\prod_{n=1}^{\infty}(7n+3)}{\prod_{n=2}^{\infty}(7n-4)}\cdot 10\cdot\frac{\prod_{n=2}^{\infty}(7n+3)}{\prod_{n=1}^{\infty}(7n+10)}=$$

$$=\frac{10}{3}\cdot\frac{\prod_{n=1}^{\infty}(7n+3)}{\prod_{n=1}^{\infty}(7n+3)}\cdot\frac{\prod_{n=1}^{\infty}(7n+10)}{\prod_{n=1}^{\infty}(7n+10)}=\frac{10}{3}$$

$$Therefore,\Omega=\prod_{n=1}^{\infty}\frac{49n^2+42n+9}{49n^2+42n-40}=\frac{10}{3}$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$First, we\ will\ show:$$

$$\prod_{k=0}^{\infty}\frac{(a_1+k)(a_2+k)\ldots(a_m+k)}{(b_1+k)(b_2+k)\ldots(b_m+k)}=\frac{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_m)}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_m)}\ if\ \sum_{j=1}^{m}a_j=\sum_{j=1}^{m}b_j$$

$$Recall\ the\ Euler's\ definition\ of\ the\ Gamma\ function:$$

$$\Gamma(z)=\lim_{n\to\infty}\frac{(n+1)^z z!}{z(z+1)(z+2)\ldots(z+n)}$$

$$Suppose\ that: \sum_{j=1}^{m}a_j=\sum_{j=1}^{m}b_j.\ Hence,$$

$$\frac{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_m)}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_m)}=\lim_{n\to\infty}\prod_{j=1}^{m}\left[(n+1)^{b_j-a_j}\prod_{k=0}^{n}\frac{a_j+k}{b_j+k}\right]=$$

$$=\lim_{n\to\infty}\left[(n+1)^{\sum_{j=1}^{m}(a_j-b_j)}\prod_{j=1}^{m}\prod_{k=0}^{n}\frac{a_j+k}{b_j+k}\right]=\lim_{n\to\infty}\prod_{j=1}^{m}\prod_{k=0}^{n}\frac{a_j+k}{b_j+k}=$$

$$=\lim_{n\to\infty}\prod_{k=0}^{n}\prod_{j=1}^{m}\frac{a_j+k}{b_j+k}=\prod_{k=0}^{\infty}\prod_{j=1}^{m}\frac{a_j+k}{b_j+k}$$

$$\Omega=\prod_{n=1}^{\infty}\frac{49n^2+42n+9}{49n^2+42n-40}=-\frac{40}{9}\prod_{n=0}^{\infty}\frac{49n^2+42n+9}{49n^2+42n-40}=$$

$$=-\frac{49}{9}\prod_{n=0}^{\infty}\frac{(7n+3)(7n+3)}{(7n-4)(7n+10)}=-\frac{49}{9}\prod_{n=0}^{\infty}\frac{\left(n+\frac{3}{7}\right)\left(n+\frac{3}{7}\right)}{\left(n-\frac{4}{7}\right)\left(n+\frac{10}{7}\right)}=$$

$$=-\frac{40}{9}\cdot\frac{\Gamma\left(-\frac{4}{7}\right)\Gamma\left(\frac{10}{7}\right)}{\Gamma\left(\frac{3}{7}\right)\Gamma\left(\frac{3}{7}\right)}=\frac{10}{3}$$

Therefore,

$$\Omega=\prod_{n=1}^{\infty}\frac{49n^2+42n+9}{49n^2+42n-40}=\frac{10}{3}$$

2023. Prove that:

$$I=\int_0^1 log\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}sin^{-1}x\,dx=\frac{\pi^2}{4}-2\log 2$$

Proposed by Ose Favour-Nigeria

Solution by Pham Duc Nam-Vietnam

$$*\ t=sin^{-1}x\Rightarrow x=\sin t\Rightarrow dx=\cos t\,dt$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}log\frac{1+\cos t}{1-\cos t}t.\cos t\,dt=\int_0^{\frac{\pi}{2}}t\cos t\,logcot^2\frac{t}{2}dt$$

$$=2\int_0^{\frac{\pi}{2}}t\cos t\,logcot\frac{t}{2}=-2\int_0^{\frac{\pi}{2}}t\cos t\,logtan\frac{t}{2}dt$$

$$\begin{cases}u=t\\ dv=\cos t\,logtan\frac{t}{2}dt\end{cases}\Rightarrow\begin{cases}du=dt\\ v=\sin t\,logtan\frac{t}{2}-t\end{cases}\Rightarrow I$$

$$=-2\left(\left(\sin t\,logtan\frac{t}{2}-t\right)t\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\left(\sin t\,logtan\frac{t}{2}-t\right)dt\right)$$

$$= -2\left(-\frac{\pi^2}{4}+\frac{\pi^2}{8}-\int_0^{\frac{\pi}{2}} \sin t\, logtan\frac{t}{2}dt\right)=\frac{\pi^2}{4}+2\int_0^{\frac{\pi}{2}} \sin t\, logtan\frac{t}{2}dt$$

$$=\frac{\pi^2}{4}+2\left(\int_0^{\frac{\pi}{2}} \sin t\, logsin\frac{t}{2}dt-\int_0^{\frac{\pi}{2}} \sin t\, logcos\frac{t}{2}dt\right)$$

$$\xrightarrow{u=logsin\frac{t}{2} \text{ and } v=logcos\frac{t}{2}} \frac{\pi^2}{4}$$

$$+2\left(\left(2\sin^2\frac{t}{2}logsin\frac{t}{2}-\sin^2\frac{t}{2}\right)\Big|_0^{\frac{\pi}{2}}-\left(\frac{\cos t}{2}-\cos t\, logcos\frac{t}{2}-logcos\frac{t}{2}\right)\Big|_0^{\frac{\pi}{2}}\right)$$

$$=\frac{\pi^2}{4}+2\left(\frac{1}{2}(-1-log\,2)-\frac{1}{2}(log\,2-1)\right)=\boxed{\frac{\pi^2}{4}-2\log 2}$$

2024. Prove that:

$$I=\int_0^{+\infty}\int_0^{+\infty}\frac{ln^2(xy)}{(x^2+x+1)(y^2+y+1)}dxdy=\frac{64\pi^4}{729}$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Pham Duc Nam-Vietnam

$$I=\int_0^{+\infty}\int_0^{+\infty}\frac{ln^2(xy)}{(x^2+x+1)(y^2+y+1)}dxdy=$$

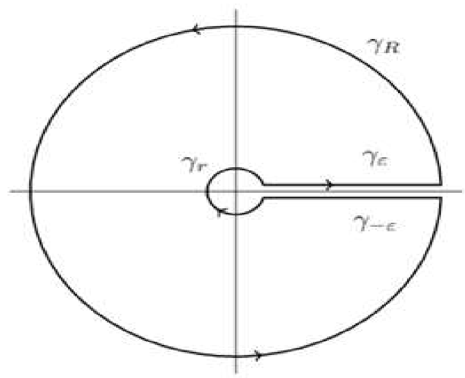
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{ln^2 x}{(x^2+x+1)(y^2+y+1)}dxdy+\int_0^{+\infty}\int_0^{+\infty}\frac{ln^2 y}{(x^2+x+1)(y^2+y+1)}dxdy$$

$$+2\int_0^{+\infty}\int_0^{+\infty}\frac{ln\,x\,ln\,y}{(x^2+x+1)(y^2+y+1)}dxdy$$

$$=2\int_0^{+\infty}\frac{ln^2 x}{(x^2+x+1)}dx\int_0^{+\infty}\frac{1}{(y^2+y+1)}dy\ (Symmetry)$$

$$=2\int_0^{+\infty}\frac{ln\,x}{(x^2+x+1)}dx\int_0^{+\infty}\frac{ln\,y}{(y^2+y+1)}dy$$

$$I_x=\int_0^{+\infty}\frac{ln^2 x}{(x^2+x+1)}dx$$



$*\ Define$: $A\ key\ hole\ contour, with\ a\ branch\ cut\ on\ real\ axis\ (positive\ direction),$

$$around\ 0, C=\gamma_R\cup\gamma_r\cup\gamma_{-\varepsilon}\cup\gamma_\varepsilon$$

$$*\ Let\ f(z)=\frac{\ln^3 z}{z^2+z+1}$$

$$*\ f(z) has\ 2\ poles\ -\frac{1}{2}\pm\frac{i\sqrt{3}}{2}, order\ 1, inside\ the\ contour.$$

$$\Rightarrow\int_C\frac{\ln^3 z}{z^2+z+1}dz=2\pi i\sum \text{Res}\left(f(z),-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)$$

$$=2\pi i\sum\lim_{z\to-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}}\left(z-\left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)\right)\frac{\ln^3 z}{z^2+z+1}$$

$$=2\pi i\ln^3\left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)\left(\pm\frac{1}{i\sqrt{3}}\right)$$

$$=2\pi i\left(i\arg-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)^3\left(\pm\frac{1}{i\sqrt{3}}\right)=2\pi i\left(\frac{2\pi i}{3}\right)^3\left(\frac{1}{i\sqrt{3}}\right)+2\pi i\left(\frac{4\pi i}{3}\right)^3\left(-\frac{1}{i\sqrt{3}}\right)=\frac{112i\pi^4\sqrt{3}}{81}$$

$$*\int_C\frac{\ln^3 z}{z^2+z+1}dz=\int_{\gamma_R}\frac{\ln^3 z}{z^2+z+1}dz+\int_{\gamma_{-\varepsilon}}\frac{\ln^3 x}{x^2+x+1}dx-\int_{\gamma_\varepsilon}\frac{(\ln x+2\pi i)^3}{x^2+x+1}dx$$

$$+\int_{\gamma_r}\frac{\ln^3 z}{z^2+z+1}dz$$

$$*\ By: ML\ inequality\Rightarrow\int_{\gamma_R}\frac{\ln^3(z)}{z^2+z+1}dz=\int_{\gamma_r}\frac{\ln^3(z)}{z^2+z+1}dz=0$$

$$\Rightarrow\lim_{\substack{\varepsilon\to0^+\\R\to+\infty}}\int_C\frac{\ln^3(z)}{z^2+z+1}dz=\frac{112i\pi^4\sqrt{3}}{81}=\int_0^{+\infty}\frac{\ln^3 x}{x^2+x+1}dx-\int_0^{+\infty}\frac{(\ln x+2\pi i)^3}{x^2+x+1}dx$$

$$=\int_0^{+\infty}\frac{\ln^3 x-(\ln x+2\pi i)^3}{x^2+x+1}dx=\int_0^{+\infty}\frac{8i\pi^3+12\pi^2\ln x-6i\pi\ln^2 x}{x^2+x+1}dx$$

$$=\int_0^{+\infty}\frac{12\pi^2\ln x}{x^2+x+1}dx+i\int_0^{+\infty}\frac{8\pi^3-6\pi\ln^2 x}{x^2+x+1}dx$$

$$*\ Taking\ imaginary\ part:\Rightarrow\frac{112\pi^4\sqrt{3}}{81}=\int_0^{+\infty}\frac{8\pi^3-6\pi\ln^2 x}{x^2+x+1}dx$$

$$=8\pi^3\int_0^{+\infty}\frac{1}{x^2+x+1}dx-6\pi\int_0^{+\infty}\frac{\ln^2 x}{x^2+x+1}dx=8\pi^3.\frac{2\pi}{3\sqrt{3}}-6\pi I$$

$$=\frac{48\pi^4\sqrt{3}}{27}-6\pi I$$

$$\Rightarrow I_x=\frac{\frac{48\pi^4\sqrt{3}}{27}-\frac{112\pi^4\sqrt{3}}{81}}{6\pi}=\frac{16\pi^4\sqrt{3}}{243}$$

$$*\ Taking\ real\ part:\Rightarrow0=\int_0^{+\infty}\frac{12\pi^2\ln x}{x^2+x+1}dx\Rightarrow\int_0^{+\infty}\frac{\ln x}{x^2+x+1}dx=0$$

$$\Rightarrow I = 2\int_0^{+\infty}\frac{\ln^2 x}{(x^2+x+1)}dx\int_0^{+\infty}\frac{1}{(y^2+y+1)}dy$$
$$+2\int_0^{+\infty}\frac{\ln x}{(x^2+x+1)}dx\int_0^{+\infty}\frac{\ln y}{(y^2+y+1)}dy = 2.\frac{16\pi^4\sqrt{3}}{243}.\frac{2\pi}{3\sqrt{3}}+2.0.0$$
$$=\boxed{\frac{64\pi^4}{729}}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

We know that: $\log^2(xy) = \log^2 x + \log^2 y + 2\log x \log y$

then we can write:

$$I = 2\int_0^{\infty}\int_0^{\infty}\frac{\log^2 x}{(x^2+x+1)(y^2+y+1)}dxdy$$
$$+2\int_0^{\infty}\int_0^{\infty}\frac{\log x\log y}{(x^2+x+1)(y^2+y+1)}dxdy = 2I_1+2I_2$$
$$I_1 = \int_0^{\infty}\int_0^{\infty}\frac{\log^2 x}{(x^2+x+1)(y^2+y+1)}dxdy = \int_0^{\infty}\frac{\log^2 x}{x^2+x+1}\int_0^{\infty}\frac{dy}{y^2+y+1}dx =$$
$$=\frac{2\pi}{3\sqrt{3}}\int_0^{\infty}\frac{\log^2 x}{x^2+x+1}dx = \frac{2\pi}{3\sqrt{3}}\cdot\frac{16\sqrt{3}\pi^3}{243}=\frac{32\pi^4}{729}$$

Note: $\int_0^{\infty}\frac{\log y}{y^2+y+1}dy = 0$, then: $I_2 = \int_0^{\infty}\int_0^{\infty}\frac{\log x\log y}{(x^2+x+1)(y^2+y+1)}dxdy = 0$, and thus,

$$I = 2I_1+2I_2 = \frac{64\pi^4}{729}$$

2025.Prove that:

$$\Omega = \int_0^1\int_0^1\int_0^1\frac{x^2\log(xy^2)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz = \frac{\pi^4}{96}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$*\ Symmetry: \int_0^1\int_0^1\int_0^1\frac{x^2\log(xy^2)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz$$
$$=\int_0^1\int_0^1\int_0^1\frac{x^2\log(xz^2)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz \Rightarrow 2\Omega$$
$$=\int_0^1\int_0^1\int_0^1\frac{x^2\log(x^2y^2z^2)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz$$

$$\Rightarrow \Omega = \int_0^1\int_0^1\int_0^1 \frac{x^2\log(xyz)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz$$
$$= \int_0^1\int_0^1\int_0^1 \frac{y^2\log(xyz)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz$$
$$= \int_0^1\int_0^1\int_0^1 \frac{z^2\log(xyz)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz$$
$$\Rightarrow 3\Omega = \int_0^1\int_0^1\int_0^1 \frac{(x^2+y^2+z^2)\log(xyz)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz \Rightarrow \Omega$$
$$= \frac{1}{3}\int_0^1\int_0^1\int_0^1 \frac{\log(xyz)}{(x^2y^2z^2-1)}dxdydz = \int_0^1\int_0^1\int_0^1 \frac{\log(x)}{(x^2y^2z^2-1)}dxdydz$$
$$= -\int_0^1\int_0^1\int_0^1 \log(x)\sum_{k=0}^{+\infty}(xyz)^{2k}\,dxdydz$$
$$= -\sum_{k=0}^{+\infty}\frac{1}{2k+1}\cdot\frac{1}{2k+1}\cdot\int_0^1 x^{2k}\ln x\,dx = -\sum_{k=0}^{+\infty}\frac{1}{2k+1}\cdot\frac{1}{2k+1}\left(-\frac{1}{(2k+1)^2}\right) = \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^4}$$
$$*\,Riemann\ zeta\ function: \zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s} \Rightarrow \zeta(4) = \sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$
$$= \sum_{n\ is\ odd}^{+\infty}\frac{1}{n^4} + \sum_{n\ is\ even}^{+\infty}\frac{1}{n^4} = \sum_{n\ is\ odd}^{+\infty}\frac{1}{n^4} + \sum_{k=1}^{+\infty}\frac{1}{(2k)^4}$$
$$= \sum_{n\ is\ odd}^{+\infty}\frac{1}{n^4} + \frac{1}{16}\sum_{k=1}^{+\infty}\frac{1}{k^4} \Rightarrow \sum_{n\ is\ odd}^{+\infty}\frac{1}{n^4} = \frac{15}{16}\zeta(4) = \frac{15}{16}\cdot\frac{\pi^4}{90} = \frac{\pi^4}{96} = \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^4}$$
$$\Rightarrow \Omega = \int_0^1\int_0^1\int_0^1 \frac{x^2\log(xy^2)}{(x^2+y^2+z^2)(x^2y^2z^2-1)}dxdydz = \boxed{\frac{\pi^4}{96}}$$

2026. Prove that:

$$\Re\int_0^\pi \cosh\left(\frac{1}{2}e^{ix}\right)\cos\left(\frac{\sqrt{3}}{2}e^{ix}\right)dx = \pi$$

Proposed by Asmat Qatea-Afghanistan

Solution by Le Thu-Vietnam

$$\cosh(\alpha) = \cos(\alpha i);\ \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)];\ (\forall)\alpha,\beta\in\mathbb{R}$$
$$I = \int_0^\pi \cosh\left(\frac{1}{2}e^{ix}\right)\cos\left(\frac{\sqrt{3}}{2}e^{ix}\right)dx = \int_0^\pi \cos\left(\frac{\sqrt{3}}{2}e^{ix}\right)\cos\left(\frac{i}{2}e^{ix}\right)dx =$$
$$= \frac{1}{2}\left[\int_0^\pi \cos\left(\frac{\sqrt{3}-i}{2}e^{ix}\right)dx + \int_0^\pi \cos\left(\frac{\sqrt{3}+i}{2}e^{ix}\right)dx\right] = \frac{1}{2}I_2$$

$$(*): \int \cos\left(\frac{\sqrt{3}-i}{2}e^{ix}\right)e^{ix}\,dx \overset{u=\frac{\sqrt{3}+i}{2}e^{ix}}{=} -i\int \frac{\cos u}{u}du + C = -iCi(u) + C$$

$$= -iCi\left(\frac{\sqrt{3}-i}{2}e^{ix}\right) + C$$

$$(**): \int \cos\left(\frac{\sqrt{3}+i}{2}e^{ix}\right)dx = -iCi\left(\frac{\sqrt{3}+i}{2}\right) + C$$

Summing both of them, we obtain:

$$I_1 = -i\left[Ci\left(\frac{\sqrt{3}-i}{2}e^{ix}\right) + Ci\left(\frac{\sqrt{3}+i}{2}\right)\right]_0^{\pi} =$$

$$= -i\left[Ci\left(\frac{\sqrt{3}-i}{2}\right) + \log\left(-\frac{\sqrt{3}-i}{2}\right) - \log\left(\frac{\sqrt{3}-i}{2}\right) + Ci\left(\frac{\sqrt{3}+i}{2}\right) + \log\left(-\frac{\sqrt{3}+i}{2}\right)\right.$$

$$\left. - \log\left(\frac{\sqrt{3}+i}{2}\right)\right] + i\left[Ci\left(\frac{\sqrt{3}-i}{2}\right) + Ci\left(\frac{\sqrt{3}+i}{2}\right)\right] =$$

$$= -i\left[\log\left(-\frac{\sqrt{3}-i}{2}\right) - \log\left(\frac{\sqrt{3}-i}{2}\right) + \log\left(-\frac{\sqrt{3}-i}{2}\right) - \log\left(\frac{\sqrt{3}+i}{2}\right)\right] =$$

$$= -i\left[\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \pi\right] + i\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) - i\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \pi\right] + i\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) =$$

$$= -i\cdot 2\pi i = 2\pi$$

$$Ci(x) = \gamma + \log x + \int_0^{\pi} \frac{\cos u - 1}{u}du$$

$$Ci(-x) = \gamma + \log(-x) + \int_0^{-x} \frac{\cos u - 1}{u}du = \gamma + \log(-x) + \int_0^{x} \frac{\cos v - 1}{v}dv =$$

$$= \gamma + \log(-x) + Ci(x) - \gamma - \log x = Ci(x) + \log(-x) - \log(x)$$

$$\text{Hence, } I = \frac{1}{2}I\cdot 2\pi = \pi \Rightarrow \Re(I) = \Re(\pi) = \pi$$

2027. Solve for real numbers:

$$\int_1^{x^2-5x+7} \frac{t^8 - t^6}{(t^2+t+1)^2}dt = 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vişescu-Romania

$$\int_1^{x^2-5x+7}\frac{t^8-t^6}{(t^2+t+1)^2}dt=\int_1^{1+(x-2)(x-3)}\frac{t^6(t^2-1)}{(t^2+t+1)^2}dt$$

1) If $x\in(-\infty,2)\cup(3,\infty)\Rightarrow 1+(x-2)(x-3)>1$ and

$$\int_1^{1+(x-2)(x-3)}\frac{t^6(t^2-1)}{(t^2+t+1)^2}dt>0$$

2) If $\in(2,3)\Rightarrow\frac{3}{4}<1+(x-2)(x-3)<1$ and

$$\int_1^{1+(x-2)(x-3)}\frac{t^6(t^2-1)}{(t^2+t+1)^2}dt>0$$

3) If $x\in\{2,3\}\Rightarrow 1+(x-2)(x-3)=1$, then:

$$\int_1^{x^2-5x+7}\frac{t^8-t^6}{(t^2+t+1)^2}dt=0$$

$$S=\{2,3\}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$a=x^2-5x+7$$

$$\text{Let: } g(a)=\int_1^a\frac{t^8-t^6}{(t^2+t+1)^2}dt$$

$$\min\{x^2-5x+7\}\Rightarrow(x^2-5x+7)'=2x-5\Rightarrow x=\frac{5}{2}$$

$$\min\{x^2-5x+7\}=\left(\frac{5}{2}\right)^2-5\left(\frac{5}{2}\right)+7=\frac{3}{4},\qquad a\ge\frac{3}{4}$$

$$g'(a)=\frac{a^6(a^2-1)}{(a^2+a+1)^2}\Rightarrow g(a)-\text{has a local minimum point at } a=1.$$

$$g(1)=0$$

$$x^2-5x+7=1\Rightarrow x^2-5x+6=0\Rightarrow(x-2)(x-3)=0\Rightarrow x\in\{2,3\}$$

$$\frac{7t^2+6t+5}{(t^2+t+1)^2}=\frac{At+B}{t^2+t+1}+\frac{Ct+D}{(t^2+t+1)^2}$$

$$7t^2+6t+5=(At+B)(t^2+t+1)+Ct+D$$

$$7t^2+6t+5=At^3+(A+B)t^2+(A+B)t+B+Ct+D$$

$$A=0\Rightarrow A+B=7\Rightarrow B=7\Rightarrow B+D=5\Rightarrow D=-2 \text{ and}$$

$$A+B+C=6\Rightarrow C=-1$$

$$t^4-2t^3+4t-5+\frac{7}{t^2+t+1}-\frac{t+2}{(t^2+t+1)^2}$$

$$\int_1^a(t^4-2t^3+4t-5)\,dt=\frac{a^5}{5}-\frac{a^4}{2}+2a^2-5a+\frac{33}{10}$$

$$\int_1^a\frac{7}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}dt=\frac{7}{\sqrt{\frac{3}{4}}}\tan^{-1}\frac{t+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\Bigg|_1^a=\frac{14}{\sqrt{3}}\tan^{-1}\left(\frac{2a+1}{\sqrt{3}}\right)-\frac{14}{\sqrt{3}}\tan^{-1}\sqrt{3}=$$

$$=\frac{14}{\sqrt{3}}\tan^{-1}\left(\frac{2a+1}{\sqrt{3}}\right)-\frac{14\pi}{\sqrt{3}}$$

$$\int_1^a\frac{1}{2}\cdot\frac{2t+1+3}{(t^2+t+1)^2}dt=\frac{3}{2}\int_1^a\frac{1}{(t^2+t+1)^2}dt+\frac{1}{2}\int_1^a\frac{d(t^2+t+1)}{t^2+t+1}=$$

$$=\frac{3}{2}\int_1^a\frac{1}{\left(\left(t+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}dt+\frac{1}{2(t^2+t+1)}\Bigg|_a^1;\left(t+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan z\right)$$

$$=\frac{3}{2}\int_{\frac{\pi}{3}}^{\tan^{-1}\left(\frac{2a+1}{\sqrt{3}}\right)}\frac{\frac{\sqrt{3}}{2}\sec^2 z}{\frac{9}{16}\sec^4 z}dz+\frac{1}{6}-\frac{1}{2(a^2+a+1)}=$$

$$=\frac{1}{6}-\frac{2}{\sqrt{3}}\cdot\frac{\pi}{3}-\frac{1}{2(a^2+a+1)}+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2a+1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\int_{\frac{\pi}{3}}^{\tan^{-1}\frac{2a+1}{\sqrt{3}}}\cos(2z)\,dz=$$

$$=\frac{1}{6}-\frac{a^2-a+1}{2(a^2+a+1)^2}+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2a+1}{\sqrt{3}}\right)-\frac{2}{\sqrt{3}}\cdot\frac{\pi}{3}=$$

$$=\frac{a}{a^2+a+1}-\frac{1}{3}-\frac{2\pi}{3\sqrt{3}}+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2a+1}{\sqrt{3}}\right)$$

$$\int_1^a\frac{t^8-t^6}{t^2+t+1}dt=$$

$$\frac{a^5}{5}-\frac{a^4}{2}+2a^2-5a+\frac{109}{30}-\frac{4\pi}{\sqrt{3}}-\frac{a}{a^2+a+1}+4\sqrt{3}\tan^{-1}\left(\frac{2a+1}{\sqrt{3}}\right)=g(a)$$

2028. Prove that:

$$\int_0^{\frac{\pi}{4}} \frac{\cos(12x)}{\cos^{14} x}\, dx = -\frac{64}{13}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Ravi Prakash-New Delhi-India

$$\cos(12x) + i\sin(12x) = (\cos x + i\sin x)^{12}$$

$$\cos(12x) = \binom{12}{0}\cos^{12} x - \binom{12}{2}\cos^{10} x \sin^2 x + \binom{12}{4}\cos^8 x \sin^4 x$$

$$-\binom{12}{6}\cos^6 x \sin^6 x$$

$$+\binom{12}{8}\cos^4 x \sin^8 x - \binom{12}{10}\cos^2 x \sin^{10} x + \binom{12}{12}\sin^{12} x$$

$$\int_0^{\frac{\pi}{4}} \frac{\cos(12x)}{\cos^{14} x}\, dx =$$

$$= \int_0^{\frac{\pi}{4}} \left[\sec^2 x - \binom{12}{2}\tan^2 x \sec^2 x + \binom{12}{4}\tan^4 x \sec^2 x - \binom{12}{8}\tan^6 x \sec^2 x\right.$$

$$\left. + \binom{12}{8}\tan^8 x \sec^2 x - \binom{12}{10}\tan^{10} x \sec^2 x + \binom{12}{12}\tan^{12} x \sec^2 x\right] dx =$$

$$= 1 - \frac{1}{3}\binom{12}{2} + \frac{1}{5}\binom{12}{4} - \frac{1}{7}\binom{12}{6} + \frac{1}{9}\binom{12}{9} - \frac{1}{11}\binom{12}{10} + \frac{1}{13}\binom{12}{12} = -\frac{64}{13}$$

Solution 2 by Bamidele Benjamin-New Delhi-India

$$\cos(12x) = \sum_{n=0}^{6} \binom{12}{2n} (-1)^n (-1)^n \cos^{12-2n} x \sin^{2n} x$$

$$I = \sum_{n=0}^{6} \binom{12}{2n} (-1)^n \int_0^{\frac{\pi}{4}} \tan^{2n} x \sec^2 x\, dx \overset{u=\tan x}{=}$$

$$= \sum_{n=0}^{6} \binom{12}{2n} (-1)^n \int_0^1 u^{2n}\, du = \sum_{n=0}^{6} \binom{12}{2n} (-1)^n \frac{u^{2n+1}}{2n+1}\bigg|_0^1 = \sum_{n=0}^{6} \binom{12}{2n} \frac{(-1)^n}{2n+1} = -\frac{64}{13}$$

Solution 3 by Saboor Halimi-Afghanistan

$$\int_0^{\frac{\pi}{4}} \frac{\cos(12x)}{\cos^{14} x} dx =$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^{12} x - 66\cos^2 x \sin^{10} x + 495\cos^4 x \sin^8 x - 924\cos^6 x \sin^6 x + 495\cos^8 x \sin^4 x - 66\cos^{10} x \sin^2 x + \cos^{12} x}{\cos^{14} x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^{12} x - \cos^{10} x (67\sin^2 x - 1) - 33\cos^6 x \sin^4 x (43\sin^2 x - 15) - 33\cos^2 x \sin^8 x (17\sin^2 x - 15)}{\cos^{14} x} dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x (\tan^{12} x - 66\tan^2 x + 33\tan^8 x (15 - 2\tan^2 x) + 33\tan^4 x (15 - 28\tan^2 x + 1)) dx \overset{u=\tan x}{=}$$

$$= \int_0^1 (u^{12} - 66u^2 + 33u^8(15 - 2u^2) + 33u^4(15 - 28u^2 + 1)) du = -\frac{64}{13}$$

2029. Prove that:

$$I = \int_0^1 \frac{x}{\sqrt{1-x^2}} \log\frac{1+\sqrt{1-x^2}}{x} \sin^{-1} x\, dx = \frac{\pi}{2}(1 - \log 2)$$

Proposed by Ose Favour-Nigeria

Solution 1 by Pham Duc Nam-Vietnam

$$I = \int_0^1 \frac{x}{\sqrt{1-x^2}} ln\frac{1+\sqrt{1-x^2}}{x} sin^{-1} x\, dx = \frac{\pi}{2}(1 - ln\, 2)$$

$$* \ t = sin^{-1} x \Rightarrow x = sin\, t,\ dt = \frac{dx}{\sqrt{1-x^2}} \Rightarrow I = \int_0^{\frac{\pi}{2}} t\, sin\, t\, ln\frac{1+cos\, t}{sin\, t} dt$$

$$\begin{cases} u = ln\frac{1+cos\, t}{sin\, t} \\ dv = t\, sin\, t\, dt \end{cases} \Rightarrow \begin{cases} du = -\frac{1}{sin\, t} dt \\ v = sin\, t - t\, cos\, t \end{cases} \Rightarrow I$$

$$= \underbrace{(sin\, t - t\, cos\, t)\, ln\frac{1+cos\, t}{sin\, t}\Big|_0^{\frac{\pi}{2}}}_{=0} + \int_0^{\frac{\pi}{2}} (sin\, t - t\, cos\, t)\frac{1}{sin\, t} dt$$

$$= \int_0^{\frac{\pi}{2}} (1 - t\, cot\, t) dt = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} t\, cot\, t\, dt, \begin{cases} u = t \\ dv = cot\, t\, dt \end{cases} \Rightarrow \begin{cases} du = dt \\ v = ln(sin\, t) \end{cases}$$

$$\Rightarrow I = \frac{\pi}{2} - t\, ln(sin\, t)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} ln(sin\, t)\, dt = \frac{\pi}{2} - \frac{\pi}{2} ln\, 2 = \frac{\pi}{2}(1 - ln\, 2)$$

***NOTES*:**

$$*\lim_{t\to 0}(\sin t - t\cos t)\ln\frac{1+\cos t}{\sin t} = 0$$

$$*\lim_{t\to 0} t\ln(\sin t) = 0$$

$$*\,Well-known\ result: \int_0^{\frac{\pi}{2}} \ln(\sin t)\,dt = -\frac{\pi}{2}\ln 2$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega = \int_0^1 \frac{x}{\sqrt{1-x^2}}\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)\sin^{-1}x\,dx \overset{\sin^{-1}x=y}{=}$$

$$= \int_0^{\frac{\pi}{2}} y\sin y\log\left(\frac{1+\cos y}{\sin y}\right)dy \overset{y=2x}{=} -4\int_0^{\frac{\pi}{4}} x\sin(2x)\log(\tan x)\,dx$$

$$\frac{1}{4}\Omega = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}[\int_0^{\frac{\pi}{4}} x\sin(2x)\cos(4nx+2x)\,dx =$$

$$= \sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\int_0^{\frac{\pi}{4}} x\sin(4nx+4x)\,dx - \int_0^{\frac{\pi}{4}} x\sin(4nx)\,dx\right] =$$

$$= \sum_{n=0}^{\infty}\frac{1}{2n+1}\left[-\left(\frac{x\cos(4nx+4x)}{4(n+1)}\right)_0^{\frac{\pi}{4}} + \frac{1}{4(n+1)}\int_0^{\frac{\pi}{4}}\cos(4nx+4x)\,dx\right.$$

$$\left. + \left(\frac{x\cos(4nx)}{4n}\right)_0^{\frac{\pi}{4}} - \frac{1}{4n}\int_0^{\frac{\pi}{4}}\cos(4nx)\,dx\right] =$$

$$= \sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{\pi\cos(n\pi)}{16(n+1)} - \frac{\sin(n\pi)}{16(n+1)^2} + \frac{\pi\cos(n\pi)}{16n} - \frac{\sin(n\pi)}{16n^2}\right] =$$

$$= \lim_{n\to 0}\left\{\frac{\pi\cos(n\pi)}{16n(n+1)} - \frac{\sin(n\pi)}{16(2n+1)}\left[\frac{2n^2+1+2n}{n^2(n+1)^2}\right] + \frac{\pi}{16}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}\right\} =$$

$$= \frac{\pi}{16} + \frac{\pi}{16}\left(-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots\right)$$

$$Therefore,\qquad \Omega = \frac{\pi}{2}(1-\log 2)$$

Solution 3 by Togrul Ehmedov-Azebaijan

$$\text{Let } x=\sin y\text{, then: }\Omega=\int_0^{\frac{\pi}{2}} y\sin y\log\left(\frac{1+\cos y}{\sin y}\right)dy=$$

$$=\int_0^{\frac{\pi}{2}} y\sin y\log\left(\cot\left(\frac{y}{2}\right)\right)dy=-\int_0^{\frac{\pi}{2}} y\sin y\log\left(\tan\left(\frac{y}{2}\right)\right)dy\overset{y=2z}{=}$$

$$=-4\int_0^{\frac{\pi}{4}} z\sin(2z)\log(\tan z)\,dz\overset{IBP}{=}$$

$$=-4\left(\frac{1}{4}\sin(2z)-\frac{z}{2}\cos(2z)\right)\log(\tan z)\Big|_0^{\frac{\pi}{4}}+4\int_0^{\frac{\pi}{4}}\frac{\sin(2z)-2z\cos(2z)}{2\sin(2z)}dz=$$

$$=2\int_0^{\frac{\pi}{4}}dz-4\int_0^{\frac{\pi}{4}} z\cot(2z)\,dz=\frac{\pi}{2}-\int_0^{\frac{\pi}{2}} z\cot z\,dz=\frac{\pi}{2}(1-\log 2)$$

Solution 4 by Ose Favour-Nigeria

$$\Omega=\int_0^1\frac{x}{\sqrt{1-x^2}}\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)\sin^{-1}x\,dx;$$

$$\begin{cases} dv=\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}} \\ u=\log\dfrac{1+\sqrt{1-x^2}}{x}\end{cases}\Rightarrow\begin{cases} v=x-\sqrt{1-x^2}\sin^{-1}x \\ du=-\dfrac{dx}{x\sqrt{1-x^2}}\end{cases}$$

$$\Omega\overset{IBP}{=}\left(x-\sqrt{1-x^2}\sin^{-1}x\right)\log\frac{1+\sqrt{1-x^2}}{x}\Big|_0^1+\int_0^1\frac{1}{\sqrt{1-x^2}}dx-\int_0^1\frac{\sin^{-1}x}{x}dx=$$

$$\overset{x=\sin y}{=}\frac{\pi}{2}-\int_0^{\frac{\pi}{2}} y\cot y\,dy\overset{IBP}{=}\frac{\pi}{2}+\int_0^{\frac{\pi}{2}}\log(\sin y)\,dy=\frac{\pi}{2}(1-\log 2)$$

Solution 5 by Le Thu-Vietnam

$$\Omega=\int_0^1\frac{x}{\sqrt{1-x^2}}\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)\sin^{-1}x\,dx=$$

$$=\int_0^1\frac{x\log\left(1+\sqrt{1-x^2}\right)\sin^{-1}x}{\sqrt{1-x^2}}dx-\int_0^1\frac{x\log x\sin^{-1}x}{\sqrt{1-x^2}}dx=I_1-I_2$$

$$I_1=\int_0^1\frac{x\log\left(1+\sqrt{1-x^2}\right)\sin^{-1}x}{\sqrt{1-x^2}}dx\overset{u=\sin^{-1}x}{=}\int_0^{\frac{\pi}{2}} u\sin u\log(1+\cos u)\,du\overset{IBP}{=}$$

$$= -u\cos u\log(1+\cos u)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos u\log(1+\cos u)\,du - \int_0^{\frac{\pi}{2}}\frac{u\cos u\sin u}{1+\cos u}du$$

$$(i)\ \int_0^{\frac{\pi}{2}}\cos u\log(1+\cos u)\,du \overset{IBP}{=} [u-\sin u+\sin u\log(1+\cos u)]_0^{\frac{\pi}{2}} = \frac{\pi}{2}-1$$

$$(ii)\ \int_0^{\frac{\pi}{2}}\frac{u\sin u\cos u}{1+\cos u}du = \int_0^{\frac{\pi}{2}}u\sin u\,du - \int_0^{\frac{\pi}{2}}\frac{u\frac{e^{iu}-e^{-iu}}{2i}}{1+\frac{e^{-iu}+e^{iu}}{2}}du =$$

$$= \sin u - u\cos u - 2iLi_2(-e^{iu})$$

$$Note: Li_2(-i) = \sum_{n=0}^{\infty}\frac{i^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2} + i\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2} = -\frac{\pi^2}{48} + iK, where\ K\ is$$

Catalan's constant.

$$2I_2 \overset{IBP}{=} [x(\sin^{-1}x)^2\log x]_0^1 - \int_0^1(\sin^{-1}x)^2\,dx - \int_0^1(\sin^{-1}x)^2\log x\,dx$$

$$(iii)\ \int_0^1(\sin^{-1}x)^2dx \overset{IBP}{=} \left[2\sqrt{1-x^2}\sin^{-1}x - 2x + x(\sin^{-1}x)^2\right]_0^1 = \frac{\pi^2}{4}-2$$

$$(iv)\ \int_0^1(\sin^{-1}x)^2\log x\,dx \overset{\sin^{-1}x=v}{=} \int_0^{\frac{\pi}{2}}v^2\cos v\log(\sin v)\,dv \overset{IBP}{=}$$

$$= [v^2\sin v\log(\sin v)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}v^2\cos v\,dv - 2\int_0^{\frac{\pi}{2}}v\log(\sin v)\sin v\,dv$$

$$(v)\ \int_0^{\frac{\pi}{1}}v\sin v\log(\sin v)\,dv \overset{IBP}{=} [-v\cos v\log(\sin v)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos v\log(\sin v)$$

$$+\int_0^{\frac{\pi}{2}}\frac{v\cos^2 v}{\sin v}dv$$

$$(vi)\ \int_0^{\frac{\pi}{2}}\frac{v\cos^2 v}{\sin v}dv = \int_0^{\frac{\pi}{2}}\frac{v}{\sin v}dv - \int_0^{\frac{\pi}{2}}v\sin v\,dv = 2K-1$$

$$K = \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{v}{\sin v}dv$$

Summing all of them, we obtain:

$$\begin{cases} I_1 = 2K - 2 + \pi\frac{1-\log 2}{2} \\ I_2 = 2K + 2 \end{cases} \Rightarrow \Omega = I_1 - I_2 = \frac{\pi}{2}(1 - \log 2)$$

2030. If $a \in (-1, 1)$ then prove:

$$\int_0^{\pi} \frac{\cos(a\sin x)}{1 + a\cos x} e^{a\cos x}\, dx = \frac{\pi}{e} \cdot \frac{e^{\sqrt{1-a^2}}}{\sqrt{1-a^2}}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$\Omega = \int_0^{\pi} \frac{\cos(a\sin x)}{1 + a\cos x} e^{a\cos x}\, dx = \frac{1}{2}\int_0^{\pi} \frac{e^{ia\sin x} + e^{-ia\sin x}}{1 + a\cos x} \cdot e^{a\cos x}\, dx =$$

$$= \frac{1}{2}\int_0^{\pi} \frac{e^{a(\cos x + i\sin x)}}{1 + a\cos x} dx + \frac{1}{2}\int_0^{\pi} \frac{e^{a(\cos x - i\sin x)}}{1 + a\cos x} dx = I_1 + I_2$$

$$I_2 = \frac{1}{2}\int_0^{\pi} \frac{e^{a(\cos x - i\sin x)}}{1 + a\cos x} dx \overset{x \to -x}{=} \frac{1}{2}\int_{-\pi}^{0} \frac{e^{a(\cos x + i\sin x)}}{1 + a\cos x} dx$$

$$\Omega = \frac{1}{2}\int_{-\pi}^{\pi} \frac{e^{a(\cos x + i\sin x)}}{1 + a\cos x} dx$$

$$Let\ z(x) = \cos x + i\sin x;\ -\pi \le x \le \pi;\ dx = \frac{dz}{iz};\ \cos x = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\Omega = \frac{1}{2}\oint \frac{e^{az}}{1 + \frac{a}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{iz} = \frac{1}{2i}\oint \frac{2e^{az}dz}{az^2 + 2z + a} =$$

$$= \frac{1}{i}\oint \frac{e^{az}dz}{a\left(z - \frac{1-\sqrt{1-a^2}}{a}\right)\left(z - \frac{1+\sqrt{1-a^2}}{a}\right)} =$$

$$= \frac{1}{i} \cdot 2\pi i \Re\left[\frac{e^{az}dz}{a\left(z - \frac{1-\sqrt{1-a^2}}{a}\right)\left(z - \frac{1+\sqrt{1-a^2}}{a}\right)}\right]\Bigg|_{z = \frac{-1+\sqrt{1-a^2}}{a}} =$$

$$= 2\pi \cdot \frac{e^{a\left(\frac{-1+\sqrt{1-a^2}}{a}\right)}}{a\left(\frac{-1+\sqrt{1-a^2}}{a} + \frac{1+\sqrt{1-a^2}}{a}\right)} = \frac{2\pi e^{\sqrt{1-a^2}}}{2e\sqrt{1-a^2}} = \frac{\pi}{e} \cdot \frac{e^{\sqrt{1-a^2}}}{\sqrt{1-a^2}}$$

