Some additional results related to Hyder Series

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Abstract

This paper will be dealing with some supplementary results related to Hyder Series. We will also be looking at a functional version of Hyder Series.

1 Introduction

Hyder Series[2]as introduced in my First paper is a special type of multiple infinite series.In this paper we will be looking at some extra results and some special cases related to it. The Hyder series is defined as

$$\mathcal{H}^q(\alpha_1, \alpha_2, \alpha_3 ... \alpha_k; p_1, p_2, p_3 ... p_k; \beta) = \sum_{m_1, m_2, m_3, ... m_k = 0}^{\infty} \frac{\prod_{1 \le i \le k} \alpha_i^{m_i}}{\left(\sum_{n=1}^{k} p_n m_n + \beta \right)^q}$$

Theorem 1.

$$\mathcal{H}^{q+1}\left(\left(\frac{1}{a} \right)_{r(m)}; p_{r(m)}; p(m-1) \right) = \frac{a^{m-1}}{p^{q+1}(m-1)!} \sum_{j=1}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} \mathrm{Li}_{q+1-j}\left(\frac{1}{a} \right)$$

$m \in \mathbb{N}, m \ge 2 \wedge q \ge m$

Here the square bracket $\begin{bmatrix} m-1 \\ j \end{bmatrix}$ *represents for the stirling number of first kind[6]. and* $\mathrm{Li}_{q+1-j}\left(\frac{1}{a} \right)$ *is the polylogarithm function[4].*

Before proceeding to the proof of theorem 1 we'll be proving Some Important Lemma's related to it.

Lemma 1.

$$\int_0^\infty \frac{ax^{n-1}}{e^x - a} dx = \Gamma(n)\mathrm{Li}_n(a) \tag{1}$$

$$\frac{\partial^m}{\partial a^m}\mathrm{Li}_n(a) = \frac{1}{a^m}\sum_{j=1}^{m} \begin{bmatrix} m \\ j \end{bmatrix} \mathrm{Li}_{n-j}(a) \tag{2}$$

Proof.

$$\begin{aligned}\int_0^\infty \frac{ax^{n-1}}{e^x - a} dx &= a\int_0^\infty \frac{e^{-x}x^{n-1}}{1 - ae^{-x}} dx \\ &= \sum_{k=0}^{\infty} a^{k+1} \int_0^\infty e^{-x(k+1)}x^{n-1}dx \\ &= \Gamma(n)\sum_{k=1}^{\infty} \frac{a^k}{k^n}\end{aligned}$$

$\Rightarrow$

$$\int_0^\infty \frac{ax^{n-1}}{e^x - a} dx = \Gamma(n)\mathrm{Li}_n(a)$$

proof of the 2nd Equation

$$\frac{\partial^m}{\partial a^m}\mathrm{Li}_n(a) = \sum_{k=1}^{\infty} \frac{1}{k^n}\frac{\partial^m}{\partial a^m}(a^k) \tag{3}$$

here $\frac{\partial^m}{\partial a^m}(a^k) = k(k-1)(k-2)...(k-(m-1))a^{k-m}$.
the term $k(k-1)(k-2)...(k-(m-1)$ can be expanded in terms of stirling numbers of first kind .i.e

$$k(k-1)(k-2)...(k-(m-1)) = \sum_{j=1}^{m} \begin{bmatrix} m \\ j \end{bmatrix} k^j \tag{4}$$

On substituting Equation (4) in Equation (3) we'll get

$$\frac{\partial^m}{\partial a^m}\mathrm{Li}_n(a) = \frac{1}{a^m}\sum_{j=1}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\mathrm{Li}_{n-j}(a)$$

□

Proof of Theorem 1

Proof. As per the definition of Hyder Series we can write

$$\mathcal{H}^{q+1}\left(\left(\frac{1}{a}\right)_{r(m)}; p_{r(m)}; p(m-1)\right) = \sum_{k_1,k_2,\dots k_m=0}^{\infty}\frac{1}{a^{k_1+k_2+\dots.k_m}(pk_1+pk_2+\dots.pk_m+p(m-1))^{q+1}}$$

