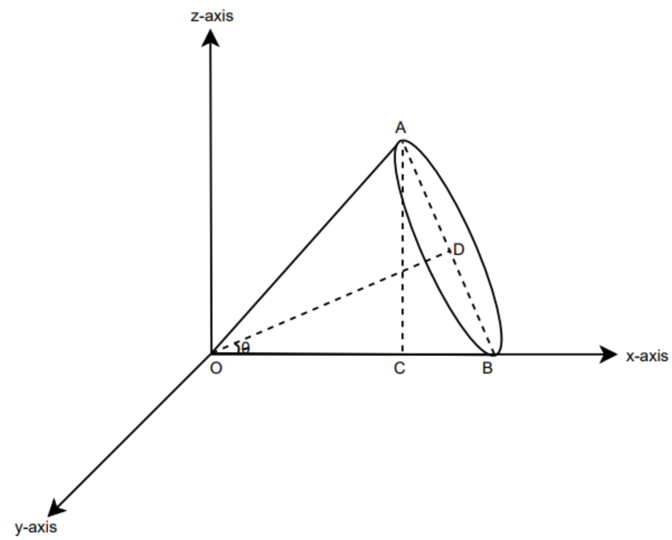
A Counter Intuitive Constant

Introduction

In this article, we will find a mathematical constant related to the geometry of a right circular cone.

Description of the Cone

Consider a right circular cone C_1, of radius R ($AD = DB = R$), height H ($OD = H$), slant height L ($OA = OB = L$) and semi-vertical angle θ ($\angle AOD = \angle DOB = \theta$), lying on the xy-plane (the point B lies on the xy-plane) on one of its generators whose vertex lies at the origin O as shown below.



C_1 is rotated about O keeping B on the xy-plane. The locus of the point A is a circle of radius r ($OC = r$) which forms the base of another cone C_2 of height h ($AC = h$) and semi-vertical angle $(90° - 2\theta)$.
If V_1 and V_2 be the volumes of C_1 and C_2 respectively, then in this article, we will find the value of λ ($\lambda = R/H = \tan(\theta)$), such that V_1 becomes equal to V_2.

Finding the Constant

Let us first find the value of r and h in terms of R and λ by finding the area of the triangle ΔOAB in two different ways, that is,

$$ar(\Delta OAB) = \frac{1}{2}(2R)(H) = \frac{1}{2}(L)(h) \implies h = \frac{2RH}{L} = \frac{2R}{\sqrt{1+\lambda^2}}$$

From ΔOAC, we have,

$$r = \frac{h}{\tan(2\theta)} = \frac{h}{\tan(2\tan^{-1}(\lambda))} = \frac{\frac{2R}{\sqrt{1+\lambda^2}}}{\frac{2\lambda}{1-\lambda^2}} = \frac{R(1-\lambda^2)}{\lambda\sqrt{1+\lambda^2}}$$

Using some algebra, we obtain the value of the expression V_1/V_2 as,

$$\frac{V_1}{V_2} = \frac{\frac{1}{3}\pi r^2 h}{\frac{1}{3}\pi R^2 H} = \frac{R^2(1-\lambda^2)^2}{\lambda^2(1+\lambda^2)}\frac{2R}{\sqrt{1+\lambda^2}}\frac{\lambda}{R^3} = \frac{2(1-\lambda^2)^2}{\lambda(1+\lambda^2)^{3/2}}$$

Since, $\lambda \in \mathbb{R}^+$, squaring the above equation, substituting $t = \lambda^2$ and equating it to 1, we obtain,

$$\frac{4(1-t)^4}{t(1+t)^3} = 1 \implies 3t^4 - 19t^3 + 21t^2 - 17t + 4 = 0$$

using the rational root test, the only possible rational roots of the above equation are $\pm 1, \pm 2, \pm 4, \pm 1/3, \pm 2/3, \pm 4/3$. Of these, $1/3$ makes the above equation true, therefore, we have,

$$3t^4 - 19t^3 + 21t^2 - 17t + 4 = 0 \implies (3t-1)(t^3 - 6t^2 + 5t - 4) = 0$$

To solve the above cubic equation, let us convert it into a depressed cubic equation by substituting $t = s + 2$, which then gives,

$$s^3 - 7s - 10 = 0$$

If s is of the form $u^{1/3} + v^{1/3}$, then,

$$s = u^{1/3} + v^{1/3} \implies s^3 - 3(uv)^{1/3}s - (u+v) = 0$$

Comparing the above equation with $s^3 - 7s - 10 = 0$, we obtain,

$$uv = \frac{343}{27} \text{ and } u + v = 10$$

which finally gives,

$$u = 5 + \frac{2\sqrt{249}}{9} \text{ and } v = 5 - \frac{2\sqrt{249}}{9}$$

Therefore, all positive real values of λ which makes V_1 equal to V_2 are given by,

$$\lambda_1 = \frac{1}{\sqrt{3}} \text{ and } \lambda_2 = \sqrt{2 + \sqrt[3]{5 + \frac{2\sqrt{249}}{9}} + \sqrt[3]{5 - \frac{2\sqrt{249}}{9}}}$$

The semi-vertical angles for which the volumes of C_1 and C_2 are equal are then given by

$$tan^{-1}(\lambda_1) = 30°$$

and

$$tan^{-1}(\lambda_2) = 66.28945...°$$

where, 66.28945...° is the counter intuitive constant.

Conclusion

The semi-vertical angle obtained at the end 66.28945...° looks slightly counter intuitive. If we look at the conditions imposed on the rotation of C_1, the angles which naturally comes to our mind are 30° and 60°. Getting an angle of 30° confirms our intuition even more which eventually turns out to be false for the second semi-vertical angle. Thus, instead of going for the “obvious”, one should always seek a well thought out proof.

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Web: http://www.ssmrmh.ro

ANGAD SINGH
email-id: angadsingh1729@gmail.com