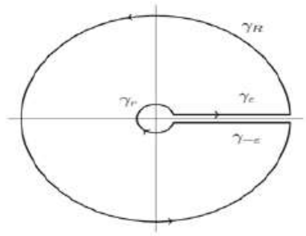
$$\textit{Therefore,}\quad \int_0^{\pi}\frac{\cos(a\sin x)}{1+a\cos x}e^{a\cos x}\,dx=\frac{\pi}{e}\cdot\frac{e^{\sqrt{1-a^2}}}{\sqrt{1-a^2}}$$

2031. Find a closed form:

$$I=\int_0^{+\infty}\frac{\ln^2 x}{x^2+x+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Pham Duc Nam-Vietnam



$$*\ \textit{Define: A key hole contour, with a branch cut on real axis (positive direction),}$$

$$\textit{around } 0, C=\gamma_R\cup\gamma_r\cup\gamma_{-\varepsilon}\cup\gamma_\varepsilon$$

$$*\ \textit{Let } f(z)=\frac{\ln^3 z}{z^2+z+1}$$

$$*\ f(z)\textit{has 2 poles } -\frac{1}{2}\pm\frac{i\sqrt{3}}{2}, \textit{order 1, inside the contour.}$$

$$\Rightarrow \int_C\frac{\ln^3 z}{z^2+z+1}dz=2\pi i\sum \operatorname{Res}\left(f(z),-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)$$

$$=2\pi i\sum \lim_{z\to-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}}\left(z-\left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)\right)\frac{\ln^3 z}{z^2+z+1}$$

$$=2\pi i\ln^3\left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)\left(\pm\frac{1}{i\sqrt{3}}\right)$$

$$=2\pi i\left(i\,arg-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)^3\left(\pm\frac{1}{i\sqrt{3}}\right)=2\pi i\left(\frac{2\pi i}{3}\right)^3\left(\frac{1}{i\sqrt{3}}\right)+2\pi i\left(\frac{4\pi i}{3}\right)^3\left(-\frac{1}{i\sqrt{3}}\right)=\frac{112i\pi^4\sqrt{3}}{81}$$

$$*\int_C\frac{\ln^3 z}{z^2+z+1}dz=\int_{\gamma_R}\frac{\ln^3 z}{z^2+z+1}dz+\int_{\gamma_{-\varepsilon}}\frac{\ln^3 x}{x^2+x+1}dx-\int_{\gamma_\varepsilon}\frac{(\ln x+2\pi i)^3}{x^2+x+1}dx$$

$$+\int_{\gamma_r}\frac{\ln^3 z}{z^2+z+1}dz$$

$$*\ \textit{By: ML inequality} \Rightarrow$$

$$\int_{\gamma_R}\frac{\ln^3(z)}{z^2+z+1}dz=\int_{\gamma_r}\frac{\ln^3(z)}{z^2+z+1}dz=0$$

$$\Rightarrow \lim_{\substack{\varepsilon\to 0^+\\ R\to+\infty}} \int_C \frac{ln^3(z)}{z^2+z+1}dz = \frac{112i\pi^4\sqrt{3}}{81} = \int_0^{+\infty} \frac{ln^3 x}{x^2+x+1}dx - \int_0^{+\infty} \frac{(ln\,x+2\pi i)^3}{x^2+x+1}dx$$

$$= \int_0^{+\infty} \frac{ln^3 x - (ln\,x+2\pi i)^3}{x^2+x+1}dx = \int_0^{+\infty} \frac{8i\pi^3 + 12\pi^2\, ln\,x - 6i\pi\, ln^2 x}{x^2+x+1}dx$$

$$= \int_0^{+\infty} \frac{12\pi^2\, ln\,x}{x^2+x+1}dx + i\int_0^{+\infty} \frac{8\pi^3 - 6\pi\, ln^2 x}{x^2+x+1}dx$$

$* \, Taking\ imaginary\ part: \Rightarrow$

$$\frac{112\pi^4\sqrt{3}}{81} = \int_0^{+\infty} \frac{8\pi^3 - 6\pi\, ln^2 x}{x^2+x+1}dx = 8\pi^3 \int_0^{+\infty} \frac{1}{x^2+x+1}dx - 6\pi \int_0^{+\infty} \frac{ln^2 x}{x^2+x+1}dx$$

$$= 8\pi^3 . \frac{2\pi}{3\sqrt{3}} - 6\pi I = \frac{48\pi^4\sqrt{3}}{27} - 6\pi I$$

$$\Rightarrow I = \frac{\frac{48\pi^4\sqrt{3}}{27} - \frac{112\pi^4\sqrt{3}}{81}}{6\pi} = \boxed{\frac{16\pi^3\sqrt{3}}{243}}$$

Solution 2 by Samir Zaakouni-Morocco

$$\Omega = \int_0^{\infty} \frac{\log^2 x}{x^2+x+1}dx = \int_0^1 \frac{\log^2 x}{x^2+x+1}dx + \int_1^{\infty} \frac{\log^2 x}{x^2+x+1}dx \overset{\frac{1}{x}=u}{=}$$

$$= \int_0^1 \frac{\log^2 x}{x^2+x+1}dx + \int_0^1 \frac{\log^2 u}{1+u+u^2}du$$

$$\Omega = 2\int_0^1 \frac{\log^2 x}{x^2+x+1}dx = 2\int_0^1 \frac{(1-x)\log^2 x}{1-x^3}dx \overset{t=x^3}{=}$$

$$= \frac{2}{27}\int_0^1 \frac{\left(t^{\frac{1}{3}-1} - t^{\frac{2}{3}-1}\right)\log^2 t}{1-t}dt = \frac{2}{27}\left(\psi^{(2)}\left(\frac{2}{3}\right) - \psi^{(2)}\left(\frac{1}{3}\right)\right)$$

$$\Omega = \frac{2}{27}\left(\pi \frac{d^2(\cot(\pi z))}{dz^2}\bigg|_{z=\frac{1}{3}}\right) = \frac{4\pi^3 \cos\left(\frac{\pi}{3}\right)}{27\sin^3\left(\frac{\pi}{3}\right)} = \frac{16\pi^3}{81\sqrt{3}}$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega = \int_0^{\infty} \frac{\log^2 x}{x^2+x+1}dx = \int_0^1 \frac{\log^2 x}{x^2+x+1}dx + \int_1^{\infty} \frac{\log^2 x}{x^2+x+1}dx \overset{\frac{1}{x}=y}{=}$$

$$= \int_0^1 \frac{\log^2 x}{x^2+x+1}dx + \int_0^1 \frac{\log^2 y}{1+y+y^2}du$$

$$= 2\int_0^1 \frac{\log^2 x}{x^2+x+1}dx = 2\int_0^1 \frac{(1-x)\log^2 x}{1-x^3}dx$$

$$\frac{\Omega}{2} = \sum_{n=0}^{\infty}\int_0^1 x^{3n}\log^2 x\,dx - \sum_{n=0}^{\infty} x^{3n+1}\log^2 x\,dx$$

$$\because \int_0^1 x^m \log^n x\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, n > -1, m \neq -1$$

$$\frac{\Omega}{4} = \sum_{n=0}^{\infty}\frac{1}{(3n+1)^3} - \sum_{n=0}^{\infty}\frac{1}{(3n+2)^3} \Rightarrow$$

$$\frac{27}{4}\Omega = \zeta\left(3,\frac{1}{3}\right) - \zeta\left(3,\frac{2}{3}\right) \Rightarrow \frac{27}{2}\Omega = \psi^{(2)}\left(\frac{2}{3}\right) - \psi^{(2)}\left(\frac{1}{3}\right)$$

$$\because \psi^{(m)}(z) = (-1)^{m+1} m! \cdot \zeta(m+1,z)$$

$$\frac{27}{2}\Omega = \pi\left(\frac{d^2}{dz^2}\cot(\pi z)\right)_{z=\frac{1}{3}} = \frac{2\pi^3\cot\left(\frac{\pi}{3}\right)}{\sin^2\left(\frac{\pi}{3}\right)} \Rightarrow \Omega = \frac{16\sqrt{3}}{3^5}\pi^3$$

$$\because \psi^{(n)}(1-z) = (-1)^n\psi^{(n)}(z) + (-1)^n\pi\frac{d^n}{dz^n}\cot(\pi z)$$

$$\textit{Therefore,} \quad \Omega = \int_0^{\infty}\frac{\log^2 x}{x^2+x+1}dx = \frac{16\pi^3\sqrt{3}}{243}$$

Solution 4 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^{\infty}\frac{\log^2 x}{x^2+x+1}dx = \int_0^1 \frac{\log^2 x}{x^2+x+1}dx + \int_1^{\infty}\frac{\log^2 x}{x^2+x+1}dx \overset{\frac{1}{x}=y}{=}$$

$$= \int_0^1 \frac{\log^2 x}{x^2+x+1}dx + \int_0^1 \frac{\log^2 y}{1+y+y^2}du$$

$$= 2\int_0^1 \frac{\log^2 x}{x^2+x+1}dx = 2\int_0^1 \frac{(1-x)\log^2 x}{1-x^3}dx =$$

$$= 2\int_0^1 \frac{\log^2 x}{1-x^3}dx - 2\int_0^1 \frac{x\log^2 x}{1-x^3}dx \overset{y=x^3}{=}$$

$$= \frac{2}{27}\int_0^1 \frac{y^{-\frac{2}{3}}}{1-y}\log^2 y\,dy - \frac{2}{27}\int_0^1 \frac{y^{-\frac{1}{3}}}{1-y}\log^2 y\,dy = \frac{2}{27}(I_1 - I_2)$$

$$\because \int_0^1 \frac{t^{z-1}}{1-t}\log^m t\,dt = -\psi^{(m)}(z)$$

$$I_1=\int_0^1\frac{y^{-\frac{2}{3}}}{1-y}\log^m t\,dt=-\psi^{(2)}\left(\frac{1}{3}\right)$$

$$I_2=\int_0^1\frac{y^{-\frac{1}{3}}}{1-y}\log^2 y\,dy=-\psi^{(2)}\left(\frac{2}{3}\right)$$

$$\Omega=\frac{2}{27}\left(\psi^{(2)}\left(\frac{2}{3}\right)-\psi^{(2)}\left(\frac{1}{3}\right)\right)$$

$$\because(-1)^m\psi^{(m)}(1-z)-\psi^{(m)}(z)=\pi\frac{\partial^m}{\partial z^m}\cot(\pi z)$$

$$\mathit{Therefore},\qquad \Omega=\int_0^\infty\frac{\log^2 x}{x^2+x+1}dx=\frac{16\pi^3\sqrt{3}}{243}$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_0^\infty\frac{\log^2 x}{x^2+x+1}dx\overset{x=e^{-y}}{=}\int_{-\infty}^\infty\frac{\log^2(e^{-y})}{e^{-2y}+e^{-y}+1}e^{-y}\,dy=$$

$$=\int_{-\infty}^\infty\frac{y^2}{e^y+1+e^{-y}}dy=2\int_0^\infty\frac{e^{-y}y^2}{e^{-2y}+e^y+1}dy=$$

$$=2\int_0^\infty e^{-y}y^2\sum_{n=0}^\infty((e^{-y})^{3n}-(e^{-y})^{3n+1})dy=$$

$$=2\sum_{n=0}^\infty\left(\int_0^\infty y^2e^{-(3n+1)y}dy-\int_0^\infty y^2e^{-(3n+2+y)}dy\right)$$

$$\int_0^\infty y^2e^{-ay}\,dy\overset{a>0}{=}-\frac{1}{a}\int_0^\infty y^2d(e^{-ay})=\frac{2}{a}\int_0^\infty e^{-ay}\,dy=\frac{2}{a^3}=$$

$$=-\frac{2}{a^2}\int_0^\infty yd(e^{-ay})\,y=\frac{2}{a^2}\int_0^\infty e^{-ay}\,dy=\frac{2}{a^3}$$

$$\Omega=\Omega=4\sum_{n=0}^\infty\left(\frac{1}{(3n+1)^3}-\frac{1}{(3n+2)^3}\right)=4\sum_{n=0}^\infty\frac{1}{(3n+1)^3}-4\sum_{n=0}^\infty\frac{1}{(3n-1)^3}=$$

$$=4\pi^3\left(\frac{2}{3}\right)^3\sum_{n=0}^\infty\frac{1}{\left(2n+\frac{2}{3}\right)^3\pi^3}-4\pi^3\left(\frac{2}{3}\right)^3\sum_{n=1}^\infty\frac{1}{\left(2n-\frac{2}{3}\right)^3\pi^3}=$$

$$= 4\pi^3\left(\frac{2}{3}\right)^3\left(\frac{2}{\sqrt{3}}\right)^3\sum_{n=0}^{\infty}\frac{\sin\left(\left(2n+\frac{2}{3}\right)\pi\right)^3}{\left(\left(2n+\frac{2}{3}\right)\pi\right)^3} + 4\pi^3\left(\frac{2}{3}\right)^3\left(\frac{2}{\sqrt{3}}\right)^3\sum_{n=1}^{\infty}\frac{\sin\left(\left(2n-\frac{2}{3}\right)\pi\right)^3}{\left(\left(2n-\frac{2}{3}\right)\pi\right)^3} =$$

$$= \frac{256}{81\sqrt{3}}\pi^3\left(\sum_{m=0}^{\infty}\left(2m+\frac{2}{3}\right) + \sum_{n=1}^{\infty}\sin c^3\left(2n-\frac{2}{3}\right)\right) =$$

$$= \frac{256\pi^3}{81\sqrt{3}}\sum_{n=-\infty}^{\infty}\sin c^3\left(2n-\frac{2}{3}\right)$$

The sum $\sum_{n=-\infty}^{\infty}\sin c^3\left(2n-\frac{2}{3}\right)$ *can be obtained using the Poison's summation formula*: $\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} G(n)$, *where*

$$G(\beta) = \text{GTFT}\{g(\alpha)\}(\beta) = \int_{-\infty}^{\infty} g(\alpha)e^{-i2\pi\beta\alpha}\,d\alpha$$

The GTFT *of* $\sin c^3(\alpha)$ *is the convoluting of* ${}_{-\frac{1}{2}}{}_{\frac{1}{2}}^{1}$ *and* ${}_{-1}\Delta_1^1$ *and is given by*:

$$h(\beta) = \begin{cases} 0; & \beta \le -\frac{3}{2} \\ \frac{1}{8}(2\beta+3)^2; & -\frac{3}{2} \le \beta \le -\frac{1}{2} \\ \frac{1}{4}(3-4\beta^2); & -\frac{1}{2} < \beta < \frac{1}{2} \\ \frac{1}{8}(2\beta-3)^2; & \frac{1}{2} \le \beta < \frac{3}{2} \\ 0; & \beta \ge \frac{3}{2} \end{cases}$$

$$\text{CTFT}\left\{\sin c^3\left(2\alpha-\frac{2}{3}\right)\right\} = G(\beta) = \frac{1}{2}e^{-i2\pi\beta\frac{1}{3}}h\left(\frac{\beta}{2}\right)$$

$\sum_{n\in\mathbb{Z}} G(n)$ *has only* 5 *nonzero terms corresponding to* $n = 0, n = \pm 1, n = \pm 2$

$$\sum_{n=-\infty}^{\infty}\sin c^3\left(2n-\frac{2}{3}\right) = \frac{3}{8} + \frac{1}{2}e^{-i2\pi\frac{1}{3}}\left(\frac{1}{2}\right) + \frac{1}{2}e^{i2\pi\frac{1}{3}}\left(\frac{1}{2}\right) + \frac{1}{2}e^{-i2\pi\frac{2}{3}}\left(\frac{1}{8}\right) + \frac{1}{2}e^{i2\pi\frac{2}{3}}\left(\frac{1}{8}\right) =$$

$$= \frac{3}{8} - \frac{1}{4} - \frac{1}{16} = \frac{1}{16}$$

$$Therefore,\ \ \Omega=\int_0^\infty\frac{\log^2 x}{x^2+x+1}dx=\frac{16\pi^3\sqrt{3}}{243}$$

2032. Prove that:

$$\int_0^1\int_0^1\left(\frac{x+y}{xy}\right)\log(1+xy+x^2y^2)\,dxdy=\frac{2\pi^2}{9}-\frac{\pi}{\sqrt{3}}-3\log 3+4$$

Proposed by Asmat Qatea-Afghanistan

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^1\int_0^1\left(\frac{x+y}{xy}\right)\log(1+xy+x^2y^2)\,dxdy\overset{xy=z}{=}$$

$$=\int_0^1\int_0^x\frac{\log(1+z+z^2)}{z}dz+\int_0^1\frac{1}{x^2}\int_0^x\log(1+z+z^2)\,dz=I_1+I_2$$

$$I_1\overset{IBP}{=}\left[x\int_0^x\frac{\log(1+z+Z^2)}{z}dz\right]_0^1-\int_0^1\log(1+x+x^2)\,dx=$$

$$=\int_0^1\frac{\log(1+z+z^2)}{z}dz-\int_0^1\log(1+x+x^2)\,dx$$

$$I_2\overset{IBP}{=}-\left[\frac{1}{x}\int_0^x\log(1+z+z^2)\,dz\right]_0^1+\int_0^2\frac{\log(1+x+x^2)}{x}dx=$$

$$=-\int_0^1\log(1+z+z^2)\,dz+\int_0^1\frac{\log(1+x+x^2)}{x}dx$$

$$\Omega=I_1+I_2=2\int_0^1\frac{\log(1+x+x^2)}{x}dx-2\int_0^1\log(1+x+x^2)\,dx=2I_3-2I_4$$

$$I_3\overset{IBP}{=}-2\int_0^1\frac{x\log x}{1+x+x^2}dx-\int_0^1\frac{\log x}{x^2+x+1}dx=-2I_{3a}-I_{3b}$$

$$We\ know\ that: I_{3a}=-\frac{1}{9}\left(\frac{7\pi^2}{6}-\psi'\left(\frac{1}{3}\right)\right)\ and\ I_{3b}=\frac{2}{9}\left(\frac{2\pi^2}{3}-\psi'\left(\frac{1}{3}\right)\right), then$$

$$I_3=-2I_{3a}-I_{3b}=\frac{\pi^2}{9}\ and\ I_4=\frac{3}{2}\log 3+\frac{\pi\sqrt{3}}{6}-2$$

$$\Omega=2(I_3-I_4)=\frac{2\pi^2}{9}-\frac{\pi}{\sqrt{3}}-3\log 3+4$$

2033. If $a\in[-1,1]$ then prove:

$$\int_0^1\frac{1}{x}\log((1-x)^2+2x(1-x)a+x^2)\,dx=-\frac{1}{2}(\cos^{-1}a)^2$$

Proposed by Asmat Qatea-Afghanistan

Solution by Le Thu-Vietnam

$$\textit{We will differentiate LHS with respect to } a:$$

$$LHS'(a)=\int_0^1 \frac{2(x-1)}{2(a-1)(x-1)x-1}dx$$

$$\textit{Let } b=2a-1;\ b\in[-4,0] \textit{ and } db=3da$$

$$LHS'(b)=-2\int_0^1 \frac{x}{bx(x-1)-1}dx$$

$$LHS'(b)=-\int_0^1 \frac{1}{bx(x-1)-1}dx \overset{b<0}{=} \left[\frac{2\tan^{-1}\left(\sqrt{-\frac{b}{b+4}}(2x-1)\right)}{\sqrt{-(b+4)b}}\right]_0^1=$$

$$=4\frac{\tan^{-1}\left(\sqrt{-\frac{b}{b+4}}\right)}{\sqrt{-b(b+4)}}$$

$$\textit{Since: } \tan^{-1}(-\theta)=-\tan^{-1}\theta, \textit{ then}$$

$$LHS=\int LHS'(b)\frac{db}{2}=2\frac{b\left(\tan^{-1}\left(\sqrt{-\frac{b}{b+4}}\right)\right)^2}{\sqrt{-\frac{b}{b+4}}\cdot\sqrt{-(b+4)b}}+C=$$

$$=-2\left(\tan^{-1}\left(\sqrt{-\frac{b}{b+4}}\right)\right)^2+C=-2\left(\tan^{-1}\left(\sqrt{\frac{1-a}{1+a}}\right)\right)^2+C$$

$$\textit{Chose } a=1\Rightarrow LHS_{a=1}=0\Rightarrow C=0.$$

$$LHS=-2\left(\tan^{-1}\left(\sqrt{\frac{1-a}{1+a}}\right)\right)^2;\ (\tan^{-1}t\ge 0;\ t\in[0,1])$$

$$\sqrt{-\frac{1}{2}LHS}=-2\tan^{-1}\left(\sqrt{\frac{1-a}{1+a}}\right)$$

$$\tan\left(\sqrt{-\frac{LHS}{2}}\right)=\sqrt{\frac{1-a}{1+a}}$$

$$a=\frac{1-\tan^2\left(\sqrt{-\frac{LHS}{2}}\right)}{1+\tan^2\left(\sqrt{-\frac{LHS}{2}}\right)}=\cos\left(2\sqrt{-\frac{LHS}{2}}\right)=\cos\left(\sqrt{-2LHS}\right)$$

$$(\cos^{-1} a)^2 = -2LHS \Leftrightarrow LHS = -\frac{1}{2}(\cos^{-1} a)^2 = RHS$$

2034. *Find* $x \in (0, 2\pi)$:

$$I(x) = \int_0^1 \frac{log(z^2 - 2z\cos x + 1)}{z} dz = 0$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Togrul Ehmedov-Azerbaijan

$$We\ know\ that: \log(1 - 2z\cos x + z^2) = -2\sum_{k=1}^{\infty} \frac{\cos(kx)}{k} z^k$$

$$then\ we\ can\ write:$$

$$-2\int_0^1 \sum_{k=1}^{\infty} \frac{\cos(kx)}{k} \frac{z^k}{z} dz = -2\sum_{k=1}^{\infty} \frac{\cos(kx)}{k} \int_0^1 z^{k-1} dz = -2\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

$$We\ also\ know\ that: \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \zeta(2) - \frac{\pi}{2}x + \frac{1}{4}x^2; x \in [0, 2\pi]$$

$$then: -2\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = 0\ and\ hence\ \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = 0$$

$$Therefore, \quad \zeta(2) - \frac{\pi}{2}x + \frac{1}{4}x^2 = 0\ or\ x_{1,2} = \pi \pm \frac{\pi}{\sqrt{3}} = \left(1 \pm \frac{1}{\sqrt{3}}\right)\pi$$

Solution 2 by Pham Duc Nam-Vietnam

$$* x \in (0, 2\pi) \Rightarrow \cos x \in (-1, 1). If: \cos x \in (-1, 0) \Rightarrow \frac{log(z^2 - 2z\cos x + 1)}{z} > 0\ \forall z \in (0, 1)$$

$$\Rightarrow \int_0^1 \frac{log(z^2 - 2z\cos x + 1)}{z} dz > 0 \Rightarrow 0 < \cos x < 1$$

$$I'(x) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{log(z^2 - 2z\cos x + 1)}{z}\right) dz = 2\sin x \int_0^1 \frac{1}{z^2 - 2z\cos x + 1} dz$$

$$= 2\sin x . \frac{1}{\sqrt{1 - \cos^2 x}} arctan \frac{z - \cos x}{\sqrt{1 - \cos^2 x}}\Big|_0^1 = 2\, arctan \frac{z - \cos x}{\sin x}\Big|_0^1$$

$$= 2\left(arctan \frac{1 - \cos x}{\sin x} - arctan \frac{-\cos x}{\sin x}\right) = 2\left(arctantan \frac{x}{2} + arctancot\, x\right)$$

$$= 2\left(\frac{x}{2} + \frac{\pi}{2} - arctan \frac{1}{\cot x}\right) = 2\left(\frac{x}{2} + \frac{\pi}{2} - x\right) = 2\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

$$= \pi - x \Rightarrow I(x) = \int (\pi - x)\, dx = \pi x - \frac{x^2}{2} + C.$$

$$I\left(\frac{\pi}{2}\right) = \int_0^1 \frac{log(z^2 + 1)}{z} dz = \frac{\pi^2}{24} = \pi . \frac{\pi}{2} - \frac{\pi^2}{8} + C \Rightarrow C = -\frac{\pi^2}{3} \Rightarrow I(x) = -\frac{x^2}{2} + \pi x - \frac{\pi^2}{3}$$

$$\Rightarrow I(x)=0 \Leftrightarrow -\frac{x^2}{2}+\pi x-\frac{\pi^2}{3}=0 \Leftrightarrow x_{1,2}=\pi\left(1\pm\frac{\sqrt{3}}{3}\right) \Rightarrow 0<\cos x_{1,2}<1(Satisfied)$$

$$\Rightarrow x_{1,2}=\pi\left(1\pm\frac{\sqrt{3}}{3}\right) \text{ are solutions.}$$

2035. Prove that:

$$I=\int_0^{\pi}\cot\left(\frac{x}{2}\right)(2\pi^2x-3\pi x^2+x^3)dx=12\pi\zeta(3)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Pham Duc Nam-Vietnam

$$I=\int_0^{\pi}\cot\left(\frac{x}{2}\right)(2\pi^2x-3\pi x^2+x^3)dx=12\pi\zeta(3)$$

$$\begin{cases}u=2\pi^2x-3\pi x^2+x^3\\ dv=\cot\left(\frac{x}{2}\right)dx.\end{cases} \Rightarrow \begin{cases}du=3x^2-6\pi x+2\pi^2\\ v=2\log\sin\frac{x}{2}\end{cases} \Rightarrow I$$

$$=\underbrace{2\log\sin\frac{x}{2}(2\pi^2x-3\pi x^2+x^3)\Big|_0^{\pi}}_{=0}-2\int_0^{\pi}\log\sin\frac{x}{2}(3x^2-6\pi x+2\pi^2)\,dx$$

$$=-2\int_0^{\pi}\log\sin\frac{x}{2}(3x^2-6\pi x+2\pi^2)\,dx$$

$$=-2\int_0^{\pi}\left(3x^2\log\sin\frac{x}{2}-6\pi x\log\sin\frac{x}{2}+2\pi^2\log\sin\frac{x}{2}\right)dx$$

$$*\ 2\pi^2\int_0^{\pi}\log\sin\frac{x}{2}dx\xrightarrow{t=\frac{x}{2}}4\pi^2\int_0^{\frac{\pi}{2}}\log\sin x\,d\,x=4\pi^2\left(-\frac{1}{2}\pi\log 2\right)=-2\pi^3\log 2$$

$$*\ 6\pi\int_0^{\pi}x\log\sin\frac{x}{2}dx\xrightarrow{t=\frac{x}{2}}24\pi\int_0^{\frac{\pi}{2}}t\log\sin t\,dt$$

$$=24\pi\left(-\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{2}}\frac{t\cos(2kt)}{k}dt-\int_0^{\frac{\pi}{2}}t\log 2\,dt\right)$$

$$=-3\pi^3\log 2-24\pi\sum_{k=1}^{+\infty}\left(\frac{\pi}{4}\cdot\frac{\sin(\pi k)}{k^2}+\frac{\cos(\pi k)}{4k^3}-\frac{1}{4k^3}\right)=-3\pi^3\log 2+\frac{21}{2}\pi\zeta(3)$$

$$*\ 3\int_0^{\pi}x^2\log\sin\frac{x}{2}dx\xrightarrow{t=\frac{x}{2}}24\int_0^{\frac{\pi}{2}}t^2\log\sin t\,dt$$

$$=24\left(-\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{2}}\frac{t^2\cos(2kt)}{k}dt-\int_0^{\frac{\pi}{2}}t^2\log 2\,dt\right)$$

$$=-\pi^3\log 2-24\sum_{k=1}^{+\infty}\left(\frac{(\pi^2k^2-2)\sin(\pi k)}{8k^4}+\frac{\pi\cos(\pi k)}{4k^3}\right)=-\pi^3\log 2+24.\frac{3\pi}{16}\zeta(3)$$

$$=-\pi^3\log 2+\frac{9}{2}\pi\zeta(3)$$

$$\Rightarrow I=-2\left(-\pi^3\log 2+\frac{9}{2}\pi\zeta(3)-\left(-3\pi^3\log 2+\frac{21}{2}\pi\zeta(3)\right)-2\pi^3\log 2\right)=-2(-6\pi\zeta(3))$$
$$=\boxed{12\pi\zeta(3)}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\mathrm{I}=\int_0^{\pi}\csc x\,(2\pi^2x-3\pi x^2+x^3)\,\mathrm{dx}\overset{IBP}{=}\left[(2\pi^2x-3\pi x^2+x^3)\log\left(tan\left(\frac{x}{2}\right)\right)\right]_0^{\pi}$$
$$-\int_0^{\pi}(2\pi^2-6\pi+3x^2)\log\left(tan\left(\frac{x}{2}\right)\right)dx$$
$$=-\int_0^{\pi}(2\pi^2-6\pi+3x^2)\log\left(tan\left(\frac{x}{2}\right)\right)dx$$
$$12\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^2}=2\pi^2-6\pi+3x^2$$
$$I=-12\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\pi}\cos(kx)\log\left(tan\left(\frac{x}{2}\right)\right)dx=-24\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\tan x)\,dx$$
$$=-24\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\sin x)\,dx-\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\cos x)\,dx\right]$$
$$=-24\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\sin x)\,dx-(-1)^k\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\sin x)\,dx\right]$$
$$=-24\sum_{k=1}^{\infty}\frac{\left(1+(-1)^{k+1}\right)}{k^2}\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\sin x)\,dx$$
$$=-24\sum_{k=1}^{\infty}\frac{\left(1+(-1)^{k+1}\right)}{k^2}\left[\left[\frac{\sin(2kx)}{2k}\log(sinx)\right]_0^{\frac{\pi}{2}}-\frac{1}{2k}\int_0^{\frac{\pi}{2}}\sin(2kx)\,ctgx\,dx\right]=$$
$$=12\sum_{k=1}^{\infty}\frac{\left(1+(-1)^{k+1}\right)}{k^3}\int_0^{\frac{\pi}{2}}\sin(2kx)\,ctgx\,dx=6\pi\sum_{k=1}^{\infty}\left[\frac{1}{k^3}+\frac{(-1)^{k+1}}{k^3}\right]$$
$$=\frac{21\pi}{2}\zeta(3)$$
$$Note: I_k=\int_0^{\frac{\pi}{2}}\sin(2kx)\,ctgx\,dx=\frac{\pi}{2}$$

2036. Find a closed form:

$$\Omega=\int_0^1\frac{dx}{(1-xy)\sqrt{1-x}}$$

Proposed by Ngulmun George Baite-India

Solution 1 by Toubal Fethi-Algerie

$$\Omega=\int_0^1\frac{dx}{(1-xy)\sqrt{1-x}}=\int_0^1\sum_{n=0}^{\infty}(xy)^n(1-x)^{-\frac{1}{2}}dx=$$

$$=\sum_{n=1}^{\infty}y^n\int_0^1 x^{(n+1)-1}(1-x)^{\frac{1}{2}-1}dx=\sum_{n=0}^{\infty}y^nB\left(n+1,\frac{1}{2}\right)=$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)y^n}{\Gamma(n+1)\left(\frac{3}{2}\right)_n}={}_2F\left(1,1;\frac{3}{2};y\right)$$

$$We\ know\ that:\frac{\sin^{-1}t}{\sqrt{1-t^2}}={}_tF\left(1,1;\frac{3}{2};t^2\right)$$

$$Let: y=t^2, we\ get:\ \Omega=\frac{2\sin^{-1}(\sqrt{y})}{\sqrt{y-y^2}}, where\ 0\le y\le 1$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\int_0^1\frac{dx}{(1-xy)\sqrt{1-x}}\int_0^1\frac{-2d(\sqrt{1-x})}{1-y+y-xy}=-2\int_0^1\frac{d(\sqrt{1-x})}{1-y+\left(\sqrt{y(1-y)}\right)^2}=$$

$$=\frac{-2}{\sqrt{y(1-y)}}\int_0^1\frac{d\left(\sqrt{\frac{(1-x)y}{1-y}}\right)}{1+\left[\sqrt{\frac{(1-x)y}{1-y}}\right]^2}=-\frac{2}{\sqrt{y(1-y)}}\cdot\tan^{-1}\left(\sqrt{\frac{(1-x)y}{1-y}}\right)\Bigg|_0^1=$$

$$=\frac{2}{\sqrt{y(1-y)}}\cdot\tan^{-1}\left(\sqrt{\frac{y}{1-y}}\right)$$

Solution 3 by Saboor Halimi-Afghanistan

$$\Omega=\int_0^1\frac{dx}{(1-xy)\sqrt{1-x}}\overset{s=\sqrt{1-x}}{=}\int_0^1\frac{2ds}{s^2y-y+1}=\int_0^1\frac{2ds}{(\sqrt{y}s)^2+(\sqrt{1-y})^2}\overset{k=\sqrt{y}}{=}$$

$$=\frac{1}{\sqrt{y}}\int_0^{\sqrt{y}}\frac{2dk}{k^2+\left(\sqrt{1-y}\right)^2}=\frac{2}{\sqrt{y(1-y)}}\cdot\tan^{-1}\left(\frac{k}{\sqrt{1-y}}\right)\Bigg|_0^{\sqrt{y}}=$$

$$=\frac{2}{\sqrt{y(1-y)}}\cdot\tan^{-1}\left(\sqrt{\frac{y}{1-y}}\right)$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega(y)=\int_0^1\frac{dx}{(1-xy)\sqrt{1-x}};y<1;\ \Omega(0)=2$$

$$\Omega(y)=\int_0^1\frac{dx}{(1-xy)\sqrt{1-x}}\overset{z=1-x}{=}\int_0^1\frac{1}{z(1-y+yz^2)}2z\,dz\overset{z\neq 0}{=}$$

$$=\frac{2}{y}\int_0^1\frac{1}{z^2+\frac{1-y}{y}}dz=\frac{2}{y}\cdot\frac{1}{\sqrt{\frac{1-y}{y}}}\cdot\tan^{-1}\sqrt{\frac{y}{1-y}}=\frac{2}{\sqrt{y(1-y)}}\cdot\tan^{-1}\left(\sqrt{\frac{y}{1-y}}\right)$$

$$\text{If } y<0\Rightarrow\Omega(y)=\frac{2}{i\sqrt{|y|(1+|y|)}}\tan^{-1}\left(i\sqrt{\frac{|y|}{1+|y|}}\right)$$

$$\tan^{-1}(iu)=v\Rightarrow iu=\tan v=-i\tanh(iv)$$
$$u=-\tanh(iv)\Rightarrow iv=-\tanh^{-1}(u)\Rightarrow v=i\tanh^{-1}(u)$$

$$\Omega(y)=\begin{cases}\dfrac{2}{\sqrt{y(1-y)}}\tan^{-1}\sqrt{\dfrac{y}{1-y}}; & 0<y<1\\ 2; & y=0\\ \dfrac{2}{\sqrt{|y|\sqrt{1+|y|}}}\tanh^{-1}\sqrt{\dfrac{|y|}{1+|y|}}; & y<0\end{cases}$$

2037. Find a closed form:

$$I=\int_0^1 log(x)\,log(x^2+x+1)\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

$$\Omega=\int_0^1\log(x)\log(x^2+x+1)\,dx=\int_0^1\log(x)\left[\log(1-x^3)-\log(1-x)\right]dx$$

$$\because \sum_{n=1}^{\infty} \frac{x^n}{n} - \log|1-x|\,;\,|x|<1$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \log(x)\,dx - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{3n} \log(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n(1+3n)^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}$$

$$\because \int_0^1 x^m \log^n(x)\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, n > -1; m \neq -1$$

$$\Omega = \sum_{n=1}^{\infty} \frac{1}{n} - 3\sum_{n=1}^{\infty} \frac{1}{1+3n} - 3\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

$$\because \psi^{(0)}(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+z}$$

$$\Omega = \psi^{(0)}\left(1+\frac{1}{3}\right) + \gamma + 3 - \frac{1}{3}\zeta\left(2, \frac{1}{3}\right) - 2 + \frac{\pi^2}{6}$$

$$-\gamma + \int_0^1 \frac{1-x^{\frac{1}{3}}}{1-x}\,dx + \gamma + 1 - \frac{1}{3}\zeta\left(2, \frac{1}{3}\right) + \zeta(2)$$

$$\because \psi^{(0)}(1+z) = -\gamma + \int_0^1 \frac{1-x^z}{1-x}\,dx$$

$$\Omega = 3\int_0^1 \frac{(1-y)y^2}{1-y^3}\,dy + 1 - \frac{1}{3}\psi^{(1)}\left(\frac{1}{3}\right) + \frac{\pi^2}{6}$$

$$3 - \frac{3}{2}\int_0^1 \frac{2y+1}{y^2+y+1}\,dy - \frac{3}{2}\int_0^1 \frac{dy}{y^2+y+1} + 1 - \frac{1}{3}\psi^{(1)}\left(\frac{1}{3}\right) + \frac{\pi^2}{6}$$

$$\Omega = 3 - 3[\log(y^2+y+1)]_0^1 - \frac{3}{2}\int_0^1 \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + 1 - \frac{1}{3}\psi^{(1)}\left(\frac{1}{3}\right) + \frac{\pi^2}{6}$$

$$3 - \frac{3}{2}\log(3) - \sqrt{3}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] + 1 - \frac{1}{3}\psi^{(1)}\left(\frac{1}{3}\right) + \frac{\pi^2}{6}$$

$$\Omega = \int_0^1 \log(x)\log(x^2+x+1)\,dx = 4 - \frac{3}{2}\log(3) - \frac{\pi\sqrt{3}}{6} - \frac{1}{3}\psi^{(1)}\left(\frac{1}{3}\right) + \frac{\pi^2}{6}$$

Solution 2 by Pham Duc Nam-Vietnam

$$I = \int_0^1 log(x)\,log(x^2+x+1)\,dx$$

$$\begin{cases} u = \log(x^2+x+1) \\ dv = \log(x)\,dx \end{cases} \Rightarrow \begin{cases} du = \dfrac{2x+1}{x^2+x+1} \\ v = x\log(x) - x \end{cases} \Rightarrow I$$

$$= (x\log(x)-x)\log(x^2+x+1)\Big|_0^1 + \int_0^1 \frac{2x^2+x}{x^2+x+1}dx - \int_0^1 \frac{x\log(x)\,(2x+1)}{x^2+x+1}dx$$

$$= -\log 3 + \left(-\frac{1}{2}\log(x^2+x+1) + 2x - \sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}}\Big|_0^1\right) - \int_0^1 \frac{x\log(x)}{x^2+x+1}dx - \int_0^1 \frac{2x^2\log(x)}{x^2+x+1}dx$$

$$= -\frac{3}{2}\log 3 + 2 - \frac{\pi\sqrt{3}}{6} - \int_0^1 \frac{x\log(x)}{x^2+x+1}dx - \int_0^1 \frac{2x^2\log(x)}{x^2+x+1}dx$$

$$* If: \theta \in \mathbb{R}, x \in (-1,1): \sum_{k=0}^{+\infty} \sin(k\theta)\,x^k = \frac{x\sin\theta}{x^2 - 2x\cos\theta + 1}, let: \theta = \frac{2\pi}{3} \Rightarrow \frac{\sqrt{3}}{2}.\frac{x}{x^2+x+1}$$

$$= \sum_{k=0}^{+\infty} \sin\left(\frac{2k\pi}{3}\right)x^k \Rightarrow \frac{x}{x^2+x+1} = \frac{2}{\sqrt{3}}\sum_{k=0}^{+\infty} \sin\left(\frac{2k\pi}{3}\right)x^k$$

$$\Rightarrow \int_0^1 \frac{x\log(x)}{x^2+x+1}dx = \frac{2}{\sqrt{3}}\int_0^1 \log(x)\sum_{k=0}^{+\infty}\sin\left(\frac{2k\pi}{3}\right)x^k\,dx$$

$$= \frac{2}{\sqrt{3}}\sum_{k=0}^{+\infty}\sin\left(\frac{2k\pi}{3}\right)\int_0^1 x^k\log(x)\,dx = -\frac{2}{\sqrt{3}}\sum_{k=0}^{+\infty}\frac{\sin\left(\frac{2k\pi}{3}\right)}{(k+1)^2} \xrightarrow{j=k+1}$$

$$-\frac{2}{\sqrt{3}}\sum_{j=1}^{+\infty}\frac{\sin\left(\frac{2\pi j}{3} - \frac{2\pi}{3}\right)}{(j)^2}$$

$$= -\frac{2}{\sqrt{3}}\sum_{j=1}^{+\infty}\frac{-\frac{\sqrt{3}}{2}\cos\left(\frac{2\pi j}{3}\right) - \frac{1}{2}\sin\left(\frac{2\pi j}{3}\right)}{(j)^2} = \sum_{j=1}^{+\infty}\frac{\cos\left(\frac{2\pi j}{3}\right)}{(j)^2} + \frac{1}{\sqrt{3}}\sum_{j=1}^{+\infty}\frac{\sin\left(\frac{2\pi j}{3}\right)}{(j)^2}$$

$$= \Re\left(Li_2\left(e^{\frac{2\pi i}{3}}\right)\right) + \frac{1}{\sqrt{3}}Cl_2\left(\frac{2\pi}{3}\right)$$

$$= -\frac{\pi^2}{18} + \frac{1}{\sqrt{3}}\left(\frac{1}{36}\sum_{n=1}^{3}\sin\frac{2\pi n}{3}\left(\psi^{(1)}\left(\frac{n}{6}\right) + (-1)^2\psi^{(1)}\left(\frac{n+3}{6}\right)\right)\right)$$

$$= -\frac{\pi^2}{18} + \frac{1}{36\sqrt{3}}\left(\frac{\sqrt{3}}{2}\left(\psi^{(1)}\left(\frac{1}{6}\right) + \psi^{(1)}\left(\frac{2}{3}\right)\right) - \frac{\sqrt{3}}{2}\left(\psi^{(1)}\left(\frac{1}{3}\right) + \psi^{(1)}\left(\frac{5}{6}\right)\right)\right)$$

$$= -\frac{\pi^2}{18} + \frac{1}{36\sqrt{3}}\left(2\sqrt{3}\psi^{(1)}\left(\frac{1}{3}\right) - 2\sqrt{3}\psi^{(1)}\left(\frac{2}{3}\right)\right) = -\frac{\pi^2}{18} + \frac{1}{18}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)\right)$$

$$* \textbf{Similiarly}: \int_0^1 \frac{2x^2 \log(x)}{x^2+x+1}dx = -2 - \frac{\pi^2}{27} + \frac{2\psi^{(1)}\left(\frac{1}{3}\right)}{9}$$

$$\Rightarrow I = -\frac{3}{2}\log 3 + 2 - \frac{\pi\sqrt{3}}{6} + \frac{\pi^2}{18} - \frac{1}{18}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{2}{3}\right)\right) + 2 + \frac{\pi^2}{27} - \frac{2\psi^{(1)}\left(\frac{1}{3}\right)}{9}$$

$$= 4 - \frac{\pi\sqrt{3}}{6} + \frac{5\pi^2}{54} - \frac{3}{2}\log 3 - \frac{5\psi^{(1)}\left(\frac{1}{3}\right)}{18} + \frac{\psi^{(1)}\left(\frac{2}{3}\right)}{18}$$