It is known that $\int_0^\infty e^{-st}t^q dt = \frac{q!}{s^{q+1}}$. In our's case $s = p(k_1+k_2+\dots.k_m+m-1)$. Therefore we can write it as

$$\begin{aligned}
&= \frac{1}{q!}\sum_{k_1,k_2,\dots k_m=0}^{\infty}\frac{1}{a^{k_1+k_2+\dots.k_m}}\int_0^\infty e^{-tp(k_1+k_2+\dots.k_m+m-1)}t^q dt \\
&= \frac{1}{q!}\int_0^\infty t^q e^{-tp(m-1)}\sum_{k_1,k_2,\dots k_m=0}^{\infty}\frac{e^{-tp(k_1+k_2+\dots.k_m)}}{a^{k_1+k_2+\dots.k_m}}dt \\
&= \frac{1}{q!}\int_0^\infty \frac{t^q e^{tp}}{\left(e^{tp}-\frac{1}{a}\right)^m}dt \quad (\text{Let } u = tp) \\
&= \frac{1}{p^{q+1}q!}\int_0^\infty \frac{u^q e^u}{\left(e^u-\frac{1}{a}\right)^m}du
\end{aligned}$$

The integral can be solved by using Differentiation under Integral Sign[1] in Equation 1 i.e

$$\frac{\partial^m}{\partial a^m}\int_0^\infty \frac{ax^{n-1}}{e^x-a}dx = \Gamma(n)\frac{\partial^m}{\partial a^m}\mathrm{Li}_n(a)$$

$$m!\int_0^\infty \frac{x^{n-1}e^x}{(e^x-a)^{m+1}}dx = \frac{\Gamma(n)}{a^m}\sum_{j=1}^{m}\begin{bmatrix} m \\ j \end{bmatrix}\mathrm{Li}_{n-j}(a)$$

when a is $\frac{1}{a}$, $n=q+1$,$x=u$ and m as $m-1$ we can rewrite the integral as

$$\int_0^\infty \frac{u^q e^u}{\left(e^u-\frac{1}{a}\right)^m}du = \frac{a^{m-1}q!}{(m-1)!}\sum_{j=1}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix}\mathrm{Li}_{q+1-j}\left(\frac{1}{a}\right)$$

$\Rightarrow$

$$\mathcal{H}^{q+1}\left(\left(\frac{1}{a}\right)_{r(m)};p_{r(m)};p(m-1)\right) = \frac{a^{m-1}}{p^{q+1}(m-1)!}\sum_{j=1}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix}\mathrm{Li}_{q+1-j}\left(\frac{1}{a}\right)$$

Corollary 1.1. *For $a=p=1$ in* **Theorem 1** we have

$$\mathcal{H}^{q+1}(1_{r(m)};1_{r(m)};m-1) = \frac{1}{(m-1)!}\sum_{j=1}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix}\zeta(q+1-j)$$

where $\zeta(q)$ is the zeta function[5]

Proof. Since $\mathrm{Li}_{q+1-j}(1)=\zeta(q+1-j)$ Therefore

$$\mathcal{H}^{q+1}(1_{r(m)};1_{r(m)};m-1) = \frac{1}{(m-1)!}\sum_{j=1}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix}\zeta(q+1-j)$$

□

Example 1. *For $m=2$,$m=3$ and $m=5$ we have*

$$\mathcal{H}^{q+1}(1_{r(2)};1_{r(2)};1) = \zeta(q)$$

$$\mathcal{H}^{q+1}(1_{r(3)};1_{r(3)};2) = \frac{1}{2}\left(\zeta(q-1)-\zeta(q)\right)$$

$$\mathcal{H}^{q+1}(1_{r(5)};1_{r(5)};4) = \frac{1}{24}\left(11\zeta(q-1)+\zeta(q-3)-6\zeta(q)-6\zeta(q-2)\right)$$

1.1 A Functional type version of Hyder Series

$$\mathcal{H}^q(\{f_i(k)\}_{i=1}^n;\{p_i\}_{i=1}^n;\beta) = \sum_{k_1,k_2,k_3,\dots k_n=0}^{\infty} \frac{\prod\limits_{1\le i\le n} f_i(k_i)}{\left(\sum\limits_{m=1}^{n} p_m k_m + \beta\right)^q} \tag{5}$$

Here $f_i(k)$ is the desired functions . If the function is $f_i(k)$ is being repeated n times. i.e $f_1(k) = f_2(k) = ... = f_n(k) = f(k)$. Then it can be denoted as $\mathcal{H}^q(\{f(k)\}_{r(n)};\{p_i\}_{i=1}^n;\beta)$

1.2 Some Harmonic Sums of Hyder Series Type

In this section we'll be looking at some unique combination of Harmonic sums related to Hyder functional Series Type.

