Shortest Distance between Two Points on a Right Circular Cone

Introduction

In this article, we will derive an expression, using elementary ways, for the shortest distance between two points on the surface of a right circular cone

Description and Theory

Let us consider a right circular cone of height H, radius R and slant height L. The base of the cone rests on the xy-plane and the vertex V lies at $(0, 0, H)$. Let there be two points A and B on the surface of the cone with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Let us call the projections of A and B along their respective generator lines (the line passing through V and A and the line passing through V and B) as P_A and P_B having coordinates $(x_1H/(H - z_1), y_1H/(H - z_1))$ and $(x_2H/(H - z_2), y_2H/(H - z_2))$ respectively. Let us now cut the cone in the direction of any generator of the cone and then flatten it on the xy-plane in the form of a circular sector with endpoints Q (coordinates $(L\cos(\alpha), L\sin(\alpha))$ and S (coordinates $(L, 0)$) such that the center of the sector (vertex V of the cone) coincides with the origin O of the xy-plane (i.e., one side of the sector (having endpoint S) lies on the x-axis and the other side (having endpoint Q) lies in some quadrant making an angle α (central angle of the sector) with the positive x-axis in the anti-clockwise direction). Thus, the points A, B, P_A and P_B can be mapped onto the points A', B', P'_A and P'_B having the coordinates (a_1, b_1), (a_2, b_2), $(L\cos(\theta'_A + \alpha), L\sin(\theta'_A + \alpha))$ and $(L\cos(\theta'_B + \alpha), L\sin(\theta'_B + \alpha))$ respectively (θ'_A and θ'_B are explained below). The generator should be chosen in such a way so that a straight line could pass through the points A' and B' without crossing any of the sides of the sector.

The Shortest Distance

Since distances are preserved in the process of cutting/bending, therefore,

$$|\overline{VA}| = |\overline{OA'}| \tag{1}$$

where, $|\overline{VA}|$ is the length of the line joining V and A and $|\overline{OA'}|$ is the length of the line joining O and A'.
Using (1) and the $3D$ distance formula, we have,

$$|\overline{OA'}| = \sqrt{(0 - x_1)^2 + (0 - y_1)^2 + (H - z_1)^2}$$

Similarly, for B, we have,

$$|\overline{OB'}| = \sqrt{(0 - x_2)^2 + (0 - y_2)^2 + (H - z_2)^2}$$

Since the sector is formed from a cone, the circumference of the circular sector (excluding the length of the sides) must be equal to the circumference of the base of the cone, hence,

$$\frac{2\pi L(360 - \alpha)}{360} = 2\pi R \implies \alpha = 360\left(1 - \frac{R}{L}\right)$$

where, $L = \sqrt{R^2 + H^2}$ is the radius of the circular sector.
Observe that,

$$|\overparen{P_A Q}| = |\overparen{P'_A Q}| \tag{2}$$

where, $|\overparen{P_A Q}|$ is the length of the arc $\overparen{P_A Q}$ in $3D$ and $|\overparen{P'_A Q}|$ is the length of the arc $\overparen{P'_A Q}$ in $2D$.
It can be shown that,

$$|\overparen{P_A Q}| = \frac{2\pi R \theta_A}{360}$$

and

$$|\overparen{P'_A Q}| = \frac{2\pi L \theta'_A}{360}$$

where, θ_A is the angle subtended by the arc $\overparen{P_A Q}$ in $3D$ and θ'_A is the angle subtended by the arc $\overparen{P'_A Q}$ in $2D$.
Since V, A and P_A are collinear points, therefore,

$$\tan(\theta_A) = \frac{y_1}{x_1}$$

using (2), we have,

$$\theta'_A = \frac{R}{L}\tan^{-1}\left(\frac{y_1}{x_1}\right)$$

Thus, the coordinates of A' could be resolved as,

$$a_1 = |\overline{OA'}|\cos(\theta'_A + \alpha) = |\overline{OA'}|\cos\left(\frac{R}{L}\tan^{-1}\left(\frac{y_1}{x_1}\right) + 360\left(1 - \frac{R}{L}\right)\right)$$

and

$$b_1 = |\overline{OA'}|\sin(\theta'_A + \alpha) = |\overline{OA'}|\sin\left(\frac{R}{L}\tan^{-1}\left(\frac{y_1}{x_1}\right) + 360\left(1 - \frac{R}{L}\right)\right)$$

Similarly, the coordinates of B' could be written as,

$$a_2 = |\overline{OB'}|\cos(\theta'_B + \alpha) = |\overline{OB'}|\cos\left(\frac{R}{L}\tan^{-1}\left(\frac{y_2}{x_2}\right) + 360\left(1 - \frac{R}{L}\right)\right)$$

and

$$b_2 = |\overline{OB'}|\sin(\theta'_B + \alpha) = |\overline{OB'}|\sin\left(\frac{R}{L}\tan^{-1}\left(\frac{y_2}{x_2}\right) + 360\left(1 - \frac{R}{L}\right)\right)$$

Since it is known that the shortest path between two points lying on the surface of a plane is a straight line, therefore, using the $2D$ distance formula, the shortest distance D_{AB} between the points A' and B' as well as A and B is given by,

$$D_{AB} = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

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