NOTE:

$$* Cl_2(\varphi) = \sum_{k=1}^{+\infty} \frac{\sin(k\varphi)}{k^2} \text{:Clausen function, and: } Cl_2\left(\frac{q\pi}{p}\right)$$

$$= \frac{1}{4p^2}\sum_{j=1}^{p} \sin\frac{q\pi j}{p}\left(\psi^{(1)}\left(\frac{j}{2p}\right) + (-1)^q\psi^{(1)}\left(\frac{j+p}{2p}\right)\right)\left(0 < \frac{q}{p} < 1, q, p \in \mathbb{Z}^+\right)$$

$$* Li_2(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^2} \text{:Spence's function or dilogarithm}$$

Solution 3 by proposer

$$\int_0^1 \log(y)\log(1+y+y^2)\,dy = \left[(y\log(y) - y)\log(1+y+y^2)\right]_0^1 - \int_0^1 \frac{(y\log(y)-y)(1+2y)}{1+y+y^2}dy$$

$$= -\log(3) - \int_0^1 \frac{y\log(y)}{1+y+y^2}dy - 2\int_0^1 \frac{y^2\log(y)}{1+y+y^2}dy + \int_0^1 \frac{y}{1+y+y^2}dy$$

$$+ 2\int_0^1 \frac{y^2}{1+y+y^2}dy$$

$$= -\log(3) + \frac{1}{9}\left(\frac{7\pi^2}{6} - \varphi^{(1)}\left(\frac{1}{3}\right)\right) + 2 + \frac{\pi^2}{27} - \frac{2}{9}\varphi^{(1)}\left(\frac{1}{3}\right) - \frac{\log(3)}{2} - \frac{\pi}{2\sqrt{3}} + 2$$

$$= 4 - \frac{3\log(3)}{2} + \frac{\pi^2}{6} - \frac{\pi}{2\sqrt{3}} - \frac{1}{3}\varphi^{(1)}\left(\frac{1}{3}\right)$$

$$\int_0^1 \frac{y\log(y)}{1+y+y^2}dy = -\frac{1}{9}\left(\frac{7\pi^2}{6} - \varphi^{(1)}\left(\frac{1}{3}\right)\right)$$

$$\int_0^1 \frac{y^2\log(y)}{1+y+y^2}dy = -1 - \frac{\pi^2}{54} + \frac{1}{9}\varphi^{(1)}\left(\frac{1}{3}\right)$$

2038. Prove that:

$$\sin\frac{\pi}{13}\sin\frac{3\pi}{13}\sin\frac{4\pi}{13}=\frac{1}{8}\sqrt{\frac{13-3\sqrt{13}}{2}}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Pham Duc Nam-Vietnam

$$*\sum_{k=1}^{n}\cos\frac{2k\pi}{2n+1}=-\frac{1}{2}$$

PROOF:

$$S=\sum_{k=1}^{n}\cos\frac{2k\pi}{2n+1}\Rightarrow 2\sin\frac{\pi}{2n+1}S=\sum_{k=1}^{n}2\sin\frac{\pi}{2n+1}\cos\frac{2k\pi}{2n+1}$$

$$Apply: \sin a\cos b=\frac{1}{2}(\sin(a+b)+\sin(a-b))\Rightarrow 2\sin\frac{\pi}{2n+1}S$$

$$=\sum_{k=1}^{n}\left(\sin\frac{\pi(2k+1)}{2n+1}-\sin\frac{\pi(2k-1)}{2n+1}\right)=-\sin\frac{\pi}{2n+1}$$

$$\Rightarrow S=\frac{-\sin\frac{\pi}{2n+1}}{2\sin\frac{\pi}{2n+1}}=-\frac{1}{2}$$

$$*\ Let: n=6\Rightarrow \cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{4\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}=-\frac{1}{2}$$

$$*\left(\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}\right)\left(\cos\frac{4\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}\right)$$

$$=\frac{1}{2}\left(3\left(\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{4\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}\right)\right)$$

$$=\frac{1}{2}.3.-\frac{1}{2}=-\frac{3}{4}$$

$$\Rightarrow\left(\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}\right),\left(\cos\frac{4\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}\right)\ are\ roots\ of: t^2$$

$$+\frac{1}{2}t-\frac{3}{4}=0\Rightarrow\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}$$

$$=\frac{\sqrt{13}-1}{4},\ \boxed{\cos\frac{4\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}=\frac{-\sqrt{13}-1}{4}}$$

$$*\ \boxed{\begin{gathered}\cos\frac{4\pi}{13}\cos\frac{10\pi}{13}+\cos\frac{10\pi}{13}\cos\frac{12\pi}{13}+\cos\frac{4\pi}{13}\cos\frac{12\pi}{13}=\\ \frac{1}{2}\left(\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{4\pi}{13}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}\right)=-\frac{1}{4}\end{gathered}}$$

$$\cos\frac{4\pi}{13}\cos\frac{12\pi}{13}\cos\frac{10\pi}{13}=\cos\frac{4\pi}{13}\cos\frac{\pi}{13}\cos\frac{3\pi}{13}=$$

$$*\ \frac{1}{2}\left(\cos\frac{8\pi}{13}+\cos\frac{10\pi}{13}\right)\cos\frac{10\pi}{13}=\frac{1}{2}\left(\frac{1}{2}\left(\cos\frac{8\pi}{13}+\cos\frac{2\pi}{13}\right)+\frac{1+\cos\frac{6\pi}{13}}{2}\right)=$$

$$\frac{1}{4}\left(1+\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}\right)=\frac{1}{4}\left(1+\frac{\sqrt{13}-1}{4}\right)=\frac{3+\sqrt{13}}{16}$$

$$*\ Apply: 1-\cos x=2\sin^2\frac{x}{2}\Rightarrow \sin\frac{2\pi}{13}=\sqrt{\frac{1-\cos\frac{4\pi}{13}}{2}},\sin\frac{6\pi}{13}$$

$$=\sqrt{\frac{1-\cos\frac{12\pi}{13}}{2}},\sin\frac{8\pi}{13}=\sin\frac{5\pi}{13}=\sqrt{\frac{1-\cos\frac{10\pi}{13}}{2}}$$

$$\Rightarrow \sin\frac{2\pi}{13}\sin\frac{6\pi}{13}\sin\frac{8\pi}{13}=\sqrt{\frac{\left(1-\cos\frac{4\pi}{13}\right)\left(1-\cos\frac{12\pi}{13}\right)\left(1-\cos\frac{10\pi}{13}\right)}{8}}$$

$$=\sqrt{\frac{\left(1-\cos\frac{12\pi}{13}-\cos\frac{4\pi}{13}+\cos\frac{4\pi}{13}\cos\frac{12\pi}{13}\right)\left(1-\cos\frac{10\pi}{13}\right)}{8}}$$

$$=\sqrt{\frac{1-\cos\frac{4\pi}{13}-\cos\frac{10\pi}{13}-\cos\frac{12\pi}{13}+\cos\frac{4\pi}{13}\cos\frac{10\pi}{13}+\cos\frac{10\pi}{13}\cos\frac{12\pi}{13}+\cos\frac{4\pi}{13}\cos\frac{12\pi}{13}-\cos\frac{4\pi}{13}\cos\frac{12\pi}{13}\cos\frac{10\pi}{13}}{8}}$$

$$=\sqrt{\frac{1+\frac{\sqrt{13}+1}{4}-\frac{1}{4}-\frac{3+\sqrt{13}}{16}}{8}}=\sqrt{\frac{13+3\sqrt{13}}{128}}$$

$$*\ \sin\frac{2\pi}{13}\sin\frac{6\pi}{13}\sin\frac{8\pi}{13}=8\sin\frac{\pi}{13}\sin\frac{3\pi}{13}\sin\frac{4\pi}{13}\cos\frac{\pi}{13}\cos\frac{3\pi}{13}\cos\frac{4\pi}{13}=\sqrt{\frac{13+3\sqrt{13}}{128}}$$

$$\Leftrightarrow 8\sin\frac{\pi}{13}\sin\frac{3\pi}{13}\sin\frac{4\pi}{13}.\frac{3+\sqrt{13}}{16}=\sqrt{\frac{13+3\sqrt{13}}{128}}$$

$$\Rightarrow \sin\frac{\pi}{13}\sin\frac{3\pi}{13}\sin\frac{4\pi}{13}=\frac{1}{8}.\sqrt{\frac{13+3\sqrt{13}}{128}}.\frac{16}{3+\sqrt{13}}$$

$$=\boxed{\frac{1}{8}\sqrt{\frac{13-3\sqrt{13}}{2}}}$$

2039. **Prove that:**

$$\int_0^1\int_0^1\int_0^1 \frac{\log(x^2)\log(y^2)\log(z^2)\log(xyz)}{1+xyz}dxdydz = 48\eta(7)$$

where $\eta(v)$ −Dirichlet's function.

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{\log(x^2)\log(y^2)\log(z^2)\log(xyz)}{1+xyz}dxdydz =$$

$$= 8\int_0^1\int_0^1\int_0^1 \frac{\log(x)\log(y)\log(z)\log(xyz)}{1+xyz}dxdydz =$$

$$= 8\int_0^1\int_0^1\int_0^1 \frac{\log^2(x)\log(y)\log(z)}{1+xyz}dxdydz$$

$$+8\int_0^1\int_0^1\int_0^1 \frac{\log(x)\log^2(y)\log(z)}{1+xyz}dxdydz +$$

$$+8\int_0^1\int_0^1\int_0^1 \frac{\log(x)\log(y)\log^2(z)}{1+xyz}dxdydz = 24\int_0^1\int_0^1 \frac{\log^2(x)\log(y)\log(z)}{1+xyz}dxdydz$$

$$= 24\sum_{n=0}^{\infty}(-1)^n\int_0^1 z^n\log(z)\,dz\int_0^1 y^n\log(y)\,dy\int_0^1 x^n\log^2(x)\,dx$$

$$Since: \int_0^1 t^n\log^m(t)\,dt = (-1)^m\frac{m!}{(n+1)^{m+1}}$$

$$\Omega = 48\sum_{n=}^{\infty}\frac{(-1)^n}{(n+1)^7} = 48\eta(7)$$

$$Therefore,$$

$$\int_0^1\int_0^1\int_0^1 \frac{\log(x^2)\log(y^2)\log(z^2)\log(xyz)}{1+xyz}dxdydz = 48\eta(7)$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{\log(x^2)\log(y^2)\log(z^2)\log(xyz)}{1+xyz}dxdydz =$$

$$= 24\sum_{n=0}^{\infty}(-1)^n\int_0^1\int_0^1\int_0^1 (xyz)^n \log^2 x \log y \log z\, dxdydz$$

$$\because \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$$

$$Since: \int_0^1 t^n \log^m(t)\, dt = (-1)^m \frac{m!}{(n+1)^{m+1}}$$

$$\Omega = 48\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}\cdot\frac{-1}{(n+1)^2}\cdot\frac{-1}{(n+1)^2} = 48\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^7} = 48\eta(7)$$

$$Therefore,$$

$$\int_0^1\int_0^1\int_0^1 \frac{\log(x^2)\log(y^2)\log(z^2)\log(xyz)}{1+xyz}\,dxdydz = 48\eta(7)$$

2040. **If $x \in \left(0,\frac{\pi}{2}\right)$, then :**

$$\sin x\tan x \cdot \arctan(\sqrt{2}\sin x) + \cos x\cot x \cdot \arctan(\sqrt{2}\cos x) \ge \frac{\pi\sqrt{2}}{4\sin(2x)}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sin x.\tan x.\arctan(\sqrt{2}\sin x) + \cos x.\cot x.\arctan(\sqrt{2}\cos x) \ge \frac{\pi\sqrt{2}}{4\sin(2x)}$$

$$\Leftrightarrow 4\sin x.\frac{\sin x}{\cos x}.\sin x.\cos x.\arctan(\sqrt{2}\sin x)$$

$$+4\cos x.\frac{\cos x}{\sin x}.\sin x.\cos x.\arctan(\sqrt{2}\cos x) \ge \frac{\pi}{\sqrt{2}}$$

$$\Leftrightarrow 4\sin^3 x.\arctan(\sqrt{2}\sin x) + 4\cos^3 x.\arctan(\sqrt{2}\cos x) \overset{(*)}{\ge} \frac{\pi}{\sqrt{2}}$$

Let $f(x) = 4\sin^3 x.\arctan(\sqrt{2}\sin x) + 4\cos^3 x.\arctan(\sqrt{2}\cos x)\ \forall x \in \left(0,\frac{\pi}{2}\right)$

$$\therefore f'(x) = 12\sin x.\cos x.\left(\sin x.\arctan(\sqrt{2}\sin x) - \cos x.\arctan(\sqrt{2}\cos x)\right)$$

$$+4\sqrt{2}\sin x.\cos x.\left(\frac{\sin^2 x}{2\sin^2 x+1} - \frac{\cos^2 x}{2\cos^2 x+1}\right)$$

$$= 12\sin x.\cos x.\left(\sin x.\arctan(\sqrt{2}\sin x) - \cos x.\arctan(\sqrt{2}\cos x)\right)$$

$$+\frac{4\sqrt{2}\sin x.\cos x(2\sin^2 x.\cos^2 x + \sin^2 x - 2\sin^2 x.\cos^2 x - \cos^2 x)}{(2\sin^2 x+1)(2\cos^2 x+1)}$$

$$\Rightarrow f'(x) \overset{(**)}{=} 12\sin x.\cos x.\left(\sin x.\arctan(\sqrt{2}\sin x) - \cos x.\arctan(\sqrt{2}\cos x)\right)$$

$$+\frac{4\sqrt{2}\sin x.\cos x(\sin^2 x-\cos^2 x)}{(2\sin^2 x+1)(2\cos^2 x+1)}$$

$\boxed{\text{Case 1}}\ x\in\left(0,\frac{\pi}{4}\right]\ \therefore\ \sqrt{2}\sin x\le\sqrt{2}\cos x$

$$\Rightarrow \arctan(\sqrt{2}\sin x)\le\arctan(\sqrt{2}\cos x)\ (\because f(t)=\arctan(t)\text{ is }\uparrow\ \forall\, t\in\mathbb{R})$$

$$\Rightarrow \sin x.\arctan(\sqrt{2}\sin x)\le\sin x.\arctan(\sqrt{2}\cos x)$$

$$\Rightarrow \sin x.\arctan(\sqrt{2}\sin x)-\cos x.\arctan(\sqrt{2}\cos x)$$

$$\le\sin x.\arctan(\sqrt{2}\cos x)-\cos x.\arctan(\sqrt{2}\cos x)$$

$$=\arctan(\sqrt{2}\cos x)(\sin x-\cos x)\le 0\ (\because \sin x-\cos x\le 0)$$

$$\Rightarrow 12\sin x.\cos x.\left(\sin x.\arctan(\sqrt{2}\sin x)-\cos x.\arctan(\sqrt{2}\cos x)\right)\overset{\text{(i)}}{\le}0$$

$$\text{Also, }\sin^2 x-\cos^2 x\le 0\Rightarrow\frac{4\sqrt{2}\sin x.\cos x}{(2\sin^2 x+1)(2\cos^2 x+1)}.(\sin^2 x-\cos^2 x)\overset{\text{(ii)}}{\le}0$$

$$\therefore\ \text{(i)}+\text{(ii)}\Rightarrow 12\sin x.\cos x.\left(\sin x.\arctan(\sqrt{2}\sin x)-\cos x.\arctan(\sqrt{2}\cos x)\right)$$

$$+\frac{4\sqrt{2}\sin x.\cos x}{(2\sin^2 x+1)(2\cos^2 x+1)}.(\sin^2 x-\cos^2 x)\le 0\overset{\text{via }(**)}{\Rightarrow} f'(x)\le 0\ \forall\, x\in\left(0,\frac{\pi}{4}\right]$$

$$\Rightarrow f(x)\text{ is }\downarrow\ \forall\, x\in\left(0,\frac{\pi}{4}\right]\Rightarrow\forall\, x\in\left(0,\frac{\pi}{4}\right], f(x)\ge f\left(\frac{\pi}{4}\right)=2.\frac{4}{2\sqrt{2}}.\arctan(1)=\frac{\pi}{\sqrt{2}}$$

$$\Rightarrow (*)\text{ is true }\forall\, x\in\left(0,\frac{\pi}{4}\right]$$

$\boxed{\text{Case 2}}\ x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)\ \therefore\ \sqrt{2}\sin x\ge\sqrt{2}\cos x\Rightarrow\arctan(\sqrt{2}\sin x)$

$$\ge\arctan(\sqrt{2}\cos x)\ (\because f(t)=\arctan(t)\text{ is }\uparrow\ \forall\, t\in\mathbb{R})$$

$$\Rightarrow \sin x.\arctan(\sqrt{2}\sin x)\ge\sin x.\arctan(\sqrt{2}\cos x)$$

$$\Rightarrow \sin x.\arctan(\sqrt{2}\sin x)-\cos x.\arctan(\sqrt{2}\cos x)$$

$$\ge\sin x.\arctan(\sqrt{2}\cos x)-\cos x.\arctan(\sqrt{2}\cos x)$$

$$=\arctan(\sqrt{2}\cos x)(\sin x-\cos x)\ge 0\ (\because \sin x-\cos x\ge 0)$$

$$\Rightarrow 12\sin x.\cos x.\left(\sin x.\arctan(\sqrt{2}\sin x)-\cos x.\arctan(\sqrt{2}\cos x)\right)\overset{(1)}{\ge}0$$

$$\text{Also, }\sin^2 x-\cos^2 x\ge 0\Rightarrow\frac{4\sqrt{2}\sin x.\cos x}{(2\sin^2 x+1)(2\cos^2 x+1)}.(\sin^2 x-\cos^2 x)\overset{(2)}{\ge}0$$

$$\therefore\ (1)+(2)\Rightarrow 12\sin x.\cos x.\left(\sin x.\arctan(\sqrt{2}\sin x)-\cos x.\arctan(\sqrt{2}\cos x)\right)$$

$$+\frac{4\sqrt{2}\sin x.\cos x}{(2\sin^2 x+1)(2\cos^2 x+1)}.(\sin^2 x-\cos^2 x)\ge 0\overset{\text{via }(**)}{\Rightarrow} f'(x)\ge 0\ \forall\, x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)$$

$$\Rightarrow f(x)\text{ is }\uparrow\ \forall\, x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)\Rightarrow\forall\, x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right), f(x)\ge f\left(\frac{\pi}{4}\right)=2.\frac{4}{2\sqrt{2}}.\arctan(1)=\frac{\pi}{\sqrt{2}}$$

$$\Rightarrow (*)\text{ is true }\forall\, x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)\ \therefore\text{ combining cases }1,2,(*)\text{ is true }\forall\, x\in\left(0,\frac{\pi}{2}\right)$$

$$\therefore\ \sin x.\tan x.\arctan(\sqrt{2}\sin x)+\cos x.\cot x.\arctan(\sqrt{2}\cos x)\ge\frac{\pi\sqrt{2}}{4\sin(2x)}$$

$$\forall\, x\in\left(0,\frac{\pi}{2}\right),\text{ equality iff }x=\frac{\pi}{4}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Let } a=\sqrt{2}\sin x \textit{ and } b=\sqrt{2}\cos x.\ \textit{We have}: a,b>0 \textit{ and } a^2+b^2=2.$$

The problem becomes to prove :

$$\frac{a^2}{b}.\arctan(a)+\frac{b^2}{a}.\arctan(b)\ge\frac{\pi}{2ab}\ \text{ or }\ a^3.\arctan(a)+b^3.\arctan(b)\ge\frac{\pi}{2}\quad(1)$$

$$\textit{Let } f(t)=t^2.\arctan(t),\ \ t>0.\ \textit{We have}:\ f'(t)=2t.\arctan(t)+\frac{t^2}{1+t^2},$$

$$\textit{and } f''(t)=2\arctan(t)+\frac{2t(2+t^2)}{(1+t^2)^2}>0 \textit{ then } f \textit{ is convex on } (0,\infty).$$

$$\textit{Then}:\ LHS_{(1)}=a.f(a)+b.f(b)\overset{Jensen}{\ge}(a+b).f\left(\frac{a^2+b^2}{a+b}\right)=(a+b).f\left(\frac{2}{a+b}\right)=$$

$$=\frac{4}{a+b}.\arctan\left(\frac{2}{a+b}\right)\overset{a+b\le\sqrt{2(a^2+b^2)}=2}{\ge}\frac{4}{2}.\arctan\left(\frac{2}{2}\right)=\frac{\pi}{2}=RHS_{(1)}.$$

$$\textit{So the proof is completed. Equality holds iff } a=b=1 \Leftrightarrow x=\frac{\pi}{4}.$$

2041. Find:

$$\Omega=\int_0^1\int_0^1\frac{x^2\log(x)\log^2(y)}{1-xy}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Pham Duc Nam-Vietnam

$$\Omega=\int_0^1\int_0^1\frac{x^2\log(x)\log^2(y)}{1-xy}dxdy$$

$$=\int_0^1\int_0^1 x^2\log(x)\log^2(y)\sum_{k=0}^{+\infty}(xy)^k\,dxdy=\sum_{k=0}^{+\infty}\int_0^1 y^k\log^2(y)\,dy\int_0^1 x^{k+2}\log(x)\,dx$$

$$=\sum_{k=0}^{+\infty}\frac{2}{(k+1)^3}\cdot\frac{-1}{(k+3)^2}=-2\sum_{k=0}^{+\infty}\frac{1}{(k+3)^2(k+1)^3}$$

$$=-2\sum_{k=0}^{+\infty}\left(\frac{3}{16}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)-\frac{1}{4(k+1)^2}+\frac{1}{4(k+1)^3}-\frac{1}{8(k+3)^2}\right)$$

$$=-\frac{3}{8}\sum_{k=0}^{+\infty}\left(\frac{1}{k+1}-\frac{1}{k+3}\right)+\frac{1}{2}\sum_{k=0}^{+\infty}\frac{1}{(k+1)^2}-\frac{1}{2}\sum_{k=0}^{+\infty}\frac{1}{(k+1)^3}+\frac{1}{4}\sum_{k=0}^{+\infty}\frac{1}{(k+3)^2}$$

$$=-\frac{3}{8}\cdot\frac{3}{2}+\frac{1}{2}\cdot\frac{\pi^2}{6}-\frac{1}{2}\zeta(3)+\frac{1}{4}\left(\frac{\pi^2}{6}-1-\frac{1}{4}\right)=-\frac{9}{16}+\frac{\pi^2}{12}-\frac{1}{2}\zeta(3)+\frac{\pi^2}{24}-\frac{5}{4}\cdot\frac{1}{4}$$

$$=\frac{\pi^2}{8}-\frac{1}{2}\zeta(3)-\frac{7}{8}$$

$$=\boxed{\frac{\pi^2-4\zeta(3)-7}{8}}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

Let xy=m

$$\int_0^1\int_0^x\frac{x\log(x)\log^2\left(\frac{m}{x}\right)}{1-m}dm\,dx=$$

$$=\int_0^1 x\log(x)\int_0^x\frac{\log^2(m)}{1-m}dm\,dx-2\int_0^1 x\log^2(x)\int_0^x\frac{\log(m)}{1-m}dm\,dx+\int_0^1 x\log^3(x)\int_0^x\frac{1}{1-m}dm\,dx$$

$$I_1=\int_0^1 x\log(x)\int_0^x\frac{\log^2(m)}{1-m}dm\,dx\overset{IBP}{=}-\frac{1}{4}\int_0^1\frac{\log^2(x)}{1-x}dx-\frac{1}{2}\int_0^1\frac{x^2\log^3(x)}{1-x}dx+\frac{1}{4}\int_0^1\frac{x^2\log^2(x)}{1-x}dx$$

$$I_2=\int_0^1 x\log^2(x)\int_0^x\frac{\log(m)}{1-m}dm\,dx=$$

$$\overset{IBP}{=}\frac{1}{4}\int_0^1\frac{\log(x)}{1-x}dx-\frac{1}{2}\int_0^1\frac{x^2\log^3(x)}{1-x}dx+\frac{1}{2}\int_0^1\frac{x^2\log^2(x)}{1-x}dx-\frac{1}{4}\int_0^1\frac{x^2\log(x)}{1-x}dx$$

$$I_3=\int_0^1 x\log^3(x)\int_0^x\frac{1}{1-m}dm\,dx=$$

$$\overset{IBP}{=}-\frac{3}{8}\int_0^1\frac{1}{1-x}dx-\frac{1}{2}\int_0^1\frac{x^2\log^3(x)}{1-x}dx+\frac{3}{4}\int_0^1\frac{x^2\log^2(x)}{1-x}dx-\frac{3}{4}\int_0^1\frac{x^2\log(x)}{1-x}dx+\frac{3}{8}\int_0^1\frac{x^2}{1-x}dx$$

$$=-\frac{1}{2}\int_0^1\frac{x^2\log^3(x)}{1-x}dx+\frac{3}{4}\int_0^1\frac{x^2\log^2(x)}{1-x}dx-\frac{3}{4}\int_0^1\frac{x^2\log(x)}{1-x}dx-\frac{9}{16}$$

$$I = I_1 - 2I_2 + I_3 = -\frac{1}{4}\int_0^1 \frac{\log^2(x)}{1-x}dx - \frac{1}{2}\int_0^1 \frac{\log(x)}{1-x}dx - \frac{9}{16} - \frac{1}{4}\int_0^1 \frac{x^2\log(x)}{1-x}dx =$$

$$= -\frac{9}{16} - \frac{1}{4}\sum_{k=1}^{\infty}\int_0^1 x^{k-1}\log^2(x)\,dx - \frac{1}{2}\sum_{k=1}^{\infty}\int_0^1 x^{k-1}\log(x)\,dx$$

$$-\frac{1}{4}\sum_{k=1}^{\infty}\int_0^1 x^{k+1}\log(x)\,dx = -\frac{9}{16} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^3} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{(k+2)^2}$$

$$= -\frac{9}{16} - \frac{1}{2}\zeta(3) + \frac{1}{2}\zeta(2) + \frac{1}{4}\left(\zeta(2) - \frac{5}{4}\right) = -\frac{1}{2}\zeta(3) + \frac{3}{4}\zeta(2) - \frac{7}{8}$$

$$= \frac{\pi^2 - 4\zeta(3) - 7}{8}$$

Solution 3 by Le Thu-Vietnam

By Maclaurin series:

$$\Omega = \sum_{n=0}^{\infty}\left(\int_0^1 y^n \log^2 y\left(\int_0^1 x^{n+2}\log x\,dx\right)dy\right) =$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{(n+3)^2}\int_0^1 y^n\log^2 y\,dy\right)$$

$$\textit{Since}: \int_0^1 x^a \log x\,dx = -\frac{1}{(a+1)^2};\,(\forall)a\in\mathbb{R}, a > -1$$

$$\Omega = \sum_{n=0}^{\infty}\left[-\frac{1}{(n+3)^2}\cdot\frac{2}{(n+1)^3}\right] = -2\sum_{n=0}^{\infty}\frac{1}{(n+3)^2(n+1)^3}$$

$$\textit{Since}: \int_0^1 y^\beta\log^2 y\,dy = \left[\frac{y^{\beta+1}[(\beta+1)^2\log^2 y - 2(\beta+1)\log y + 2]}{(\beta+1)^3}\right]_0^1 = \frac{2}{(\beta+1)^3};$$

$$(\forall)\beta\in\mathbb{R},\qquad \beta > 1$$

$$1)\ \frac{3}{8}\sum_{n=0}^{\infty}\left(\frac{1}{n+3} - \frac{1}{n+1}\right) =$$

$$= \frac{3}{8}\left[\sum_{n=0}^{\infty}\left(\frac{1}{n+3} - \frac{1}{n+2}\right) + \sum_{n=0}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+2}\right)\right] = -\frac{9}{16}$$

$$2)\ \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{12}$$

$$3)\ \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+3)^2}=\frac{1}{4}\left(\frac{\pi^2}{6}-\frac{5}{4}\right)=\frac{\pi^2}{24}-\frac{5}{6}$$

$$4)\ -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)^3}=-\frac{1}{2}\zeta(2)$$

where $\zeta(3)$ is Apery's constant. By summing all of them, we obtain:

$$\Omega=\frac{\pi^2-3\zeta(3)-7}{8}$$

2042. Find a closed form:

$$\Omega=\int_0^{\pi}\cot^{-1}\left(2022^{\cos(\pi-x)}\right)dx$$

Proposed by Saboor Halimi-Afghanistan

Solution 1 by Adrian Popa-Romania

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$\Omega=\int_0^{\pi}\cot^{-1}(2022^{\cos x})\,dx;\ \ (1)$$

$$\cos(\pi-x)=-\cos x\Rightarrow\Omega=\int_0^{\pi}\cot^{-1}(2022^{-\cos x})\,dx=$$

$$=\int_0^{\pi}\cot^{-1}\left(\frac{1}{2022^{\cos x}}\right)dx=\int_0^{\pi}\tan^{-1}(2022^{\cos x})\,dx;\ \ (2)$$

From (1) and (2):

$$2\Omega=\int_0^{\pi}\left(\cot^{-1}(2022^{-\cos x})+\tan^{-1}(2022^{\cos x})\right)dx=\frac{\pi}{2}\int_0^{\pi}dx=\frac{\pi^2}{2}$$

$$\textit{Therefore,}\qquad \Omega=\frac{\pi^2}{4}$$

Solution 2 by Ankush Kumar Parcha-India

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)=\frac{\pi}{2}$$

$$\Omega=\int_0^{\pi}\cot^{-1}\left(2022^{\cos(\pi-x)}\right)dx=\int_0^{\pi}\left(\frac{\pi}{2}-\tan^{-1}\left(2022^{\cos(\pi-x)}\right)\right)dx=$$

$$=\frac{\pi^2}{2}-\int_0^{\pi}\tan^{-1}\left(\frac{1}{2022^{\cos x}}\right)dx;\quad (1)$$

$$\Omega=\frac{\pi^2}{2}-\int_0^{\pi}\tan^{-1}(2022^{\cos x})\,dx;\quad (2)$$

$$\textit{By adding } (1) \textit{ and } (2), \textit{we get}:$$

$$2\Omega=\pi^2-\left[\int_0^{\pi}\tan^{-1}(2022^{\cos x})\,dx+\int_0^{\pi}\tan^{-1}\left(\frac{1}{2022^{\cos x}}\right)dx\right]=\pi^2-\frac{\pi^2}{2}$$

$$\textit{Therefore,}\qquad \Omega=\frac{\pi^2}{4}$$

Solution 3 by Le Thu-Vietnam

$$\textit{Note}: \tan^{-1}\left(\frac{1}{x}\right)=\cot^{-1}x$$

$$\tan^{-1}x+\cot^{-1}x=\begin{cases}\frac{\pi}{2};\ \textit{if } x>0\\ -\frac{\pi}{2};\ \textit{if } x<0\end{cases}$$

$$\tan^{-1}(2022^{\cos x})+\cot^{-1}(2022^{\cos x})=\frac{\pi}{2},\ \textit{since } 2022^{\cos x}>0$$

$$\Omega=\int_0^{\pi}\cot^{-1}\left(2022^{\cos(\pi-x)}\right)dx$$

$$2\Omega=\int_0^{\pi}\left[\cot^{-1}\left(2022^{\cos(\pi-x)}\right)+\cot^{-1}(2022^{\cos x})\right]dx=$$

$$=\int_0^{\pi}[\tan^{-1}(2022^{\cos x})+\cot^{-1}(2022^{\cos x})]\,dx=\frac{\pi}{2}\int_0^{\pi}dx=\frac{\pi^2}{2}$$

$$\textit{Therefore,}\qquad \Omega=\frac{\pi^2}{4}$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int_0^{\pi}\cot^{-1}\left(2022^{\cos(\pi-x)}\right)dx\overset{y=\pi-x}{=}$$

$$=\frac{1}{2}\int_0^{\pi}\cot^{-1}(2022^{-\cos x})\,dx+\frac{1}{2}\int_0^{\pi}\cot^{-1}(2022^{-\cos y})\,dy=$$

$$=\frac{1}{2}\int_0^{\pi}\cot^{-1}(2022^{-\cos x})\,dx+\frac{1}{2}\int_0^{\pi}\cot^{-1}(2022^{\cos x})\,dx=$$

$$=\frac{1}{2}\int_0^{\pi}(\cot^{-1}(2022^{-\cos x})+\cot^{-1}(2022^{\cos x}))dx=\frac{1}{2}\int_0^{\pi}\frac{\pi}{2}dx=\frac{\pi^2}{4}$$

2043. Prove that:

$$I=\int_0^{\pi}csc(x)\,(2\pi^2x-3\pi x^2+x^3)dx=\frac{21}{2}\pi\zeta(3)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\pi}\frac{2\pi^2x-3\pi x^2+x^3}{\sin x}dx\overset{x\to\pi-x}{=}\int_0^{\pi}\frac{\pi^2x-x^3}{\sin x}dx\overset{IBP}{=}$$

$$=\left[(\pi^2x-x^3)\log\left(tan\left(\frac{x}{2}\right)\right)\right]_0^{\pi}-\int_0^{\pi}(\pi^2-3x^2)\log\left(\tan\left(\frac{x}{2}\right)\right)dx=$$

$$=-\pi^2\int_0^{1}\log\left(\tan\left(\frac{x}{2}\right)\right)dx+3\int_0^{\pi}x^2\log\left(\tan\left(\frac{x}{2}\right)\right)dx$$

$$\int_0^{\pi}\log\left(\tan\left(\frac{x}{2}\right)\right)dx=2\int_0^{\frac{\pi}{2}}\log\left(tan\left(\frac{x}{2}\right)\right)dx=0$$

$$\Omega=3\int_0^{\pi}x^2\log\left(\tan\left(\frac{x}{2}\right)\right)dx=-6\sum_{n=1}^{\infty}\frac{1}{2n-1}\int_0^{\pi}x^2\cos((2n-1)x)\,dx=$$

$$=-6\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[\frac{x^2\sin((2n-1)x)}{2n-1}-\frac{2\sin((2n-1)x)}{(2n-1)^3}+\frac{2x\cos((2n-1)x)}{(2n-1)^2}\right]_0^{\pi}=$$

$$=-12\pi\sum_{n=1}^{\infty}\frac{\cos((2n-1)\pi)}{(2n-1)^3}=12\pi\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}=12\pi\left(\frac{7}{8}\zeta(3)\right)=\frac{21\pi}{2}\zeta(3)$$

$$\textit{Therefore,}\quad\int_0^{\pi}\frac{2\pi^2x-3\pi x^2+x^3}{\sin x}dx=\frac{21\pi}{2}\zeta(3)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^{\pi}\frac{2\pi^2x-3\pi x^2+x^3}{\sin x}dx\overset{IBP}{=}\left[(2\pi^2x-3\pi x^2+x^3)\log\left(tan\left(\frac{x}{2}\right)\right)\right]_0^{\pi}-$$

$$-\int_0^{\pi}(2\pi^2-6\pi x+3x^2)\log\left(tan\left(\frac{x}{2}\right)\right)dx=$$

$$= -\int_0^{\pi} (2\pi^2 - 6\pi x + 3x^2)\log\left(tan\left(\frac{x}{2}\right)\right)dx$$

$$We\ know\ that:\ 12\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^2} = 2\pi^2 - 6\pi x + 3x^2$$

$$\Omega = -12\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\pi}\cos(kx)\log\left(\tan\left(\frac{x}{2}\right)\right)dx = -24\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\tan x)\,dx$$

$$= -24\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\int_0^{\frac{\pi}{2}}cos(2kx)\log(\sin x)\,dx - \int_0^{\frac{\pi}{2}}cos(2kx)\log(cos\,x)\,dx\right] =$$

$$= -24\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\int_0^{\frac{\pi}{2}}\cos(2kx)\log(\sin x)\,dx - (-1)^k\int_0^{\frac{\pi}{2}}\cos(2k\pi)\log(sin\,x)\,dx\right] =$$

$$= -24\sum_{k=1}^{\infty}\frac{1+(-1)^{k+1}}{k^2}\int_0^{\frac{\pi}{2}}\cos(2k\pi)\log(\sin x)\,dx =$$

$$= -24\sum_{k=1}^{\infty}\frac{1+(-1)^{k+1}}{k^2}\left[\frac{\sin(2kx)}{2k}\log(\sin x)\Big|_0^{\frac{\pi}{2}} - \frac{1}{2k}\int_0^{\frac{\pi}{2}}\sin(2kx)\cot x\,dx\right] =$$

$$= 12\sum_{k=1}^{\infty}\frac{1+(-1)^{k+1}}{k^3}\int_0^{\frac{\pi}{2}}\sin(2kx)\cot x\,dx = 6\pi\sum_{k=1}^{\infty}\left[\frac{1}{k^3}+(-1)^{k+1}\right] = \frac{21\pi}{2}\zeta(3)$$

$$Note: I_k = \int_0^{\frac{\pi}{2}}\sin(2kx)\cot x\,dx = \frac{\pi}{2}$$

Solution 3 by Pham Duc Nam-Vietnam

$$I = \int_0^{\pi} csc(x)\left(2\pi^2 x - 3\pi x^2 + x^3\right)dx = \frac{21}{2}\pi\zeta(3)$$

$$*\ 2\pi^2 x - 3\pi x^2 + x^3 = x(\pi - x)(2\pi - x) \Rightarrow I = \int_0^{\pi} csc(x)\,x(\pi - x)(2\pi - x)dx$$

$$*\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \Rightarrow I$$

$$= \int_0^{\pi} csc(\pi - x)\,(\pi - x)\big(\pi - (\pi - x)\big)\big(2\pi - (\pi - x)\big)dx$$

$$= \int_0^{\pi} csc(x)\,x(\pi - x)(\pi + x)dx$$

$$\Rightarrow 2I=\int_0^{\pi} csc(x)\,x(\pi-x)(\pi+x)dx+\int_0^{\pi} csc(x)\,x(\pi-x)(2\pi-x)dx$$

$$=3\pi\int_0^{\pi} csc(x)\,x(\pi-x)dx$$

$$*\begin{cases}u=x(\pi-x)\\ dv=csc(x)\,dx\end{cases}\Rightarrow\begin{cases}du=-2x+\pi\\ v=logtan\frac{x}{2}\end{cases}\Rightarrow 2I$$

$$=3\pi\left(\underbrace{x(\pi-x)\,logtan\frac{x}{2}\Big|_0^{\pi}}_{=0}-\int_0^{\pi} logtan\frac{x}{2}(-2x+\pi)dx\right)$$

$$*\ \pi\int_0^{\pi} logtan\frac{x}{2}dx\xrightarrow{t=\frac{x}{2}}2\pi\int_0^{\frac{\pi}{2}} logtan\,t\,dt=0$$

$$*\ 2\int_0^{\pi} x\,logtan\frac{x}{2}dx\xrightarrow{t=\frac{x}{2}}8\int_0^{\frac{\pi}{2}} t\,logtan\,t\,dt=8\int_0^{\frac{\pi}{2}} t(logsin\,t-logcos\,t)\,dt$$

$$=8\int_0^{\frac{\pi}{2}} t\left(-log\,2-\sum_{k=1}^{+\infty}\frac{cos(2kt)}{k}+log\,2+\sum_{k=1}^{+\infty}(-1)^k\frac{cos(2kt)}{k}\right)dt$$

$$=8\left(-\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{2}} t\frac{cos(2kt)}{k}dt+\sum_{k=1}^{+\infty}\int_0^{\frac{\pi}{2}}(-1)^k t\frac{cos(2kt)}{k}dt\right)$$

$$=8\left(-\sum_{k=1}^{+\infty}\left(\frac{\pi}{4}.\frac{sin(\pi k)}{k^2}+\frac{cos(\pi k)}{4k^3}-\frac{1}{4k^3}\right)\right.$$

$$\left.+\sum_{k=1}^{+\infty}(-1)^k\left(\frac{\pi}{4}.\frac{sin(\pi k)}{k^2}+\frac{cos(\pi k)}{4k^3}-\frac{1}{4k^3}\right)\right)$$

$$=8\left(\frac{3}{16}\zeta(3)+\frac{1}{4}\zeta(3)+\frac{1}{4}\zeta(3)+\frac{3}{16}\zeta(3)\right)=8.\frac{7}{8}\zeta(3)=7\zeta(3)$$

$$\Rightarrow 2I=3\pi(7\zeta(3))=21\pi\zeta(3)\Rightarrow I=\boxed{\frac{21}{2}\pi\zeta(3)}$$

Solution 4 by Max Wong –Hong Kong

$$\Omega=\int_0^{\pi}\csc x\,(2\pi^2x-3\pi x^2+x^3)\,dx=$$

$$=\int_0^{\pi}\csc(\pi-x)\,(2\pi^2(\pi-x)-3\pi(\pi-x)^2+(\pi-x)^2)\,dx=$$

$$=\int_0^{\pi}\csc x\,(\pi^2x-x^3)\,dx$$

$$2\Omega=\int_0^{\pi}\csc x\left((2\pi^2x-3\pi x^2-x^3)+(\pi^2x-x^3)\right)dx$$

$$\Omega=\frac{3}{2}\int_0^{\pi}\frac{\pi^2x-\pi x^2}{\sin x}dx=\frac{3\pi}{2}\int_0^{\pi}\frac{x(\pi-x)}{\sin x}dx=$$

$$=3\pi i\int_0^{\pi}\frac{x(\pi-x)}{e^{ix}-e^{-ix}}dx=3\pi i\int_0^{\pi}x(\pi-x)\sum_{k=0}^{\infty}e^{-ix}(e^{-2ix})^k\,dx$$

$$\textit{Note that }\int_0^{\pi}x(\pi-x)\left|\sum_{k=0}^{\infty}e^{-ix}(e^{-2ix})^k\right|dx\ \textit{ exists, then by Fubini's theorem:}$$

$$\Omega=3\pi i\sum_{k=0}^{\infty}x(\pi-x)e^{-(2k+1)ix}\,dx=$$

$$=-\int_0^{\pi}\frac{e^{-(2k+1)ix}}{-(2k+1)i}\cdot(\pi-2x)dx=$$

$$=\frac{1}{(2k+1)i}\left((\pi-2x)(\frac{e^{-(2k+1)ix}}{-(2k+1)i}\bigg|_0^{\pi}-\int_0^{\pi}\frac{e^{-2(2k+1)ix}}{-(2k+1)i}(-2dx)\right)=$$

$$=\frac{1}{(2k+1)i}\left(-\pi(-1)-\pi(1)\right)+\frac{2}{(2k+1)^2}\frac{e^{-(2k+1)ix}}{-(2k+1)i}\bigg|_0^{\pi}=\frac{4}{(2k+1)^3i}$$

$$\Omega=12\pi\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}=12\pi\left(\frac{7}{8}\right)\zeta(3)=\frac{21\pi}{3}\zeta(3)$$