1.2.1 Notations

When we include the Harmonic Number[7] in to Hyder Series, for example if its in the case of a Double Sum Type where $f_i(k) = \dfrac{H_k}{a^k}$ such that $f_1(k) = f_2(k) = \dfrac{H_k}{a^k}$.

$$\mathcal{H}^q\left(\left\{\frac{H_k}{a^k}\right\}_{r(2)};p_{r(2)};\beta\right) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{H_{k_1}H_{k_2}}{a^{k_1+k_2}(pk_1+pk_2+\beta)^q}$$

Here H_k is the Harmonic number.

Lemma 2.

$$\int_0^1 \frac{\log^m\left(1-\frac{x}{a}\right)}{\left(1-\frac{x}{a}\right)^m}dx = \frac{-m!a}{(m-1)^{m+1}} + \frac{m!a^m}{(a-1)^{m-1}(m-1)^{m+1}} + \frac{m!a^m}{(a-1)^{m-1}} \left(\sum_{k=1}^{m}\frac{\log^k\left(1-\frac{1}{a}\right)}{k!(m-1)^{m-k+1}}\right)$$

Proof.

$$\int_0^1 \frac{\log^m\left(1-\frac{x}{a}\right)}{\left(1-\frac{x}{a}\right)^m}dx = a\int_{1-\frac{1}{a}}^1 \frac{\log^m t}{t^m}dt \qquad \left(\text{Let } \beta = 1-\frac{1}{a}\right)$$

$$= a\frac{\partial^m}{\partial^m \alpha}\int_\beta^1 t^{\alpha-m}$$

$$= a\lim_{\alpha\to 0}\frac{\partial^m}{\partial^m\alpha}\left(\frac{1}{\alpha-m+1}-\frac{\beta^{\alpha-m+1}}{\alpha-m+1}\right)$$

here

$$\frac{\partial^m}{\partial^m\alpha}\left(\frac{1}{\alpha-m+1}\right) = \frac{(-1)^m m!}{(\alpha-m+1)^{m+1}}$$

By General Leibinz differentiation

$$\frac{\partial^m}{\partial^m\alpha}\left(\frac{\beta^{\alpha-m+1}}{\alpha-m+1}\right) = \frac{\beta^{\alpha-m+1}(-1)^m m!}{(\alpha-m+1)^{m+1}}+\sum_{k=1}^m\binom{m}{k}\frac{\beta^{\alpha-m+1}(-1)^{m-k}(m-k)!}{(\alpha-m+1)^{m-k+1}}\log^k\beta$$

on substituting this values and letting $\alpha = 0$ we'll get

$$\int_0^1 \frac{\log^m\left(1-\frac{x}{a}\right)}{\left(1-\frac{x}{a}\right)^m}dx = \frac{-m!a}{(m-1)^{m+1}}+\frac{m!a^m}{(a-1)^{m-1}(m-1)^{m+1}}+\frac{m!a^m}{(a-1)^{m-1}}\left(\sum_{k=1}^m\frac{\log^k\left(1-\frac{1}{a}\right)}{k!(m-1)^{m-k+1}}\right)$$

□

Theorem 2.

$$\mathcal{H}\left(\left\{\frac{H_k}{a^k}\right\}_{r(m)};1_{r(m)};1\right) = \frac{(-1)^{m+1}m!a}{(m-1)^{m+1}}+\frac{(-1)^m m!a^m}{(a-1)^{m-1}(m-1)^{m+1}}+\frac{(-1)^m m!a^m}{(a-1)^{m-1}}\left(\sum_{k=1}^m\frac{\log^k\left(1-\frac{1}{a}\right)}{k!(m-1)^{m-k+1}}\right)$$