2044. Prove that:

$$\int_0^{\pi}\csc(x)\cosh(a\cos x)\sin(a\sin x)\,dx=\pi\sinh(a)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$\Omega=\int_0^{\pi}\csc(x)\cosh(a\cos x)\sin(a\sin x)\,dx=\int_0^{\pi}\frac{\cos(ia\cos x)\cdot\sin(a\sin x)}{\sin x}dx=$$

$$=\frac{1}{2}\int_0^{\pi}\frac{1}{\sin x}\left(\sin\left(a(\sin x+i\cos x)\right)+\sin\left(a(\sin x-i\cos x)\right)\right)dx=$$

$$=\frac{1}{2}\int_0^{\pi}\frac{\sin\left(a(\sin x+i\cos x)\right)}{\sin x}dx+\frac{1}{2}\int_0^{\pi}\frac{\sin\left(a(\sin x-i\cos x)\right)}{\sin x}dx=$$

$$=I_1+I_2$$

$$I_2=\frac{1}{2}\int_0^{\pi}\frac{\sin\left(a(\sin x-i\cos x)\right)}{\sin x}dx\overset{x\to\pi-x}{=}\frac{1}{2}\int_0^{\pi}\frac{\sin\left(a(\sin x+i\cos x)\right)}{\sin x}dx=I_1$$

$$\Omega=\int_0^{\pi}\frac{\sin\big(a(\sin x+i\cos x)\big)}{\sin x}dx\overset{x\to\pi+x}{=}\int_{\pi}^{2\pi}\frac{\sin(a(\sin(\pi+x)+i\cos(\pi+x))}{\sin(\pi+x)}dx=$$

$$=\int_{\pi}^{2\pi}\frac{\sin\big(a(\sin x+i\cos x)\big)}{\sin x}dx$$

$$2\Omega=\int_0^{2\pi}\frac{\sin\big(a(\sin x+i\cos x)\big)}{\sin x}dx$$

$$\textit{Let } z=\sin x+i\cos x\,, x\in[0,2\pi]\subset C, dz=-iz\,dx, \textit{then}$$

$$dx=\frac{dz}{iz}\ \textit{and}\ \sin x=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$2\Omega=\oint_C\frac{\sin(az)}{\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}=2\oint_C\frac{\sin(az)}{i(z^2+1)}dz$$

$$\Omega=\oint_C\frac{\sin(az)}{i(z^2+1)}dz=\pi i\left(Res_{z=i}\left(\frac{\sin(az)}{i(z^2+1)}\right)+Res_{z=-i}\left(\frac{\sin(az)}{i(z^2+1)}\right)\right)=$$

$$=\pi i\left(\frac{\sin(ia)}{2i\cdot i}+\frac{\sin(-ia)}{-2i\cdot i}\right)=\pi\left(\frac{\sinh(a)}{2}+\frac{\sinh(a)}{2}\right)=\pi\sinh(a)$$

2045. Prove that:

$$\int_0^{\pi}x\arctan\left(\frac{\sin x}{2+\cos x}\right)dx=\frac{\pi^3}{12}-\frac{\pi}{2}\log^2(2)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Bui Hong Suc-Vietnam

$$\Omega=\int_0^{\pi}x\,arctan\left(\frac{\sin x}{2+\cos x}\right)dx=\frac{1}{2}\int_0^{\pi}arctan\left(\frac{\sin x}{2+\cos x}\right)d(x^2)=$$

$$=\left[\frac{1}{2}x^2\arctan\left(\frac{\sin x}{2+\cos x}\right)\right]_0^{\pi}-\frac{1}{2}\int_0^{\pi}x^2\cdot\frac{1+2\cos x}{5+4\cos x}dx=$$

$$=-\frac{1}{4}\int_0^{\pi}x^2\cdot\frac{5+4\cos x-3}{5+4\cos x}dx=-\frac{1}{4}\int_0^{\pi}x^2\,dx+\frac{3}{4}\int_0^{\pi}\frac{x^2}{5+4\cos x}dx=$$

$$=-\frac{\pi^3}{12}+\frac{3}{4}I_1, \textit{where } I_1=\int_0^{\pi}\frac{x^2}{5+4\cos x}dx$$

$$Using\ the\ identity: \frac{a^2-b^2}{a^2+b^2-2ab\cos x}=1+2\sum_{n=1}^{\infty}\left(\frac{b}{a}\right)^n\cos(nx)\,;|b|<a$$

$$for\ a=2,b=-1\ and \frac{3}{5+4\cos x}=1+2\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\cos(nx)$$

$$I_1=\frac{1}{3}\int_0^{\pi}\frac{3x^2}{5+4\cos x}dx=\frac{1}{3}\int_0^{\pi}\left[x^2+2\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n x^2\cos(nx)\right]dx=$$

$$=\frac{\pi^3}{9}+\frac{2}{3}\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\int_0^{\pi}x^2\cos(nx)\,dx=$$

$$=\frac{\pi^3}{9}+\frac{2}{3}\cdot 2\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}\cdot\pi=\frac{\pi^3}{9}+\frac{4}{3}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n^2}\cdot\pi=\frac{\pi^3}{9}+\frac{4\pi}{3}\cdot Li_2\left(\frac{1}{2}\right)$$

$$\Omega=-\frac{\pi^3}{12}+\frac{3}{4}\left(\frac{\pi^3}{9}+\frac{4\pi}{3}Li_2\left(\frac{1}{2}\right)\right)=\pi Li_2\left(\frac{1}{2}\right)=\frac{\pi^3}{12}-\frac{\pi^2}{2}\log^2(2)$$

Solution 2 by Le Thu-Vietnam

$$Define: \Omega(a)=\int_0^{\pi}x\arctan\left(\frac{\sin x}{a+\cos x}\right)dx\,;\Omega(1)=\frac{\pi^3}{6},\Omega(2)=\Omega$$

$$\Omega'(a)=\int_0^{\pi}-\frac{x\sin x}{a^2+2a\cos x+1}dx=\frac{1}{a}\int_0^{\pi}x\frac{-a\sin x}{a^2+2a\cos x+1}dx$$

$$Recall: \frac{cd\sin x}{d^2-2cd\cos x+c^2}=\sum_{n=1}^{\infty}\left(\frac{c}{d}\right)^n\sin(nx)$$

$$Proof: Consider\ \Psi=\sum_{n=1}^{\infty}\left(\frac{c}{d}e^{ix}\right)^n=\frac{ce^{ix}}{d-ce^{ix}}$$

$$Geometric\ series\ where\ |c|<|d|\ and\ c,d\in\mathbb{R}-\{0\}$$

$$Taking\ the\ imaginary\ part\ of\ \Psi, we\ will\ get\ the\ desired\ result.$$

$$Put\ c=-1\ and\ d=a, we\ obtain:$$

$$\int_0^{\pi}x\frac{-a\sin x}{a^2+2a\cos x+1}dx=\sum_{n=1}^{\infty}\left[\left(-\frac{1}{a}\right)^n\int_0^{\pi}x\sin(nx)\,dx\right]=$$

$$=\sum_{n=1}^{\infty}\left[\left(-\frac{1}{a}\right)^n\cdot\frac{\sin(nx)-nx\cos(nx)}{n^2}\right]_0^{\pi}=-\pi\sum_{n=1}^{\infty}\frac{\left(\frac{1}{a}\right)^n}{n}=\pi\log\left(1-\frac{1}{a}\right)$$

Since: $\sin(n\pi)=0;\ (\forall)n\in\mathbb{N}$ *and* $|a|>1$. *Hence*:

$$\Omega(a)=\pi\int\frac{\log\left(1-\frac{1}{a}\right)}{a}da=\pi\log\left|\frac{a-1}{a}\right|\log|a|-\pi Li_2\left(\frac{a-1}{a}\right)+C$$

$$Put\ a=1:\frac{\pi^3}{6}=\pi\cdot 0-\pi Li_2(0)+C\Rightarrow C=\frac{\pi^3}{6}.\ Hence,$$

$$\Omega=\Omega(2)=\pi\log\left(\frac{1}{2}\right)\log(2)-\pi Li_2\left(\frac{1}{2}\right)+\frac{\pi^3}{6}=\frac{\pi^3}{12}-\frac{\pi\log^2(2)}{2}$$

$$Note: Li_2(0)=0\ and\ Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{\log^2(2)}{2}$$

Solution 3 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\pi}x\arctan\left(\frac{\sin x}{2+\cos x}\right)dx\overset{IBP}{=}\left[\frac{x^2}{2}\arctan\left(\frac{\sin x}{2+\cos x}\right)\right]_0^{\pi}-\frac{1}{2}\int_0^{\pi}x^2\frac{1+2\cos x}{5+4\cos x}dx$$

$$=-\frac{1}{2}\int_0^{\pi}x^2\frac{1+2\cos x}{5+4\cos x}dx=-\frac{1}{4}\int_0^{\pi}x^2\,dx+\frac{3}{4}\int_0^{\pi}\frac{x^2}{5+4\cos x}dx$$

$$=-\frac{\pi^3}{12}+\frac{3}{4}\int_0^{\pi}\frac{x^2}{5+4\cos x}dx$$

$$I_1=\int_0^{\pi}\frac{x^2}{5+4\cos x}dx=\int_0^{\pi}\frac{(\pi-x)^2}{5+4\cos(\pi-x)}dx$$

$$=\pi^2\int_0^{\pi}\frac{dx}{5-4\cos x}-2\pi\int_0^{\pi}\frac{x}{5-4\cos x}dx+\int_0^{\pi}\frac{x^2}{5-4\cos x}dx$$

$$=\frac{\pi^3}{3}-2\pi\left(\frac{\pi^2}{6}+\frac{2}{3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(\frac{1}{2}\right)^k-\frac{2}{3}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{1}{2}\right)^k\right)$$

$$+\left(\frac{\pi^3}{9}+\frac{4\pi}{3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(\frac{1}{2}\right)^k\right)=\frac{\pi^3}{9}+\frac{4\pi}{3}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{1}{2}\right)^k=\frac{\pi^3}{9}+\frac{4\pi}{3}Li_2\left(\frac{1}{2}\right)$$

$$=\frac{2\pi^3}{9}-\frac{2\pi}{3}\log^2(2)$$

$$I=-\frac{\pi^3}{12}+\frac{3}{4}I_1=\frac{\pi^3}{12}-\frac{\pi}{2}\log^2(2)$$

We know that

$$\text{We know that } 1+2\sum_{k=1}^{\infty} p^k\cos(kx)=\frac{1-p^2}{1-2p\cos x+p^2} \textit{ then} \ldots$$

$$\frac{1}{5-4\cos x}=\frac{1}{3}+\frac{2}{3}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\cos(kx)$$

$$\textit{Note} \int_0^{\pi}\frac{dx}{5-4\cos x}=\frac{\pi}{3}$$

$$\int_0^{\pi}\frac{x}{5-4\cos x}dx=\frac{\pi^2}{6}+\frac{2}{3}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\int_0^{\pi}x\cos(kx)\,dx=\frac{\pi^2}{6}+\frac{2}{3}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\left[\frac{(-1)^k}{k^2}-\frac{1}{k^2}\right]$$

$$=\frac{\pi^2}{6}+\frac{2}{3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(\frac{1}{2}\right)^k-\frac{2}{3}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{1}{2}\right)^k$$

$$\int_0^{\pi}\frac{x^2}{5-4\cos x}dx=\frac{\pi^3}{9}+\frac{2}{3}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\int_0^{\pi}x^2\cos(kx)\,dx=\frac{\pi^3}{9}+\frac{4\pi}{3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(\frac{1}{2}\right)^k$$

2046. Prove that:

$$I=\int_0^1\frac{\log(x)\log(-\log(x))}{1-x}dx=\zeta(2)(12\log(A)-\log(2\pi)-1)$$

A-Glaisher Kinkelin constant

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Pham Duc Nam-Vietnam

$$*\, t=-\log(x)\Rightarrow x=e^{-t}\Rightarrow dx=-e^{-t}dt\Rightarrow I=\int_0^{+\infty}\frac{-t\log(t)}{1-e^{-t}}e^{-t}dt$$

$$=\int_0^{+\infty}-t\log(t)\sum_{k=0}^{+\infty}e^{-t(k+1)}\,dt=-\sum_{k=0}^{+\infty}\int_0^{+\infty}t\log(t)\,e^{-t(k+1)}\,dt$$

$$=-\sum_{k=0}^{+\infty}\int_0^{+\infty}t\log(t)\,e^{-\varphi t}\,dt\ (\varphi=k+1)$$

$$*\, K=\int_0^{+\infty}t\log(t)\,e^{-\varphi t}\,dt=\int_0^{+\infty}te^{-\varphi t}\left(\int_0^{+\infty}\frac{e^{-u}-e^{-tu}}{u}du\right)dt$$

$$=\int_0^{+\infty}\frac{1}{u}\left(\int_0^{+\infty}\left(e^{-u-\varphi t}-e^{-t(u+\varphi)}\right)tdt\right)du$$

$$=\int_0^{+\infty}\frac{1}{u}\left(\frac{e^{-u}}{\varphi^2}-\frac{1}{(u+\varphi)^2}\right)du=\frac{1}{\varphi^2}\int_\varepsilon^{+\infty}\frac{e^{-u}}{u}du-\int_\varepsilon^{+\infty}\frac{1}{u(u+\varphi)^2}du\ (We\ fix\ \varepsilon>0)$$

$$*\ \frac{1}{\varphi^2}\int_\varepsilon^{+\infty}\frac{e^{-u}}{u}du=\frac{1}{\varphi^2}\left(e^{-u}\log(u)\Big|_\varepsilon^{+\infty}+\int_\varepsilon^{+\infty}e^{-u}\log(u)\,du\right)$$

$$=\frac{1}{\varphi^2}\left(-e^{-\varepsilon}\log(\varepsilon)+\int_\varepsilon^{+\infty}e^{-u}\log(u)\,du\right)$$

$$*\int_\varepsilon^{+\infty}\frac{1}{u(u+\varphi)^2}du=\frac{\log\left(\frac{u}{u+\varphi}\right)}{\varphi^2}+\frac{1}{\varphi(u+\varphi)}\Bigg|_\varepsilon^{+\infty}=-\frac{\log\frac{\varepsilon}{\varepsilon+\varphi}}{\varphi^2}-\frac{1}{\varphi(\varepsilon+\varphi)}$$

$$\Rightarrow\int_\varepsilon^{+\infty}\frac{1}{u}\left(\frac{e^{-u}}{\varphi^2}-\frac{1}{(u+\varphi)^2}\right)du$$

$$=\frac{1}{\varphi^2}\left(-e^{-\varepsilon}\log(\varepsilon)+\int_\varepsilon^{+\infty}e^{-u}\log(u)\,du\right)+\frac{\log\frac{\varepsilon}{\varepsilon+\varphi}}{\varphi^2}+\frac{1}{\varphi(\varepsilon+\varphi)}$$

$$=\frac{-e^{-\varepsilon}\log(\varepsilon)+\log\frac{\varepsilon}{\varepsilon+\varphi}}{\varphi^2}+\frac{1}{\varphi^2}\int_\varepsilon^{+\infty}e^{-u}\log(u)\,du+\frac{1}{\varphi(\varepsilon+\varphi)}$$

$$\Rightarrow Let:\varepsilon\to 0^+\Rightarrow K=\frac{-\log(\varphi)-\gamma+1}{\varphi^2}\Rightarrow I=\sum_{k=0}^{+\infty}\frac{\log(k+1)+(\gamma-1)}{(k+1)^2}$$

$$=\frac{\pi^2}{6}(\gamma-1)+\sum_{k=0}^{+\infty}\frac{\log(k+1)}{(k+1)^2}=\frac{\pi^2}{6}(\gamma-1)+\sum_{n=2}^{+\infty}\frac{\log(n)}{n^2}=\frac{\pi^2}{6}(\gamma-1)$$

$$+\frac{\pi^2}{6}(12\log A-\gamma-\log(2\pi))$$

$$=\frac{\pi^2}{6}(12\log A-\log(2\pi)-1)=\boxed{\zeta(2)(12\log(A)-\log(2\pi)-1)}$$

NOTES:

$$*\ \log(x)=\int_0^{+\infty}\frac{e^{-u}-e^{-xu}}{u}du\ \ \text{(Frullani integral)}$$

$$*\int_0^{+\infty}e^{-x}\log(x)\,dx=-\gamma$$

$$*\sum_{n=2}^{+\infty}\frac{\log(n)}{n^2}=\frac{\pi^2}{6}(12\log A-\gamma-\log(2\pi))$$

$$* \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1 \frac{\log(x)\log(-\log(x))}{1-x} dx \overset{x=e^{-t}}{=} -\int_0^\infty \frac{t\log(t)}{1-e^{-t}} e^{-t}\, dt =$$

$$= -\sum_{n=1}^\infty t\log(t)\, e^{-nt}\, dt = -\sum_{n=1}^\infty L\{t\log(t)\}(n)$$

$$L\{t\log(t)\}(n) = -\frac{d}{dn} L\{\log(t)\}(n) = \frac{d}{dn}\left(\frac{\gamma+\log(n)}{n}\right) = \frac{1-\gamma-\log(n)}{n^2}$$

$$\Omega = \sum_{n=1}^\infty \frac{\gamma-1+\log(n)}{n^2} = (\gamma-1)\sum_{n=1}^\infty \frac{1}{n^2} + \sum_{n=1}^\infty \frac{\log(n)}{n^2} =$$

$$= (\gamma-1)\zeta(2) + \sum_{n=2}^\infty \frac{\log(n)}{n^2}$$

$$\sum_{n=2}^\infty \frac{\log(n)}{n^2} = -\lim_{s\to 2} \frac{d}{ds} \sum_{n=2}^\infty \frac{1}{n^s} = -\zeta'(2)$$

$$\zeta'(2) = (\gamma + \log(2\pi) - 12\log(A))\zeta(2)$$

$$= (\gamma-1)\zeta(2) - (\gamma + \log(2\pi) - 12\log(A))\zeta(2)$$

$$\Omega = \int_0^1 \frac{\log(x)\log(-\log(x))}{1-x} dx = \zeta(2)(12\log(A) - \log(2\pi) - 1)$$

Solution 3 by Ngulmun George Baite-India

$$\Omega = \int_0^1 \frac{\log(x)\log(-\log(x))}{1-x} dx = \sum_{n=0}^\infty \int_0^1 x^n \log(x)\log(-log(x))\, dx \overset{-\log(x)=u}{=}$$

$$= \sum_{n=0}^\infty \int_0^\infty e^{-nu}\log(e^{-u})\log(u)\, e^{-u} du = -\sum_{n=0}^\infty \int_0^\infty u e^{-(n+1)u} \log(u)\, du \overset{(n+1)u=x}{=}$$

$$= -\sum_{n=0}^\infty \int_0^\infty \frac{x}{n+1} e^{-x} \log\left(\frac{x}{n+1}\right) \frac{dx}{n+1} =$$

$$= -\sum_{n=0}^\infty \frac{1}{(n+1)^2}\left[\int_0^\infty x e^{-x}\log(x)\, dx - \log(n+1)\int_0^\infty x e^{-x} dx\right] =$$

$$= -\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}[\Gamma(2)\psi(2) - \log(n+1)\,\Gamma(2)] =$$

$$= -\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}[1-\gamma-\log(n+1)] =$$

$$= -\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} + \gamma\sum_{n=0}^{\infty}\frac{1}{(n+1)^2} + \sum_{n=0}^{\infty}\frac{\log(n+1)}{(n+1)^2} =$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n^2} + \gamma\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{\log(n)}{n^2} =$$

$$= -\zeta(2) + \gamma\zeta(2) - \zeta'(2) = \zeta(2)(12\log(A) - \log(2\pi) - 1)$$

$$\Omega = \int_0^1 \frac{\log(x)\log(-\log(x))}{1-x}dx = \zeta(2)(12\log(A) - \log(2\pi) - 1)$$

Solution 4 by Le Thu-Vietnam

$$\Omega = \int_0^1 \frac{\log(x)\log(-\log(x))}{1-x}dx \overset{u=-\log(x)}{=} \int_0^{\infty} -\frac{u\log(u)}{1-e^{-u}}e^{-u}\,du =$$

$$= -\int_0^{\infty}\frac{u\log(u)}{e^u-1}du$$

$$\textit{Recall}: \Gamma(z)\zeta(z) = \int_0^{\infty}\frac{t^{z-1}}{e^t-1}dt;\ (\forall)\Re(z) > 1$$

$$\Omega = -\int_0^{\infty}\lim_{s\to 0_+}\frac{d}{ds}\left(\frac{u^{s+1}}{e^u-1}\right)du = -\lim_{s\to 0_+}\frac{d}{ds}\left(\int_0^{\infty}\frac{u^{s+1}}{e^u-1}du\right) =$$

$$= -\lim_{s\to 0_+}\frac{d}{ds}[\zeta(s+2)\Gamma(s+2)] =$$

$$= \lim_{s\to 0_+}[\Gamma'(s+2)\zeta(s+2) + \zeta'(s+2)\Gamma(s+2)] =$$

$$= -\lim_{s\to 0_+}\Gamma(s+2)[\zeta'(s+2) + \zeta(s+2)\psi_0(s+2)] =$$

$$= -\Gamma(2)\zeta'(2) - \Gamma(2)\zeta(2)\psi_0(2)$$

$$\Omega = \int_0^1 \frac{\log(x)\log(-\log(x))}{1-x}dx = \zeta(2)(12\log(A) - \log(2\pi) - 1)$$

$$\textit{Note}: \frac{d\Gamma(z)}{dz} = \Gamma(z)\psi_0(z);\ \psi_0(2) = 1-\gamma$$

Solution 5 by Sakthi Vel-India

$$\Omega=\int_0^1\frac{\log(x)\log(-\log(x))}{1-x}dx\overset{-\log(x)=y}{=}-\sum_{n=0}^{\infty}\int_0^{\infty}e^{-y(n+1)}\log(y)\,ydy\overset{y(n+1)=z}{=}$$

$$=-\sum_{n=0}^{\infty}\int_0^{\infty}e^{-z}\log\left(\frac{z}{n+1}\right)\frac{z}{n+1}\frac{dz}{n+1}=$$

$$=-\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\int_0^{\infty}e^{-z}\log(z)\,zdz-\int_0^{\infty}e^{-z}\log(n+1)\,zdz=$$

$$=-\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\left[\psi^{(0)}(2)-\log(n+1)\,\Gamma(2)\right]=$$

$$=-\sum_{n=1}^{\infty}\frac{1}{n^2}[1-\gamma-\log(2n)]=-\zeta(2)(1-\gamma)+\zeta'(2)=$$

$$=\frac{\pi^2}{6}(\gamma-1+12\log(A)-\gamma-\log(2\pi))=\frac{\pi^2}{6}(12\log(A)-\gamma\log(2\pi)-1)$$

2047.

$$S=\sum_{k=1}^{+\infty}\frac{\cos\frac{2k\pi}{3}}{k^3}=-\frac{4}{9}\zeta(3)$$

Proposed by Le Thu-Vietnam

Solution 1 by Pham Duc Nam-Vietnam

$$S=\sum_{k=1}^{+\infty}\frac{\cos\frac{2k\pi}{3}}{k^3}=-\frac{4}{9}\zeta(3)$$

$$*\cos\frac{2k\pi}{3}=\begin{cases}1, if\ k\equiv 0\ mod\ 3\\ -\frac{1}{2}, every\ k\ else\end{cases}$$

$$\Rightarrow S=-\frac{1}{2}\cdot\frac{1}{1^3}-\frac{1}{2}\cdot\frac{1}{2^3}+1.\frac{1}{3^3}-\frac{1}{2}\cdot\frac{1}{4^3}-\frac{1}{2}\cdot\frac{1}{5^3}+1.\frac{1}{6^3}-\dots$$

$$=\frac{1}{1^3}-\frac{3}{2}\cdot\frac{1}{1^3}+\frac{1}{2^3}-\frac{3}{2}\cdot\frac{1}{2^3}+1.\frac{1}{3^3}+\frac{1}{4^3}-\frac{3}{2}\cdot\frac{1}{4^3}+\frac{1}{5^3}-\frac{3}{2}\cdot\frac{1}{5^3}+1.\frac{1}{6^3}+\dots$$

$$=\zeta(3)-\frac{3}{2}\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{4^3}+\frac{1}{5^3}+\dots\right)$$

$$=\zeta(3)-\frac{3}{2}\left(\zeta(3)-\sum_{j=1}^{+\infty}\frac{1}{(3j)^3}\right)=\zeta(3)-\frac{3}{2}\left(\zeta(3)-\frac{1}{27}\sum_{j=1}^{+\infty}\frac{1}{j^3}\right)=\zeta(3)-\frac{3}{2}\left(\zeta(3)-\frac{1}{27}\zeta(3)\right)$$

$$=\zeta(3)-\frac{3}{2}\cdot\frac{26}{27}\zeta(3)=\zeta(3)-\frac{13}{9}\zeta(3)=-\frac{4}{9}\zeta(3)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\sum_{k=1}^{\infty}\frac{\cos\left(\frac{2\pi k}{3}\right)}{k^3}=\sum_{k=1}^{\infty}\frac{\cos\left(\frac{2\pi(3k)}{3}\right)}{(3k)^3}+\sum_{k=1}^{\infty}\frac{\cos\left(\frac{2\pi(3k-1)}{3}\right)}{(3k-1)^3}+\sum_{k=1}^{\infty}\frac{\cos\left(\frac{2\pi(3k-2)}{3}\right)}{(3k-2)^3}$$

$$=\frac{1}{27}\sum_{k=1}^{\infty}\frac{\cos(2\pi k)}{k^3}+\sum_{k=1}^{\infty}\frac{\cos\left(2\pi k-\frac{2\pi}{3}\right)}{(3k-1)^3}+\sum_{k=1}^{\infty}\frac{\cos\left(2\pi k-\frac{4\pi}{3}\right)}{(3k-2)^3}$$

$$=\frac{1}{27}\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{k=1}^{\infty}\frac{\cos\left(2\pi k+\frac{\pi}{3}\right)}{(3k-1)^3}-\sum_{k=1}^{\infty}\frac{\cos\left(2\pi k-\frac{\pi}{3}\right)}{(3k-2)^3}$$

$$=\frac{1}{27}\sum_{k=1}^{\infty}\frac{1}{k^3}-\cos\left(\frac{\pi}{3}\right)\sum_{k=1}^{\infty}\frac{1}{(3k-1)^3}-\cos\left(\frac{\pi}{3}\right)\sum_{k=1}^{\infty}\frac{1}{(3k-2)^3}$$

$$=\frac{1}{27}\sum_{k=1}^{\infty}\frac{1}{k^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k-1)^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k-2)^3}$$

$$=\frac{1}{27}\sum_{k=1}^{\infty}\frac{1}{k^3}-\frac{1}{2}\left[\sum_{k=1}^{\infty}\frac{1}{(3k)^3}+\sum_{k=1}^{\infty}\frac{1}{(3k-1)^3}+\sum_{k=1}^{\infty}\frac{1}{(3k-2)^3}\right]+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k)^3}$$

$$=\frac{1}{27}\sum_{k=1}^{\infty}\frac{1}{k^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^3}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k)^3}=-\frac{4}{9}\sum_{k=1}^{\infty}\frac{1}{k^3}=-\frac{4}{9}\zeta(3)$$

Solution 3 by Asmat Qatea-Afghanistan

$$S=\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\left(\frac{2n\pi}{3}\right)=\Re\sum_{n=1}^{\infty}\frac{1}{n^3}e^{\frac{2n\pi i}{3}}=$$

$$=\Re\sum_{n=1}^{\infty}\left(\frac{1}{(3n)^2}e^{\frac{6n\pi i}{3}}+\frac{1}{(3n-1)^3}e^{\frac{2(3n-1)\pi i}{3}}+\frac{1}{(3n-2)^3}e^{\frac{2(3n-2)\pi i}{3}}\right)=$$

$$=\Re\sum_{n=1}^{\infty}\left(\frac{1}{27n^3}+\frac{1}{(3n-1)^3}e^{\frac{2\pi i}{3}}+\frac{1}{(3n-2)^3}e^{-\frac{4\pi i}{3}}\right)=$$

$$=\frac{1}{27}\zeta(3)-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(3n-1)^3}+\frac{1}{(3n-2)^3}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\sum_{n=1}^{\infty}\frac{1}{(3n)^3}+\sum_{n=1}^{\infty}\frac{1}{(3n-1)^3}+\sum_{n=1}^{\infty}\frac{1}{(3n-2)^3}$$

$$S=\frac{1}{27}\zeta(3)-\frac{1}{2}\left(\zeta(3)-\frac{1}{27}\zeta(3)\right)=-\frac{4}{9}\zeta(3)$$

Solution 4 by Ankush Kumar Parcha-India

$$A=\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\left(\frac{2n\pi}{3}\right)$$

$$=\sum_{k=1}^{\infty}\frac{\cos\left(\frac{2\pi(3k-2)}{3}\right)}{(3k-2)^3}+\sum_{k=1}^{\infty}\frac{\cos\left(\frac{2\pi(3k-1)}{3}\right)}{(3k-1)^3}+\sum_{k=1}^{\infty}\frac{\cos\left(\frac{6k\pi}{3}\right)}{(3k)^3}=$$

$$=\sum_{k=1}^{\infty}\frac{\cos(2k\pi)\cos\left(\frac{4\pi}{3}\right)}{(3k-2)^3}+\sum_{k=1}^{\infty}\frac{\cos(2k\pi)\cos\left(\frac{2\pi}{3}\right)}{(3k-1)^3}+\sum_{k=1}^{\infty}\frac{\cos(2k\pi)}{(3k)^3}=$$

$$=-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k-2)^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k-1)^3}+\frac{1}{27}\sum_{k=1}^{\infty}\frac{1}{k^3}$$

$$Therefore,\qquad A=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{3}\right)}{n^3}=-\frac{4}{9}\zeta(3)$$

Solution 5 by Syed Shahabudeen-India

$$S=\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\left(\frac{2n\pi}{3}\right)$$

$$The\ cube\ of\ the\ unity\ are\ 1,\omega,\omega^2, where\ \omega=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$$

$$\omega^2=-\frac{1}{2}-\frac{i\sqrt{3}}{2}, then\ S=\Re\sum_{n=1}^{\infty}\frac{\omega^n}{n^3}$$

$$S=\sum_{k=1}^{\infty}\frac{1}{(3k)^3}+\Re\left(\omega^2\sum_{k=1}^{\infty}\frac{1}{(3k-1)^3}\right)+\Re\left(\omega\sum_{k=1}^{\infty}\frac{1}{(3k-2)^3}\right)$$

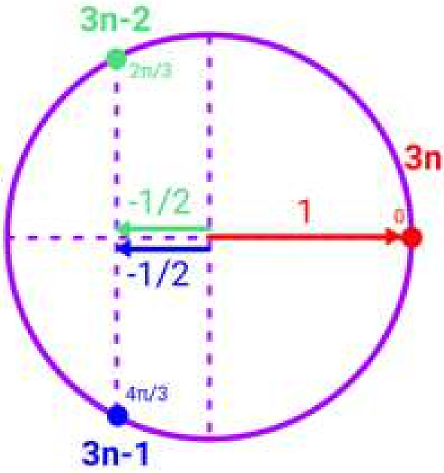
$$\Re(\omega)=\Re(\omega^2)=-\frac{1}{2}$$

$$S=\sum_{k=1}^{\infty}\frac{1}{(3k)^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k-1)^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(3k-2)^3}=$$

$$=\frac{\zeta(3)}{27}-\frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{1}{(3k-1)^3}+\sum_{k=1}^{\infty}\frac{1}{(3k-2)^3}+\sum_{k=1}^{\infty}\frac{1}{(3k)^3}-\sum_{k=1}^{\infty}\frac{1}{(3k)^3}\right)=$$

$$=\frac{\zeta(3)}{27}-\frac{1}{2}\left(\zeta(3)-\frac{\zeta(3)}{27}\right)=-\frac{4}{9}\zeta(3)$$

Solution 6 by Izumi Ainsworth-Tokyo-Japan



$$S=\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\left(\frac{2n\pi}{3}\right)=$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(3n-2)^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(3n-1)^3}+\sum_{n=1}^{\infty}\frac{1}{(3n)^3}=$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(3n-2)^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(3n-1)^3}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(3n)^3}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{(3n)^3}=$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{(3n)^3}=\left(-\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{3^3}\right)\zeta(3)=-\frac{4}{9}\zeta(3)$$

Solution 7 by Hikmat Mammadov-Azerbaijan

$$\Psi=\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\left(\frac{2n\pi}{3}\right)=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\left(\frac{2\pi\cdot 3n}{3}\right)+\sum_{n=1}^{\infty}\frac{1}{(3n-2)^3}\cos\left(\frac{2\pi(3n-2)}{3}\right)+\sum_{n=1}^{\infty}\frac{1}{(3n-1)^3}\cos\left(\frac{2\pi(3n-1)}{3}\right)$$

$$=\frac{1}{27}\zeta(3)-\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{(3n+1)^3}+\frac{1}{(3n+2)^3}\right)\right)$$

$$\text{Note: } \zeta(3)-\sum_{n=1}^{\infty}\frac{1}{(3n)^3}-\frac{1}{27}\zeta(3)=\frac{26}{27}\zeta(3)$$

$$\Psi=\left(\frac{1}{27}-\frac{13}{27}\right)\zeta(3)=-\frac{12}{27}\zeta(3)=-\frac{4}{9}\zeta(3)$$

2048. Find:

$$\Omega=\lim_{n\to\infty}\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\cdots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{[\sqrt[3]{1}]+[\sqrt[3]{2}]+[\sqrt[3]{3}]+\cdots+[\sqrt[3]{n-1}]},[*]-\mathrm{GIF},\alpha\in\mathbb{R}$$

Proposed by Florică Anastase-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\Omega=\lim_{n\to\infty}\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\cdots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{[\sqrt[3]{1}]+[\sqrt[3]{2}]+[\sqrt[3]{3}]+\cdots+[\sqrt[3]{n-1}]}=$$

$$=\lim_{n\to\infty}\frac{n^{\alpha}}{[\sqrt[3]{1}]+[\sqrt[3]{2}]+[\sqrt[3]{3}]+\cdots+[\sqrt[3]{n-1}]}\cdot\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\cdots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{n^{\alpha}}$$

$$m\le\sqrt[3]{k}<m+1,(\forall)m\in\mathbb{Z}\Rightarrow m^3\le k<(m+1)^3$$

$$\sum_{m=1}^{n-1}\sum_{j=m^3}^{(m+1)^3-1}[\sqrt[3]{k}]=\sum_{m=1}^{n-1}\sum_{j=m^3}^{(m++1)^3-1}m=\sum_{m=1}^{n-1}m((m+1)^3-m^3)=$$

$$=\sum_{m=1}^{n-1}(3m^3+3m^2+m)=-\left(3\left(\frac{n(n-1)}{2}\right)^2+3\frac{n(n-1)(2n-1)}{6}+\frac{n(n-1)}{2}\right)=$$

$$=\frac{1}{4}n(n-1)(3n(n-1)+2(2n-1)+2)=\frac{1}{4}n^2(n-1)(3n+1)$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{1-\frac{1}{2^m}}}=1\Rightarrow\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\sqrt{1-\frac{1}{2^m}}}=1$$

$$\Omega = \lim_{n\to\infty} \frac{4n^\alpha}{n^2(n-1)(3n+1)} = \frac{4}{3}\lim_{n\to\infty}\frac{n^\alpha}{n^4} = \begin{cases} 0; & \alpha < 4 \\ \frac{4}{3}; & \alpha = 4 \\ \infty; & \alpha > 4 \end{cases}$$

2049. Prove that:

$$\int_0^1\int_0^1\int_0^1 \frac{1}{\sqrt{-\log(xyz)}}dxdydz = \frac{3\sqrt{\pi}}{8}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{1}{\sqrt{-\log(xyz)}}dxdydz \overset{t=xyz}{=} \int_0^1 \frac{1}{z}\int_0^1 \frac{1}{y}\int_0^{yz}\frac{dt}{\sqrt{-\log(t)}}dydz =$$

$$= \int_0^1 \frac{1}{z}\left[\log(y)\int_0^{yz}\frac{dt}{\sqrt{-\log(t)}}\right]_0^1 dz - \int_0^1\int_0^1 \frac{\log(y)}{\sqrt{-\log(yz)}}dydz =$$

$$= -\int_0^1\int_0^1 \frac{\log(y)}{\sqrt{-\log(yz)}}dydz \overset{t=yz}{=} -\int_0^1 \frac{\log(y)}{y}\int_0^y \frac{dt}{\sqrt{-\log(t)}}dy =$$

$$= -\left[\frac{1}{2}\log^2(y)\int_0^y \frac{dt}{\sqrt{-\log(t)}}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{\log^2(y)}{\sqrt{-\log(y)}}dy =$$

$$= \frac{1}{2}\int_0^1 \frac{\log^2(y)}{\sqrt{-\log(y)}}dy \overset{y=e^{-u}}{=} \frac{1}{2}\int_0^\infty u^{\frac{5}{2}-1}e^{-u}\,du = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{8}$$

Solution 2 by Probal Chrakraborty-India

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{x^s}{\sqrt{-\log(xyz)}}dxdydz$$

$$Take: w = \frac{1}{z}; u = xyz; v = \frac{y}{xz} \Rightarrow dudvdw = \frac{\partial(u;v;w)}{\partial(x;y;z)} \Rightarrow \frac{dudvdw}{2vw} = dxdydz$$

$$\int_0^1\int_u^{\frac{1}{u}}\int_u^{\frac{1}{u}} \frac{u^s}{\sqrt{-\log(u)}}\frac{dudvdw}{2vw} = \frac{1}{2}\int_0^1 \frac{u^s}{\sqrt{-\log(u)}}(-\log(u))^2du =$$

$$= \frac{1}{2}u^s(-\log(u))^{\frac{3}{2}}du \overset{(*)}{=}$$

$$Take: \sqrt{-\log(u)} = w \Rightarrow e^{-w^2} = u \Rightarrow -2we^{-w^2}dw = du$$

$$\overset{(*)}{=}\int_{\infty}^{0} e^{-sw^2}w^3\left(-2we^{-w^2}\right)dw=\int_0^{\infty} e^{-(s+1)w^2}w^4dw\overset{(**)}{=}$$

$$Let: (s+1)w^2=t\Rightarrow dw=\frac{dt}{2\sqrt{s+1}}\ and\ w=\frac{\sqrt{t}}{\sqrt{s+1}}$$

$$\overset{(**)}{=}\frac{1}{2(s+1)^{\frac{3}{2}}}\int_0^{\infty} e^{-t}t^{\frac{5}{2}-1}dt=\frac{1}{2(s+1)^{\frac{3}{2}}}\Gamma\left(\frac{5}{2}\right)=\frac{3\sqrt{\pi}}{8}$$

$$For\ s=0:\ \int_0^1\int_0^1\int_0^1 \frac{1}{\sqrt{-\log(xyz)}}dxdydz=\frac{3\sqrt{\pi}}{8}$$

2050. Prove that:

$$I=\int_0^1\int_0^1\int_0^1 \frac{1}{1-\phi xyz+x^2y^2z^2}dxdydz=3\left(\frac{\pi}{5}\right)^3\sqrt{2+\frac{2}{5}\sqrt{5}}$$

where: ϕ is golden ratio

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$*\ Known: \phi=\frac{\sqrt{5}+1}{2}\ and: \cos\frac{\pi}{5}=\frac{\sqrt{5}+1}{4}\Rightarrow \phi=2\cos\frac{\pi}{5}$$

$$\Rightarrow I=\int_0^1\int_0^1\int_0^1 \frac{1}{1-2\cos\frac{\pi}{5}xyz+x^2y^2z^2}dxdydz$$

$$=\int_0^1\int_0^1\int_0^1 \frac{1}{xyz\sin\frac{\pi}{5}}\cdot\frac{xyz\sin\frac{\pi}{5}}{1-2\cos\frac{\pi}{5}xyz+x^2y^2z^2}dxdydz$$

$$*\sum_{k=1}^{\infty}\sin(k\varphi)x^n=\frac{x\sin(\varphi)}{1-2\cos(\varphi)x+x^2}\left(x\in(-1,1),\varphi\in\mathbb{R}\right)\Rightarrow I$$

$$=\frac{1}{\sin\frac{\pi}{5}}\int_0^1\int_0^1\int_0^1 \frac{1}{xyz}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{5}\right)(xyz)^k dxdydz$$

$$=\frac{1}{\sin\frac{\pi}{5}}\sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{5}\right)\int_0^1\int_0^1\int_0^1 (xyz)^{k-1}dxdydz=\frac{1}{\sin\frac{\pi}{5}}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{5}\right)}{k^3}=\frac{1}{\sin\frac{\pi}{5}}Sl_3\left(\frac{\pi}{5}\right)$$

$$*\ Use\ indentity\ of\ Clausen\ function\ order 3: Sl_3(\varphi)=\frac{\pi^2}{6}\varphi-\frac{\pi}{4}\varphi^2+\frac{1}{12}\varphi^3$$

$$\Rightarrow Sl_3\left(\frac{\pi}{5}\right)=\frac{\pi^3}{30}-\frac{\pi^3}{100}+\frac{\pi^3}{1500}=\frac{3\pi^3}{125}=3\left(\frac{\pi}{5}\right)^3$$

$$*\ \frac{1}{\sin\frac{\pi}{5}}=\frac{1}{\sqrt{\frac{5-\sqrt{5}}{8}}}=\sqrt{\frac{8}{5-\sqrt{5}}}=\sqrt{\frac{8}{20}(5+\sqrt{5})}=\sqrt{\frac{2}{5}(5+\sqrt{5})}=\sqrt{2+\frac{2}{5}\sqrt{5}}$$

$$\Rightarrow I=\boxed{3\left(\frac{\pi}{5}\right)^3\sqrt{2+\frac{2}{5}\sqrt{5}}}$$

2051. Find:

$$\int_0^1\frac{\arctan\left(x\pm\sqrt{x^2+1}\right)}{x+1}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution 1 by Pham Duc Nam-Vietnam

$$*\ Denote: I=\int_0^1\frac{\arctan\left(x+\sqrt{x^2+1}\right)}{x+1}dx\ and$$

$$J=\int_0^1\frac{\arctan\left(x-\sqrt{x^2+1}\right)}{x+1}dx$$

$$*\left(x+\sqrt{x^2+1}\right)\left(x-\sqrt{x^2+1}\right)=-1\Rightarrow\left(x-\sqrt{x^2+1}\right)=\frac{-1}{\left(x+\sqrt{x^2+1}\right)}$$

$$\Rightarrow \arctan\left(x-\sqrt{x^2+1}\right)=-\arctan\left(\frac{1}{x+\sqrt{x^2+1}}\right)$$

$$*\ I+J=\int_0^1\frac{\arctan\left(x+\sqrt{x^2+1}\right)+\arctan\left(x-\sqrt{x^2+1}\right)}{x+1}dx$$

$$=\int_0^1\frac{\arctan\left(\frac{x+\sqrt{x^2+1}+x-\sqrt{x^2+1}}{1-\left(x+\sqrt{x^2+1}\right)\left(x-\sqrt{x^2+1}\right)}\right)}{x+1}dx$$

$$=\int_0^1\frac{\arctan\left(\frac{2x}{1-\left(x^2-(x^2-1)\right)}\right)}{x+1}dx=\int_0^1\frac{\arctan(x)}{x+1}dx$$

$$*\ I-J=\int_0^1 \frac{\arctan\left(x+\sqrt{x^2+1}\right)-\arctan\left(x-\sqrt{x^2+1}\right)}{x+1}dx$$

$$=\int_0^1 \frac{\arctan\left(x+\sqrt{x^2+1}\right)+\arctan\left(\frac{1}{x+\sqrt{x^2+1}}\right)}{x+1}dx$$

$$=\frac{\pi}{2}\int_0^1 \frac{1}{x+1}dx=\frac{\pi}{2}\log(2)$$

$$*\ K=\int_0^1 \frac{\arctan(x)}{x+1}dx, let: x=\tan(t)\Rightarrow dx=\frac{dt}{\cos^2(t)}\Rightarrow K$$

$$=\int_0^{\frac{\pi}{4}} \frac{t}{\cos^2(t)\,(1+\tan(t))}dt=\int_0^{\frac{\pi}{4}} \frac{t}{\cos(t)\,(\cos(t)+\sin(t))}dt$$

$$\begin{cases} u=t \\ dv=\dfrac{dt}{\cos(t)\,(\cos(t)+\sin(t))} \end{cases}\Rightarrow \begin{cases} du=dt \\ v=\log(\sin(t)+\cos(t))-\log(\cos(t)) \end{cases}$$

$$\Rightarrow K$$

$$=t(\log(\sin(t)+\cos(t))-\log(\cos(t)))\Big|_0^{\frac{\pi}{4}}$$

$$-\int_0^{\frac{\pi}{4}}(\log(\sin(t)+\cos(t))-\log(\cos(t)))\,dt$$

$$=\frac{\pi}{4}\log(2)-\int_0^{\frac{\pi}{4}}\left(\log\left(\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)\right)-\log(\cos(t))\right)dt$$

$$=\frac{\pi}{4}\log(2)-\frac{\pi}{8}\log(2)$$

$$-\int_0^{\frac{\pi}{4}}\log\left(\sin\left(t+\frac{\pi}{4}\right)\right)dt+\int_0^{\frac{\pi}{4}}\log(\cos(t))\,dt$$

$$=\frac{\pi}{8}\log(2)-\int_0^{\frac{\pi}{4}}\log\left(\sin\left(\frac{\pi}{4}-t+\frac{\pi}{4}\right)\right)dt+\int_0^{\frac{\pi}{4}}\log(\cos(t))\,dt$$

$$=\frac{\pi}{8}\log(2)-\int_0^{\frac{\pi}{4}}\log\left(\sin\left(\frac{\pi}{2}-t\right)\right)dt+\int_0^{\frac{\pi}{4}}\log(\cos(t))\,dt$$

$$=\frac{\pi}{8}\log(2)-\int_0^{\frac{\pi}{4}}\log(\cos(t))\,dt+\int_0^{\frac{\pi}{4}}\log(\cos(t))\,dt=\frac{\pi}{8}\log(2)$$

$$*\begin{cases} I+J=\dfrac{\pi}{8}\log(2) \\ I-J=\dfrac{\pi}{2}\log(2) \end{cases}\Rightarrow \begin{cases} I=\dfrac{5\pi}{16}\log(2) \\ J=-\dfrac{3\pi}{16}\log(2) \end{cases}\Rightarrow \int_0^1 \frac{\arctan\left(x\pm\sqrt{x^2+1}\right)}{x+1}dx$$

$$=(1\pm 4)\frac{\pi}{16}\log(2)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{\tan^{-1}\left(x\pm\sqrt{1+x^2}\right)}{1+x}dx\overset{x=\tan\theta}{=}\int_0^{\frac{\pi}{4}}\frac{\tan^{-1}\left(\frac{\sin\theta\pm1}{\cos\theta}\right)}{\cos\theta\,(\sin\theta+\cos\theta)}d\theta=$$

$$=\int_0^{\frac{\pi}{4}}\frac{\frac{\theta}{2}\pm\frac{\pi}{4}}{\cos\theta\,(\sin\theta+\cos\theta)}d\theta=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\theta}{\cos\theta\,(\sin\theta+\cos\theta)}d\theta\pm\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{d\theta}{\cos\theta\,(\sin\theta+\cos\theta)}$$

$$\int\frac{d\theta}{\cos\theta\,(\sin\theta+\cos\theta)}d\theta=\int\frac{d\theta}{\cos^2\theta\,(1+\tan\theta)}=\int\frac{d(\tan\theta)}{1+\tan\theta}=\log(1+tan\,\theta)$$

$$\Omega=\frac{1}{2}[\theta\log(1+\tan\theta)]_0^{\frac{\pi}{4}}-\frac{1}{2}\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta\pm\frac{\pi}{4}[\log(1+\tan\theta)]_0^{\frac{\pi}{4}}=$$

$$=\frac{\pi}{8}\log(2)\pm\frac{\pi}{4}\log(2)-\frac{1}{2}\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta,\ \int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta=\frac{\pi}{8}\log(2)$$

$$\Omega=\frac{\pi}{16}\log(2)\pm\frac{\pi}{4}\log(2),\quad\Omega=\int_0^1\frac{\tan^{-1}\left(x\pm\sqrt{1+x^2}\right)}{1+x}dx=(1\pm4)\frac{\pi}{16}\log(2)$$

2052. Prove that:

$$\int_0^1\int_0^1\int_0^1\frac{\log^n(xyz)}{1-xyz}dx\,dy\,dz=\frac{1}{2}(-1)^n\Gamma(n+3)\zeta(n+3)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\text{Let } xyz=m$$

$$I=\int_0^1\int_0^1\int_0^1\frac{\log^n(xyz)}{1-xyz}dx\,dy\,dz=\int_0^1\frac{1}{x}\int_0^1\frac{1}{y}\int_0^{xy}\frac{\log^n(m)}{1-m}dm\,dy\,dx$$

$$\int_0^1\frac{1}{y}\int_0^{xy}\frac{\log^n(m)}{1-m}dm\,dy\overset{IBP}{=}\left[\log(y)\int_0^{xy}\frac{\log^n(m)}{1-m}dm\right]_0^1-x\int_0^1\frac{\log(y)\log^n(xy)}{1-xy}dy$$

$$=-x\int_0^1\frac{\log(y)\log^n(xy)}{1-xy}dy$$

$$I=-\int_0^1\int_0^1\frac{\log(y)\log^n(xy)}{1-xy}dy\,dx$$

$$\text{Let } xy=t$$

$$I=-\int_0^1\frac{\log(y)}{y}\int_0^y\frac{\log^n(t)}{1-t}dt\,dy=-\left[\frac{1}{2}\log^2(y)\int_0^y\frac{\log^n(t)}{1-t}dt\right]_0^1+\frac{1}{2}\int_0^1\frac{\log^{n+2}(y)}{1-y}dy$$

$$=\frac{1}{2}\int_0^1\frac{\log^{n+2}(y)}{1-y}dy=\frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 y^k\log^{n+2}(y)\,dy=\frac{1}{2}\sum_{k=0}^{\infty}(-1)^{n+2}\frac{1}{(k+1)^{n+3}}$$

$$=\frac{1}{2}(-1)^n\sum_{k=0}^{\infty}\frac{\Gamma(n+3)}{(k+1)^{n+3}}=\frac{1}{2}(-1)^n\Gamma(n+3)\sum_{k=1}^{\infty}\frac{1}{k^{n+3}}$$

$$=\frac{1}{2}(-1)^n\Gamma(n+3)\zeta(n+3)$$

Solution 2 by Le Thu-Vietnam

$$\Omega=\frac{d^n}{ds^n}\left[\sum_{m=0}^{\infty}\left[\int_0^1\int_0^1\int_0^1(xyz)^{s+m}\,dxdydz\right]\right]_{s=0}=$$

$$=\frac{d^n}{ds^n}\left[\sum_{m=0}^{\infty}\frac{1}{(s+m+1)^3}\right]_{s=0}=\frac{d^n}{ds^n}\left[-\frac{1}{2}\psi_2(s+1)\right]_{s=0}$$

$$=-\frac{1}{2}(-1)^n\psi_{n+2}(s+1)|_{s=0}=-\frac{1}{2}\psi_{n+2}(1)=-\frac{1}{2}(-1)^{n+3}(n+2)!\sum_{k=0}^{\infty}\frac{1}{(k+1)^{n+3}}=$$

$$=\frac{1}{2}(-1)^n\Gamma(n+3)\zeta(n+3)$$

$$Note:\ \psi_m(z)=(-1)^{m+1}m!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{m+1}}$$

$$\frac{d^n}{dz^n}[\psi_m(z)]=\psi_{m+n}(z);\ (\forall)m,n\in\mathbb{N},z\notin\mathbb{Z}_-$$

Solution 3 by Kartick Chandra Betal-India

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{\log^n(xyz)}{1-xyz}dxdydz=\int_0^1\int_0^1\frac{1}{y}\int_0^y\frac{\log^n(xz)}{1-xz}dxdydz=$$

$$=\int_0^1\int_0^1\frac{\log^n(xz)}{1-xz}\int_x^1\frac{1}{y}dy\,dxdy=-\int_0^1\int_0^1\frac{\log^n(xz)}{1-xz}\log(x)\,dxdz=$$

$$= -\int_0^1 \frac{1}{z}\int_0^z \frac{\log^n(x)}{1-x}(\log(x)-\log(z))dxdz =$$

$$= \int_0^1 \frac{\log(z)}{z}\int_0^z \frac{\log^n(z)}{1-x}dx - \int_0^1 \frac{1}{z}\int_0^z \frac{\log^{n+1}(x)}{1-x}dxdz =$$

$$= \int_0^1 \frac{\log^n(x)}{1-x}\int_x^1 \frac{\log(z)}{z}dzdx - \int_0^1 \frac{\log^{n+1}(x)}{1-x}\int_x^1 \frac{1}{z}dzdx =$$

$$= -\frac{1}{2}\int_0^1 \frac{\log^{n+2}(x)}{1-x}dx + \int_0^1 \frac{\log^{n+2}(x)}{1-x}dx =$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\int_0^1 x^{m-1}\log^{n+2}(x)\,dx = \frac{1}{2}\sum_{m=1}^{\infty}\int_{\infty}^0 e^{-(m-1)x}(-x)^{n+2}(-e^{-x})dx =$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}(-1)^n\int_0^{\infty} e^{-mx}x^{n+3-1}dx\,;(x\to e^{-x})$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}(-1)^n\cdot\frac{\Gamma(n+3)}{m^{n+3}} = \frac{1}{2}(-1)^n\Gamma(n+3)\sum_{m=1}^{\infty}\frac{1}{m^{n+3}} = \frac{1}{2}(-1)^n\Gamma(n+3)\zeta(n+3)$$

2053. Prove that:

$$I = \int_0^1\int_0^1\int_0^1 \frac{arcsin(xyz)}{log(xyz)}dxdydz = 1-\frac{\pi}{4}-\frac{1}{2}log(2)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$*\ Let: I(a) = \int_0^1\int_0^1\int_0^1 \frac{(xyz)^a\,arcsin(xyz)}{log(xyz)}dxdydz \Rightarrow I = I(0), I(\infty) = 0$$

$$\Rightarrow I'(a) = \int_0^1\int_0^1\int_0^1 \frac{log(xyz)\,(xyz)^a\,arcsin(xyz)}{log(xyz)}dxdydz$$

$$= \int_0^1\int_0^1\int_0^1 (xyz)^a\,arcsin(xyz)\,dxdydz$$

$$Use: arcsin(x)=\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}x^{2k+1}\Rightarrow I'(a)$$
$$=\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\int_0^1\int_0^1\int_0^1(xyz)^{2k+1+a}\,dxdydz$$
$$=\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\cdot\frac{1}{(2k+2+a)^3}$$
$$\Rightarrow I(a)=\int\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\cdot\frac{1}{(2k+2+a)^3}da=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\cdot\frac{1}{(2k+2+a)^2}$$
$$+C, a\to\infty\Rightarrow C=0$$
$$\Rightarrow I(a)=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\cdot\frac{1}{(2k+2+a)^2}\Rightarrow I=I(0)$$
$$=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\cdot\frac{1}{(2k+2)^2}=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\cdot\frac{1}{4(k+1)^2}\ (1)$$
$$*\ arcsin(x)=\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}x^{2k+1}\Rightarrow log(x)\,arcsin(x)$$
$$=\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}log(x)\,x^{2k+1}\text{ ,integrating both sides from 0 to 1}$$
$$\Rightarrow\int_0^1 log(x)\,arcsin(x)\,dx=\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\int_0^1 log(x)\,x^{2k+1}\,dx\Rightarrow 2-\frac{\pi}{2}-log(2)$$
$$=-\sum_{k=0}^{\infty}\frac{C_{2k}^{k}}{4^k(2k+1)}\cdot\frac{1}{4(k+1)^2}\ (2)$$
$$From: (1)\ and\ (2)\Rightarrow I=\frac{1}{2}\left(2-\frac{\pi}{2}-log(2)\right)=\boxed{1-\frac{\pi}{4}-\frac{1}{2}log(2)}$$

2054. Prove that:

$$\int_0^1\int_0^1\int_0^1 \log\left[log\left(\frac{1}{xyz}\right)\right]dxdydz=\frac{3}{2}-\gamma$$

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$Let\ be\ \Omega_t=\int_0^1\int_0^1\int_0^1(xyz)^t\,dxdydz=(t+1)^{-3};\quad (1)$$

$$\Omega_t^{(q-1)} = \int_0^1\int_0^1\int_0^1 (xyz)^t \log^{q-1}\left(\frac{1}{xyz}\right) dxdydz = \frac{\Gamma(q+2)}{2}(t+1)^{-3-q+1};\ \ (2)$$

$$\left(\Omega_t^{(q-1)}\right)^{(q)} = \int_0^1\int_0^1\int_0^1 (xyz)^t \log\left(log\left(\frac{1}{xyz}\right)\right)\log^{(q-1)}\left(\frac{1}{xyz}\right) dxdydz =$$

$$= \frac{\Gamma'(q+2)}{2}(t+1)^{-3-q+1} - \frac{\Gamma(q+2)}{2}\log(t+1)\,(t+1)^{-3-q+1};\quad (3)$$

$$Let\ t=0\ and\ q=1\ into\ (3):$$

$$\Omega = \int_0^1\int_0^1\int_0^1 \log\left[log\left(\frac{1}{xyz}\right)\right] dxdydz = \frac{\Gamma'(3)}{2} = \frac{\Gamma'(3)}{\Gamma(3)}$$

$$We\ know\ that:\ \psi(3) = \frac{\Gamma'(3)}{\Gamma(3)} = -\gamma + H_2 = -\gamma + 1 + \frac{1}{2} = \frac{3}{2} - \gamma$$

$$Therefore,\qquad \int_0^1\int_0^1\int_0^1 \log\left[log\left(\frac{1}{xyz}\right)\right] dxdydz = \frac{3}{2} - \gamma$$

2055. Prove that:

$$\int_0^1\int_0^1\int_0^1 \frac{1}{(xyz-1)\log(xyz)} dxdydz = \frac{\pi^2}{12}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{1}{(xyz-1)\log(xyz)} dxdydz$$

$$Using:\ \int_0^1 (xyz)^t\,dt = \frac{xyz-1}{\log(xyz)}$$

$$\Omega = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{(xyz)^t}{(1-xyz)^2} dxdydzdt$$

$$If\ |u|<1:\ \frac{1}{(1-u)^2} = \frac{d}{du}\left(\frac{1}{1-u}\right) = \frac{d}{du}\sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} nu^{n-1} = \sum_{n=1}^{\infty} nu^{n-1}$$

$$\Omega = \sum_{n=1}^{\infty} n\int_0^1\left(\int_0^1 z^{n+t-1}dz\int_0^1 y^{n+t-1}dy\int_0^1 x^{n+t-1}dx\right)dt = \sum_{n=1}^{\infty} n\int_0^1 \frac{dt}{(n+t)^3}$$

$$\Omega = \sum_{n=1}^{\infty} n\left[-\frac{1}{2(n+t)^2}\right]_0^1 = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{(n+1)^2}\right)$$

$$\Omega = \frac{1}{2}\underbrace{\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}}_{\zeta(2)-1} + \frac{1}{2}\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{Telescopic\ terms=1} = \frac{1}{2}\zeta(2)$$

$$Therefore,\ \ \Omega = \int_0^1\int_0^1\int_0^1 \frac{1}{(xyz-1)\log(xyz)}dxdydz = \frac{\pi^2}{12}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$Let\ \varphi(k) = \int_0^1\int_0^1\int_0^1 \frac{(xyz)^k}{(xyz-1)\log(xyz)}dxdydz\,, then:$$

$$\varphi'(k) = -\int_0^1\int_0^1\int_0^1 \frac{(xyz)^k\log(xyz)}{(1-xyz)\log(xyz)}dxdydz = -\int_0^1\int_0^1\int_0^1 \frac{(xyz)^k}{1-xyz}dxdydz =$$

$$= -\int_0^1\int_0^1\int_0^1 (xyz)^k\sum_{m=0}^{\infty}(xyz)^m\,dxdydz = -\sum_{m=0}^{\infty}\int_0^1\int_0^1\int_0^1 (xyz)^{m+k}dxdydz =$$

$$= -\sum_{m=0}^{\infty}\left(\int_0^1 x^{m+k}dx\right)\left(\int_0^1 y^{m+k}dy\right)\left(\int_0^1 z^{m+k}dz\right) = -\sum_{m=0}^{\infty}\frac{1}{(m+k+1)^3}$$

$$\varphi(k) = -\sum_{m=0}^{\infty}\int\frac{dk}{(m+k+1)^3} + C = \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{(m+k+1)^2} + C$$

$$\lim_{k\to-\infty}\varphi(k) = \frac{1}{2}\sum_{m=0}^{\infty}\lim_{k\to-\infty}\frac{1}{(m+k+1)^2} + C; C = 0$$

$$Therefore,$$

$$\int_0^1\int_0^1\int_0^1 \frac{1}{(xyz-1)\log(xyz)}dxdydz = \varphi(0) = \frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{(m+1)^2} = \frac{\zeta(2)}{2} = \frac{\pi^2}{12}$$

2056. Prove that:

$$I = \int_0^1\int_0^1\int_0^1 \frac{1}{(1+x^2y^2z^2)\,log^2(xyz)}dxdydz = \frac{\pi}{8}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^1\int_0^1 \frac{1}{xy}\int_0^{xy}\frac{1}{(1+t^2)\log^2 t}dtdydx \overset{IBP}{=}$$

$$=\int_0^1 \frac{1}{x}\int_0^1 \frac{-\log y\cdot x}{(1+x^2y^2)\log^2(xy)}dydx = -\underbrace{\int_0^1\int_0^1 \frac{\log y}{(1+x^2y^2)\log^2(xy)}dydx}_{I_1}$$

$$I_1=\int_0^1\int_0^1 \frac{\log y}{(1+x^2y^2)\log^2(xy)}dydx=\int_0^1\int_0^1 \frac{\log x}{(1+x^2y^2)\log^2(xy)}dxdy=$$

$$=\frac{1}{2}\int_0^1\int_0^1 \frac{dydx}{(1+x^2y^2)\log(xy)} \Rightarrow \Omega = -\frac{1}{2}\int_0^1\int_0^1 \frac{dydx}{(1+x^2y^2)\log(xy)}$$

$$Let\ xy=u \Rightarrow \Omega = -\frac{1}{2}\int_0^1 \frac{1}{x}\int_0^x \frac{dudx}{(1+u^2)\log u} \overset{IBP}{=} \frac{1}{2}\int_0^1 \frac{\log x}{(1+x^2)\log x}dx=$$

$$=\frac{1}{2}\int_0^1 \frac{dx}{1+x^2}=\frac{\pi}{8}$$

Solution 2 by Pham Duc Nam Vietnam

$$I=\int_0^1\int_0^1\int_0^1 \frac{1}{(1+x^2y^2z^2)\,log^2(xyz)}dxdydz=\frac{\pi}{8}$$

$$*\ Let: I(a)=\int_0^1\int_0^1\int_0^1 \frac{(xyz)^a}{(1+x^2y^2z^2)\,log^2(xyz)}dxdydz \Rightarrow I=I(0)$$

$$*\ Differentiating\ with\ respect\ to\ a:$$

$$I'(a)=\int_0^1\int_0^1\int_0^1 \frac{(xyz)^a\,log(xyz)}{(1+x^2y^2z^2)\,log^2(xyz)}dxdydz \Rightarrow I''(a)$$

$$=\int_0^1\int_0^1\int_0^1 \frac{(xyz)^a\,log^2(xyz)}{(1+x^2y^2z^2)\,log^2(xyz)}dxdydz=\int_0^1\int_0^1\int_0^1 \frac{(xyz)^a}{(1+x^2y^2z^2)}dxdydz$$

$$*\ Known: \sum_{k=0}^{+\infty}(-1)^k x^k=\frac{1}{x+1}\,(|x|<1) \Rightarrow I''(a)=\int_0^1\int_0^1\int_0^1 (xyz)^a\sum_{k=0}^{+\infty}(-1)^k(xyz)^{2k}\,dxdydz$$

$$=\sum_{k=0}^{+\infty}(-1)^k\int_0^1\int_0^1\int_0^1 (xyz)^{2k+a}\,dxdydz$$

$$=\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+a+1)^3}$$

$$*\ Integrating\ with\ respect\ to\ a:$$

$$\Rightarrow I'(a)=\int\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+a+1)^3}da=\sum_{k=0}^{+\infty}-\frac{(-1)^k}{2(2k+a+1)^3}+C_1, let: a\to\infty \Rightarrow C_1=0 \Rightarrow I'(a)$$

$$=\sum_{k=0}^{+\infty}-\frac{(-1)^k}{2(2k+a+1)^3}$$

$$\Rightarrow I(a)=\int\sum_{k=0}^{+\infty}-\frac{(-1)^k}{2(2k+a+1)^3}da=\sum_{k=0}^{+\infty}\frac{(-1)^k}{2(2k+a+1)}+C_2, let: a\to\infty\Rightarrow C_2=0\Rightarrow I(a)$$

$$=\sum_{k=0}^{+\infty}\frac{(-1)^k}{2(2k+a+1)}$$

$$*\ Let: a=0\Rightarrow I=I(0)=\frac{1}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)}, known: arctan(x)$$

$$=\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)}(|x|\le 1), let: x=1$$

$$\Rightarrow I=\frac{1}{2}.arctan(1)=\frac{1}{2}.\frac{\pi}{4}=\frac{\pi}{8}$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\Psi(t)_{\lim_{t\to 0}\Psi(t)=0}=\int_0^1\int_0^1\int_0^1\frac{(xyz)^t}{1+(xyz)^2}\cdot\frac{1}{\log^2(xyz)}dxdydz;$$

$$\psi'(t)_{\lim_{t\to 0}\Psi'(t)=0}=\int_0^1\int_0^1\int_0^1\frac{(xyz)^t}{(1+(xyz)^2)\log(xyz)}dxdydz$$

$$\psi''(t)_{\lim_{t\to 0}\Psi''(t)=0}=\int_0^1\int_0^1\int_0^1\frac{(xyz)^t}{1+(xyz)^2}dxdydz$$

$$\Psi''(t)=\int_0^1\int_0^1\int_0^1(xyz)^t\sum_{k=0}^{\infty}(-1)^k(xyz)^{2k}dxdydz=$$

$$=\sum_{k=0}^{\infty}(-1)^k\int_0^1\int_0^1\int_0^1(xyz)^{2k+t}dxdydz\overset{z=\frac{w}{xy}}{=}\sum_{k=0}^{\infty}(-1)^k\int_0^1\int_0^1\int_0^{xy}w^{2k+t}\frac{1}{xy}dxdydz=$$

$$=\sum_{k=0}^{\infty}(-1)^k\int_0^1\int_0^1\frac{1}{xy}\cdot\frac{xy^{2k+t+1}}{2k+t+1}dxdy\overset{x=\frac{v}{y}}{=}\sum_{k=0}^{\infty}(-1)^k\int_0^1\frac{1}{y}\int_0^y\frac{v^{2k+t}}{2k+t+1}dvdy=$$

$$=\sum_{k=0}^{\infty}(-1)^k\int_0^1\frac{1}{y}\frac{y^{2k+t+1}}{(2k+t+1)^2}dy=\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+t+1)^3}$$

$$\Psi''(t)=\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+t+1)^3}$$

$$\int_u^{\infty}\Psi''(t)dt=-\Psi'(u)=\sum_{k=0}^{\infty}\left[(-1)^k\frac{1}{2}\frac{1}{(2k+t+1)^2}\right]_{\infty}^{u}\Rightarrow$$

$$\Psi'(u) = -\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+u+1)^2}$$

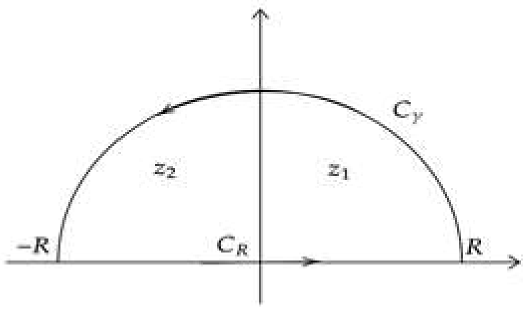
$$\int_s^{\infty}\Psi'(u)\,du = -\Psi'(s) = -\frac{1}{2}\sum_{k=0}^{\infty}\left[(-1)^k\frac{1}{2k+u+1}\right]_{\infty}^{s} \Rightarrow$$

$$\Psi(s) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+s+1} \Rightarrow \Omega = \Psi(0) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \Rightarrow$$

$$I = \int_0^1\int_0^1\int_0^1\frac{1}{(1+x^2y^2z^2)\,log^2(xyz)}dxdydz = \frac{\pi}{8}$$

2057. Prove that:

$$\Omega = \int_0^{+\infty}\frac{(2-x^2)\cos(x)+2x\sin(x)}{x^4+4}dx = \frac{\pi}{e}\sin(1)$$



Proposed by Asmat Qatea-Afghanistan

Solution by Pham Duc Nam-Vietnam

$$\Omega = \int_0^{+\infty}\frac{2\cos(x)}{x^4+4}dx - \int_0^{+\infty}\frac{x^2\cos(x)}{x^4+4}dx + \int_0^{+\infty}\frac{2x\sin(x)}{x^4+4}dx$$

$$= \int_{-\infty}^{+\infty}\frac{\cos(x)}{x^4+4}dx - \frac{1}{2}\int_{-\infty}^{+\infty}\frac{x^2\cos(x)}{x^4+4}dx + \int_{-\infty}^{+\infty}\frac{x\sin(x)}{x^4+4}dx$$

$$= \Re\int_{-\infty}^{+\infty}\frac{e^{iz}}{z^4+4}dz - \frac{1}{2}\Re\int_{-\infty}^{+\infty}\frac{z^2e^{iz}}{z^4+4}dz + \Im\int_{-\infty}^{+\infty}\frac{ze^{iz}}{z^4+4}dz$$

$$* \, Let: f_1(z) = \frac{e^{iz}}{z^4+4}, f_2(z) = \frac{z^2e^{iz}}{z^4+4}, f_3(z) = \frac{ze^{iz}}{z^4+4}$$

$* \, Let: C = C_\gamma \cup C_R$ *is a contour, counter − clockwise direction where* $\Im(z) \geq 0, R > 2 \to \infty$

$* \, f_k(z)$ *has two poles*: $z_1 = 1+i, z_2 = -1+i$ (*order* 1) *inside the contour*

$$\Rightarrow \int_C f_k(z)\,dz = 2\pi i \sum \mathrm{Res}\big(f_k(z), z_{1,2}\big)$$

$$and: \int_C f_k(z)\,dz = \int_{C_\gamma} f_k(z)\,dz + \int_{C_R} f_k(x)\,dx, by: ML\ inequality \Rightarrow \lim_{R\to\infty} \int_{C_\gamma} f_k(z)\,dz = 0$$

$$\Rightarrow \int_{C_R} f_k(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f_k(x)dx = \int_C f_k(z)\,dz$$

$$* \int_C \frac{e^{iz}}{z^4+4}dz = 2\pi i \sum \mathrm{Res}\left(\frac{e^{iz}}{z^4+4}, z_{1,2}\right)$$

$$= 2\pi i\left(\lim_{z\to 1+i} \frac{(z-(1+i))e^{iz}}{z^4+4} + \lim_{z\to -1+i} \frac{(z-(-1+i))e^{iz}}{z^4+4}\right)$$

$$= 2\pi i\left(\left(-\frac{1}{16}-\frac{i}{16}\right)e^{-1+i} + \left(\frac{1}{16}-\frac{i}{16}\right)e^{-1-i}\right) = \frac{\pi}{4e}(sin(1)+cos(1))$$

$$* \int_C \frac{z^2e^{iz}}{z^4+4}dz = 2\pi i \sum \mathrm{Res}\left(\frac{z^2e^{iz}}{z^4+4}, z_{1,2}\right)$$

$$= 2\pi i\left(\lim_{z\to 1+i} \frac{(z-(1+i))z^2e^{iz}}{z^4+4} + \lim_{z\to -1+i} \frac{(z-(-1+i))z^2e^{iz}}{z^4+4}\right)$$

$$= 2\pi i\left(\left(-\frac{1}{8}-\frac{i}{8}\right)e^{-1-i} + \left(\frac{1}{8}-\frac{i}{8}\right)e^{-1+i}\right) = \frac{\pi}{2e}(cos(1)-sin(1))$$

$$* \int_C \frac{ze^{iz}}{z^4+4}dz = 2\pi i \sum \mathrm{Res}\left(\frac{ze^{iz}}{z^4+4}, z_{1,2}\right)$$

$$= 2\pi i\left(\lim_{z\to 1+i} \frac{(z-(1+i))ze^{iz}}{z^4+4} + \lim_{z\to -1+i} \frac{(z-(-1+i))ze^{iz}}{z^4+4}\right)$$

$$= 2\pi i\left(-\frac{1}{8}ie^{-1+i} + \frac{1}{8}ie^{-1-i}\right) = i\frac{\pi}{2e}sin(1)$$

$* \ Let: R$

$\to \infty$, *taking real part of* $\int_C \frac{e^{iz}}{z^4+4}dz$ *and* $\int_C \frac{z^2e^{iz}}{z^4+4}dz$, *imaginary part of* $\int_C \frac{ze^{iz}}{z^4+4}dz$

$$\Rightarrow \Omega = \frac{\pi}{4e}(sin(1)+cos(1)) - \frac{1}{2}\cdot\frac{\pi}{2e}(cos(1)-sin(1)) + \frac{\pi}{2e}sin(1)$$

$$= \frac{\pi}{4e}(sin(1)+cos(1)) - \frac{\pi}{4e}(cos(1)-sin(1)) + \frac{\pi}{2e}sin(1) = \boxed{\frac{\pi}{e}sin(1)}$$

2058. Find:

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{n}\cdot 3^{\frac{k}{n}}\right)\left(1+\frac{1}{n}\cdot 5^{\frac{k}{n}}\right)\left(1+\frac{1}{n}\cdot 7^{\frac{k}{n}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$Let's\ focus\ on \lim_{n\to\infty}\sum_{k=1}^{n}\log\left(1+\frac{1}{n}\cdot a^{\frac{k}{n}}\right), where\ a>1.$$

$$For\ z\in\mathbb{C}, |z|<\frac{1}{2}\ and\ |\log(1+z)-z|\le|z|^2.$$

$$\log(1+z)-z=-\frac{z^2}{2}\left(1-\frac{3}{2}z+\frac{2}{4}z^2-\frac{2}{5}z^3\dots\right)$$

$$Applying\ triangle\ inequality: |\log(1+z)-z|\le\frac{1}{2}|z|^2\left(1+\frac{2}{3}|z|+\frac{2}{4}|z|^2+\cdots\right)\le$$

$$\le\frac{1}{2}|z|^2(1+|z|+|z|^2+\cdots)=\frac{1}{2}\cdot\frac{|z|^2}{1-|z|}\le|z|^2, when\ |z|\le\frac{1}{2}$$

$$Note\ that\ \frac{1}{n}a^{\frac{k}{n}}\le\frac{1}{n}a\le\frac{1}{2}\Rightarrow n\ge 2a. This\ is\ satisfied\ in\ our\ case\ n\to\infty.$$

$$When\ z\ is\ real-valued\ and\ nonnegative,$$

$$z-z^2\le\log(1+z)\le z+z^2$$

$$\sum_{k=1}^{n}\frac{1}{n}a^{\frac{k}{n}}=\frac{a^{\frac{1}{n}}(a-1)}{a^{\frac{1}{n}}-1}\cdot\frac{1}{n}\ and\ \sum_{k=1}^{n}\frac{1}{n^2}a^{\frac{2k}{n}}=\frac{a^{\frac{2}{n}}(a^2-1)}{a^{\frac{2}{n}}-1}\cdot\frac{1}{n^2}$$

$$\lim_{x\to\infty}\frac{a^{\frac{1}{x}}-1}{\frac{1}{x}}\overset{L'H}{=}\lim_{x\to\infty}\frac{a^{\frac{1}{x}}\log a\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=\log(a)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}a^{\frac{k}{n}}=\frac{a-1}{\log(a)}$$

$$\lim_{x\to\infty}\frac{a^{\frac{2}{x}}-1}{\frac{1}{x^2}}\overset{L'H}{=}\lim_{x\to\infty}\frac{a^{\frac{2}{x}}\log a\cdot\left(-\frac{2}{x^2}\right)}{-\frac{2}{x^3}}=+\infty$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n^2}a^{\frac{2k}{n}}=0$$

$$By\ the\ squeeze\ theorem,\lim_{n\to\infty}\sum_{k=1}^{n}\log\left(1+\frac{1}{n}a^{\frac{k}{n}}\right)=\frac{a-1}{\log(a)}.$$

$$Therefore,\qquad \Omega=e^{\frac{2}{\log(3)}+\frac{4}{\log(5)}+\frac{6}{\log(7)}}\cong 161842421123813$$

2059. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}(2k-1)\binom{n}{k-1}\binom{n}{k}}$$

Proposed by Marian Ursărescu-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$Let's\ first\ evaluate\ the\ sum\ in\ closed\ form,$$

$$\binom{m}{v}=\frac{1}{2\pi i}\oint_{|z|=\epsilon}\frac{(1+z)^m}{z^{v+1}}dz\Rightarrow\Psi=\sum_{k=1}^{n}(2k-1)\binom{n}{k-1}\frac{1}{2\pi i}\oint_{|z|=\epsilon}\frac{(1+z)^n}{z^{k+1}}dz$$

$$\Psi=\frac{1}{2\pi i}\oint_{|z|=\epsilon}(1+z)^n\underbrace{\sum_{k=1}^{n}(2k-1)\binom{n}{k-1}\frac{1}{z^{k+1}}}_{\bar{\psi}}dz$$

$$\sum_{k=1}^{n}\binom{n}{k-1}\frac{1}{z^{k+1}}\overset{m=k-1}{=}\sum_{m=0}^{n-1}\binom{n}{m}\frac{1}{z^{m+2}}=-z^{-n-2}+\sum_{m=0}^{n}\binom{n}{m}z^{-m}z^{-2}=$$

$$=-z^{-n-2}+z^{-2}(1+z^{-1})^n$$

$$\Rightarrow\sum_{k=1}^{n}\binom{n}{k-1}z^{-k}=z^{-1}(1+z^{-1})^n-z^{-n+1}$$

$$\bar{\Psi}=z^{-n-2}-z^{-2}(1+z^{-1})^n-2\frac{d}{dz}\left(z^{-1}(1+z^{-1})^n-z^{-n-1}\right)=$$

$$=z^{-n-2}-z^{-2}(1+z^{-1})^n+2z^{-2}(1+z^{-1})^n+2z^{-3}n(1+z^{-1})^{n-1}-2(n+1)z^{-n-2}$$

$$\Psi=\frac{1}{2\pi i}\oint_{|z|=\epsilon}\left[\frac{(1+z)^n}{z^{n+2}}(-1-2n)+\frac{(1+z^{-1})^{2n}}{z^{n+2}}+2n\frac{(1+z)^{2n-1}}{z^{n+2}}\right]dz=$$

$$=0+\binom{2n}{n+1}+2n\binom{2n-1}{n+1}=\binom{2n}{n+1}+(2n)\binom{2n-1}{n+1}=$$

$$=\frac{(2n)!}{(n+1)!\,(n-1)!}+\frac{(2n)(2n-1)!}{(n+1)!\,(n-2)!}=\frac{(2n)!}{(n+1)!}\cdot\frac{1}{(n-2)!}\left(\frac{1}{n-1}+1\right)=$$

$$=\frac{n\cdot(2n)!}{(n+1)!\,(n-1)!}=\frac{(2n)!}{\left((n-1)!\right)^2}\cdot\frac{1}{n+1},\lim_{x\to\infty}\left(\frac{1}{x+1}\right)^{\frac{1}{x}}=e^{\lim_{x\to\infty}\frac{\log(1+x)}{x}}=e^{\lim_{x\to\infty}\frac{1}{1+x}}=e^0=1$$

$$\Omega=\lim_{n\to\infty}\left(\frac{\Gamma(2n+1)}{\Gamma(n)^2}\right)^{\frac{1}{n}}=$$

$$=\lim_{x\to\infty}\left(\frac{1}{\sqrt{2\pi}}\cdot\frac{\exp\left((2n+1)\log(2n+1)-(2n+1)-\frac{1}{2}\log(2n+1)+O\left(\frac{1}{n^2}\right)\right)^{\frac{1}{n}}}{\exp\left(2n\log(n)-2n-\log(n)+O\left(\frac{1}{n^2}\right)\right)}\right)^{\frac{1}{n}}$$

$$=e^{2\log(2)}=4$$

2060. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{1-2\sum_{1\le k<i\le n}\binom{n}{k}\binom{n}{i}\cos\frac{2(i-k)\pi}{3}}$$

Proposed by Daniel Sitaru-Romania

Solution by Ruxandra Daniela Tonilă-Romania

$$\sum_{1\le k<i\le n}\binom{n}{k}\binom{n}{i}\cos\frac{2(i-k)\pi}{3}=\sum_{1\le k<i\le n}\binom{n}{k}\binom{n}{i}\cos\left(\frac{2\pi i}{3}-\frac{2\pi k}{3}\right)=$$

$$=\sum_{1\le k<i\le n}\binom{n}{k}\binom{n}{i}\left(\cos\frac{2\pi i}{3}\cdot\cos\frac{2\pi k}{3}+\sin\frac{2\pi i}{3}\cdot\sin\frac{2\pi k}{3}\right);\quad (1)$$

We have:

$$\sum_{k=1}^{n}\binom{n}{k}\cos\left(\frac{2k\pi}{3}\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{e^{\frac{2\pi ki}{3}}+e^{-\frac{2\pi ki}{3}}}{2}-1=$$

$$=\frac{1}{2}\left(\sum_{k=0}^{n}e^{\frac{2\pi ki}{3}}\cdot\binom{n}{k}+\sum_{k=0}^{n}e^{-\frac{2\pi ki}{3}}\binom{n}{k}\right)-1==\frac{\left(1+e^{\frac{2\pi i}{3}}\right)^n+\left(1+e^{-\frac{2\pi i}{3}}\right)^n}{2}-1;\quad (2)$$

$$1+e^{\frac{2\pi i}{3}}=1+\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=\frac{1}{2}+\frac{i\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$\left(1+e^{\frac{2\pi i}{3}}\right)^n=\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\ and\ \left(1+e^{-\frac{2\pi i}{3}}\right)^n=\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3};\quad (3)$$

From (2) *and* (3) *it follows that*:

$$\sum_{k=1}^{n}\binom{n}{k}\cos\left(\frac{2k\pi}{3}\right)=\frac{\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}+\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}}{2}-1$$

$$\sum_{k=1}^{n}\binom{n}{k}\cos\left(\frac{2k\pi}{3}\right)=\cos\frac{n\pi}{2}-1;\quad (4)$$

Also,

$$\sum_{k=1}^{n}\binom{n}{k}\sin\left(\frac{2n\pi}{3}\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{e^{\frac{2k\pi i}{3}}-e^{-\frac{2k\pi i}{3}}}{2i}=$$

$$=\frac{\left(1+e^{\frac{2\pi i}{3}}\right)^n-\left(1+e^{-\frac{2\pi i}{3}}\right)^n}{2i}=\frac{\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}-\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}}{2i}=\sin\frac{n\pi}{3};\quad (5)$$

$$\left(\sum_{k=1}^{n}\binom{n}{k}\cos\left(\frac{2k\pi}{3}\right)\right)^2=$$

$$=\sum_{k=1}^{n}\binom{n}{k}^2\cos^2\left(\frac{2k\pi}{3}\right)+2\sum_{1\le k<j\le n}\binom{n}{k}\binom{n}{j}\cdot\cos\left(\frac{2k\pi}{3}\right)\cdot\cos\left(\frac{2j\pi}{3}\right)$$

and

$$\left(\sum_{k=1}^{n}\binom{n}{k}\sin\left(\frac{2k\pi}{3}\right)\right)^2=$$

$$=\sum_{k=1}^{n}\binom{n}{k}^2\sin^2\left(\frac{2k\pi}{3}\right)+2\sum_{1\le k<j\le n}\binom{n}{k}\binom{n}{j}\cdot\sin\left(\frac{2k\pi}{3}\right)\cdot\sin\left(\frac{2j\pi}{3}\right)$$

By adding:

$$\left(\sum_{k=1}^{n}\binom{n}{k}\cos\left(\frac{2k\pi}{3}\right)\right)^2+\left(\sum_{k=1}^{n}\binom{n}{k}\sin\left(\frac{2k\pi}{3}\right)\right)^2=$$

$$=\sum_{k=1}^{n}\left(\cos^2\left(\frac{2k\pi}{3}\right)+\sin^2\left(\frac{2k\pi}{3}\right)\right)\cdot\binom{n}{k}^2+$$

$$+2\sum_{1\le k<j\le n}\binom{n}{k}\binom{n}{j}\left(\cos\left(\frac{2k\pi}{3}\right)\cos\left(\frac{2j\pi}{3}\right)+\sin\left(\frac{2k\pi}{3}\right)\sin\left(\frac{2j\pi}{3}\right)\right);\quad(6)$$

$$\sum_{1\le k<i\le n}\binom{n}{k}\binom{n}{i}\cos\frac{2(i-k)\pi}{3}=$$

$$=\frac{1}{2}\left(\cos^2\left(\frac{n\pi}{3}\right)+1-2\cos\left(\frac{n\pi}{3}\right)+\sin^2\left(\frac{n\pi}{3}\right)-\sum_{k=1}^{n}\binom{n}{k}^2\right)=$$

$$=\frac{1}{2}\left(2-2\cos\left(\frac{n\pi}{3}\right)-\binom{2n}{n}+1\right)=\frac{3}{2}-\cos\left(\frac{n\pi}{3}\right)-\frac{1}{2}\binom{2n}{n}$$