Proof. It's known

$$\mathcal{H}\left(\left\{\frac{H_k}{a^k}\right\}_{r(m)};1_{r(m)};1\right) = \sum_{k_1,k_2,\ldots k_m\geq 0}\frac{H_{k_1}H_{k_2}\ldots.H_{k_m}}{a^{k_1+k_2\ldots.+k_m}(k_1+k_2\ldots.+k_m+1)}$$

$$= \sum_{k_1,k_2,\ldots k_m\geq 0}\frac{H_{k_1}H_{k_2}\ldots.H_{k_m}}{a^{k_1+k_2\ldots.+k_m}}\int_0^1 x^{k_1+k_2+..+k_m}dx$$

Since the Sum is Convergent we can interchange the order of Integral and Sum . Therefore

$$\mathcal{H}\left(\left\{\frac{H_k}{a^k}\right\}_{r(m)};1_{r(m)};1\right)=\int_0^1\sum_{k_1,k_2,\dots k_m\geq 0}H_{k_1}H_{k_2}....H_{k_m}\left(\frac{x}{a}\right)^{k_1+k_2+..+k_m}dx$$

The above harmonic sum can be evaluated by making use of the Generating Function i.e

$$\sum_{n=1}^{\infty}z^nH_n=\frac{-\log(1-z)}{1-z}$$

Therefore

$$\mathcal{H}\left(\left\{\frac{H_k}{a^k}\right\}_{r(m)};1_{r(m)};1\right)=(-1)^m\int_0^1\frac{\log^m\left(1-\frac{x}{a}\right)}{\left(1-\frac{x}{a}\right)^m}dx$$

The Integral has been evaluated in **Lemma 2**. Therefore the final result becomes

$\square$

Example 2. *For $a=2$ and $m=3$ we have*

$$\mathcal{H}\left(\left\{\frac{H_k}{2^k}\right\}_{r(3)};1_{r(3)};1\right)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{H_{k_1}H_{k_2}H_{k_3}}{2^{k_1+k_2+k_3}(k_1+k_2+k_3+1)}$$

Therefore from **Theorem 2** we'll get

$$\mathcal{H}\left(\left\{\frac{H_k}{2^k}\right\}_{r(3)};1_{r(3)};1\right)=4\log^3(2)+6\log(2)-6\log^2(2)-\frac{9}{4}$$

Lemma 3.

$$\int_0^1\frac{\log^2\left(1-\frac{x}{a}\right)}{\left(1-\frac{x}{a}\right)^2}\log(x)\,dx=a\left(-2\zeta(3)+\log^2\left(1-\frac{1}{a}\right)(\log(a)+1)+2\text{Li}_3\left(1-\frac{1}{a}\right)\right.$$
$$\left.-2\text{Li}_2\left(1-\frac{1}{a}\right)\log\left(1-\frac{1}{a}\right)+\frac{\log^3\left(1-\frac{1}{a}\right)}{3}+2\text{Li}_2\left(\frac{1}{a}\right)+2\log\left(1-\frac{1}{a}\right)\right.$$

Proof.

$$I=\int_0^1 \frac{\log^2\left(1-\frac{x}{a}\right)}{\left(1-\frac{x}{a}\right)^2}\log(x)\,dx = a\int_{1-\frac{1}{a}}^1 \frac{\log^2 t}{t^2}\log(a(1-t))\,dt \qquad \left(\text{Let } \beta = 1-\frac{1}{a}\right)$$

$$= a\int_\beta^1 \frac{\log^2 t}{t^2}(\log a+\log(1-t))dt$$

$$= a\log(a)\underbrace{\int_\beta^1 \frac{\log^2 t}{t^2}dt}_{I_1} + a\underbrace{\int_\beta^1 \frac{\log^2 t\log(1-t)}{t^2}dt}_{I_2}$$

Here I_1 can be easily evaluated by using Differentiation under Integral Rule therefore

$$I_1 = \frac{\log^2\beta+2\log\beta+2}{\beta}-2$$

To evaluate I_2 we'll Indefinitely Integrate it by using Integral by Parts

$$I_2 \overset{IBP}{=} -\log(1-t)\left(\frac{\log^2 t+2\log t+2}{t}\right)+\underbrace{\int \frac{\log^2 t+2\log t+2}{t(t-1)}dt}_{A}$$

here

$$A \overset{PFD}{=} \int \frac{\log^2 t}{(t-1)}dt - \frac{\log^3 t}{3}+2\int \frac{\log t}{(t-1)}dt - \log^2 t + 2(\log(t-1)-\log(t))$$

(PFD=Partial Fraction Decomposition)

where

$$\int \frac{\log^2 t}{(t-1)}dt \overset{IBP}{=} \log^2 t\log(1-t) - 2\int \frac{\log t\log(1-t)}{t}dt$$

$$= \log^2 t\log(1-t) - 2\mathrm{Li}_3(t) + 2\mathrm{Li}_2(t)\log(t)$$

and

$$\int \frac{\log t}{(t-1)}dt = -\mathrm{Li}_2(1-t)$$

On substituting the respective values we'll get

$$I_2 = \log(1-t)\left(2-\frac{\log^2 t+2\log t+2}{t}\right)+\log^2 t(\log(1-t)-1)-2\mathrm{Li}_3(t)+2\mathrm{Li}_2(t)\log(t)$$

$$-\frac{\log^3 t}{3}-2\mathrm{Li}_2(1-t)-2\log(t)$$

On applying the Integral Limits in I_2 we'll get

$$I_2 = -2\zeta(3) + \log(1-\beta)\left(2 - \frac{\log^2\beta - 2\log\beta + 2}{\beta}\right) - \log^2\beta(\log(1-\beta) - 1)$$
$$+2\mathrm{Li}_3(\beta) - 2\mathrm{Li}_2(\beta)\log(\beta) + \frac{\log^3\beta}{3} + 2\mathrm{Li}_2(1-\beta) + 2\log(\beta)$$

Finally upon substituting the values of I_1 and I_2 in the Integral I we'll get the result as

$$I = a\left(-2\zeta(3) + \log^2\left(1-\frac{1}{a}\right)(\log(a)+1) + 2\mathrm{Li}_3\left(1-\frac{1}{a}\right) - 2\mathrm{Li}_2\left(1-\frac{1}{a}\right)\log\left(1-\frac{1}{a}\right)\right.$$
$$\left. + \frac{\log^3\left(1-\frac{1}{a}\right)}{3} + 2\mathrm{Li}_2\left(\frac{1}{a}\right) + 2\log\left(1-\frac{1}{a}\right)\right)$$

□

Theorem 3.

$$\mathcal{H}^2\left(\left\{\frac{H_k}{a^k}\right\}_{r(2)}; 1_{r(2)}; 1\right) = a\left(2\zeta(3) - \log^2\left(1-\frac{1}{a}\right)(\log(a)+1) - 2\mathrm{Li}_3\left(1-\frac{1}{a}\right)\right.$$
$$\left. +2\mathrm{Li}_2\left(1-\frac{1}{a}\right)\log\left(1-\frac{1}{a}\right) - \frac{\log^3\left(1-\frac{1}{a}\right)}{3} - 2\mathrm{Li}_2\left(\frac{1}{a}\right) - 2\log\left(1-\frac{1}{a}\right)\right)$$

Proof.

$$\mathcal{H}^2\left(\left\{\frac{H_k}{a^k}\right\}_{r(2)}; 1_{r(2)}; 1\right) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{H_{k_1}H_{k_2}}{a^{k_1+k_2}(k_1+k_2+1)^2}$$
$$= (-1)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{H_{k_1}H_{k_2}}{a^{k_1+k_2}}\int_0^1 x^{k_1+k_2}\log(x)\,dx$$
$$= (-1)\underbrace{\int_0^1 \frac{\log^2\left(1-\frac{x}{a}\right)}{\left(1-\frac{x}{a}\right)^2}\log(x)\,dx}_{I}$$

As we can see that the underlined Integral has already been proved in **Lemma 3** therefore our final result becomes

$$\mathcal{H}^2\left(\left\{\frac{H_k}{a^k}\right\}_{r(2)}; 1_{r(2)}; 1\right) = a\left(2\zeta(3) - \log^2\left(1-\frac{1}{a}\right)(\log(a)+1) - 2\mathrm{Li}_3\left(1-\frac{1}{a}\right)\right.$$
$$\left. +2\mathrm{Li}_2\left(1-\frac{1}{a}\right)\log\left(1-\frac{1}{a}\right) - \frac{\log^3\left(1-\frac{1}{a}\right)}{3} - 2\mathrm{Li}_2\left(\frac{1}{a}\right) - 2\log\left(1-\frac{1}{a}\right)\right)$$

□

Example 3. *For* $a = 2$ *in* **Theorem 3** we have the result as

$$\mathcal{H}^2\left(\left\{\frac{H_k}{2^k}\right\}_{r(2)};1_{r(2)};1\right) = 2\left(2\zeta(3) - \log^2\left(1-\frac{1}{2}\right)(\log(2)+1) - 2\mathrm{Li}_3\left(1-\frac{1}{2}\right)\right.$$
$$\left.+2\mathrm{Li}_2\left(1-\frac{1}{2}\right)\log\left(1-\frac{1}{2}\right) - \frac{\log^3\left(1-\frac{1}{2}\right)}{3} - 2\mathrm{Li}_2\left(\frac{1}{2}\right) - 2\log\left(1-\frac{1}{2}\right)\right)$$

On simplyfing the above expression we get the result as

$$\mathcal{H}^2\left(\left\{\frac{H_k}{2^k}\right\}_{r(2)};1_{r(2)};1\right) = \frac{\zeta(3)}{2} - \frac{\pi^2}{3} + 4\log(2)$$

Lemma 4.

$$\int_0^{\frac{1}{a}} \frac{\mathrm{Li}_2(x)\ln(1-x)}{(1-x)^2}dx = \mathrm{Li}_2\left(\frac{1}{a}\right)\left(\frac{\log(1-\frac{1}{a})+1}{(1-\frac{1}{a})}\right) + \log^2\left(1-\frac{1}{a}\right)\left(\log\left(\frac{1}{a}\right)-\frac{1}{2}\right)$$
$$+2\log\left(1-\frac{1}{a}\right)\mathrm{Li}_2\left(1-\frac{1}{a}\right) - 2\mathrm{Li}_3\left(1-\frac{1}{a}\right) + 2\zeta(3) - \frac{1}{3}\log^3\left(1-\frac{1}{a}\right) - \mathrm{Li}_2\left(\frac{1}{a}\right)$$

Proof.

$$\begin{aligned} I &= \int_0^{\frac{1}{a}} \frac{\mathrm{Li}_2(x)\log(1-x)}{(1-x)^2}dx \\ &= \mathrm{Li}_2\left(\frac{1}{a}\right)\left(\frac{\log(1-\frac{1}{a})+1}{(1-\frac{1}{a})}\right) + \underbrace{\int_0^{\frac{1}{a}}\left(\frac{\log(1-x)}{x}\right)\left(\frac{\log(1-x)+1}{(1-x)}\right)dx}_{I_1} \end{aligned}$$

here

$$I_1 = \underbrace{\int_0^{\frac{1}{a}} \frac{\log^2(1-x)}{x}dx}_{I_{1,1}} + \int_0^{\frac{1}{a}} \frac{\log^2(1-x)}{(1-x)}dx + \int_0^{\frac{1}{a}} \frac{\log(1-x)}{x}dx + \int_0^{\frac{1}{a}} \frac{\log(1-x)}{(1-x)}dx$$

here $I_{1,1}$ is a known special Integral and its beautiful proof can also be seen in the book of Almost impossible Integral[3] and Series page 3.

$\Rightarrow I_{1,1} = \log\left(\frac{1}{a}\right)\log^2\left(1-\frac{1}{a}\right)+2\log\left(1-\frac{1}{a}\right)\mathrm{Li}_2\left(1-\frac{1}{a}\right)-2\mathrm{Li}_3\left(1-\frac{1}{a}\right)+$
$2\zeta(3)$
The remaining three integrals can be easily evaluated by elementary method. therefore the final result becomes

$$I = \mathrm{Li}_2\left(\frac{1}{a}\right)\left(\frac{\log(1-\frac{1}{a})+1}{(1-\frac{1}{a})}\right)+\log^2\left(1-\frac{1}{a}\right)\left(\log\left(\frac{1}{a}\right)-\frac{1}{2}\right)+2\log\left(1-\frac{1}{a}\right)\mathrm{Li}_2\left(1-\frac{1}{a}\right)$$
$$-2\mathrm{Li}_3\left(1-\frac{1}{a}\right)+2\zeta(3)-\frac{1}{3}\log^3\left(1-\frac{1}{a}\right)-\mathrm{Li}_2\left(\frac{1}{a}\right)$$

Theorem 4.

$$\mathcal{H}\left(\left\{\frac{H_k^2}{a^k},\frac{H_k}{a^k}\right\};1_{r(2)};1\right) = -a\left(\mathrm{Li}_2\left(\frac{1}{a}\right)\left(\frac{\log(1-\frac{1}{a})+1}{(1-\frac{1}{a})}\right)+\log^2\left(1-\frac{1}{a}\right)\left(\log\left(\frac{1}{a}\right)-\frac{1}{2}\right)+\right.$$
$$\left.2\log\left(1-\frac{1}{a}\right)\mathrm{Li}_2\left(1-\frac{1}{a}\right)-2\mathrm{Li}_3\left(1-\frac{1}{a}\right)+2\zeta(3)-\frac{1}{3}\log^3\left(1-\frac{1}{a}\right)-\mathrm{Li}_2\left(\frac{1}{a}\right)\right)$$

Proof.

$$\mathcal{H}\left(\left\{\frac{H_k^2}{a^k},\frac{H_k}{a^k}\right\};1_{r(2)};1\right) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{H_{k_1}^2 H_{k_2}}{a^{k_1+k_2}(k_1+k_2+1)}$$
$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{H_{k_1}^2 H_{k_2}}{a^{k_1+k_2}}\int_0^1 x^{k_1+k_2}dx$$

It is known that

$$\sum_{n=1}^{\infty} z^n H_n^2 = \frac{\mathrm{Li}_2(z)}{1-z}$$

$\Rightarrow$

$$\mathcal{H}\left(\left\{\frac{H_k^2}{a^k},\frac{H_k}{a^k}\right\};1_{r(2)};1\right) = -a\int_0^{\frac{1}{a}}\frac{\mathrm{Li}_2(x)\ln(1-x)}{(1-x)^2}dx$$

The proof for the integral is given in **Lemma 2** . Upon substituting the integral value the final result is

$$\mathcal{H}\left(\left\{\frac{H_k^2}{a^k},\frac{H_k}{a^k}\right\};1_{r(2)};1\right)=-a\left(\mathrm{Li}_2\left(\frac{1}{a}\right)\left(\frac{\log(1-\frac{1}{a})+1}{(1-\frac{1}{a})}\right)+\log^2\left(1-\frac{1}{a}\right)\left(\log\left(\frac{1}{a}\right)-\frac{1}{2}\right)+\right.$$
$$\left.2\log\left(1-\frac{1}{a}\right)\mathrm{Li}_2\left(1-\frac{1}{a}\right)-2\mathrm{Li}_3\left(1-\frac{1}{a}\right)+2\zeta(3)-\frac{1}{3}\log^3\left(1-\frac{1}{a}\right)-\mathrm{Li}_2\left(\frac{1}{a}\right)\right)$$

□

Example 4. *For $a=2$ in* **Theorem 3** we have

$$\mathcal{H}\left(\left\{\frac{H_k^2}{2^k},\frac{H_k}{2^k}\right\};1_{r(2)};1\right)=-\frac{\zeta(3)}{2}-\frac{\pi^2}{6}-2\log^3(2)+2\log^2(2)+\frac{\pi^2\log(2)}{3}$$

2 Conclusion

This is the second paper related to Hyder Series. In this paper we came to see the Functional Type version of Hyder Series,here the function that was introduced was a sequential type function of Harmonic Series and We derived some special results related to it. The Functional Type version can be experimented with different special functions and can give out some unique results. I hope there might be some intresting results related to this series which is still yet to be discovered.

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