Hence,

$$\Omega=\lim_{n\to\infty}\sqrt[n]{1-2\sum_{1\le k<i\le n}\binom{n}{k}\binom{n}{i}\cos\frac{2(i-k)\pi}{3}}=$$

$$=\lim_{n\to\infty}\sqrt[n]{\binom{2n}{n}}\cdot\sqrt[n]{1+\frac{2\cos\left(\frac{n\pi}{3}\right)}{\binom{2n}{n}}-\frac{2}{\binom{2n}{n}}}=$$

$$=\lim_{n\to\infty}\sqrt[n]{\binom{2n}{n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{\binom{2n+2}{n+1}}{\binom{2n}{n}}=\lim_{n\to\infty}\frac{(2n+2)!}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!}=\lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)^2}=4$$

2061. For $x\in(-1,1)$ prove that:

$$\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{\Gamma(k)}x^k=\Gamma(n+1)\frac{x}{(1-x)^{n+1}}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Fao Ler-Iraq

$$\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{\Gamma(k)}x^k=\sum_{k=1}^{\infty}\frac{x^k}{\Gamma(k)}\int_0^\infty t^{n+k-1}e^{-t}\,dt=$$

$$=\int_0^\infty t^{n-1}e^{-t}\sum_{k=1}^{\infty}\frac{(xt)^k}{\Gamma(k)}dt=\int_0^\infty t^{n-1}e^{-t}\sum_{k=0}^{\infty}\frac{(xt)^{k+1}}{k!}dt=$$

$$=\int_0^\infty t^{n-1}e^{-t}xte^{xt}\,dt=x\int_0^\infty t^ne^{-t(1-x)}\,dt=\Gamma(n+1)\frac{x}{(1-x)^{n+1}}$$

Solution 2 by Vuk Stojiljkovik-Serbie

$$\because\ \Gamma(n+1)=n\Gamma(n);\Gamma(n+1)=n!$$

Shifting the index backwards and multiplying top and bottom with $\Gamma(n+1)$

and using the identity $\frac{\Gamma(x+n)}{\Gamma(x)}=(x)_n$, *we get*:

$$x\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(k+1)}\frac{\Gamma(n+1)}{\Gamma(n+1)}x^k=x\Gamma(n+1)\sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}x^k$$

using the famous representation with a substitution $x\to -x$

$$(1+x)^{-a}=\sum_{n=0}^{\infty}(-1)^n\frac{(a)_n}{n!}x^n\text{, gives us the result}$$

$$\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{\Gamma(k)}x^k=\Gamma(n+1)\frac{x}{(1-x)^{n+1}}$$

2062. For $a\in\mathbb{Z}, b\in\mathbb{N}$ prove:

$$\int_0^\pi\frac{\sin(ax)}{\cos(ax)+\cosh(bx)}dx=\frac{2a}{a^2+b^2}\log\left(\frac{2e^{\pi b}}{e^{\pi b}+(-1)^a}\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$\Omega=\int_0^\pi\frac{\sin(ax)}{\cos(ax)+\cosh(bx)}dx=\int_0^\pi\frac{\sin(ax)}{\cos(ax)+\cos(ibx)}dx=$$

$$=\int_0^\pi\frac{\sin(ax)}{2\cos\left(\frac{(a+ib)x}{2}\right)\cos\left(\frac{(a-ib)x}{2}\right)}dx\overset{v=\frac{\pi}{2}}{=}\int_0^{\frac{\pi}{2}}\frac{\sin(2av)}{\cos\big((a+ib)v\big)\cos\big((a-ib)v\big)}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin[(a+ib+a-ib)v]}{\cos\big((a+ib)v\big)\cos\big((a-ib)v\big)}dx=$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin\big((a+ib)v\big)\cos\big((a-ib)v\big)+\sin\big((a-ib)v\big)\cos\big((a+ib)v\big)}{\cos\big((a+ib)v\big)\cos\big((a-ib)v\big)}dv=$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sin((a+ib)v)}{\cos((a+ib)v)} + \frac{\sin((a-ib)v)}{\cos((a-ib)v)} \right) dv =$$

$$= \left[\frac{-1}{a+ib} \log|cos((a+ib)v)| + \frac{-1}{a-ib} \log|cos((a-ib)v)| \right]_0^{\frac{\pi}{2}} =$$

$$= \frac{-1}{a^2+b^2} \left[a\log\left(\cos\left((a+ib)\frac{\pi}{2} \right) \cos\left((a-ib)\frac{\pi}{2} \right) \right) + ib\log\left(\frac{\cos\left((a-ib)\frac{\pi}{2} \right)}{\cos\left((a+ib)\frac{\pi}{2} \right)} \right) \right] =$$

$$= \frac{-1}{a^2+b^2} a\log\left(\frac{cos(a\pi) + cos(ib\pi)}{2} \right) = \frac{-a}{a^2+b^2} \log\left(\frac{\left(e^{\pi b} + (-1)^a\right)^2}{\left(2e^{\frac{\pi b}{2}}\right)^2} \right) =$$

$$= \frac{-2a}{a^2+b^2} \log\left(\frac{e^{\pi b} + (-1)^a}{2e^{\frac{\pi b}{2}}} \right) = \frac{2a}{a^2+b^2} \log\left(\frac{2e^{\frac{\pi b}{2}}}{e^{\pi b} + (-1)^a} \right)$$

2063. Prove that:

$$I = \int_1^{+\infty} \frac{arctan(x)}{x^\alpha} dx = \frac{\psi\left(\frac{\alpha+2}{4}\right) - \psi\left(\frac{\alpha}{4}\right) + \pi}{4(\alpha-1)} \quad (\Re(\alpha) > 1)$$

Proposed by Le Thu-Vietnam

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \int_1^{+\infty} \frac{\arctan x}{x^a} dx \overset{x\to\frac{1}{x}}{=} \int_0^1 \frac{\frac{\pi}{2} - \arctan x}{x^{2-a}} dx = \frac{\pi}{2}\int_0^1 x^{a-2}\,dx - \int_0^1 x^{a-2}\arctan x\,dx$$

$$\Omega \overset{IBP}{=} \frac{\pi}{2(a-1)} - \left[\frac{x^{a-1}\arctan x}{a-1} \right]_0^1 + \frac{1}{a-1}\int_0^1 \frac{x^{a-1}}{1+x^2} dx \overset{x\to x^2}{=}$$

$$= \frac{\pi}{4(a-1)} + \frac{1}{2(a-1)} \int_0^1 \frac{x^{\frac{a}{2}-1}}{1+x} dx$$

$$\int_0^1 \frac{x^{\frac{a}{2}-1}}{1+x} dx = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{n+\frac{a}{2}-1} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+\frac{a}{2}} =$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2n+\frac{\alpha}{2}}-\frac{1}{2n+1+\frac{\alpha}{2}}\right)=\frac{1}{2}\left\{\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha}{4}\right)\right\}$$

$$Therefore,\qquad \Omega=\int_1^{+\infty}\frac{\arctan x}{x^\alpha}dx=\frac{\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha}{4}\right)+\pi}{4(\alpha-1)}$$

Solution 2 by Pham Duc Nam-Vietnam

$$I=\int_1^{+\infty}\frac{arctan(x)}{x^\alpha}dx=\frac{\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha}{4}\right)+\pi}{4(\alpha-1)}\,(\Re(\alpha)>1)$$

$$*\,t=\frac{1}{x}\Rightarrow dt=-\frac{dx}{x^2}\Rightarrow dx=-\frac{dt}{t^2}\Rightarrow I=\int_0^1\frac{arctan\left(\frac{1}{t}\right)}{\left(\frac{1}{t}\right)^\alpha}\frac{dt}{t^2}=\int_0^1\frac{\frac{\pi}{2}-arctan(t)}{t^{2-\alpha}}dt$$

$$=\frac{\pi}{2}\int_0^1\frac{dt}{t^{2-\alpha}}dt-\int_0^1\frac{arctan(t)}{t^{2-\alpha}}dt=\frac{\pi}{2(\alpha-1)}-\int_0^1\frac{arctan(t)}{t^{2-\alpha}}dt$$

$$*\int_0^1\frac{arctan(t)}{t^{2-\alpha}}dt=\int_0^1\frac{1}{t^{2-\alpha}}\left(\sum_{k=0}^{+\infty}\frac{(-1)^kt^{2k+1}}{(2k+1)}\right)dt=\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)}\int_0^1t^{2k+\alpha-1}\,dt$$

$$=\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)}\cdot\frac{1}{(2k+\alpha)}$$

$$=\frac{1}{(\alpha-1)}\sum_{k=0}^{+\infty}\left(\frac{(-1)^k}{(2k+1)}-\frac{(-1)^k}{(2k+\alpha)}\right)=\frac{\pi}{4(\alpha-1)}-\frac{1}{(\alpha-1)}\sum_{k=0}^{+\infty}\left(\frac{(-1)^k}{(2k+\alpha)}\right)$$

$$=\frac{\pi}{4(\alpha-1)}-\frac{1}{2(\alpha-1)}\sum_{k=0}^{+\infty}\left(\frac{(-1)^k}{\left(k+\frac{\alpha}{2}\right)}\right)$$

$$=\frac{\pi}{4(\alpha-1)}-\frac{1}{2(\alpha-1)}\sum_{k=0}^{+\infty}\left(\frac{1}{2k+1+\frac{\alpha}{2}}-\frac{1}{2k+\frac{\alpha}{2}}\right)$$

$$=\frac{\pi}{4(\alpha-1)}+\frac{1}{4(\alpha-1)}\left(\psi\left(\frac{\alpha}{4}\right)-\psi\left(\frac{\alpha+2}{4}\right)\right)$$

$$\Rightarrow I=\frac{\pi}{2(\alpha-1)}-\frac{\pi}{4(\alpha-1)}-\frac{1}{4(\alpha-1)}\left(\psi\left(\frac{\alpha}{4}\right)-\psi\left(\frac{\alpha+2}{4}\right)\right)$$

$$=\frac{\pi}{4(\alpha-1)}+\frac{1}{4(\alpha-1)}\left(\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha}{4}\right)\right)$$

$$=\frac{\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha}{4}\right)+\pi}{4(\alpha-1)}\,(\Re(\alpha)>1)$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_1^{+\infty}\frac{\arctan x}{x^\alpha}dx\overset{y\to\frac{1}{x}}{=}-\int_1^0\frac{y^\alpha\arctan\left(\frac{1}{y}\right)}{y^2}dy=\int_0^1y^{\alpha-2}\arctan\left(\frac{1}{y}\right)dy=$$

$$=\left[\frac{\arctan\left(\frac{1}{y}\right)y^{a-1}}{a-1}\right]_0^1+\frac{1}{a-1}\int_0^1\frac{y^2}{1+y^2}y^{a-1}\frac{dy}{y^2}=$$

$$=\frac{\pi}{4(a-1)}+\frac{1}{a-1}\sum_{n=0}^{\infty}(-1)^n\int_0^1 y^{2n+a-1}\,dy$$

$$\because\ \frac{1}{z}\Phi(-1,1,z^{-1})=\frac{1}{2z}\left[\psi^{(0)}\left(\frac{z+1}{2z}\right)-\psi^{(0)}\left(\frac{1}{2z}\right)\right]$$

$$\Omega=\frac{\pi}{4(\alpha-1)}+\frac{1}{2(\alpha-1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{\frac{2}{\alpha}}}=$$

$$=\frac{\pi}{4(2\alpha-1)}+\frac{1}{2(\alpha-1)}\cdot\frac{1}{2}\left[\psi^{(0)}\left(\frac{\frac{2}{\alpha}+1}{2\cdot\frac{2}{\alpha}}\right)-\psi^{(0)}\left(\frac{1}{2\cdot\frac{2}{\alpha}}\right)\right]$$

$$\textit{Therefore,}\qquad \Omega=\int_1^{+\infty}\frac{\arctan x}{x^{\alpha}}dx=\frac{\psi\left(\frac{\alpha+2}{4}\right)-\psi\left(\frac{\alpha}{4}\right)+\pi}{4(\alpha-1)}$$

2064. Find:

$$\Omega=\lim_{n\to\infty}\left(n\left(\log 2-\sum_{i=1}^{n}\frac{(n+i)^4}{3+(n+i)^5+\cot^{-1}(n+i)}\right)\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\frac{1}{\frac{(n+i)^5}{(n+i)^4}+\frac{3+\cot^{-1}(n+i)}{(n+i)^4}}=\frac{1}{n+i+O\left(\frac{1}{n^4}\right)}$$

$$\frac{1}{n+i}=\frac{1}{n\left(1+\frac{i}{n}\right)}=\sum_{j=0}^{\infty}\frac{(-1)^j}{n}\left(\frac{i}{n}\right)^j$$

$$\sum_{i=1}^{n}\frac{1}{n+i}=\sum_{m=1}^{\infty}\frac{1}{n+m}-\sum_{k=n+1}^{\infty}\frac{1}{n+k}$$

$$\sum_{i=1}^{n}\frac{1}{n+i}=\sum_{i=0}^{\infty}\left(\frac{1}{i+n+1}-\frac{1}{i+2n+1}\right)=\sum_{i=0}^{\infty}\left(\frac{1}{i+n+1}-\frac{1}{i+1}+\frac{1}{i+1}-\frac{1}{i+2n+1}\right)$$

Since the digamma function, $\Psi(z)=-\gamma+\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+z}\right)$; $(z\neq 0,-1,-2,\dots)$.

$$\textit{Then},\sum_{i=1}^{n}\frac{1}{n+i}=(\Psi(2n+1)+\gamma)-(\Psi(n+1)+\gamma)=\Psi(2n+1)-\Psi(n+1)$$

$$\textit{Asymptotically},\Psi(z)\sim\log(z)-\frac{1}{2z}\Rightarrow$$

$$\sum_{i=1}^{n}\frac{1}{n+i}\sim\log(2n+1)-\frac{1}{2(2n+1)}-\log(n+1)+\frac{1}{2(n+1)}$$

$$\Omega=\lim_{n\to\infty}n\left(\log 2-\log\left(\frac{2n+1}{n+1}\right)\right)-\frac{n}{2(n+1)}+\frac{n}{2(2n+1)}=$$

$$=-\frac{1}{4}+\lim_{n\to\infty}n\left(\log 2-\log\left(\frac{2n+1}{n+1}\right)\right)$$

$$\lim_{x\to\infty}\left(\log 2-\log\left(\frac{2x+1}{x+1}\right)\right)=\lim_{x\to\infty}\frac{\log 2-\frac{2x+1}{x+1}}{\frac{1}{x}}\overset{L'H}{=}\lim_{x\to\infty}\frac{-\frac{x+1}{2x+1}\cdot\frac{1}{(x+1)^2}}{-\frac{1}{x^2}}=$$

$$=\lim_{x\to\infty}\frac{1}{\left(2+\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}=\frac{1}{2}$$

$$\textit{Therefore},\quad \Omega=\frac{1}{4}.$$

2065. For $a\in(-1,1)$ prove that:

$$\sum_{k=1}^{n}\frac{1-a\cos\left(\frac{2k\pi}{n}\right)}{1-2a\cos\left(\frac{2k\pi}{n}\right)+a^2}=\frac{n}{1-a^2}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$S=\sum_{k=1}^{n}\frac{1-a\cos\left(\frac{2k\pi}{n}\right)}{1-2a\cos\left(\frac{2k\pi}{n}\right)+a^2}=\frac{1}{2}\sum_{k=1}^{n}\frac{2-a\left(e^{\frac{2\pi ik}{n}}+e^{-\frac{2\pi ik}{n}}\right)}{1-a\left(e^{\frac{2\pi ik}{n}}+e^{-\frac{2\pi ik}{n}}\right)+a^2}=$$

$$=\frac{1}{2}\sum_{k=1}^{n}\frac{\left(1-ae^{\frac{2\pi ik}{n}}\right)+\left(1-ae^{-\frac{2\pi ik}{n}}\right)}{\left(1-ae^{\frac{2\pi ik}{n}}\right)\left(1-ae^{-\frac{2\pi ik}{n}}\right)}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{1-ae^{\frac{2\pi ik}{n}}}+\frac{1}{1-ae^{-\frac{2\pi ik}{n}}}\right)=$$

$$=\frac{1}{2}\sum_{k=1}^{n}\sum_{j=0}^{\infty}a^j\left(e^{\frac{2\pi ijk}{n}}+e^{-\frac{2\pi ijk}{n}}\right)=\frac{1}{2}\sum_{j=0}^{\infty}a^j\left(\sum_{k=1}^{n}e^{\frac{2\pi ijk}{n}}+\sum_{k=1}^{n}e^{-\frac{2\pi ijk}{n}}\right)$$

We know that:

$$\sum_{k=1}^{n}e^{\frac{2\pi ijk}{n}}=\begin{cases}n,\text{if } n|j\\0\text{ others}\end{cases},\text{and}\sum_{k=1}^{n}e^{-\frac{2\pi ijk}{n}}=\begin{cases}n\text{ if } n|j\\0\text{ others}\end{cases}$$

Hence, let $j=mn$:

$$S=\frac{1}{2}\sum_{j=0}^{\infty}a^j\left(\sum_{k=1}^{n}e^{\frac{2\pi ijk}{n}}+\sum_{k=1}^{n}e^{-\frac{2\pi ijk}{n}}\right)=\frac{1}{2}\sum_{m=0}^{\infty}a^{mn}2n=n\sum_{m=0}^{\infty}a^{mn}=\frac{n}{1-a^n}$$

2066. For $|a|\geq 1,|b|\geq 1$ prove that:

$$\int_0^{\pi}\arctan\left(\frac{\sin x}{a+\cos x}\right)\arctan\left(\frac{\sin x}{b+\cos x}\right)dx=\frac{\pi}{2}Li_2\left(\frac{1}{ab}\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\pi}\arctan\left(\frac{\sin x}{a+\cos x}\right)\arctan\left(\frac{\sin x}{b+\cos x}\right)dx\,;|a|,|b|\geq 1$$

$$\arctan\left(\frac{\sin x}{r+\cos x}\right)=-\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{r}\right)^n\sin(nx)}{n}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{a}\right)^n}{n}\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{b}\right)^k}{k}\int_0^{\pi}\sin(nx)\sin(kx)\,dx$$

$$\int_0^{\pi}\sin(nx)\sin(kx)\,dx=\frac{\pi}{2}\delta_{nk}=\begin{cases}\frac{\pi}{2}\text{ if } k=n\\0\text{ if } k\neq n\end{cases}$$

$$\Omega=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{ab}\right)^n}{n^2}=\frac{\pi}{2}Li_2\left(\frac{1}{ab}\right)$$

$$\int_0^{\pi} \arctan\left(\frac{\sin x}{a+\cos x}\right)\arctan\left(\frac{\sin x}{b+\cos x}\right)dx = \frac{\pi}{2}Li_2\left(\frac{1}{ab}\right)$$

2067. Prove that:

$$I = \int_0^{\pi} (\pi - x)^2 \cos(x)\log(\sin(x))\,dx = 2\pi(\log(2) - 2)$$

Proposed by Ankush Kumar Parcha-India

Solution by Pham Duc Nam-Vietnam

$$* \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \Rightarrow I = \int_0^{\pi}\left(\pi - (\pi - x)\right)^2 \cos(\pi - x)\log(\sin(\pi - x))\,dx$$

$$= -\int_0^{\pi} x^2\cos(x)\log(\sin(x))\,dx$$

$$\Rightarrow 2I = \int_0^{\pi}\left((\pi-x)^2 - x^2\right)\cos(x)\log(\sin(x))\,dx = \int_0^{\pi}\left(\pi^2 - 2\pi x\right)\cos(x)\log(\sin(x))\,dx$$

$$* \pi^2\int_0^{\pi}\cos(x)\log(\sin(x))\,dx = \pi^2\int_0^{\pi}\log(\sin(x))\,d(\sin(x))$$

$$= \pi^2(\sin(x)\log(\sin x) - \sin(x))|_0^{\pi} = 0$$

$$* 2\pi\int_0^{\pi} x\cos(x)\log(\sin(x))\,dx, \begin{cases} u = x \\ dv = \cos(x)\log(\sin(x))\,dx \end{cases}$$

$$\Rightarrow \begin{cases} du = dx \\ v = \sin(x)\log(\sin x) - \sin(x) \end{cases}$$

$$\Rightarrow 2\pi\int_0^{\pi} x\cos(x)\log(\sin(x))\,dx$$

$$= \underbrace{2\pi x(\sin(x)\log(\sin x) - \sin(x))|_0^{\pi}}_{=0} - 2\pi\int_0^{\pi}(\sin(x)\log(\sin x) - \sin(x))\,dx$$

$$= -2\pi(\log(4) - 4)$$

$$\Rightarrow 2I = 2\pi(\log(4) - 4) \Rightarrow I = 2\pi(\log(2) - 2)$$

2068. Prove that:

$$\int_0^1 \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{1 - x}}}}\,dx = \frac{2^9}{3}\cdot\frac{839}{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Adrian Popa-Romania

$$I = \int_0^1 \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{1 - x}}}}\,dx = \int_0^1 \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{x}}}}\,dx \overset{\sqrt{x}=t}{=}$$

$$\int_0^1 \sqrt{1-\sqrt{1-\sqrt{1-t}}}\cdot 2t\,dt = 2\int_0^1 \sqrt{1-\sqrt{1-\sqrt{t}}}(1-t)dt \overset{t=u^2}{=}$$

$$= 2\int_0^1 \sqrt{1-\sqrt{1-u}}(1-u^2)\cdot 2u\,du = 2^2\int_0^1 \sqrt{1-\sqrt{u}}(1-(1-u)^2)(1-u)\,du =$$

$$\overset{\sqrt{u}=z}{=} 2^2\int_0^1 \sqrt{1-z}(1-(1-z^2)^2)(1-z^2)\cdot 2z\,dz =$$

$$= 2^3\int_0^1 \sqrt{1-z}(1-(1-z^2)^2)(1-z^2)\cdot z\,dz =$$

$$= 2^3\int_0^1 \sqrt{1-z}(1-1+2z^2-z^4)(1-z^2)z\,dz =$$

$$= 2^3\int_0^1 \sqrt{1-z}\cdot z^3(2-z^2)(1-z^2)\,dz = 2^3\int_0^1 \sqrt{1-z}\cdot z^3(z^4-3z^2+2)\,dz =$$

$$= 2^3\int_0^1 \sqrt{z}(1-z)^3((1-z)^4-3(1-z)^2+2)\,dz \overset{\sqrt{z}=y}{=}$$

$$= 2^3\int_0^1 y(1-y^2)^3((1-y^2)^4-3(1-y^2)^2+2)\cdot 2y\,dy =$$

$$= 2^4\int_0^1 y^2(1-3y^2+3y^4-y^6)(1-4y^2+6y^4-4y^6+y^8-3+6y^2-3y^4+2)\,dy$$

$$= 2^4\int_0^1 (-y^{16}+7y^{14}-18y^{12}+20y^{10}-7y^8-3y^6+2y^4)dy =$$

$$= 2^4\left(-\frac{1}{17}+\frac{2}{15}-\frac{18}{13}+\frac{20}{11}-\frac{7}{9}-\frac{3}{7}+\frac{2}{5}\right) = \frac{2^9}{3}\cdot\frac{839}{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17}$$

2069. Find:

$$\Omega = \int_0^\infty \frac{x\arctan x}{x^3+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^\infty \frac{x\arctan x}{x^3+1}dx = \int_0^1 \frac{x\arctan x}{1+x^3}dx + \int_1^\infty \frac{x\arctan x}{1+x^3}dx =$$

$$=\int_0^1 \frac{x\arctan x}{1+x^3}dx+\int_0^1 \frac{\arctan\left(\frac{1}{x}\right)}{1+x^3}dx=$$

$$=\int_0^1 \frac{x\arctan x}{1+x^3}dx+\int_0^1 \frac{\frac{\pi}{2}-\arctan x}{1+x^3}dx=\frac{\pi}{2}\int_0^1 \frac{dx}{1+x^3}+\int_0^1 \frac{x-1}{1+x^3}\arctan x\,dx$$

$$I_1=\int_0^1 \frac{x-1}{1+x^3}\arctan x\,dx=\frac{1}{3}\int_0^1 \frac{2x-1}{x^2-x+1}\arctan x\,dx-\frac{2}{3}\int_0^1 \frac{\arctan x}{1+x}dx=$$

$$=\frac{1}{3}\int_0^1 \frac{2x-1}{x^2-x+1}\arctan x\,dx-\frac{\pi}{12}\log(2)=$$

$$=\frac{1}{3}\left(\frac{5}{3}G-\frac{\pi}{3}\log(2+\sqrt{3})\right)-\frac{\pi}{12}\log(2)=\frac{5}{9}G-\frac{\pi}{9}\log(2+\sqrt{3})-\frac{\pi}{12}\log(2)$$

$$\Omega=\frac{\pi}{2}\left(\frac{1}{3}\log(2)+\frac{\pi}{3\sqrt{3}}\right)+I_1=\frac{\pi}{12}\log(2)+\frac{\pi^2}{6\sqrt{3}}+\frac{5}{9}G-\frac{\pi}{9}\log(2+\sqrt{3})$$

$$Therefore,\Omega=\int_0^\infty \frac{x\arctan x}{x^3+1}dx=\frac{5}{3}G-\frac{\pi}{3}\log(2+\sqrt{3})$$

2070. Prove that:

$$\int_0^1 \sqrt{x^2+\sqrt{\phi+x^4}}\,dx=\frac{\phi}{2}+\frac{\sqrt{2\phi}}{2}\arctan\left(\sqrt{1+2\phi}+\sqrt{2\phi}\right)-\frac{\pi}{8}\sqrt{2\phi}$$

where ϕ is golden ratio.

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$\Omega=\int_0^1 \sqrt{x^2+\sqrt{\phi+x^4}}\,dx\overset{x^2=\sqrt{\phi}\tan u}{=}$$

$$=\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\sqrt{\sqrt{\phi}\tan u+\sqrt{\phi+\phi\tan^2 u}}\,\frac{\sqrt[4]{\phi}du}{2\sqrt{\tan u}\cos^2 u}$$

$$=\frac{\sqrt{\phi}}{2}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\sqrt{\frac{\sin u}{\cos u}+\frac{1}{\cos u}}\cdot\sqrt{\frac{\cos u}{\sin u}}\cdot\frac{du}{\cos^2 u}=\frac{\sqrt{\phi}}{2}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{\sqrt{1+\sin u}}{\sqrt{\sin u}}\frac{du}{\cos^2 u}=$$

$$=\frac{\sqrt{\phi}}{2}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{\sqrt{\left(\cos\frac{u}{2}+\sin\frac{u}{2}\right)^2}}{\sqrt{2\sin\frac{u}{2}\cos\frac{u}{2}}}\cdot\frac{du}{\left(\cos^2\frac{u}{2}-\sin^2\frac{u}{2}\right)^2}=$$

$$=\frac{\sqrt{2\phi}}{4}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{\cos\frac{u}{2}+\sin\frac{u}{2}}{\sqrt{\sin\frac{u}{2}\cos\frac{u}{2}}}\cdot\frac{\left(1+\tan^2\frac{u}{2}\right)^2du}{\left(1-\tan^2\frac{u}{2}\right)^2}=$$

$$=\frac{\sqrt{2\phi}}{4}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{1+\tan\frac{u}{2}}{\sqrt{\tan\frac{u}{2}}}\cdot\frac{\left(1+\tan^2\frac{u}{2}\right)^2du}{\left(1-\tan^2\frac{u}{2}\right)^2}=$$

$$=\sqrt{2\phi}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{1+\tan\frac{u}{2}}{\left(1+\tan\frac{u}{2}\right)^2}\cdot\frac{\left(1+\tan^2\frac{u}{2}\right)d\left(\sqrt{\tan\frac{u}{2}}\right)}{\left(1-\tan\frac{u}{2}\right)^2}=$$

$$=\sqrt{2\phi}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{\left(1+\tan^2\frac{u}{2}\right)d\left(\sqrt{\tan\frac{u}{2}}\right)}{\left(1+\tan\frac{u}{2}\right)\left(1-\tan\frac{u}{2}\right)^2}\overset{t=\sqrt{\tan\frac{u}{2}}}{=}\sqrt{2\phi}\int_0^{\sqrt{\phi-\sqrt{\phi}}}\frac{(1+t^4)dt}{(1+t^2)(1-t^2)^2}=$$

$$=\sqrt{2\phi}\int_0^{\sqrt{\phi-\sqrt{\phi}}}\left[\frac{1}{4}\left(\frac{1}{(1-t)^2}+\frac{1}{(1+t)^2}\right)+\frac{1}{2}\cdot\frac{1}{1+t^2}\right]dt=$$

$$=\frac{\sqrt{2\phi}}{2}\left[\frac{1}{2}\left(\frac{1}{1-t}-\frac{1}{1+t}\right)+\arctan t\right]_0^{\sqrt{\phi-\sqrt{\phi}}}=$$

$$=\frac{\sqrt{2\phi}}{2}\left[\frac{\sqrt{\phi-\sqrt{\phi}}}{1+\sqrt{\phi}-\phi}+\arctan\left(\sqrt{\phi-\sqrt{\phi}}\right)\right]$$

2071. Prove that:

$$\int_0^{\frac{\pi}{2}}\frac{1}{(1+\sin x)^2\sqrt{\log(\tan x+\sec x)}}dx=\frac{\sqrt{\pi}}{2}\left(1+\frac{1}{\sqrt{3}}\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Samir Zaakouni-Morocco

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{1}{(1+\sin x)^2\sqrt{\log(\tan x+\sec x)}}dx=$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{(\tan x+\sec x)^2\sqrt{\log(\tan x+\sec x)}}dx$$

$$let\ u=\log(\tan x+\sec x)\Rightarrow e^u=\tan x+\sec x;$$

$$du=\sec x\,dx;\ \sec x=\frac{e^u+e^{-u}}{2}$$

$$\Omega=\frac{1}{2}\int_0^\infty\frac{e^u+e^{-u}}{e^{2u}\sqrt{u}}du=\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}}e^{-u}\,du+\frac{1}{2}\int_0^\infty u^{-\frac{1}{2}}e^{-3u}\,du=$$

$$=\frac{1}{2}\int_0^\infty u^{1-\frac{1}{2}}e^{-u}\,du+\frac{\sqrt{3}}{6}\int_0^\infty t^{1-\frac{1}{2}}e^{-t}\,dt;\ (3u=t)$$

$$\Omega=\left(\frac{1}{2}+\frac{\sqrt{3}}{6}\right)\int_0^\infty t^{1-\frac{1}{2}}e^{-t}\,dt=\left(\frac{1}{2}+\frac{\sqrt{3}}{6}\right)\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}\left(1+\frac{1}{\sqrt{3}}\right)$$

$$Therefore,\int_0^{\frac{\pi}{2}}\frac{1}{(1+\sin x)^2\sqrt{\log(\tan x+\sec x)}}dx=\frac{\sqrt{\pi}}{2}\left(1+\frac{1}{\sqrt{3}}\right)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{1}{(1+\sin x)^2\sqrt{\log(\tan x+\sec x)}}dx=$$

$$\overset{x\to\frac{\pi}{2}-x}{=}\int_0^{\frac{\pi}{2}}\frac{1}{(1+\cos x)^2\sqrt{\log(\cot x+\csc x)}}dx$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1}{\cos^2\left(\frac{x}{2}\right)\sqrt{-\log\left(\tan\frac{x}{2}\right)}}\frac{dx}{\cos^2\left(\frac{x}{2}\right)}\overset{2x\to x}{=}\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2 x\sqrt{-\log(\tan x)}}\frac{dx}{\cos^2 x}=$$

$$\overset{\tan x=e^{-t}}{=}\frac{1}{2}\int_0^\infty\frac{e^{-t}(1+e^{-2t})}{\sqrt{t}}dt=\frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt+\frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1}e^{-3t}dt=$$

$$=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)+\frac{1}{2\sqrt{3}}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}\left(1+\frac{1}{\sqrt{3}}\right)$$

$$Therefore,\int_0^{\frac{\pi}{2}}\frac{1}{(1+\sin x)^2\sqrt{\log(\tan x+\sec x)}}dx=\frac{\sqrt{\pi}}{2}\left(1+\frac{1}{\sqrt{3}}\right)$$

2072. Prove that:

$$\int_0^\infty \int_0^\infty \frac{(xy)^2(e^{x+y}-1)}{(e^{2x}-1)(e^{2y}-1)}dxdy = 3\zeta^2(3)$$

where $\zeta(3)$ is the Apery's constant.

Proposed by Ankush Kumar Parcha-India

Solution by Le Thu-Vietnam

$$\Omega = \int_0^\infty \int_0^\infty \frac{(xy)^2(e^{x+y}-1)}{(e^{2x}-1)(e^{2y}-1)}dxdy =$$

$$= \int_0^\infty \int_0^\infty \frac{x^2y^2e^xe^y}{(e^{2x}-1)(e^{2y}-1)}dxdy - \int_0^\infty \int_0^\infty \frac{x^2y^2}{(e^{2x}-1)(e^{2y}-1)}dxdy =$$

$$= \left(\int_0^\infty \frac{z^2e^z}{e^{2z}-1}dz\right)^2 - \left(\int_0^\infty \frac{z^2}{e^{2z}-1}dz\right)^2 = \Omega_1^2 - \Omega_2^2$$

$$\Omega_1 = \frac{1}{2}\int_0^\infty z^{3-1}\operatorname{csch} z\, dz = \frac{1}{2}\cdot\frac{7}{4}\Gamma(3)\zeta(3) = \frac{7}{4}\zeta(3)$$

$$\Omega_2 \overset{u=2z}{=} \frac{1}{8}\int_0^\infty \frac{u^{3-1}}{e^u-1}du = \frac{1}{8}\Gamma(3)\zeta(3) = \frac{1}{4}\zeta(3)$$

$$Hence, \Omega = \left(\frac{7}{4}\zeta(3)\right)^2 - \left(\frac{1}{4}\zeta(3)\right)^2 = 3\zeta^2(3), where$$

$$2\Gamma(y)\zeta(y) = \frac{2^y}{2^y-1}\int_0^\infty t^{y-1}\operatorname{csch} t\, dt$$

$$\Gamma(y)\zeta(y) = \int_0^\infty \frac{t^{y-1}}{e^t-1}dt \;\; for\; \Re(y) > 1$$

$$\frac{e^z}{e^{2z}-1} = \frac{\operatorname{csch} z}{2}$$

2073. Prove that:

$$\int_0^1 \frac{x\log(x^{1296})}{(1-x^2)(1+x^3)}dx = 12\left(\log(2^{18}) + \psi^{(1)}\left(\frac{5}{6}\right) - \psi^{(1)}\left(\frac{2}{6}\right)\right) - (7\pi)^2$$

where $\psi^{(1)}$ is the trigamma function.

Proposed by Ankush Kumar Parcha-India

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^1\frac{x\log(x^{1296})}{(1-x^2)(1+x^3)}dx=1296\int_0^1\frac{x\log x}{(1-x^2)(1+x^3)}dx=1296\Omega_1$$

$$\Omega_1=\frac{1}{3}\int_0^1\frac{x\log x}{x^2-x+1}dx-\frac{1}{12}\int_0^1\frac{\log x}{1+x}dx-\frac{1}{6}\int_0^1\frac{\log x}{(1+x)^2}dx-\frac{1}{4}\int_0^1\frac{\log x}{x-1}dx=$$

$$=\frac{1}{3}\Omega_{1a}-\frac{1}{12}\Omega_{1b}-\frac{1}{6}\Omega_{1c}-\frac{1}{4}\Omega_{1d}$$

$$\Omega_{1a}=\int_0^1\frac{x\log x}{x^2-x+1}dx=\frac{1}{6}\left(\frac{6\pi^2}{6}-\psi'\left(\frac{1}{3}\right)\right)$$

$$\Omega_{1b}=\int_0^1\frac{\log x}{x+1}dx=-\frac{\pi^2}{12}$$

$$\Omega_{1c}=\int_0^1\frac{\log x}{(x+1)^2}dx=-\log 2$$

$$\Omega_{1d}=\int_0^1\frac{\log x}{x-1}dx=\frac{\pi^2}{6},\qquad \Omega_1=\frac{5\pi^2}{432}-\frac{1}{18}\psi'\left(\frac{1}{3}\right)+\frac{1}{6}\log 2$$

$$We\ have:\ \psi_1\left(\frac{1}{3}\right)=\frac{8\pi^2}{9}-\frac{1}{6}\left\{\psi_1\left(\frac{5}{6}\right)-\psi_1\left(\frac{1}{3}\right)\right\},\ then:$$

$$\Omega_1=-\frac{49\pi^2}{1296}+\frac{1}{108}\left\{\psi_1\left(\frac{5}{6}\right)-\psi_1\left(\frac{1}{3}\right)\right\}+\frac{1}{6}\log 2\ \ and$$

$$\Omega=1296\Omega_1=12\left\{\log(2^{18})-\psi_1\left(\frac{5}{6}\right)-\psi_1\left(\frac{1}{3}\right)\right\}-(7\pi)^2$$

Solution 2 by Le Thu-Vietnam

$$\Omega=432\int_0^1\frac{x\log x\,dx}{x^2-x+1}-216\int_0^1\frac{\log x\,dx}{(x+1)^2}+324\int_0^1\frac{\log x\,dx}{1-x}-108\int_0^1\frac{\log x\,dx}{x+1}=$$

$$=432\Omega_4-216\Omega_3+324\Omega_2-108\Omega_1$$

$$\Omega_1=\sum_{n=0}^{\infty}\left[(-1)^n\int_0^1 x^n\log x\,dx\right]=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2}=(2^{1-2}-1)\zeta(2)=-\frac{\pi^2}{12};\left(\zeta(2)=\frac{\pi^2}{6}\right)$$

$$Similarly,\Omega_2=-\sum_{m=1}^{\infty}\frac{1}{m^2}=-\zeta(2)=-\frac{\pi^2}{6}$$

$$\Omega_3\overset{IBP}{=}\left[-\frac{\log x}{x+1}+\log x-\log(x+1)\right]_0^1=-\log 2$$

$$\textit{Using the relation: } \frac{x\sin\theta}{x^2-2\cos\theta\, x+1}=\sum_{m=0}^{\infty}\sin(m\theta)\, x^n$$

$$\textit{Hence, } \Omega_4=\frac{2}{\sqrt{3}}\int_0^1\frac{x\sin\left(\frac{\pi}{3}\right)}{x^2-2\cos\left(\frac{\pi}{3}\right)x+1}\log x\,dx=$$

$$=\frac{2}{\sqrt{3}}\sum_{m=0}^{\infty}\left[\sin\left(\frac{n\pi}{3}\right)\int_0^1 x^n\log x\,dx\right]=-\sum_{m=0}^{\infty}\frac{\sin\left(\frac{m\pi}{3}\right)}{(m+1)^2}=\frac{5\pi^2}{72\sqrt{3}}-\frac{\sqrt{3}}{12}\psi_1\left(\frac{1}{3}\right)$$

$$\textit{We can easily show that: } \psi_1\left(\frac{1}{3}\right)=\frac{16\pi^2}{15}-\frac{1}{5}\psi_1\left(\frac{5}{6}\right)$$

Summing all of them, we obtain the desired result.

2074. Prove that:

$$\int_0^{\pi}\arctan^2\left(\frac{\sin x}{\phi+\cos x}\right)dx=\frac{\pi^3}{30}-\frac{\pi}{2}\log^2\phi,\ \phi-\text{golden ratio.}$$

Proposed by Ty Halpen-USA

Solution by Rana Ranino-Setif-Algerie

$$\arctan\left(\frac{\sin x}{a+\cos x}\right)=-\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{a}\right)^n\sin(nx)}{n}$$

$$\Omega=\int_0^{\pi}\arctan^2\left(\frac{\sin x}{\phi+\cos x}\right)dx=\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{\phi}\right)^n}{n}\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{\phi}\right)^k}{k}\int_0^{\pi}\sin(nx)\sin(kx)\,dx$$

$$\int_0^{\pi}\sin(nx)\sin(kx)\,dx=\begin{cases}\frac{\pi}{2}\text{ if }k=n\\ 0\text{ if }k\neq n\end{cases}$$

$$\Omega=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\phi^2}\right)^n}{n^2}=\frac{\pi}{2}Li_2\left(\frac{1}{\phi^2}\right)$$

$$\textit{From Landen relation: } Li_2(z)+Li_2\left(\frac{z}{z-1}\right)=-\frac{1}{2}\log^2(1-z)\ \textit{and also}$$

$$Li_2(z)+Li_2(-z)=\frac{1}{2}Li_2(z^2)$$

$$\textit{We obtain: } \frac{1}{2}Li_2(z^2)-Li_2(-z)+Li_2\left(\frac{z}{z-1}\right)=-\frac{1}{2}\log^2(1-z)$$

$$\textit{By taking } z=-\frac{1}{\phi}\Rightarrow\frac{1}{2}Li_2\left(\frac{1}{\phi^2}\right)-Li_2\left(\frac{1}{\phi}\right)+Li_2\left(\frac{1}{\phi+1}\right)=-\frac{1}{2}\log^2(\phi)$$

$$\textit{Since: } \phi^2=\phi+1\Rightarrow\frac{1}{\phi}=1-\frac{1}{\phi^2}$$

$$\frac{3}{2}Li_2\left(\frac{1}{\phi^2}\right)-Li_2\left(\frac{1}{\phi}\right)=-\frac{1}{2}\log^2(\phi);\quad (1)$$

$$\textit{From Euler relation: } Li_2(z)-Li_2(1-z)=\frac{\pi^2}{6}-\log(1-z)\log z$$

$$\textit{and taking } z=\frac{1}{\phi^2}$$

$$Li_2\left(\frac{1}{\phi^2}\right)+Li_2\left(\frac{1}{\phi}\right)=\frac{\pi^2}{6}-2\log^2(\phi);\quad (2)$$

$$\textit{From (1) and (2): }\frac{5}{2}Li_2\left(\frac{1}{\phi^2}\right)=\frac{\pi^2}{6}-\frac{5}{2}\log^2(\phi)$$

$$Li_2\left(\frac{1}{\phi^2}\right)=\frac{\pi^2}{15}-\log^2(\phi)$$

$$\textit{Therefore, }\int_0^{\pi}\arctan^2\left(\frac{\sin x}{\phi+\cos x}\right)dx=\frac{\pi^3}{30}-\frac{\pi}{2}\log^2\phi$$

2075.Prove that:

$$\int_0^1\int_0^1\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\frac{x}{y}+\frac{y}{x}}}}\,dx\,dy=\frac{2\cdot 4^n}{4^n-1}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Tapas Das-India

$$\sqrt{2+\frac{x}{y}+\frac{y}{x}}=\sqrt{\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)^2}=\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}$$

$$\sqrt{2+\sqrt{2+\frac{x}{y}+\frac{y}{x}}}=\sqrt{2+\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}}=\sqrt{\left(\sqrt[4]{\frac{x}{y}}+\sqrt[4]{\frac{y}{x}}\right)^2}=\sqrt[4]{\frac{x}{y}}+\sqrt[4]{\frac{y}{x}}$$

$$\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\frac{x}{y}+\frac{y}{x}}}}=\sqrt[2^n]{\frac{x}{y}}+\sqrt[2^n]{\frac{y}{x}}$$

$$\int_0^1\int_0^1\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\frac{x}{y}+\frac{y}{x}}}}dxdy=\int_0^1\int_0^1\left(\sqrt[2^n]{\frac{x}{y}}+\sqrt[2^n]{\frac{y}{x}}\right)dxdy=$$

$$=\int_0^1\left[\frac{x^{\frac{1}{2^n}+1}}{\frac{1}{2^n}+1}\cdot\frac{1}{y^{\frac{1}{2^n}}}+y^{\frac{1}{2^n}}\cdot\frac{x^{-\frac{1}{2^n}+1}}{-\frac{1}{2^n}+1}\right]_0^1dy=\int_0^1\left[\frac{2^n}{2^n+1}\cdot y^{-\frac{1}{2^n}}+\frac{2^n}{2^n-1}\cdot y^{\frac{1}{2^n}}\right]dy=$$

$$=\left[\frac{2^n}{2^n+1}\cdot\frac{y^{-\frac{1}{2^n}+1}}{-\frac{1}{2^n}+1}+\frac{2^n}{2^n-1}\cdot\frac{y^{\frac{1}{2^n}+1}}{\frac{1}{2^n}+1}\right]_0^1=\frac{2^n}{2^n+1}\cdot\frac{2^n}{2^n-1}+\frac{2^n}{2^n-1}\cdot\frac{2^n}{2^n+1}=$$

$$=\frac{2^{2n}}{2^{2n}-1}+\frac{2^{2n}}{2^{2n}-1}=2\cdot\frac{4^n}{4^n-1}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$Let\ \frac{y}{x}=z$$

$$I=\int_0^1\int_0^1\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\frac{x}{y}+\frac{y}{x}}}}dx\,dy$$

$$=\int_0^1 x\int_0^{\frac{1}{x}}\sqrt{2+\sqrt{2+\cdots+\sqrt{2+z+\frac{1}{z}}}}dz\,dx\overset{İBP}{=}$$

$$\left[\frac{x^2}{2}\int_0^{\frac{1}{x}}\sqrt{2+\sqrt{2+\cdots+\sqrt{2+z+\frac{1}{z}}}}dz\right]_{x=0}^{x=1}+\frac{1}{2}\int_0^1\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x+\frac{1}{x}}}}dx$$

$$=\frac{1}{2}\int_0^1\sqrt{2+\sqrt{2+\cdots+\sqrt{2+z+\frac{1}{z}}}}dz+\frac{1}{2}\int_0^1\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x+\frac{1}{x}}}}dx$$

$$=\int_0^1\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x+\frac{1}{x}}}}dx=\int_0^1\left(\sqrt[2^n]{x}+\sqrt[2^n]{\frac{1}{x}}\right)dx=\frac{2*(4^n)}{4^n-1}$$

$$\textit{Note}:\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x+\frac{1}{x}}}}=\sqrt[2^n]{x}+\sqrt[2^n]{\frac{1}{x}}$$

2076. Prove that:

$$\int_0^1\int_0^1\sqrt{-\log|x-y|}\,dxdy=\sqrt{\pi}\left(1-\frac{1}{\sqrt{8}}\right)$$

Proposed by Asmat Qatea-Afghanistan

Solution by Kartick Chandra Betal-India

$$\int_0^1\int_0^1\sqrt{-\log|x-y|}\,dxdy=\int_0^1\left(\int_0^y\sqrt{-\log(y-x)}\,dx+\int_y^1\sqrt{-\log(x-y)}\,dx\right)dy=$$

$$=\int_0^1\left(\int_0^y\sqrt{-\log x}\,dx+\int_y^1\sqrt{-\log(1-x)}\,dx\right)dy=$$

$$=\int_0^1(1-x)\sqrt{-\log x}\,dx+\int_0^1 x\sqrt{-\log(1-x)}\,dx=$$

$$=\int_0^1(1-x)\sqrt{-\log x}\,dx+\int_0^1(1-x)\sqrt{-\log x}\,dx=$$

$$=2\int_0^1(1-x)\sqrt{-\log x}\,dx=2\int_\infty^0(1-e^{-x})\sqrt{x}(-e^{-x})\,dx=$$

$$=2\int_0^\infty\sqrt{x}e^{-x}\,dx-2\int_0^\infty\sqrt{x}e^{-2x}\,dx=2\int_0^\infty\sqrt{x}e^{-x}\,dx-\frac{2}{2\sqrt{2}}\int_0^\infty\sqrt{x}e^{-x}\,dx=$$

$$=\left(2-\frac{1}{\sqrt{2}}\right)\int_0^{\infty} x^{\frac{3}{2}-1}e^{-x}\,dx=\left(2-\frac{1}{\sqrt{2}}\right)\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}\left(1-\frac{1}{2\sqrt{2}}\right)$$

2077. For $|a|<1$ prove that:

$$\int_0^{\pi}\cos x\arctan^2\left(\frac{a\sin x}{1+a\cos x}\right)dx=\frac{\pi}{2}\left(a-\frac{1}{a}\right)\log(1-a^2)-\frac{\pi}{2}a$$

Proposed by Asmat Qatea-Afghanistan

Solution by Rana Ranino-Setif-Algerie

$$\arctan\left(\frac{a\sin x}{1+a\cos x}\right)=-\sum_{n=1}^{\infty}\frac{(-a)^n\sin(nx)}{n}$$

$$\Omega=\int_0^{\pi}\cos x\arctan^2\left(\frac{a\sin x}{1+a\cos x}\right)dx=$$

$$=\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\sum_{k=1}^{\infty}\frac{(-a)^k}{k}\int_0^{\pi}\cos x\sin(nx)\sin(kx)\,dx=$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\sum_{k=1}^{n}\frac{(-a)^k}{k}\int_0^{\pi}\sin\big((n+1)x\big)\sin(kx)\,dx+$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\sum_{k=1}^{n}\frac{(-a)^k}{k}\int_0^{\pi}\sin\big((n-1)x\big)\sin(kx)\,dx$$

$$\sin\big((n+1)x\big)\sin(kx)=\begin{cases}\frac{\pi}{2}\ if\ k=n+1\\ 0\ if\ k\neq n+1\end{cases};$$

$$\sin\big((n-1)x\big)\sin(kx)=\begin{cases}\frac{\pi}{2}\ if\ k=n+1\\ 0\ if\ k\neq n+1\end{cases}$$

$$\Omega=\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-a)^n(-a)^{n+1}}{n(n+1)}+\frac{\pi}{4}\sum_{k=1}^{\infty}\frac{(-a)^k(-a)^{k+1}}{k(k+1)}=-\frac{a\pi}{2}\sum_{n=1}^{\infty}\frac{a^{2n}}{n(n+1)}=$$

$$=\frac{a\pi}{2}\sum_{n=1}^{\infty}\frac{a^{2n}}{n+1}-\frac{a\pi}{2}\sum_{n=1}^{\infty}\frac{a^{2n}}{n}=\frac{a\pi}{2}\left(-1-\frac{\log(1-a^2)}{a^2}\right)+\frac{a\pi}{2}\log(1-a^2)=$$

$$=\frac{\pi}{2}a+\frac{\pi}{2}\left(a\log(1-a^2)-\frac{\log(1-a^2)}{a}\right)$$

Therefore,

$$\int_0^\pi \cos x \arctan^2\left(\frac{a\sin x}{1+a\cos x}\right)dx = \frac{\pi}{2}\left(a-\frac{1}{a}\right)\log(1-a^2)-\frac{\pi}{2}a$$

2078. Find:

$$\Omega = \int_0^1 \log(x)\left(\prod_{n=0}^{\infty}\left(1+\left(\frac{2x}{3+\sqrt{5}}\right)^{2^n}\right)\right)dx$$

Proposed by Bui Hong Suc-Vietnam

Solution by Pham Duc Nam-Vietnam

$$*\ Let: P = \lim_{N\to\infty}\prod_{n=0}^{N}\left(1+\left(\frac{2x}{3+\sqrt{5}}\right)^{2^n}\right), x\in(0,1) \Rightarrow \frac{2x}{3+\sqrt{5}} < 1, let: k = \frac{2x}{3+\sqrt{5}}$$

$$\Rightarrow P = \lim_{N\to\infty}\prod_{n=0}^{N}\left(1+k^{2^n}\right) = \lim_{N\to\infty}(1+k)(1+k^2)(1+k^4)\dots\left(1+k^{2^N}\right)$$

$$\Rightarrow (1-k)P = \lim_{N\to\infty}(1-k)(1+k)(1+k^2)(1+k^4)\dots\left(1+k^{2^N}\right)$$

$$= \lim_{N\to\infty}(1-k^2)(1+k^2)(1+k^4)\dots\left(1+k^{2^N}\right)$$

$$= \lim_{N\to\infty}(1-k^4)(1+k^4)\dots\left(1+k^{2^N}\right) = \dots = \lim_{N\to\infty}\left(1-k^{2^{N+1}}\right) \Rightarrow P = \lim_{N\to\infty}\frac{1-k^{2^{N+1}}}{1-k}$$

$$= \frac{1}{1-k}\left(k<1 \Rightarrow \lim_{N\to\infty}\left(1-k^{2^{N+1}}\right) = 1\right)$$

$$*\Rightarrow \Omega = \int_0^1 \log(x).\frac{1}{1-\frac{2x}{3+\sqrt{5}}}dx = \int_0^1 \log(x)\sum_{k=0}^{\infty}\left(\frac{2}{3+\sqrt{5}}\right)^k x^k dx$$

$$= \sum_{k=0}^{\infty}\left(\frac{2}{3+\sqrt{5}}\right)^k\int_0^1 x^k\log(x)\,dx = -\sum_{k=0}^{\infty}\left(\frac{2}{3+\sqrt{5}}\right)^k.\frac{1}{(k+1)^2}$$

$$= -\frac{3+\sqrt{5}}{2}\sum_{k=0}^{\infty}\left(\frac{2}{3+\sqrt{5}}\right)^{k+1}.\frac{1}{(k+1)^2} = -\frac{3+\sqrt{5}}{2}Li_2\left(\frac{2}{3+\sqrt{5}}\right)$$

$$*\ \frac{2}{3+\sqrt{5}}.\frac{2}{3-\sqrt{5}} = 1 \Rightarrow Li_2\left(\frac{2}{3+\sqrt{5}}\right) = Li_2\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15}-\log^2\frac{1+\sqrt{5}}{2}(Special\ value)$$

$$\Rightarrow \Omega = -\frac{3+\sqrt{5}}{2}\left(\frac{\pi^2}{15}-\log^2\frac{1+\sqrt{5}}{2}\right)$$

2079. Prove that:

$$\int_0^1 \frac{\sqrt{-x^6+x^4\sqrt{x^4+\phi}}-\sqrt{\phi x^2+\phi\sqrt{x^4+\phi}}}{x^4+\phi}dx=\sqrt{2}\left[\arctan\left(\frac{\sqrt{\phi-1}}{\sqrt{2+\phi}}\right)-\frac{\pi}{4}\right]$$

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$\Omega=\int_0^1 \frac{\sqrt{-x^6+x^4\sqrt{x^4+\phi}}-\sqrt{\phi x^2+\phi\sqrt{x^4+\phi}}}{x^4+\phi}dx=$$

$$=\int_0^1 \frac{x^2\sqrt{-x^2+\sqrt{x^4+\phi}}-\sqrt{\phi x^2+\phi\sqrt{x^4+\phi}}}{x^4+\phi}dx\overset{x^2=\sqrt{\phi}\tan t}{=}$$

$$=\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{\sqrt{\phi}\tan t\sqrt{-\sqrt{\phi}\tan t+\sqrt{\phi\tan^2 t+\phi}}-\sqrt{\phi\sqrt{\phi}\tan t+\phi\sqrt{\phi\tan^2 t+\phi}}}{\sqrt{\tan t}}\frac{\sqrt[4]{\phi}(1+\tan^2 t)dt}{2\sqrt{\tan t}}$$

$$=\frac{\sqrt[4]{\phi}}{2\phi}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{\sqrt[4]{\phi^3}\tan t\sqrt{-\frac{\sin t}{\cos t}+\frac{1}{\cos t}}-\sqrt[4]{\phi^3}\sqrt{\frac{\sin t}{\cos t}+\frac{1}{\cos t}}}{\sqrt{\tan t}}dt=$$

$$=\frac{1}{2}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{\tan t\sqrt{-\frac{\sin t}{\cos t}+\frac{1}{\cos t}}-\sqrt{\frac{\sin t}{\cos t}+\frac{1}{\cos t}}}{\sqrt{\tan t}}dt=$$

$$=\frac{1}{2}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\left(\sqrt{\frac{2\tan\frac{t}{2}}{1-\tan^2\frac{t}{2}}}\sqrt{\frac{\cos\frac{t}{2}-\sin\frac{t}{2}}{\cos\frac{t}{2}+\sin\frac{t}{2}}}-\frac{\sqrt{\frac{\cos\frac{t}{2}+\sin\frac{t}{2}}{\cos\frac{t}{2}-\sin\frac{t}{2}}}}{\sqrt{\frac{2\tan\frac{t}{2}}{1-\tan^2\frac{t}{2}}}}\right)dt=$$

$$=\frac{1}{2}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\left(\frac{\sqrt{2\tan\frac{t}{2}}}{1+\tan\frac{t}{2}}+\frac{1+\tan\frac{t}{2}}{\sqrt{2\tan\frac{t}{2}}}\right)dt=$$

$$=\frac{1}{2}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\left(\frac{2\tan\frac{t}{2}+\left(1-\tan\frac{t}{2}\right)^2}{\left(1+\tan\frac{t}{2}\right)\sqrt{2\tan\frac{t}{2}}}\right)dt=$$

$$=-\frac{1}{2\sqrt{2}}\int_0^{\arctan\frac{1}{\sqrt{\phi}}}\frac{1+\tan^2\frac{t}{2}}{\left(1+\tan\frac{t}{2}\right)\sqrt{\tan\frac{t}{2}}}dt \overset{u^2=\tan\frac{t}{2}}{=}$$

$$=-\frac{1}{2\sqrt{2}}\int_0^{\sqrt{\phi-\sqrt{\phi}}}\frac{4u}{(1+u)^2}\frac{du}{u}=-\sqrt{2}\int_0^{\sqrt{\phi-\sqrt{\phi}}}\frac{du}{1+u^2}=-\sqrt{2}\arctan t\Big|_0^{\sqrt{\phi-\sqrt{\phi}}}=$$

$$=-\sqrt{2}\arctan\left(\sqrt{\phi-\sqrt{\phi}}\right)$$

2080. Prove that

$$\int_0^\infty\frac{x+1}{\sqrt{x}(x^2+1)}dx=\pi\sqrt{2},\qquad \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin y}{\sin x}\sqrt[3]{\frac{\sin(2x)}{\sin(2y)}}dydx=\pi\sqrt{3}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Ravi Prakash-New Delhi-India

$$I_1=\int_0^\infty\frac{x+1}{\sqrt{x}}\frac{1}{1+x^2}dx \overset{x=\tan\theta;\theta\in\left(0,\frac{\pi}{2}\right)}{=}\int_0^{\frac{\pi}{2}}\frac{\tan\theta+1}{\sqrt{\tan\theta}}\frac{\sec^2\theta}{\sec^2\theta}d\theta=$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin\theta+\cos\theta}{\sqrt{\sin\theta\cos\theta}}d\theta \overset{\sin\theta-\cos\theta=t}{=}\int_{-1}^1\frac{\sqrt{2}dt}{\sqrt{1-t^2}}=2\sqrt{2}\int_0^1\frac{dt}{\sqrt{1-t^2}}=$$

$$=2\sqrt{2}\arcsin t\Big|_0^1=2\sqrt{2}\cdot\frac{\pi}{2}=\pi\sqrt{2}$$

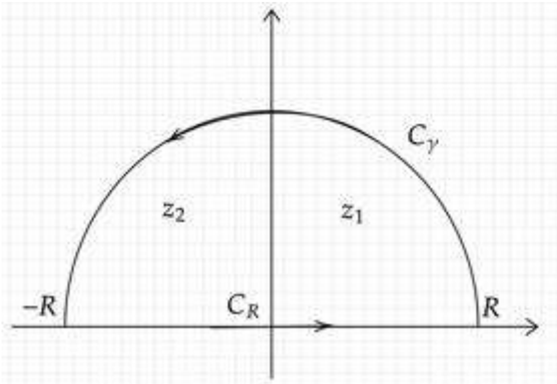
$$I_2=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin y}{\sin x}\sqrt[3]{\frac{\sin(2x)}{\sin(2y)}}dydx=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x\cos x}{\sin^3 x}\right)^{\frac{1}{3}}\left(\frac{\sin^3 y}{\sin y\cos y}\right)^{\frac{1}{3}}dydx=$$

$$=\int_0^{\frac{\pi}{2}}(\sin x)^{-\frac{2}{3}}(\cos x)^{\frac{1}{3}}dx\cdot\int_0^{\frac{\pi}{2}}(\sin y)^{\frac{2}{3}}(\cos y)^{-\frac{1}{3}}dy=$$

$$=\frac{1}{2}\beta\left(\frac{-\frac{2}{3}+1}{2},\frac{\frac{1}{3}+1}{2}\right)\frac{1}{2}\beta\left(\frac{\frac{2}{3}+1}{2},\frac{-\frac{1}{3}+1}{2}\right)=$$

$$=\frac{1}{4}\beta\left(\frac{1}{6},\frac{2}{3}\right)\beta\left(\frac{5}{6},\frac{1}{3}\right)=\frac{1}{4}\cdot\frac{\sqrt{\frac{1}{6}}\sqrt{\frac{2}{3}}}{\sqrt{\frac{5}{6}}}\cdot\frac{\sqrt{\frac{5}{6}}\sqrt{\frac{1}{3}}}{\sqrt{\frac{7}{6}}}=\frac{3}{2}\sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}=\frac{3}{2}\frac{\pi}{\sin\left(\frac{\pi}{3}\right)}=\pi\sqrt{3}$$

Solution 2 by Pham Duc Nam-Vietnam



$$I=\int_0^\infty \frac{x+1}{\sqrt{x}(x^2+1)}dx=2\int_0^\infty \frac{x+1}{x^2+1}d(\sqrt{x})=\int_{-\infty}^\infty \frac{t^2+1}{t^4+1}dt \overset{t=\sqrt{x}}{=}$$

$$\left(f(t)=\frac{t^2+1}{t^4+1}\ \text{is even function}\right)$$

Using the above contour: $C=C_\gamma \mathrm{v} C_R$, *counter − clockwise direction*, $R>1\to\infty$

$$f(z)=\frac{z^2+1}{z^4+1}\ \text{has two poles (order 1)}: z_1=\frac{1+i}{\sqrt{2}}\ \text{and}\ z_2=-\frac{1-i}{\sqrt{2}}$$

$$\int_C \frac{z^2+1}{z^4+1}dz=2\pi i\sum Res\big(f(z),z_{1,2}\big)=$$

$$=2\pi i\left(\lim_{z\to\frac{1+i}{\sqrt{2}}}\left(z-\frac{1+i}{\sqrt{2}}\right)\frac{z^2+1}{z^4+1}+\lim_{z\to-\frac{1-i}{\sqrt{2}}}\left(z+\frac{1-i}{\sqrt{2}}\right)\frac{z^2+1}{z^4+1}\right)=$$

$$=2\pi i\left(\frac{-i}{2\sqrt{2}}+\frac{-i}{2\sqrt{2}}\right)=\pi\sqrt{2}$$

$$\int_C \frac{z^2+1}{z^4+1}dz=\int_{C_R}\frac{x^2+1}{x^4+1}dx+\int_{C_\gamma}\frac{z^2+1}{z^4+1}dz,$$

$$\text{By ML inequality}: \lim_{R\to\infty}\int_{C_\gamma}\frac{z^2+1}{z^4+1}dz=0$$

$$\lim_{R\to\infty}\int_{C_R}\frac{x^2+1}{x^4+1}dx=\lim_{R\to\infty}\int_{-R}^R\frac{x^2+1}{x^4+1}dx=\int_C\frac{z^2+1}{z^4+1}dz=\pi\sqrt{2}\Rightarrow I=\pi\sqrt{2}$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$\int_0^\infty \frac{x+1}{\sqrt{x}(x^2+1)}dx \overset{y=\sqrt{x}}{=} \int_0^\infty \frac{y^2+1}{y(y^4+1)}\cdot 2y\,dy = 2\int_0^\infty \frac{y^2+1}{y^4+1}dy =$$

$$= 2\int_0^\infty \frac{1}{y^2+\frac{1}{y^2}}\left(1+\frac{1}{y^2}\right)dy = 2\int_0^\infty \frac{1}{\left(y-\frac{1}{y}\right)^2+2}d\left(y-\frac{1}{y}\right) =$$

$$= \frac{2}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(y-\frac{1}{y}\right)\Big|_0^\infty = \frac{2}{\sqrt{2}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \pi\sqrt{2}$$

2081. Find:

$$\Omega = \int_0^\infty e^{-ax}\log^6(bx)\,dx\,, a, b > 0$$

Proposed by Bui Hong Suc-Vietnam

Solution by Fao Ler-Iraq

$$\Omega = \int_0^\infty e^{-ax}\log^6(bx)\,dx = \frac{d^6}{dy^6}\int_0^\infty e^{-ax}(bx)^y\,dy; (y=0)$$

$$= \frac{d^6}{dy^6}b^y\int_0^\infty e^{-x}\left(\frac{x}{a}\right)^y d\left(\frac{x}{a}\right) = \frac{d^6}{dy^6}\frac{b^y}{a^{y+1}}\int_0^\infty x^y e^{-x}\,dx =$$

$$= \frac{1}{a}\frac{d^6}{dy^6}\left(\frac{b}{a}\right)^y\Gamma(y+1) = \frac{1}{a}\sum_{k=0}^{6}\binom{6}{k}\left(\frac{d^k}{dy^k}\Gamma(y+1)\right)\left(\frac{d^{6-k}}{dy^{6-k}}\left(\frac{b}{a}\right)^y\right) =$$

$$= \frac{1}{a}\sum_{k=0}^{6}\binom{6}{k}\Gamma^{(k)}(y+1)\left(\frac{b}{a}\right)^y\left(\ln\frac{b}{a}\right)^{6-k} = \frac{1}{a}\sum_{k=0}^{6}\binom{6}{k}\Gamma^{(k)}(1)\left(\ln\frac{b}{a}\right)^{6-k}$$

2082. Find:

$$\Omega = \int \frac{(x+1)^4}{x^4+8x^3+30x^2+64x+65+3e^x}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$x^4+8x^3+30x^2+64x+65 =$$

$$= (x^4+4x^3+6x^2+4x+1)+4(x^3+3x^2+3x+1)+12(x^2+2x+1)+24(x+1) + 24 =$$

$$= (x+1)^4+4(x+1)^3+12(x+1)^2+24(x+1)+24$$

$$\Omega=\int\frac{x^4+4x^3+6x^2+4x+1}{x^4+8x^3+30x^2+64x+65+3e^x}dx=$$

$$=\int\frac{(x+1)^4dx}{(x+1)^4+4(x+1)^3+12(x+1)^2+24(x+1)+24+3e^x}=$$

$$=\int\left[1-\frac{4(x+1)^3+12(x+1)^2+24(x+1)+3e^x}{(x+1)^4+4(x+1)^3+12(x+1)^2+24+3e^x}\right]dx=$$

$$=x-\log\left[(x+1)^4+4(x+1)^3+12(x+1)^2+24+3e^x\right]+C$$

Solution 2 by Pham Duc Nam-Vietnam

$$\Omega=\int\frac{x^4+4x^3+6x^2+4x+1}{x^4+8x^3+30x^2+64x+65+3e^x}dx$$

$$=\int\frac{x^4+8x^3+30x^2+64x+65+3e^x-(4x^3+24x^2+60x+64+3e^x)}{x^4+8x^3+30x^2+64x+65+3e^x}dx$$

$$=\int\frac{x^4+8x^3+30x^2+64x+65+3e^x}{x^4+8x^3+30x^2+64x+65+3e^x}dx$$

$$-\int\frac{4x^3+24x^2+60x+64+3e^x}{x^4+8x^3+30x^2+64x+65+3e^x}dx$$

$$=\int dx-\int\frac{d(x^4+8x^3+30x^2+64x+65+3e^x)}{x^4+8x^3+30x^2+64x+65+3e^x}$$

$$=x-\log(x^4+8x^3+30x^2+64x+65+3e^x)+C$$

2083. Prove that:

$$\int_0^1\frac{1}{x}\log^2\left(x+\sqrt{1+x^2}\right)\arccos x\,dx=\frac{3\pi}{32}\zeta(3)$$

Proposed by Ose Favour-Nigeria

Solution by Le Thu-Vietnam

$$\text{arcsin}\,h(z):=\log\left(z+\sqrt{1+z^2}\right)$$

$$\text{arcsin}\,h(z)=\frac{1}{i}\arcsin(iz)$$

$$\textit{Maclaurin series}:\arcsin^2 z=\frac{1}{2}\sum_{n=1}^{\infty}\frac{4^n(n!)^2z^{2n}}{n^2(2n)!}$$

$$\text{arcsin}\,h^2(z)=-\arcsin^2(iz)=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n4^n(n!)^2z^{2n}}{n^2(2n)!}$$

On the other hand,

$$\Omega = \int_0^1 \frac{\text{arcsin}\,h^2(x)\arccos x}{x}dx = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n 4^n \Gamma^2(n+1)}{n^2\Gamma(2n+1)}\int_0^1 x^{2n-1}\arccos x\,dx$$

$$Recall, \int_0^1 x^a \arccos x\,dx = \frac{\sqrt{\pi}\Gamma\left(\frac{a}{2}+1\right)}{(a+1)^2\Gamma\left(\frac{a+1}{2}\right)}, for\ \Re(a) > -1$$

This can be proved easily by integration by parts and using the Beta function:

$$\int_0^1 x^{2n-1}\arccos x\,dx = \frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{4n^2\Gamma(n)}, for\ n \in \mathbb{N}$$

$$By\ Legendre\ duplication\ formula, \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2n)}{2^{2n-1}\Gamma(n)}$$

$$\int_0^1 x^{2n-1}\arccos x\,dx = \frac{\sqrt{\pi}\frac{\sqrt{\pi}\Gamma(2n)}{2^{2n-1}\Gamma(n)}}{4n^2\Gamma(n)} = \frac{\pi\Gamma(2n)}{4^n 2n^2\Gamma^2(n)}$$

$$Hence, \Omega = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n 4^n\Gamma^2(n+1)}{n^2\Gamma(2n+1)}\cdot\frac{\pi\Gamma(2n)}{4^n 2n^2\Gamma^2(n)} =$$

$$= -\frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} = \frac{\pi}{8}\eta(3) = \frac{\pi}{8}(1-2^{1-3})\zeta(3) = \frac{3\pi}{32}\zeta(3)$$

2084. Prove that:

$$\Omega = \int_0^1\int_0^1 \frac{arcsin(xy)}{xy}dxdy = \frac{\pi^3}{48} + \frac{\pi}{4}log^2(2)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Pham Duc Nam-Vietnam

$$*\int_0^{\frac{\pi}{2}} log(sin(t))\,dt = \int_0^{\frac{\pi}{2}} log(cos(t))\,dt = -\frac{\pi}{2}log(2)$$

$$*\int_0^{\frac{\pi}{2}} log^2(sin(t))\,dt = \int_0^{\frac{\pi}{2}} log^2(cos(t))\,dt = \int_0^{\frac{\pi}{2}} log^2(sin(2t))\,dt$$

$$*\,arcsin(x) = \sum_{k=0}^{\infty}\frac{C_{2k}^k x^{2k+1}}{4^k(2k+1)} \text{ and } \frac{2}{(2k+1)^3} = \int_0^1 x^{2k} log^2(x)\,dx$$

$$\underline{Denote}: I = \int_0^{\frac{\pi}{2}} log^2(sin(t))\,dt, J = \int_0^{\frac{\pi}{2}} log(sin(t))\,log(cos(t))\,dt$$

$$*\,\Omega=\int_0^1\int_0^1\frac{arcsin(xy)}{xy}dxdy=\int_0^1\int_0^1\frac{1}{xy}\sum_{k=0}^{\infty}\frac{C_{2k}^k(xy)^{2k+1}}{4^k(2k+1)}dxdy$$

$$=\sum_{k=0}^{\infty}\frac{C_{2k}^k}{4^k(2k+1)}\int_0^1\int_0^1(xy)^{2k}dxdy=\sum_{k=0}^{\infty}\frac{C_{2k}^k}{4^k(2k+1)^3}$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{C_{2k}^k}{4^k}\int_0^1 x^{2k}\,log^2(x)\,dx$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}C_{2k}^k\left(\frac{x}{2}\right)^{2k}\int_0^1 log^2(x)\,dx=\frac{1}{2}\int_0^1\frac{log^2(x)}{\sqrt{1-x^2}}dx,let:x=sin(t)\Rightarrow\frac{1}{2}\int_0^{\frac{\pi}{2}}log^2(sin(t))\,dt$$

$$=\frac{1}{2}I$$

$$*\,I=\int_0^{\frac{\pi}{2}}log^2(sin(t))\,dt=\int_0^{\frac{\pi}{2}}log^2(sin(2t))\,dt$$

$$=\int_0^{\frac{\pi}{2}}(log(2)+log(sin(t))+log(cos(t)))^2\,dt$$

$$=\int_0^{\frac{\pi}{2}}\Big(log^2(2)+log^2(sin(t))+log^2(cos(t))$$

$$+2(log(2)\,log(sin(t))+log(2)\,log(cos(t))+log(sin(t))\,log(cos(t)))\Big)\,dt$$

$$=\frac{\pi}{2}log^2(2)+2I+4\,log(2)\left(-\frac{\pi}{2}log(2)\right)+2J\Leftrightarrow I+2J=\frac{3\pi}{2}log^2(2)$$

$$*\,2I-2J=\int_0^{\frac{\pi}{2}}log^2(sin(t))\,dt+\int_0^{\frac{\pi}{2}}log^2(cos(t))\,dt-2\int_0^{\frac{\pi}{2}}log(sin(t))\,log(cos(t))\,dt$$

$$=\int_0^{\frac{\pi}{2}}log^2(tan(t))\,dt,let:tan(t)=e^u\Rightarrow 2I-2J$$

$$=\int_{-\infty}^{\infty}\frac{u^2e^u}{e^{2u}+1}du=2\int_0^{\infty}\frac{u^2e^u}{e^{2u}+1}du=2\int_0^{\infty}\frac{u^2e^{-u}}{e^{-2u}+1}du$$

$$=2\int_0^{\infty}u^2\sum_{k=0}^{\infty}(-1)^ke^{-(2k+1)u}\,du=2\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}u^2\,e^{-(2k+1)u}du=4\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=4.\frac{\pi^3}{32}$$

$$=\frac{\pi^3}{8}$$

$$*\begin{cases} I+2J=\frac{3\pi}{2}log^2(2) \\ 2I-2J=\frac{\pi^3}{8}\end{cases} \Rightarrow \begin{cases} I=\frac{\pi^3}{24}+\frac{\pi}{2}log^2(2) \\ J=-\frac{\pi^3}{48}+\frac{\pi}{2}log^2(2)\end{cases} \Rightarrow \frac{1}{2}I=\frac{1}{2}\left(\frac{\pi^3}{24}+\frac{\pi}{2}log^2(2)\right)$$

$$=\boxed{\frac{\pi^3}{48}+\frac{\pi}{4}log^2(2)}$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\int_0^1\frac{\arcsin(xy)}{xy}dxdy\overset{t=xy}{=}\int_0^1\frac{1}{x}\int_0^1\frac{\arcsin t}{t}dt\overset{IBP}{=}$$

$$=\left[\log x\int_0^x\frac{\arcsin t}{t}dt\right]_0^1-\int_0^1\frac{\arcsin x\log x}{x}dx=$$

$$=-\int_0^1\frac{\arcsin x\log x}{x}dx\overset{IBP}{=}-\left[\frac{1}{2}\log^2 x\arcsin x\right]_0^1+\frac{1}{2}\int_0^1\frac{\log^2 x}{\sqrt{1-x^2}}dx\overset{x=\sin\theta}{=}$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log^2(\sin\theta)\,d\theta=\frac{1}{8}\lim_{s\to\frac{1}{2}}\frac{d^2}{ds^2}\int_0^{\frac{\pi}{2}}\sin^{2s-\frac{1}{2}}\theta\,d\theta=\frac{1}{16}\lim_{s\to\frac{1}{2}}\frac{d^2}{ds^2}B\left(s,\frac{1}{2}\right)=$$

$$=\frac{1}{16}\lim_{s\to\frac{1}{2}}\left\{\left(\psi(s)-\psi\left(s+\frac{1}{2}\right)\right)^2+\psi'(s)-\psi'\left(s+\frac{1}{2}\right)\right\}B\left(s,\frac{1}{2}\right)=$$

$$=\frac{\pi}{16}\left\{\left(\psi\left(\frac{1}{2}\right)-\psi(1)\right)^2+\psi'\left(\frac{1}{2}\right)-\psi'(1)\right\}=\frac{\pi}{16}\{(-2\log 2)^2+3\zeta(2)-\zeta(2)\}$$

$$\textit{Therefore,}\quad \Omega=\int_0^1\int_0^1\frac{\arcsin(xy)}{xy}dxdy=\frac{\pi}{4}\log^2 2+\frac{\pi^3}{48}$$

2085. For $x>0$, prove:

$$\int_0^\pi\frac{\sin(2y)\left(1+\cos\left(\frac{y}{2}\right)\right)}{1+x\cos\left(\frac{y}{2}\right)}dy<\frac{8}{15}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Adrian Popa-Romania

$$y\in[0,\pi]\Rightarrow\frac{y}{2}\in\left[0,\frac{\pi}{2}\right]\Rightarrow\cos\frac{y}{2}\in[0,1]\Rightarrow x\cos\frac{y}{2}>0$$

$$\int_0^\pi\frac{\sin(2y)\left(1+\cos\left(\frac{y}{2}\right)\right)}{1+x\cos\left(\frac{y}{2}\right)}dy<\int_0^\pi\sin 2y\left(1+\cos\frac{y}{2}\right)dy=$$

$$=\int_0^\pi 2\sin y\cos y\left(1+\cos\frac{y}{2}\right)dy=\int_0^\pi 4\sin\frac{y}{2}\cos\frac{y}{2}\left(2\cos^2\frac{y}{2}-1\right)\left(1+\cos\frac{y}{2}\right)dy=$$

$$\overset{\cos\frac{y}{2}=t}{=} 8\int_0^1 t(2t^2-1)(1+t)\,dt = 8\int_0^1 (2t^3-t)(1+t)dt =$$

$$= 8\int_0^1 (2t^4+2t^3-t-t^2)\,dt = 8\left(\frac{2}{5}+\frac{1}{2}-\frac{1}{2}-\frac{1}{3}\right) = \frac{8}{15}$$

2086. Prove that:

$$\sum_{k=0}^{\infty}\frac{\cos(xk)}{(nk)!} = \frac{1}{n}\sum_{k=1}^{n} e^{\cos\left(\frac{x+2k\pi}{n}\right)}\cos\left[\sin\left(\frac{x+2k\pi}{n}\right)\right]$$

Proposed by Asmat Qatea-Afghanistan

Solution by Bui Hong Suc-Vietnam

$$\frac{1}{n}\sum_{k=1}^{n} e^{\cos\left(\frac{x+2k\pi}{n}\right)}\cos\left[\sin\left(\frac{x+2k\pi}{n}\right)\right] = \frac{1}{2n}\sum_{k=1}^{n} e^{\cos\left(\frac{x+2k\pi}{n}\right)}\left(e^{i\sin\left(\frac{x+2k\pi}{n}\right)} + e^{-i\sin\left(\frac{x+2k\pi}{n}\right)}\right)$$

$$=$$

$$= \frac{1}{2n}\left(\sum_{k=1}^{n} e^{\cos\left(\frac{x+2k\pi}{n}\right)+i\sin\left(\frac{x+2k\pi}{n}\right)} + \sum_{k=1}^{n} e^{\cos\left(\frac{x+2k\pi}{n}\right)+i\sin\left(\frac{x+2k\pi}{n}\right)}\right) =$$

$$= \frac{1}{2n}\left(\sum_{k=1}^{n} e^{e^{i\left(\frac{x+2k\pi}{n}\right)}} + \sum_{k=1}^{n} e^{e^{-i\left(\frac{x+2k\pi}{n}\right)}}\right) = \frac{1}{2n}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_{k=1}^{n} e^{i\left(\frac{x+2k\pi}{n}\right)j} + \sum_{k=1}^{n} e^{-i\left(\frac{x+2k\pi}{n}\right)j}\right) =$$

$$= \frac{1}{2n}\sum_{j=0}^{\infty}\frac{1}{j!}\left(e^{ij\frac{x}{n}}\sum_{k=1}^{n} e^{i\frac{2k\pi}{n}j} + e^{-ij\frac{x}{n}}\sum_{k=1}^{n} e^{-i\frac{2k\pi}{n}j}\right) \overset{j=kn}{=}$$

$$= \frac{1}{2n}\sum_{k=0}^{\infty}\frac{1}{(kn)!}\left(e^{ikn\frac{x}{n}} + e^{-ikn\frac{x}{n}}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(kn)!}\left(e^{ik}+e^{-ik}\right) = \sum_{k=0}^{\infty}\frac{\cos(xk)}{(nk)!}$$

2087. Let $(x_n)_{n\geq 1}$ be sequence of real numbers such that

$$x_n = \sum_{k=1}^{n}\sin\frac{\pi}{k} - \pi\log n.\ \text{Find:}$$

$$\Omega = \lim_{n\to\infty} x_n \cdot \sum_{k=1}^{n}\frac{1}{n+\sqrt[3]{(k+1)^2(k^2+1)^2}}$$

Proposed by Florică Anastase-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$x_n = \sum_{k=1}^{n}\sin\frac{\pi}{k} - \pi\log n,\ x\in\left(0,\frac{\pi}{2}\right)\ and\ x-\frac{x^3}{3}\leq \sin x\leq x$$

$$1+\sum_{k=3}^{n}\left(\frac{\pi}{k}-\frac{\pi^3}{3k^3}\right)\le\sum_{k=1}^{n}\sin\frac{\pi}{k}=1+\sum_{k=3}^{n}\sin\frac{\pi}{k}\le 1+\pi\sum_{k=3}^{n}\frac{1}{k}\le$$

$$\le 1+\pi\left(\log n-\frac{1}{2}\right), because \sum_{k=3}^{n}\frac{1}{k}\le 1+\log n$$

$$1+\pi\left(\log n-\frac{3}{2}\right)-\frac{\pi^3}{3}\zeta(3)\le\sum_{k=1}^{n}\sin\frac{\pi}{k}\le 1+\pi\left(\log n-\frac{1}{2}\right)$$

$$1-\frac{3\pi}{2}-\frac{\pi^3}{3}\zeta(3)\le\sum_{k=1}^{n}\sin\frac{\pi}{k}\le 1-\frac{\pi}{2}; n\ge 2\Rightarrow (x_n)_{n\ge1}-bounded.$$

$$\sum_{k=1}^{n}\frac{1}{n+\sqrt[3]{(k+1)^2(k^2+1)^2}}=\sum_{k=1}^{n}\pi\{k\le n\}\frac{1}{n+\sqrt[3]{(k+1)^2(k^2+1)^2}}\le\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+\sqrt[3]{(k+1)^2(k^2+1)^2}}=\sum_{k=1}^{n}\pi\{k\le n\}\cdot\frac{1}{n+\sqrt[3]{(k+1)^2(k^2+1)^2}}$$

$$(x_n)_{n\ge1}-bouded, therefore:$$

$$\Omega=\lim_{n\to\infty}x_n\cdot\sum_{k=1}^{n}\frac{1}{n+\sqrt[3]{(k+1)^2(k^2+1)^2}}=0$$

2088. For $m\ge 0$, find:

$$\Omega(\mathrm{m})=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{(2n+1)!!}\right)^{m+1}-\left(\sqrt[n]{(2n-1)!!}\right)^{m+1}\right)\cdot\sin^m\frac{\pi}{n}$$

Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$z!!=2^{\frac{1}{4}(1+2z-\cos\pi z)}\pi^{\frac{1}{4}(\cos(\pi z)-1)}\Gamma\left(1+\frac{1}{2}z\right)$$

$$(2n+1)!!=2^{n+1}\pi^{-\frac{1}{2}}\Gamma\left(n+\frac{3}{2}\right) and\ (2n-1)!!=2^n\pi^{-\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)$$

$$\Gamma(x)=\sqrt{2}\pi e^{x\log x-x-\frac{1}{2}-\log x+O\left(\frac{1}{x^2}\right)}\left(1+O\left(\frac{1}{x}\right)\right) and \sin^m\frac{\pi}{n}=\frac{\pi^m}{n^m}+O\left(\frac{1}{n^{m+2}}\right)$$

$$\Omega(m)=\lim_{n\to\infty}2^{m+1}\pi^{-\frac{m+1}{2n}}\Gamma^{\frac{m+1}{n}}\left(n+\frac{1}{2}\right)\left(\frac{2^{m+1}\pi^{-\frac{m+1}{2(n+1)}}\Gamma^{\frac{m+1}{n+1}}\left(n+\frac{3}{2}\right)}{2^{m+1}\pi^{-\frac{m+1}{2n}}\Gamma\left(n+\frac{1}{2}\right)}\right)\cdot\frac{\pi^m}{n^m}=$$

$$= 2^{m+1}\pi^m \lim_{n\to\infty}\frac{1}{n^m}e^{(m+1)\left(1+\frac{1}{2n}\right)\log\left(n+\frac{1}{2}\right)-\left(1+\frac{1}{2n}\right)-\frac{1}{2n}\log\left(1+\frac{1}{2}\right)}\left(-1\right.$$

$$\left.+\frac{e^{(m+1)\left(\left(1+\frac{1}{2(n+1)}\right)\log\left(n+\frac{3}{2}\right)-\left(1+\frac{1}{2(n+1)}\right)-\frac{1}{2(n+1)}\log\left(n+\frac{3}{2}\right)\right)}}{e^{(m+1)\left(\left(1+\frac{1}{2n}\right)\log\left(n+\frac{1}{2}\right)-\left(1+\frac{1}{2n}\right)-\frac{1}{2n}\log\left(n+\frac{1}{2}\right)\right)}}\right)$$

$$= 2^{m+1}\pi^m \lim_{n\to\infty}\frac{1}{n^m}e^{(m+1)\left(\left(1+\frac{1}{2n}\right)\log\left(n+\frac{1}{2}\right)-\left(1+\frac{1}{2n}\right)-\frac{1}{2n}\log\left(n+\frac{1}{2}\right)\right)}\left(-1+e^{(m+1)\left(\log\left(n+\frac{3}{2}\right)-\log\left(n+\frac{1}{2}\right)\right)}\right)=$$

$$= 2^{m+1}\pi^m(m+1)\lim_{n\to\infty}\frac{1}{n^{m+1}}e^{-(m+1)}e^{(m+1)\log n} =$$

$$= (m+1)2^{m+1}\pi^m e^{-(m+1)}$$

2089. **Find $a, b \in \mathbb{R}$ such that:**

$$arctan(1+2i) = a+bi$$

Proposed by Asmat Qatea-Afghanistan

Solution by Daniel Sitaru-Romania

$$arctan(1+2i) = \frac{i}{2}log\left(\frac{1-i(1+2i)}{1+i(1+2i)}\right) = \frac{i}{2}log\left(\frac{3-i}{-1+i}\right) =$$

$$= \frac{i}{2}\left(log(3-i) - log(-1+i)\right) =$$

$$= \frac{i}{2}\left(log|3-i| + iarg(3-i) - log|-1+i| - iarg(-1+i)\right) =$$

$$= \frac{i}{2}\left(log\sqrt{10} - log\sqrt{2} + i\left(arctan\left(-\frac{1}{3}\right) + 2\pi - \frac{3\pi}{4}\right)\right) =$$

$$= \frac{i}{2}\left(log\sqrt{5} + i\left(\frac{5\pi}{4} - arctan\frac{1}{3}\right)\right) = -\frac{1}{2}\left(\frac{5\pi}{4} - arctan\frac{1}{3}\right) + \frac{i}{2}log\sqrt{5} = a+bi$$

$$a = -\frac{1}{2}\left(\frac{5\pi}{4} - arctan\frac{1}{3}\right), b = \frac{1}{2}log\sqrt{5}$$

2090. **Find:**

$$\Omega = \lim_{n\to\infty}\frac{1}{n^2}\cdot\left(\sum_{i+j+k+l=n} ijkl\right)\cdot\left(\sum_{i+j+k=n} ijk\right)^{-1}$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$S_2^{(n)}=\sum_{i+j=n} ij=\sum_{i=1}^{n-1} i(n-i)=\frac{n^2}{2}(n-1)-\frac{n}{6}(n-1)2n-1)=$$

$$=\frac{(n-1)n(n+1)}{6}=\frac{n}{6}(n^2-1);n\geq 2$$

$$S_3^{(n)}=\sum_{i+j+k=n} ijk=\sum_{m=1}^{n-1}\sum_{k=1}^{m-1} k(m-k)(n-m)=\sum_{m=1}^{n-1}\frac{(m-1)m(m+1)}{6}(n-m)=$$

$$=\frac{n}{6}\sum_{m=1}^{n-1}(m^3-m)-\frac{1}{6}\sum_{m=1}^{n-1}(m^4-m^2)=\frac{n}{120}(n^2-1)(n^2-4);n\geq 3$$

$$S_4^{(n)}=\sum_{i+j+k+l=n} ijkl=\sum_{m=1}^{n-1}\frac{1}{120}m(m^2-1)(m^2-4)(n-m)=\cdots=$$

$$=\frac{n}{5040}(n^2-1)(n^2-4)(n^2-9);n\geq 4$$

$$Therefore,$$

$$\Omega=\lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{120}{5040}\cdot\frac{n(n^2-1)(n^2-4)(n^2-9)}{n(n^2-1)(n^2-4)}=\frac{1}{42}$$

2091. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{1\leq i<j\leq n}\left((-1)^{i+j}\cdot i\cdot j\right)\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}ij=\sum_{i=1}^{n} i^2+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(-1)^{i+j}ij=$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}ij-\frac{1}{12}n(n+1)(2n+1)=$$

$$=\left(\sum_{i=1}^{n}(-1)^i i\right)^2=\left(\sum_{i=1}^{n} i+2\sum_{i=1}^{\left[\frac{n}{2}\right]}2i\right)^2=\left(-\frac{1}{2}n(n+1)+2\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)\right)^2=$$

$$=\begin{cases}\left(-\frac{1}{2}n(n+1)+\frac{1}{2}n(n+1)\right)^2, n-even\\ \left(-\frac{1}{2}n(n+1)+\frac{1}{2}(n-1)(n+1)\right)^2, n-odd\end{cases}=$$

$$=\begin{cases}\frac{1}{4}n^2, n-even\\ \frac{1}{4}(n+1)^2, n-odd\end{cases}$$

$$\sum_{1\le i<j\le n}\left((-1)^{i+j}\cdot i\cdot j\right)=-\frac{1}{12}n(n+1)(2n+1)+\frac{1}{8}(n+\prod\{n-odd\})^2$$

$$\Omega=-\frac{1}{6}$$

2092. Find:

$$\Omega=\lim_{n\to\infty}\left(\binom{n}{3}\cdot\lim_{m\to\infty}\left(\frac{1}{m}\sum_{k=0}^{m}\left(\frac{\binom{m}{k}}{\binom{m+n}{k+3}}\right)\right)\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\frac{(a-1)!\,(b-1)!}{(a+b-1)!}=\frac{1}{(a+b-1)\binom{a+b-2}{a-1}}$$

$$a,b\in\mathbb{Z}_{\ge 1}$$

$$\Psi_{m,n}=\sum_{k=0}^{m}\binom{m}{k}\int_0^1 t^{k+3}(1-t)^{m+n-k-3}\,dt=$$

$$=(m+n+1)\int_0^1 t^3(1-t)^{m+n+3}\sum_{k=0}^{m}\binom{m}{k}\left(\frac{t}{1-t}\right)^k dt=$$

$$=(m+n+1)\int_0^1 t^3(1-t)^{m+n-3}\left(1+\frac{t}{1-t}\right)^m dt=$$

$$= (m+n+1)\int_0^1 t^3(1-t)^{n-3}dt = (m+n+1)\frac{1}{(n+1)\binom{n}{3}}$$

$$\lim_{m\to\infty}\frac{\Psi_{m,n}}{m} = \frac{1}{(n+1)\binom{n}{3}}; \Omega = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

2093. Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n]{2}-1\right)\left((n+1)!\right)^{-1}\left(n+1-\frac{2}{n}\right)^{-1}\sum_{k=1}^{n}k^2(k+1)!$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$k^2(k+1)! = [(k+3)(k+2)-5(k+2)+4](k+1)! =$$

$$= (k+3)! - 5(k+2)! + 4(k+1)! = (k+3)! - (k+2)! - 4[(k+2)! - (k+1)!]$$

$$\sum_{k=1}^{n}k^2(k+1)! = (n+3)! - 3! - 4[(n+2)! - 2!] = (n+2)!(n-1) + 2$$

$$Also, \left((n+1)!\right)^{-1}\left(n+1-\frac{2}{n}\right)^{-1} = \frac{1}{(n+1)!}\cdot\frac{n}{n^2+n-2} =$$

$$= \frac{1}{(n+1)!}\cdot\frac{n}{(n-1)(n+2)} = \frac{n}{(n+2)!(n-1)}$$

$$Thus, \left((n+1)!\right)^{-1}\left(n+1-\frac{2}{n}\right)^{-1}\cdot\sum_{k=1}^{n}k^2(k+1) = \frac{[(n+2)!(n-1)+2]n}{(n+2)!(n-1)}$$

$$\Omega = \lim_{n\to\infty}\frac{2^{\frac{1}{n}}-1}{\frac{1}{n}}\cdot\lim_{n\to\infty}\left[1+\frac{2}{(n+2)!(n-1)}\right] = \log 2$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\sum_{k=1}^{n}k^2(k+1)! = \sum_{k=1}^{n}(k^2+k+1)(k+1)! - \sum_{k=1}^{n}(k+1)(k+1)!$$

$$\sum_{k=1}^{n}(k+2-1)\left((k+1)!\right) = \sum_{k=1}^{n}[(k+2)! - (k+1)!] = (n+2)! - 2 =$$

$$= 2 - (n+2)! + \sum_{k=1}^{n}(k^2+k+1)(k+1)! =$$

$$= 2-(n+2)!+\sum_{k=1}^{n}[(k+(k+2)!)-(k-1)(k+1)!] =$$
$$= 2-(n+2)!+n(n+2)!$$
$$\Omega = \lim_{n\to\infty}\frac{2^{\frac{1}{n}}-1}{(n+1)!}\cdot\frac{1}{n+1-\frac{2}{n}}[n(n+2)!-(n+2)!+2] =$$
$$= \lim_{n\to\infty}\left(2^{\frac{1}{n}}-1\right)\cdot\frac{1}{n}(n-1)(n+2) = \lim_{n\to\infty}\frac{2^{\frac{1}{n}}-1}{\frac{1}{n}} = \log 2$$

Solution 3 by Adrian Popa-Romania

$$\sum_{k=1}^{n}k^2(k+1) = \sum_{k=1}^{n}k(k+1)!\,k = \sum_{k=1}^{n}k(k+1)\cdot k\cdot k! = \sum_{k=1}^{n}k(k+1)[(k+1)!-k!] =$$
$$= \sum_{k=1}^{n}k(k+1)(k+1)! - \sum_{k=1}^{n}k(k+1)k! =$$
$$= \sum_{k=1}^{n}k(k+2)! - \sum_{k=1}^{n}k(k+1)! - \sum_{k=1}^{n}(k+1)(k+1)! + \sum_{k=1}^{n}(k+1)! =$$
$$= \sum_{k=1}^{n}(k+2)(k+2)! - 2\sum_{k=1}^{n}(k+2)! - \sum_{k=1}^{n}(k+1)(k+1)! + \sum_{k=1}^{n}(k+1)!$$
$$-\left((n+2)!-2\right) + \sum_{k=1}^{n}(k+1)! =$$
$$= \sum_{k=1}^{n}[(k+2)!-(k+1)!] - 2\left[\sum_{k=1}^{n}\left((k+2)!-(k+1)!\right)\right] =$$
$$= (n+3)! - 3(n+2)!$$
$$\Omega = \lim_{n\to\infty}\frac{\left(2^{\frac{1}{n}}-1\right)n}{(n-1)!\,(n^2+n-2)}\cdot\frac{(n-1)!\,(n^2+5n+6-3n-6)}{1} =$$
$$= \lim_{n\to\infty}\frac{2^{\frac{1}{n}}-1}{\frac{1}{n}}\cdot\frac{n^2+2n}{n^2+n-2} = \ln 2$$

2094. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n^3}\left(\int_0^1\frac{x^k}{3x+2}dx\right)^{-1}\left(\int_0^1\frac{x^k}{2x+3}dx\right)^{-1}$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$0\le x\le 1\to x^{k+1}\le x^k, 3x+2>0\to \frac{x^{k+1}}{3x+2}\le \frac{x^k}{3x+2}\to \int_0^1 \frac{x^{k+1}}{3x+2}dx\le \int_0^1 \frac{x^k}{3x+2}dx$$

$$0\le x\le 1\to x^{k+1}\le x^k, 2x+3>0\to \frac{x^{k+1}}{2x+3}\le \frac{x^k}{2x+3}\to \int_0^1 \frac{x^{k+1}}{2x+3}dx\le \int_0^1 \frac{x^k}{2x+3}dx$$

$$I_k=\int_0^1 \frac{x^k}{3x+2}dx\to I_{k+1}\le I_k,\qquad 3I_{k+1}+2I_k=\int_0^1 \frac{3x^{k+1}+2x^k}{3x+2}dx=\int_0^1 x^k dx=\frac{1}{k+1}$$

$$J_k=\int_0^1 \frac{x^k}{2x+3}dx\to J_{k+1}\le J_k,\qquad 2J_{k+1}+3J_k=\int_0^1 \frac{2x^{k+1}+3x^k}{2x+3}dx=\int_0^1 x^k dx=\frac{1}{k+1}$$

$$I_{k+1}\le I_k, 3I_{k+1}+2I_k=\frac{1}{k+1}$$

$$I_{k+1}=\frac{1}{3(k+1)}-\frac{2}{3}I_k\to \frac{1}{3(k+1)}-\frac{2}{3}I_k\le I_k\to \frac{1}{3(k+1)}\le \frac{5}{3}I_k\to I_k\ge \frac{1}{5(k+1)}$$

$$I_k=\frac{1}{2(k+1)}-\frac{3}{2}I_{k+1}\to I_{k+1}\le \frac{1}{2(k+1)}-\frac{3}{2}I_{k+1}\to \frac{5}{2}I_{k+1}\le \frac{1}{2(k+1)}$$

$$I_{k+1}\le \frac{1}{5(k+1)}\to I_k\le \frac{1}{5k}\to \frac{1}{5(k+1)}\le I_k\le \frac{1}{5k},\qquad \frac{1}{5(k+1)}\le J_k\le \frac{1}{5k}$$

$$25k^2\le I_k^{-1}J_k^{-1}\le 25(k+1)^2$$

$$25\cdot \frac{n(n+1)(2n+1)}{6}\le \sum_{k=1}^n I_k^{-1}J_k^{-1}\le 25\cdot\left(\frac{(n+1)(n+2)(2n+3)}{6}-1\right)$$

$$25\cdot \frac{n(n+1)(2n+1)}{6n^3}\le \frac{1}{n^3}\sum_{k=1}^n I_k^{-1}J_k^{-1}\le 25\cdot\left(\frac{(n+1)(n+2)(2n+3)}{6n^3}-\frac{1}{n^3}\right)$$

$$\frac{25}{3}\le \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^n I_k^{-1}J_k^{-1}\le \frac{25}{3}$$

$$\Omega=\frac{25}{3}$$

2095. Prove that:

$$\sum_{n=1}^{10}\cos\left(\frac{n\pi}{10}\right)\left(\psi\left(\frac{11-n}{20}\right)-\psi\left(\frac{21-n}{20}\right)\right)=$$

$$=\frac{9(\phi-1)\pi}{2}-5\sqrt{\phi+2}\log\left(2-\sqrt{\phi+2}\right)$$

ϕ −golden ratio.

Proposed by Asmat Qatea-Afghanistan

Solution by Izumi Ainsworth-Tokyo-Japan

$$S=\Omega(p)=\sum_{n=1}^{p}\cos\left(\frac{n\pi}{p}\right)\left(\psi\left(\frac{p+1-n}{2p}\right)-\psi\left(\frac{2p+1-n}{2p}\right)\right)=$$

$$=\sum_{n=1}^{p}\cos\left(\frac{n-1}{p}\pi\right)\left(\psi\left(\frac{n}{2p}+\frac{1}{2}\right)-\psi\left(\frac{n}{2p}\right)\right);(i)$$

$$Now,\sum_{n=1}^{p}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}\right)=$$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}\right)-\sum_{n=p+1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}\right)=$$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}\right)-\sum_{n=1}^{\infty}\cos\left(\frac{n+p-1}{p}\pi\right)\psi\left(\frac{n+p}{2p}\right)=$$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}\right)+\sum_{n=1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}+\frac{1}{2}\right)=$$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}\right)+\sum_{n=1}^{p}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}+\frac{1}{2}\right)$$

$$+\sum_{n=p+1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}+\frac{1}{2}\right)\overset{(i)}{\Rightarrow}$$

$$\Omega(p)=-\sum_{n=1}^{\infty}\cos\left(\frac{n+p-1}{p}\pi\right)\psi\left(\frac{n+p}{2p}+\frac{1}{2}\right)-\sum_{n=1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\psi\left(\frac{n}{2p}\right)=$$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{n-1}{p}\pi\right)\left(\psi\left(\frac{n}{2p}+1\right)-\psi\left(\frac{n}{2p}\right)\right)=$$

$$=2p\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n-1}{p}\pi\right)}{n}=2p\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{p}-\frac{\pi}{p}\right)}{n}=$$

$$=2p\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{p}\right)\sin\left(\frac{\pi}{p}\right)+\cos\left(\frac{n\pi}{p}\right)\cos\left(\frac{\pi}{p}\right)}{n}=$$

$$=2p\left(\sin\left(\frac{\pi}{p}\right)\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{p}\right)}{n}+\cos\left(\frac{\pi}{p}\right)\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{p}\right)}{n}\right)=$$

$$=2p\left[\sin\left(\frac{\pi}{p}\right)\left(\frac{\pi-\frac{\pi}{p}}{2}\right)+\cos\left(\frac{\pi}{p}\right)\left(-\ln\left(2\,sin\left(\frac{\pi}{2p}\right)\right)\right)\right]=$$

$$=\pi(p-1)\sin\left(\frac{\pi}{p}\right)-2p\cos\left(\frac{\pi}{p}\right)\log\left(2\,sin\left(\frac{\pi}{2p}\right)\right)$$

$$For\ p=10\Rightarrow\Omega(10)=S\Rightarrow S=9\pi\sin 18^\circ-20\cos 18^\circ\log(2\,sin\,9^\circ)=$$

$$=9\pi\left(\frac{\phi-1}{2}\right)-20\left(\frac{\sqrt{\phi+2}}{2}\right)\log(4\sin^2 9^\circ)^{\frac{1}{2}}=$$

$$=9\pi\left(\frac{\phi-1}{2}\right)-20\left(\frac{\sqrt{\phi+2}}{2}\right)\frac{1}{2}\log(2-2\,cos\,18^\circ)=$$

$$=\frac{9(\pi-1)\pi}{2}-5\sqrt{\phi+2}\log\left(2-\sqrt{\phi+2}\right)$$

$$\because\ \psi(x+1)-\psi(x)=\frac{1}{x};\ \psi(z)-Digamma\ function.$$

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}=\frac{\pi-x}{2};\ \sum_{n=1}^{\infty}\frac{\cos(nx)}{n}=-\log\left|2\sin\frac{x}{2}\right|$$

$$\cos(A-B)=\sin A\sin B+\cos A\cos B$$

$$\sin 18^\circ=\frac{\phi-1}{2};\cos 18^\circ=\frac{\sqrt{\phi+2}}{2},\phi-Golden\ ratio.$$

2096. Prove that:

$$\int_0^1 \frac{\arccos x}{\sqrt{1+ax^2}}dx = \frac{4Li_2(\sqrt{-a}) - Li_2(-a)}{4\sqrt{-a}}, a \geq -1$$

where $Li_2(z)$ is the dilogarithm function.

Proposed by Le Thu-Vietnam

Solution 1 by Bui Hong Suc-Vietnam

$$\Omega = \int_0^1 \frac{\arccos x}{\sqrt{1+ax^2}}dx = \int_0^1 \arccos x\left(1-(-ax^2)\right)^{-\frac{1}{2}}dx =$$

$$= \int_0^1 \arccos x \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}(-ax^2)^n\, dx =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}(-a)^n \int_0^1 x^{2n} \arccos x\, dx =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}(-a)^n \cdot \frac{1}{2n+1}\int_0^1 \arccos x\, d(x^{2n+1}) =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}(-a)^n \frac{1}{2n+1}\left([x^{2n+1}\arccos x]_0^1 - \int_0^1 x^{2n+1}\cdot\frac{-1}{\sqrt{1-x^2}}dx\right) =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}(-a)\cdot\frac{1}{2(2n+1)}\int_0^1 (x^2)^{n+1-1}(1-x^2)^{\frac{1}{2}-1}d(x^2) =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}(-a)^n\cdot\frac{1}{2(2n+1)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(\frac{1}{2}+n+1\right)} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{2(2n+1)\left(n+\frac{1}{2}\right)}(-a)^n = \frac{1}{\sqrt{-a}}\sum_{n=0}^{\infty}\frac{\left(\sqrt{-a}\right)^{2n+1}}{(2n+1)^2} =$$

$$= \frac{1}{2\sqrt{-a}}\left(Li_2\left(\sqrt{-a}\right) - Li_2(-a)\right)$$

We use dilogarithm identity:

$$Li_2(z)+Li_2(-z)=\frac{Li_2(z^2)}{2}\Leftrightarrow Li_2(-z)=\frac{Li_2(z^2)}{2}-Li_2(z)$$

$$Li_2(-a=\frac{Li_2(-a)}{2}-Li_2(\sqrt{-a})$$

$$\Omega=\frac{1}{2\sqrt{-a}}(Li_2(\sqrt{-a})-Li_2(-a)=\frac{1}{2\sqrt{-a}}\left(Li_2(\sqrt{-a})-\left(\frac{Li_2(-a)}{2}-Li_2(\sqrt{-a})\right)\right)=$$

$$=\frac{1}{4\sqrt{-a}}\left(4Li_2(\sqrt{-a})-Li_2(-a)\right)$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{\arccos x}{\sqrt{1+ax^2}}dx=\int_0^1\frac{\arccos x}{\sqrt{\left(1-\sqrt{-a}x\right)^2}}dx=$$

$$\because\frac{1}{(1-ax)^{k+1}}=\sum_{n=0}^{\infty}a^n\frac{(n+k)!}{k!\,n!}x^n$$

$$=\int_0^1\arccos x\sum_{n=0}^{\infty}\frac{(-a)^n\left(n-\frac{1}{2}\right)!}{\left(-\frac{1}{2}\right)!\,n!}dx=$$

$$=\sum_{n=0}^{\infty}\frac{(-a)^n\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}\int_0^1x^{2n}\arccos x\,dx\overset{IBP}{=}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)\sqrt{\pi}\Gamma(n+1)}\int_0^1\frac{x^{2n+1}}{\sqrt{1-x^2}}dx\overset{x^2=t}{=}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-a)^n\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)\sqrt{\pi}\Gamma(n+1)}\int_0^1t^{n+1-1}(1-t)^{\frac{1}{2}-1}dt=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-a)^n\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)\sqrt{\pi}\Gamma(n+1)}\frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}=$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-a)^n\left(n-\frac{1}{2}\right)!}{\left(n+\frac{1}{2}\right)!\,(2n+1)}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-a)^n}{\left(n+\frac{1}{2}\right)(2n+1)}=$$

$$=\sum_{n=0}^{\infty}\frac{(-a)^n}{(2n+1)^2}=\sum_{n=0}^{\infty}\frac{(\sqrt{-a})^{2n}}{(2n+1)^2}=\frac{1}{\sqrt{-a}}\sum_{n=0}^{\infty}\frac{(\sqrt{-a})^{2n+1}}{(2n+1)^2}=$$

$$\left(\because\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^s}=\frac{x\Phi\left(x^2;s;\frac{1}{2}\right)}{2^s}=\chi_s(x)=\frac{1}{2}\left(Li_s(x)-Li_s(-x)\right)\right)$$

$$=\frac{1}{2\sqrt{-a}}\left(Li_2(\sqrt{-a})-Li_2(-\sqrt{-a})\right)=Li_2(x)+Li_2(-x)=\frac{Li_2(x^2)}{2}$$

$$=\frac{1}{2\sqrt{-a}}\left(Li_2(\sqrt{-a})-\left(\frac{Li_2(-a)}{2}-Li_2(\sqrt{-a})\right)\right)=$$

$$=\frac{4Li_2(\sqrt{-a})-Li_2(-a)}{4\sqrt{-a}}$$

Therefore,

$$\int_0^1\frac{\arccos x}{\sqrt{1+ax^2}}dx=\frac{4Li_2(\sqrt{-a})-Li_2(-a)}{4\sqrt{-a}},a\geq -1$$

2097. **Prove that**

$$\int_0^1\log^n\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{m+1}\frac{dx}{\sqrt{x}}=$$

$$4(-1)^{2n+m}n!\left[(m+1)\eta(n+1)-\eta(n)+(m+1)\sum_{k=1}^{m}\frac{(-1)^k}{k^{n+1}}-\sum_{k=1}^{m}\frac{(-1)^k}{k^n}\right]$$

where $\eta(n)$ is Drichlet eta function, $n,m\in N$

Proposed by Syed Shahabudeen-Kerala-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\log^n\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{m+1}\frac{dx}{\sqrt{x}}\overset{x\to x^2}{=}2(-1)^n\int_0^1\log^n\left(\frac{1-x}{1+x}\right)\left(\frac{1-x}{1+x}\right)^{m+1}dx=$$

$$\overset{x\to\frac{1-x}{1+x}}{=}4(-1)^n\int_0^1\frac{x^{m+1}}{(1+x)^2}\log^n x\,dx$$

$$Using:\ \frac{1}{(1+x)^2}=-\sum_{k=0}^{\infty}(-1)^k k x^{k-1}$$

$$\Omega=-4(-1)^n\sum_{k=0}^{\infty}(-1)^k k\int_0^1 x^{m+k}\log^n x\,dx=-4(n!)\sum_{k=0}^{\infty}\frac{(-1)^k k}{(m+k+1)^{n+1}}=$$

$$=4(n!)(m+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(m+k+1)^{n+1}}-4(n!)\sum_{k=0}^{\infty}\frac{(-1)^k}{(m+k+1)^n}=$$

$$=4(n!)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(m+k)^n}-(m+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(m+k)^{n+1}}\right)$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{(m+k)^n}\overset{k+m\to k}{=}(-1)^m\sum_{k=m+1}^{\infty}\frac{(-1)^k}{k^n}=(-1)^m\sum_{k=1}^{\infty}\frac{(-1)^k}{k^n}-(-1)^m\sum_{k=1}^{m}\frac{(-1)^k}{k^n}=$$

$$=-(-1)^m\left(\eta(n)+\sum_{k=1}^{m}\frac{(-1)^k}{k^n}\right)$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{(m+k)^{m+1}}=-(-1)^m\left(\eta(n+1)+\sum_{k=1}^{m}\frac{(-1)^k}{k^{n+1}}-\sum_{k=1}^{m}\frac{(-1)^k}{k^n}\right)$$

Therefore,

$$\Omega=\int_0^1\log^n\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{m+1}\frac{dx}{\sqrt{x}}=$$

$$=4(-1)^{2n+m}n!\left((m+1)\eta(n+1)-\eta(n)+(m+1)\sum_{k=1}^{m}\frac{(-1)^k}{k^{n+1}}-\sum_{k=1}^{m}\frac{(-1)^k}{k^n}\right)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\text{Let }\frac{1-\sqrt{x}}{1+\sqrt{x}}=y$$

$$I=4\int_0^1\frac{y^{m+1}}{(1+y)^2}\log^n\left(\frac{1}{y}\right)dy=4(-1)^n\int_0^1\frac{y^{m+1}}{(1+y)^2}\log^n(y)\,dy$$

$$=4(-1)^n\int_0^1 y^{m+1}\log^n(y)\sum_{k=0}^{\infty}(-1)^{k+1}ky^{k-1}\,dy=4(-1)^n\sum_{k=0}^{\infty}(-1)^{k+1}k\int_0^1 y^{m+k}\log^n(y)\,dy$$

$$=4(-1)^n\sum_{k=0}^{\infty}(-1)^{k+1}k\left\{(-1)^n\frac{n!}{(m+k+1)^{n+1}}\right\}=4(-1)^{2n}n!\sum_{k=0}^{\infty}(-1)^{k+1}\frac{k}{(m+k+1)^{n+1}}$$

$$=4(-1)^{2n}n!\left[\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(m+k+1)^n}-(m+1)\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(m+k+1)^{n+1}}\right]=$$

$$= 4(-1)^{2n}n!\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{(m+k)^n} - (m+1)\sum_{k=1}^{\infty}\frac{(-1)^k}{(m+k)^{n+1}}\right]$$

$$= 4(-1)^{2n+m}n!\left[\left[-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^n} + \sum_{k=1}^{m}\frac{(-1)^{k+1}}{k^n}\right] - (m+1)\left[-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{n+1}} + \sum_{k=1}^{m}\frac{(-1)^{k+1}}{k^{n+1}}\right]\right]$$

$$= 4(-1)^{2n+m}n!\left[\left[-\eta(n) + \sum_{k=1}^{m}\frac{(-1)^{k+1}}{k^n}\right] - (m+1)\left[-\eta(n+1) + \sum_{k=1}^{m}\frac{(-1)^{k+1}}{k^{n+1}}\right]\right]$$

$$= 4(-1)^{2n+m}n!\left[(m+1)\eta(n+1) - \eta(n) + \sum_{k=1}^{m}\frac{(-1)^{k+1}}{k^n} - (m+1)\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k^{n+1}}\right]$$

$$= 4(-1)^{2n+m}n!\left[(m+1)\eta(n+1) - \eta(n) + (m+1)\sum_{k=1}^{m}\frac{(-1)^k}{k^{n+1}} - \sum_{k=1}^{m}\frac{(-1)^k}{k^n}\right]$$

2098. Find:

$$\Omega = \int_0^1\int_0^1 \frac{\sin^{-1}(xy)\cos^{-1}(xy)}{xy}\,dxdy$$

Proposed by Togrul Ehmedov-Azerbaijan

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^1\int_0^1 \frac{\sin^{-1}(xy)\cos^{-1}(xy)}{xy}\,dxdy \overset{xy=t}{=} \int_0^1 \frac{1}{x}\int_0^x \frac{\sin^{-1}t\cos^{-1}t}{t}\,dt\,dx \overset{IBP}{=}$$

$$= \left[\log x\int_0^x \frac{\sin^{-1}t\cos^{-1}t}{t}\,dt\right]_{x=0}^{x=1} - \int_0^1 \frac{\log x\sin^{-1}x\cos^{-1}x}{x}\,dx =$$

$$= -\int_0^1 \frac{\log x\sin^{-1}x\cos^{-1}x}{x}\,dx \overset{x=\sin x}{=} -\int_0^{\frac{\pi}{2}} x\left(\frac{\pi}{2}-x\right)\cot x\log(\sin x)\,dx \overset{IBP}{=}$$

$$= -\left[\frac{1}{2}x\left(\frac{\pi}{2}-x\right)\log^2(\sin x)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{4}-x\right)\log^2(\sin x)\,dx =$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\log^2(\sin x)\,dx - \int_0^{\frac{\pi}{2}} x\log^2(\sin x)\,dx$$

$$\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\log^2(\sin x)\,dx = \frac{\pi^2}{8}\log^2 2 + \frac{\pi^4}{96} = \frac{3}{4}\zeta(2)\log^2 2 + \frac{15}{16}\zeta(4)$$

$$\frac{\pi}{4}\int_0^{\frac{\pi}{2}} x\log^2(\sin x)\,dx = Li_4\left(\frac{1}{2}\right)+\frac{1}{2}\zeta(2)\log^2 2+\frac{1}{24}\log^4 2-\frac{19}{32}\zeta(4)$$

Therefore,

$$\Omega=\int_0^1\int_0^1 \frac{\sin^{-1}(xy)\cos^{-1}(xy)}{xy}dxdy=\frac{1}{4}\zeta(2)\log^2 2-Li_4\left(\frac{1}{2}\right)-\frac{1}{24}\log^4 2+\frac{49}{32}\zeta(4)$$

2099. For $a,b\in(-1,1)$ prove that:

$$\int_0^{\pi}\cos x\log(1+2a\cos x+a^2)\log(1+2b\cos x+b^2)\,dx=$$

$$=\frac{\pi(a+b)}{ab}\big((ab-1)\log(1-ab)-ab\big)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\pi}\cos x\log(1+2a\cos x+a^2)\log(1+2b\cos x+b^2)\,dx\,,a,b\in(-1,1)$$

$$\log(1+2r\cos x+r^2)=-2\sum_{n=1}^{\infty}\frac{(-r)^n\cos(nx)}{n}$$

$$\Omega=4\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\sum_{k=1}^{\infty}\frac{(-b)^k}{k}\int_0^{\pi}\cos x\cos(nx)\cos(kx)\,dx=$$

$$=2\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\sum_{k=1}^{n}\frac{(-b)^k}{k}\int_0^{\pi}\cos\big((n+1)x\big)\cos(kx)\,dx+$$

$$+2\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\sum_{k=1}^{n}\frac{(-b)^k}{k}\int_0^{\pi}\cos\big((n-1)x\big)\cos(kx)\,dx$$

$$\cos\big((n+1)x\big)\cos(kx)=\begin{cases}\frac{\pi}{2}\ if\ k=n+1\\ 0\ if\ k\neq n+1\end{cases};\cos\big((n-1)x\big)\cos(kx)=\begin{cases}\frac{\pi}{2}\ if\ k=n+1\\ 0\ if\ k\neq n+1\end{cases}$$

$$\Omega=\pi\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\cdot\frac{(-b)^{n+1}}{n+1}+\pi\sum_{k=1}^{\infty}\frac{(-b)^k}{k}\cdot\frac{(-a)^{k+1}}{k+1}=$$

$$=-\pi b\sum_{n=1}^{\infty}\frac{(ab)^n}{n(n+1)}-\pi a\sum_{n=1}^{\infty}\frac{(ab)^n}{n(n+1)}=-\pi(a+b)\sum_{n=1}^{\infty}\frac{(ab)^n}{n(n+1)}=$$

$$= -\pi(a+b)\sum_{n=1}^{\infty}\left(\frac{(ab)^n}{n}-\frac{(ab)^n}{n+1}\right)=\pi(a+b)\left(\log(1-ab)-\frac{ab+\log(1-ab)}{ab}\right)$$

$$Therefore,$$

$$\int_0^{\pi}\cos x\log(1+2a\,cos\,x+a^2)\log(1+2b\cos x+b^2)\,dx=$$

$$=\frac{\pi(a+b)}{ab}\left((ab-1)\log(1-ab)-ab\right)$$

Solution 2 by Bui Hong Suc-Vietnam

$$\Omega=\int_0^{\pi}\cos x\log(1+2a\cos x+a^2)\log(1+2b\cos x+b^2)\,dx$$

$$We\ have:$$

$$\log(1+2a\cos x+a^2)=-2\sum_{k=1}^{\infty}\frac{(-a)^k\cos kx}{k}$$

$$\log(1+2b\,cos\,x+b^2)=-2\sum_{n=1}^{\infty}\frac{(-b)^n\cos nx}{n}$$

$$\log(1+2a\,cos\,x+a^2)\cdot\log(1+2b\,cos\,x+b^2)=4\sum_{k=1}^{\infty}\frac{(-a)^k\cos kx}{k}\sum_{n=1}^{\infty}\frac{(-b)^n\cos nx}{n}$$

$$=4\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^k}{k}\cos nx\cos kx$$

$$\Omega=\int_0^{\pi}\cos x\log(1+2a\,cos\,x+a^2)\log(1+2b\cos x+b^2)\,dx=$$

$$=4\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^k}{k}\int_0^{\pi}\cos x\cos nx\cos kx\,dx=$$

$$=4\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^k}{k}\cdot J,where$$

$$J=\int_0^{\pi}\cos x\cos nx\cos kx\,dx=$$

$$=\frac{1}{4}\int_0^{\pi}\{\cos(n+1+k)x+\cos(n+1-k)x+\cos(n-1-k)x\}\,dx=$$

$$=\frac{1}{4}(J_1+J_2+J_3+J_4)$$

$$J_1=\int_0^{\pi}\cos(n+k+1)x\,dx=0;\ J_2=\int_0^{\pi}\cos(n-k+1)x\,dx=\begin{cases}0\ if\ n\neq k-1\\ \pi\ if\ n=k-1\end{cases}$$

$$J_3=\int_0^{\pi}\cos(n+k-1)x\,dx=0;J_4=\int_0^{\pi}\cos(n-k-1)x\,dx=\begin{cases}0\ if\ n\neq k+1\\ \pi\ if\ n=k+1\end{cases}$$

$$J=\frac{1}{4}(J_2+J_4)$$

$$\Omega=4\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^k}{k}\cdot J=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^k}{k}(J_2+J_4)=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^k}{k}\cdot J_2+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^k}{k}\cdot J_4=$$

$$=\pi\left[\sum_{n=1}^{\infty}\frac{(-b)^n}{n}\cdot\frac{(-a)^{n+1}}{n+1}+\sum_{k=1}^{\infty}\frac{(-b)^{k+1}}{k+1}\cdot\frac{(-a)^k}{k}\right]=$$

$$=\pi\left[-a\sum_{n=1}^{\infty}\left(\frac{(ab)^n}{n}-\frac{(ab)^n}{n+2}\right)-b\sum_{k=1}^{\infty}\left(\frac{(ab)^k}{k}-\frac{(ab)^k}{k+1}\right)\right]=$$

$$=-\pi(a+b)\sum_{n=1}^{\infty}\left(\frac{(ab)^n}{n}-\frac{(ab)^n}{n+1}\right)=-\pi(a+b)\left[\sum_{n=1}^{\infty}\frac{(ab)^n}{n}-\frac{1}{ab}\sum_{n=1}^{\infty}\frac{(ab)^{n+1}}{n+1}\right]=$$

$$=\pi(a+b)\left[\log(1-ab)-\frac{1}{ab}(ab+\log(1-ab))\right]=$$

$$=\frac{\pi(a+b)}{ab}[(ab-1)\log(1-ab)-ab]$$

$$Hence,\Omega=\frac{\pi(a+b)}{ab}[(ab-1)\log(1-ab)-ab]$$

2100. Prove the integral:

$$\int_0^{\pi}\int_{-1}^{1}\sqrt{\frac{1+\cos\frac{\theta}{3}}{1+x\cos\frac{\theta}{3}}}\,dx\,d\theta=2\left(\pi+3\log\left(\frac{1}{2}(2+\sqrt{3})\right)\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Tapas Das-India

$$\int_0^\pi\int_{-1}^1\sqrt{\frac{1+\cos\frac{\theta}{3}}{1+x\cos\frac{\theta}{3}}}dx\,d\theta=\int_0^\pi\frac{\sqrt{1+\cos\frac{\theta}{3}}}{\cos\frac{\theta}{3}}\int_{-1}^1\frac{\cos\frac{\theta}{3}}{\sqrt{1+x\cos\frac{\theta}{3}}}dx\,d\theta=$$

$$=\int_0^\pi\frac{\sqrt{1+\cos\frac{\theta}{3}}}{\cos\frac{\theta}{3}}\cdot 2\left[\sqrt{1+x\cos\frac{\theta}{3}}\right]_{-1}^1 d\theta$$

$$=\int_0^\pi\frac{2\sqrt{1+\cos\frac{\theta}{3}}}{\cos\frac{\theta}{3}}\left[\sqrt{1+\cos\frac{\theta}{3}}-\sqrt{1-\cos\frac{\theta}{3}}\right]d\theta=$$

$$=2\int_0^\pi\frac{1}{\cos\frac{\theta}{3}}\left[1+\cos\frac{\theta}{3}-\sin\frac{\theta}{3}\right]d\theta=2\int_0^\pi\left(\sec\frac{\theta}{3}+1-\tan\frac{\theta}{3}\right)d\theta=$$

$$=2\left|3\log\left|\sec\frac{\theta}{3}+\tan\frac{\theta}{3}\right|+\theta-3\log\left|sec\frac{\theta}{3}\right|\right|=2\left[\pi+3\log\left(\frac{1}{2}(2+\sqrt{3})\right)\right]$$

Solution 2 by Le Thu-Vietnam

$$\text{Since }\cos\frac{\theta}{3}>0;\forall\theta\in[0,\pi]$$

$$\Omega=\int_0^\pi\int_{-1}^1\sqrt{\frac{1+\cos\frac{\theta}{3}}{1+x\cos\frac{\theta}{3}}}dx\,d\theta=\int_0^\pi\sqrt{1+\cos\frac{\theta}{3}}\int_{-1}^1\frac{dx}{\sqrt{1+x\cos\frac{\theta}{3}}}d\theta=$$

$$=\int_0^\pi\sqrt{1+\cos\frac{\theta}{3}}\frac{\sqrt{x\cos\frac{\theta}{3}+1}}{\cos\frac{\theta}{3}}\Bigg|_{-1}^1 d\theta=2\int_0^\pi\frac{\sqrt{1+\cos\frac{\theta}{3}}}{\cos\frac{\theta}{3}}\left[\sqrt{1+\cos\frac{\theta}{3}}-\sqrt{1-\cos\frac{\theta}{3}}\right]d\theta$$

$$=2\int_0^\pi\left[1+\frac{1-\left|\sin\frac{\theta}{3}\right|}{\cos\frac{\theta}{3}}\right]d\theta=2\int_0^\pi d\theta+2\int_0^\pi\frac{1-\sin\frac{\theta}{3}}{\cos\frac{\theta}{3}}d\theta=$$

$$=\left[2\theta+6\,\mathrm{arctanh}\left(\sin\frac{\theta}{3}\right)+\log\left(sin\frac{\theta}{3}\right)\right]_0^\pi=2\pi+2\left[3\,\mathrm{arctanh}\left(\frac{\sqrt{3}}{2}\right)-3\log 2\right]$$

$$Using: \text{arctanh}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \Rightarrow \text{arctanh}\left(\frac{\sqrt{3}}{2}\right) = \log(2+\sqrt{3})$$

$$Hence, \Omega = 2\left[\pi + 3\log\left(\frac{1}{2}(2+\sqrt{3})\right)\right]$$

Solution 3 by Sakthi Vel-India

$$\Omega = \int_0^\pi \int_{-1}^1 \sqrt{\frac{1+\cos\frac{\theta}{3}}{1+x\cos\frac{\theta}{3}}}\,dx\,d\theta \overset{1+x\cos\frac{\theta}{3}=y}{=} \int_0^\pi \int_{-1}^1 \frac{\sqrt{1+\cos\frac{\theta}{3}}}{\cos\frac{\theta}{3}} \cdot \frac{1}{\sqrt{y}}dy =$$

$$= \int_0^\pi \frac{\sqrt{1+\cos\frac{\theta}{3}}}{\cos\frac{\theta}{3}} 2\sqrt{y}\Big|_{1-\cos\frac{\theta}{3}}^{1+\cos\frac{\theta}{3}} d\theta = \int_0^\pi 2\sqrt{1+\cos\frac{\theta}{3}}\left(\sqrt{1+\cos\frac{\theta}{3}} - \sqrt{1-\cos\frac{\theta}{3}}\right)d\theta =$$

$$= 2\int_0^\pi \frac{1+\cos\frac{\theta}{3} - \sqrt{1-\cos^2\frac{\theta}{3}}}{\cos\frac{\theta}{3}}d\theta = 2\int_0^\pi \left[\sec\frac{\theta}{3} + 1 - \tan\frac{\theta}{3}\right]d\theta =$$

$$== 2\left|3\log\left|\sec\frac{\theta}{3} + \tan\frac{\theta}{3}\right| + \theta - 3\log\left|sec\frac{\theta}{3}\right|\right| = 2\left[\pi + 3\log\left(\frac{1}{2}(2+\sqrt{3})\right)\right]$$

It’s nice to be important but more important it’s to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru