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CONTENT

A TRIBUTE TO TRAIAN LALESCU

AN OUTSTANDING ROMANIAN GREAT SCIENTIST

By D.M.Bătinețu-Giurgiu,Neculai Stanciu-Romania

(Born on July 12, 1882 – Died on June 15, 1929)

Traian Lalescu's Problem – Published in Romanian Mathematical Gazette, Vol. VI, 1900-1901, as problem 579, p. 148.

Problem 579. Compute the limit:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)$$

Solution:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n}\right)\overset{k!\cong\left(\frac{k}{e}\right)^k\Rightarrow\sqrt[k]{k!}\cong\frac{k}{e}}{=}\lim_{n\to\infty}\left(\frac{n+1}{e}-\frac{n}{e}\right)=\frac{1}{e}$$

Traian Lalescu started in this world from Bucharest on July 12, 1882 and there he would find his premature end on June 15, 1929.

His father, a modest bank clerk, had the same first name, Traian and was originally from Cornea commune, Caraş-Severin county. In 1876 he wrote a paper on the economic problems of agriculture and another, "The agenda of popular banks and the Lalescu coefficient method", which can be found at the Library of the Romanian Academy. His mother was from the Transylvanian side. The scientist presented himself as a native of the village near Caransebeș.

He begins his primary education in his native Bucharest. The first two middle school classes in Craiova. He took the next two middle school classes in Moldova, at Roman, then studied high school at the Boarding School in Iaşi. His name is inscribed on the high school honor roll. Traian Lalescu has always had the imprint of the environment so varied in which he was formed due to the childhood pilgrimages he made with his family: he was as solid as Banat, talkative as Olten, serious as Transylvanian, beautiful lover as Moldovan and with the sprinting spirit like the one from Bucharest.

Throughout his studies, Traian Lalescu was the first prize winner of the class and the honorary prize winner of the school, becoming from the 10th grade correspondent of the Mathematical Gazette.

In 1900 he was the first to enter the National School of Bridges and Roads in Bucharest. In the first year of studies, he is financially assisted by professor Andrei Ioachimescu, who took him home and treated him like his own child for a year.

In 1901, he published the first original mathematical note of the Mathematical Gazette, ''On a Summary of Series''.

In 1903 he retired and went to the Faculty of Sciences of the University of Bucharest, Department of Mathematics.

In 1905 he became a member of the Mathematical Gazette editorial office.

On June 17, 1905, he obtained a degree in mathematics with the grade "very good". Also in 1905 he obtained by competition, again succeeding the first, an "Adamachi" scholarship for further studies in Paris, Sorbonne, where he again obtained his Licence of Mathematics. Here he is also helped financially by professor Ion Ionescu-Bizeţ.

Between 1906 and 1910 he was a mathematics teacher at the gymnasium in Giurgiu. In 1906 he was attracted to Émile Picard's course of integral equations at the Sorbonne.

In 1907 he published four notes in Comptes Rendus des Séances de l'Academie des Sciences de Paris (CRASP).

In 1908 he defended his doctoral dissertation "Sur l'équation de Volterra", under the direction of Émile Picard, which he published both in the Gauthier-Villars Publishing House and in the prestigious journal, Journal de Mathématiques Pures et Appliquèes, Paris. Also in 1908 he published a work on Galois Theory. Thanks to the help provided by the Romanian Academy, he presented his results at the International Congress of Mathematicians in Rome, April 6-11, 1908. Here he met Vito Volterra. The results obtained are also presented in Romania, in the Bulletin of the Société des Sciences, Bucharest (BSS).

From the summer of 1908 to the spring of 1909, he was in another large mathematical center in Göttingen, where David Hilbert and the school he had set up were located. He attended the classes of D. Hilbert and presented a paper at the Mathematical Society of Göttingen, in a meeting chaired by Felix Klein, and on June 15, 1909 he obtained the scientific title of docent.

He made his debut as university professor on June 1, 1909, as an assistant for graphic works of Ion Ionescu-Bizeţ professor. He stayed here until May 15, 1910.

After a brief return to the country, he returned to Göttingen for 1910-1911, where he gave a series of papers on his own research, which were appreciated by David Hilbert, Erhardt Schmidt and Felix Klein. Then, he goes again to Paris, where he publishes three other articles in CRASP and in our country in BSS.

Between 1910 and 1913 he was an associate professor of higher algebra at the University of Bucharest.

Between 1911 and 1912 he was transferred from Giurgiu to Bucharest, to the Central Seminary, then to the Şincai and Dimitrie Cantemir Gymnasiums.

In 1911 he published the world's first significant monograph, before Hilbert, on integral equations (the following year it was translated into French). This monograph was translated and edited in 1918, in Polish by S. Mazurkiewicz at the Polish Academy of Sciences and Letters, and as Hugo Steinhauss said, this was the book from which Polish mathematicians learned the theory of integral equations. This book was then republished by the Romanian Academy Publishing House in 1957. Vito Volterra and Édouard Goursat emphasized in their books the importance of Traian Lalescu's research on integral equations. The echoes about Traian Lalescu's works, about the results obtained in the theory of integral equations continued long after his physical disappearance. In particular, Prof. Albrecht Pietsch from Jena, in 1980, during a visit to the Institute of the Romanian Academy told to Prof. Nicolae Popa that Traian Lalsecu, together with Şerban Gheorghiu, were the first which prove that the product of two Hilbert operators - Schmidt is a track operator.

In 1911 he was appointed full professor at the School of Roads and Bridges, at the department of analytical geometry, in place of Spiru Haret and also in 1911 he was professor of rational mechanics at the University of Bucharest.

Since 1912 he has been an assistant at the department of descriptive geometry at the University of Bucharest.

After the publication of the last issue of the 21st year of the Mathematical Gazette, the First World War begins. From the following year, only the first two issues appeared in Bucharest, the occupation of the city by German troops and the destruction of the printing house make it impossible for the magazine to appear. In December 1917, at the residence of Traian Lalescu from Iassy, it was decided to print the magazine in the capital of Moldova, at the printing house „H. Goldner" - where most of the workers were old and infirm. In order to stimulate them to print the Gazette, T. Lalescu and V. Teodoreanu brought them food from their own rations!

The number of pages per issue decreases and major dysfunctions appear in the publication of the magazine: the December 1916 issue appears in April 1917, and no. 3 of vol. XXII appears at the end of the war! And the content of the articles is different. Articles on ballistics or applications of mathematics in the military sciences are written. Number 1 of

vol. XXIII is opened by the vibrant article "To the Romanian soldiers" and is dedicated to the soldiers in the front line (Gazeta had the authorization to be distributed on the front). The editorial meetings are held regularly, under the chairmanship of the venerable professor C. Climescu, the initiator of Scientific Recreations. The construction of a Mathematical Gazette House has been planned since from 1920. N. Nicolescu donates the first 500 lei for this purpose. Three years later, Traian Lalescu proposed to Tancred Constantinescu, then General Manager of the Railways, to donate a plot of land near the North Station for the construction of the place. Started in September 1933, it was completed in August 1934, and on January 27, 1935, on a Sunday, it was inaugurated. All four "pillars" of the Gazette are present. "Of all the problems proposed in the Mathematical Gazette, none was more difficult, more beautiful and more interesting than the problem of the Mathematical Gazette House", remarked Gh. Ţiţeica on this occasion.

In 1919 he graduated as an electrical engineer after graduating from the Ecole Supérieure de Electricité in Paris.

In order to support the efforts of the Romanian delegation to the Paris Peace Conference (1919), of which Traian Lalescu was a member, the scientist wrote a monograph on the ethnographic problem of Banat, providing scientific arguments regarding this region belonging to Romania. Traian Lalescu was deputy of Caransebeş. He drafted and presented in Parliament a Report on the budget for the year 1925. He wrote philosophical dialogues on mathematical topics, being "primarily interested in the idea, the elegance of the proof, and the deep meanings of the theorems."

He campaigned for the establishment of the Polytechnic School of Timişoara, whose first rector (or director) was in 1920.

Professor Traian Lalescu played an important role in the publication on March 15, 1921 of the Journal of Mathematics from Timișoara.

Since 1990 he has been a post-mortem member of the Romanian Academy.

From the work of Traian Lalescu we present:

ARTICLES AND BOOKS IN ROMANIAN:

1. Agenda bãncilor populare și metodul de coeficient Lalescu. București, 1906.

2. Introducere la teoria ecuațiunilor integrale. Bucuresti, 1911.

3. Dl. Spiru Haret ca om de științã. În: Lui Spiru C. Haret, ale tale, dintru ale tale, la împlinirea celor șeasezeci ani. București, 1911.

4. Asupra variației valorilor caracteristice. București, Librãriile Socec și C. Sfetea; Viena, Gerold; Berlin, R. Friedlander und Sohn; Lipsca, O. Harrassowitz, 1912, Academia Românã.

5. Însumarea a doi sîmburi neortogonali. Notã: București, Librãriile Socec și C. Sfetea; Viena, Gerold; Berlin, R. Friedlaender und Sohn; Lipsca, O. Harrassowitz, 1913.

6. Raportul general asupra proiectului de buget al veniturilor și cheltuielilor Statului pe anul 1925, prezentat Adunãrii Deputaților. București, 1914.

7. Culegere de probleme de geometrie descriptivã și cosmografie (în colaborare cu Şt. N. Mirea). București, 1914.

8. Cuvântare la sãrbãtorirea ing. Constantin M. Mironescu. În vol.: Sãrbãtorirea domnului inginer inspector general Constantin M Mironescu, cu ocaziunea retragerii sale din funcțiunea de Director al școalei de Poduri și șosele. Lucrare întocmitã din initiativa Comitetului organizator de Dl. Prof. Traian Lalescu. București, Tipografia Profesionalã Dim. C. Ionescu, 1915.

9. Transcrierea dupã slove cirilice însoțitã de o notã biograficã și note explicative a cărţii Trigonometria de Gheorghe Lazăr. București, 1919. (Biblioteca Gazetei matematice).

10. Tratat de geometrie analiticã. Dreaptã, Plan, Conice, Cuadrice, Aplicatiile geometrice ale calculului infinitezimal. Editia întîi. București, 1920. (Biblioteca Gazetei matematice).

11. Tratat de geometrie analiticã. Ediția II. Fasc. I. București, 1923. (Biblioteca Gazetei Matematice).

12. Telefonia fãră fir. București, Cartea Româneascã, 1923.

13. Calculul algebric. Polinoame, fracțiuni raționale. Biblioteca manualelor științifice, București ,1924.

14. Prefațã la cartea Dunãrea dintre Bazias și Turnu-Severin, Daniil Laitin. București, Tipografiile Române Unite, 1925. (Biblioteca Academiei Bucureşti).

15. Curs de geometrie analiticã. Fascicula IV. Aplicațiile geometrice ale calculului infinitezimal. București, Tipografia F. Göbl și Fiii, 1927.(Biblioteca Gazetei Matematice)

16. Curs de geometrie analiticã. Dreaptã, plan, conice, cuadrice. București, 1931.(Biblioteca Universitarã).

17. Culegere de probleme de geometrie descriptivă. Ediția a doua. Revăzută de R. N. Racliş. București, 1935. (Publicațiunile Institutului Matematic Român).

18. Tratat de geometrie analiticã. Curs. Ediția 1938 revãzutã. Caietul I—III. Caietul 1: Dreapta, planul; 2. Conicele; 3. Cuadricele. București, 1938.

19. Tratat de geometrie analiticã. Curs profesat la Politehnica din București de Traian Lalescu. Editia 1944, revãzutã de Neculai Raclis. Cu o prefațã de D. Busilã. Caietul I. Dreapta, planul. București, Tipografia F. Göbl si Fiii, 1944.

20. Tratat de geometrie analiticã. Curs profesat. Ediția 1938, revãzutã. Caietul 2, 3, ed. 1944. Bucuresti, F. Göbl si Fiii, 1938–1947.

21. Introducere la teoria ecuațiilor integrale. București, Editura Academiei Republicii Populare Române, 1956.

22. Geometria triunghiului. Traducere îngrijitã de O. Sacter dupã ediția a 2-a apãrutã în limba francezã în anul 1937. București, Editura tineretului, 1958.

23. Tratat de geometrie analiticã. Dreaptã, Plan, Conice, Cuadrice, Aplicațiile geometrice ale calculului infinitezimal. Fasc. 3. Cuadrice. București, 1992. (Biblioteca Gazetei matematice).

24. Geometria triunghiului. Craiova, Editura Apollo, 1993.

SCIENTIFIC ARTICLES AND BOOKS IN A FOREIGN LANGUAGE:

1. Sur la composition des formes quadratiques. Extrait des Nouvelles Annales de Mathématiques, 4-e série, t. VII, Paris, avril, 1907.

2. Sur les solutions périodiques des equations différentielles linéaires. Paris, 1907.(CRASP).

3. Sur l'ordre de la fonction entière D (l) de Fredholm. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1907.(CRASP).

4. Sur le groupe des équations trinomes. Paris, 1907.(CRASP).

5. Sur une classe d'équations différentielles linéaires d'ordre infini. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1908.(CRASP).

6. Thèses présentées à la Faculté des Sciences de Paris. 1-ère Thèse sur l'équation de Volterra. 2-e Thèse. Propositions données par la faculté. Paris, 1908.

7. Sur l'équation de Volterra, 1-ère thèse. Propositions données par la Faculté, 2-e thèse. Soutenues [en] 1908, devant la commission d'examen. Thèses présentées à la Faculté des Sciences de Paris pour obtenir le grade de docteur en sciences mathématiques. Paris, Gauthier-Villars, 1908.

8. La théorie générale de Galois, Annales de la Faculté des Sciences de Toulouse, Paris, 1908.

9. Quelques remarques sur l'équation intégrale de Volterra. Bucarest,1909. (BSS).

10. Sur les solutions analytiques de l'équation In: Atti del IV Congresso internazionale dei matematici. Roma, 6–11 avrile 1908. Communicazione delle sezioni I e II. Vol. 2. Roma, 1909.

11. La théorie des équations intégrales linéaires d'ordre infini. Bucarest, 1910. (BSS).

12. Quelques remarques sur l'équation intégrale de Fredholm. Bucuresti, 1910. (BSS).

13. Sur l'équation de Lamé, nr. 1. Bucarest, 1910. (BSS).

14. Sur les noyaux résolvants. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1910.(CRASP).

15. Sur les noyaux symétriques gauches. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1910.(CRASP).

16. Théorèmes sur les valeurs caractéristiques. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1911. (CRASP).

17. Introduction à la théorie des équations intégrales. Avec une préface de M. Emile Picard. Paris, 1912. (CRASP).

18. Sur L'addition des noyaux non ortogonaux. Bucarest, 1913. (BSS).

19. Sur la notion des noyaux symétriques gauches définis. Sur une suite de noyaux remarquables. Sur une classe de noyaux brisés. Bucarest, 1915. (BSS).

20. 1. Sur un piège de la théorie des equations intégrales. 2. Un théorème sur les noyaux composés. Bucarest, 1915. (BSS).

21. Sur les solutions périodiques des equations différentielles du second ordre. Jassy, 1915.

22. Sur les problèmes bilocaux relatifs à l'équation différentielle linéaire du second ordre. Bucarest, 1916. (BSS).

23.Les classes de noyaux symétrisables. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1918. (CRASP).

24. Sur les séries trigonométriques et la théorie des équations intégrales. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1918. (CRASP).

25. Sur les fonctions polygonales périodiques. In: Comptes Rendus des Séances de l'Académie de Sciences. Paris, 1918. (CRASP).

26. Sur l'application des équations intégrales à la théorie des équations différentielles linéaires. In: Comptes Rendus des Séances de l'Académie des Sciences. Paris, 1918. (CRASP).

27. Les problèmes bilocaux pour l'équation différentielle linéaire du second ordre. Bucarest, 1918. (BSS).

28. Les équations différentielles linéaires d'ordre infini et l'équation de Fredholm. Roma, 1918.

29. Wstep do teorji równan calckowych. [Introducere în teoria ecuaþiilor integrale]. traducere din limba francezã de S. Mazurkiewicy. Warszawa, 1918.

30. Données statistiques sur le Banat. Paris, 1919.

31. Le problème éthnographique du Banat. Paris, 1919.

32. Sur l'approximation des fonctions par des séries trigonométriques. Bucarest, 1920. (BSS).

33. Sur la loi asymptotique de quelques classes de valeurs caractéristiques. Bucureşti, 1924. (BSS).

34. Sur un théorème de la théorie des noyaux simétrisables. Bucarest, Cultura Naţională, 1925. (Académie de Roumanie).

35. La géométrie du triangle. Deuxième édition. Avec une lettre de M. Émile Picard et une préface de M. Georges Tzitzeica. Bucarest, 1937.

36. La géométrie du triangle. Paris, Librairie Vuibert, 1952.

ARTICLES PUBLISHED IN MATHEMATICAL GAZETTE:

1. Asupra însumării de factoriale consecutive, anul V, 1899-1900, pp. 277-281.

2. Câteva relaţiuni trigonometrice, anul VI, 1900-1901, pp. 197-200.

3. Asupra unei chestiuni de analizăcombinatorie, anul VII, 1901-1902.

4. O generalizare, anul VIII, 1902-1903, pp. 197.

5. Proprietăţi ale cercului ortocentroidal, anul IX, 1903-1904, pp. 31-34.

6. Asupra unei integrale duble, anul X, 1904-1905, pp. 227-229.

7. Asupra substituţiilor circulare, anul XI, 1905-1906, pp. 270-273.

8. Un exemplu de aproximaţii successive, anul XIII, 1907-1908, pp. 97-102.

9. Asupra unei formule a lui Riemann, anul XIV, 1908-1909, pp. 33-35.

10. O problemă de algebră, anul XIV, 1908-1909, pp. 68-72.

11. Asupra unei formule a lui Riemann-Hadamard, anul XIV, 1908-1909, pp. 99-103.

12. Criterii pentru recunoaşterea cuadricelor, anul XIV, 1908-1909, pp. 232-234.

13. Careacterizarea conicelor date prin 5 puncte, anul XV, 1909-1910, pp. 193-194.

14. Perpendiculara comună la două drepte, anul XVI, 1910-1911, pp. 84-86.

15. Asupra pendulului lui Foucault, anul XVI, 1910-1911, pp. 404-406.

16. Privire istorică asupra teoriei numerelor, anul XVIII, 1912-1913, pp. 85-91. (Acest articol a fost tradus în limba spaniolă de Bernard Baidaff şi tipărită în revista argentiniană ''Boletin matematico''. Lucrarea a apărut la Buenos Aires, sub titlul Una mirade historica de la teoria de los numerous, vol. XIII, pp. 76-78 şi 105-111, 1940).

17. Perspectiva în studiul geometriei descriptive, anul XVIII, 1912-1913, pp. 439-443.

18. Nicolae Culianu, anul XXI, 1915-1916, pp. 161-166.

19. Asupra unui punct remarcabil al triunghiului, anul XXI, 1915-1916, pp. 241-243.

20. Viaţa şi activitatea lui Gheorghe Lazăr, anul XXII, 1916-1917, pp. 151-156, 177-185, 207-209 şi 217-221.

21. Bibliografia matematică românească, anul XXII, 1916-1917, pp. 270-271.

22. Cărţi de matematici din Transilvania, anul XXII, 1916-1917, pp. 300-306.

23. Cărţi şi manuscrise greceşti de matematică din ţările române, anul XXIII, 1917-1918, pp. 107-110, 130-132.

24. Catalogul cărţilor şi manuscriselor româneşti de matematică la expoziţia de la Iaşi din 1885, anul XXIII, 1917-1918, p. 178.

25. Câtul a două polinoame, anul XXVII, 1921-1922, pp. 105-111.

26. Asupra unei colineaţiuni a conicelor, anul XXVII, 1921-1922, pp. 272-275.

27. Unul din primii profesori de matematici: Simion Marcovici, anul XXIX, 1923-1924, pp.41-43.

28. Câteva date asupra lui Simion Marcu zis Marcovici, ca profesor de matematică, anul XXIX, 1923-1924, pp. 121-123.

ARTICLES PUBLISHED IN THE MATHEMATICAL JOURNAL OF TIMISOARA:

1. Rezolvarea ineglităţilor algebrice, anul I, 1921, pp. 3-7.

2. Probleme de Algebră superioară, anul I, nr. 2, 15 aprilie 1921, pp. 21-24.

3. Construcţii geometrice, anul I, nr. 5, iulie 1921, pp. 67-69.

4. Discuţia construcţiunilor geometrice, anul I, nr. 6, august 1921, pp. 83-85.

5. Dreapta lui Simson, anul II, nr. 10, decembrie 1922.

6. Serii convergente şi serii divergente, anul III, 1923, pp. 35-37.

7. Simetrie şi omogeneitate, anul III, nr. 6, 1923, pp. 83-85.

8. Patrulatere remarcabile, anul III, nr. 12, februarie 1924, pp. 179-180.

9. Diviziunea polinoamelor, anul V, nr. 1, martie 1926, pp. 4-6.

STATUES:

Bust of Traian Lalescu from the "Polytechnic" University of Timișoara - sculptor Corneliu Medrea

Bust of Traian Lalescu from the "Polytechnic" University of Timișoara - sculptor Peter Jecza

Bust of Traian Lalescu from the University of Bucharest - Faculty of Mathematics and Informatics - sculptor Peter Jecza

STREETS:

Traian Lalescu Street from Timișoara

Traian Lalescu Street from Reșița

Traian Lalescu Street from Oradea

Traian Lalescu Street from Craiova

HIGH SCHOOLS:

„Traian Lalescu" National College of Informatics - Hunedoara, Hunedoara County

„Traian Lalescu" National College - Reșița, Caraș-Severin County „Traian Lalescu" Theoretical High School - Orșova - Mehedinți County

„Traian Lalescu" Theoretical High School - Mehadia, Caraș-Severin County

"Traian Lalescu" Theoretical High School - Branesti, Ilfov County "Traian Lalescu" High School - Bucharest - (private high school, established in 1992, accredited in 2009)

OTHER:

The documentary film "Traian Lalescu - the right to memory", 48 min - TVR Cultural, 2008.

Anniversary Medal "Traian Lalescu - 125 years since birth"

"Traian Lalescu" presentation panel from the Faculty of Mathematics and Informatics of the University of Bucharest

Short film about Traian Lalescu – 70's years- TVR

OSTL-"Traian Lalescu" Student Association-"Politehnica" University of Timișoara- Faculty of Constructions and Department of Communication and Foreign Languages, established in 2007.

www.ostl.ro

https://www.youtube.com/watch?v=tNJfRYKb8DQ

https://www.youtube.com/watch?v=8IYbRAPAddw

EVOCATIONS:

Emile Picard: "Lalescu's very lively intelligence allowed him to immediately reach the heart of a problem; that is why his texts have that spontaneity that makes them particularly attractive. His curious spirit was interested in the most varied fields of mathematics, and we often walked together through the Luxembourg Garden, discussing various subjects of philosophy of science".

Grigore Moisil: "The prodigious activity of this great scientist (…) is for us an invaluable scientific legacy".

Edmond Nicolau: "The history of mathematics in our country places professor Traian Lalescu together with Gheorghe Ţiţeica and Dimitrie Pompei in the group of founders of the Romanian mathematics school".

Ion Ionescu-Bizeţ: "Lalescu's appearance in the world was like a comet that shone and shone wonderfully and at the same time amazed and scared with its unusually long tail".

Gheorghe Ţiţeica: "Lalescu's head was worth much more than ten estates".

The presented ones characterize the complex personality, encyclopedic spirit and the erudition of Traian Lalescu. In this sense, the finding of the academician Solomon Marcus is convincing, who classifies mathematicians in two classes: those of the ant type, who insist in a certain direction throughout their lives and those of the bee type, who do not remain in the same place, but ''flies from flower to flower''. Solomon Marcus, places Lalescu in the class of bee researchers:

''Albina Lalescu was not satisfied with the flowers offered by mathematics, but ventured to the flowers of Romanian history, finance, sociology, physics, engineering, linguistics, history of mathematics textbooks, history of mathematics, propagation in masses of scientific culture, philosophy, etc''.

REFERENCES ABOUT THE LIMIT OF TRAIAN LALESCU

1. **D.M. Bătineţu-Giurgiu, *Şiruri,* Editura Albatros, Bucureşti, 1979.**
2. **D.M. Bătineţu-Giurgiu, *Şiruri Lalescu*, R.M.T., anul XX (1989), pp. 37-38.**
3. **D.M. Bătineţu-Giurgiu, *Şirurile Lalescu şi funcţia Euler de speţa a doua. Funcţii Euler-Lalescu*, Gazeta Matematică seria A, nr.1 /1990, pp. 21-26.**
4. **D.M. Bătineţu-Giurgiu, *Asupra unei generalizări a şirului lui Traian Lalescu. Metode de abordare,* Gazeta Matematică, nr. 8-9/1990, pp. 219-224.**
5. **D.M. Bătineţu-Giurgiu, N. Stanciu – *120 Years of Lalescu Sequences*, Romanian Mathematical Magazine, February 2020.**
6. **D.M. Bătineţu-Giurgiu, D. Sitaru, N. Stanciu - *New classes of sequences-functions and famous limits*– Octogon Mathematical Magazine, Vol. 27, No.2, October, 2019, 784-797.**
7. **D.M. Bătineţu-Giurgiu, D. Sitaru, N. Stanciu -*Two classes of Lalescu's sequences* – Octogon Mathematical Magazine, Vol. 27, No. 2, October, 2019, 805-813.**
8. **D.M. Bătineţu-Giurgiu, M. Bencze, D. Sitaru, N. Stanciu - *Some limits of Traian Lalescu type with Fibonacci and Lucas numbers (II)* – Octogon Mathematical Magazine, Vol. 27, No. 1, April, 2019, 101-111.**
9. **D.M. Bătineţu-Giurgiu, M. Bencze, D. Sitaru, N. Stanciu - *Some limits of Traian Lalescu type (II)*– Octogon Mathematical Magazine, Vol. 27, No. 1, April, 2019, 184-209.**

10. **D.M. Bătineţu-Giurgiu, M. Bencze, D. Sitaru, N. Stanciu - *Some limits of definite integrals of Lalescu's type* – Octogon Mathematical Magazine, Vol. 27, No. 1, April, 2019, 312-328.**

11. **D.M. Bătineţu-Giurgiu, M. Bencze, D. Sitaru, N. Stanciu - *Some limits of Traian Lalescu type (II)*– Octogon Mathematical Magazine, Vol. 26, No. 2, October, 2018, 664-690.**

12. **D.M. Bătineţu-Giurgiu, M. Bencze, N. Stanciu - *Some limits of Traian Lalescu type with Fibonacci and Lucas numbers* – Octogon Mathematical Magazine, Vol. 26, No. 1, April, 2018, 54-87.**

13. **D.M. Bătineţu-Giurgiu, M. Bencze, N. Stanciu - *Some limits of Traian Lalescu type (I)* – Octogon Mathematical Magazine, Vol. 26, No. 1, April, 2018, 116-136.**

14. **D.M. Bătineţu-Giurgiu, M. Bencze, D. Sitaru, N. Stanciu -*Certain classes of Lalescu sequence*– Octogon Mathematical Magazine, Vol. 26, No. 1, April, 2018, 243-251.**

15. **D.M. Bătineţu-Giurgiu, N. Stanciu -*Several results of some classes of sequences*– The Pentagon – Volume 73, Number 2, Spring 2014, 10-24.**

16. **D.M. Bătineţu-Giurgiu, A. Kotronis, N. Stanciu -*Calculating the limits of some real sequences*– Math Problems, Issue 1, 2014, 252-257.**

17. **D.M. Bătineţu-Giurgiu, N. Stanciu - *New methods for calculations of some limits*– The Teaching of Mathematics, Vol. XVI, Nr. 2, 2013, 82-88.**

18. **D.M. Bătineţu-Giurgiu, N. Stanciu - *120 de ani de la limita şirului lui Traian Lalescu* – Sclipirea Minţii, Nr. 25, Mai, 2020, 6-7.**

19. **D.M. Bătineţu-Giurgiu, N. Stanciu, José Luis Díaz-Barrero - *The Last Three Decades of Lalescu Limit* – Arhimede Mathematical Journal, Vol. 7, No.1, 2020, 18-28.**

20. **D.M. Bătineţu-Giurgiu, N. Stanciu – *120 de ani de la limita lui Traian Lalescu* – Recreaţii Matematice, Nr. 2, Iulie-Decembrie, 2020, 104-106.**

21. **N. Stanciu - *Limits of Lalescu Type* – Journal of Mathematics and Statistics Research, Research Article,Volume 2, Issue 1, 2020, 1-16.**

22. **D. Sitaru, N. Stanciu – *Traian Lalescu limit - 120 years, D.M. Bătineţu-Giurgiu – 85 years*, Romanian Mathematical Magazine, July 20, 2020.**

Proposed problem for RMM

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Let the positive real sequence $(a_n)_{n\geq 1}$, such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}=a\in\mathbb{R}_+^*$. Compute:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Solution:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^n}\cdot\frac{n^n}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\cdot\left(\frac{n}{n+1}\right)^{n+1}\cdot\frac{\sqrt[n]{n!}}{n!}=$$

$$=\lim_{n\to\infty}\frac{a}{e}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}\frac{a}{e}\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{a}{e}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{a}{e^2}$$

$$\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}=\sqrt[n]{a_n}\cdot(u_n-1)=\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\ln u_n^n,\text{ where}$$

$$u_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}=\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{n}{\sqrt[n]{a_n}},\ \forall n\geq 2$$

$$\lim_{n\to\infty}u_n=1,\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1,\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n}{n+1}\cdot\frac{\sqrt[n]{n!}}{n}=a\cdot\frac{e^2}{a}\cdot 1\cdot\frac{1}{e}=e$$

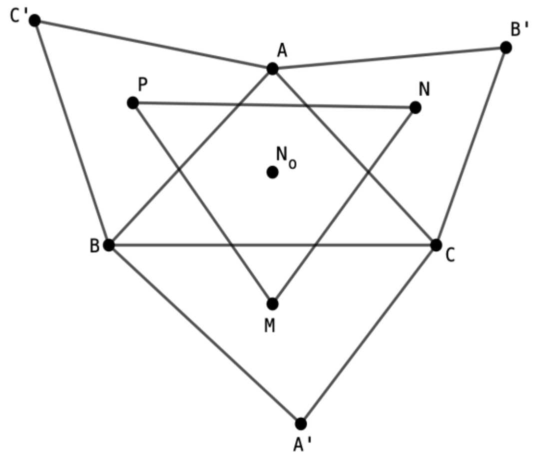
Hence, $\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\frac{a}{e^2}\cdot 1\cdot\ln e=\frac{a}{e^2}$

NAPOLEON'S OUTER TRIANGLE REVISITED

By Daniel Sitaru-Romania

Abstract: In this paper is proved Napoleon's theorem and are made connections with famous inequalities as Ionescu-Weitzenbock's.

Napoleon's theorem for outer triangle



In the figure above, ABC is any fixed triangle, BCA', CAB', ABC' are equilateral triangles constructed on sides of ABC in exterior. The lines connecting the centroids M, N, P of triangles BCA', CAB', ABC' form an equilateral triangle named Napoleon's outer triangle of ΔABC.

Proof: $BC = a, CA = b, AB = c, \mu(\measuredangle PAN) = \mu(A) + \frac{\pi}{3}, AP = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot c = \frac{\sqrt{3}c}{3}$;

$$AN = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot b = \frac{\sqrt{3}b}{3}$$

By cosine law in ΔPAN: $MN^2 = AP^2 + AN^2 - 2AP \cdot AN \cdot \cos(\measuredangle PAN)$

$$MN^2 = \frac{1}{3}c^2 + \frac{1}{3}b^2 - 2bc \cdot \frac{1}{3} \cdot \cos\left(A + \frac{\pi}{3}\right)$$

$$MN^2 = \frac{b^2 + c^2}{3} - \frac{2bc}{3}\left(\cos A \cos\frac{\pi}{3} - \sin A \sin\frac{\pi}{3}\right)$$

$$MN^2 = \frac{b^2 + c^2}{3} - \frac{2bc}{3}\left(\cos A \cdot \frac{1}{2} - \sin A \cdot \frac{\sqrt{3}}{2}\right)$$

$$MN^2 = \frac{b^2 + c^2}{3} - \frac{bc}{3} \cdot \frac{b^2 + c^2 - a^2}{2bc} + bc \sin A \cdot \frac{\sqrt{3}}{3}$$

$$MN^2 = \frac{b^2 + c^2}{3} - \frac{b^2 + c^2}{6} + \frac{a^2}{6} + 2F \cdot \frac{\sqrt{3}}{3}$$

$$MN^2 = \frac{b^2 + c^2 + a^2}{6} + \frac{2F\sqrt{3}}{3} \quad (1)$$

Expression (1) is symmetrical in terms of a, b, c hence $MN = NP = PM \Rightarrow \Delta MNP$ is an equilateral one. **Sides of Napoleon's outer triangle** are given by:

$$MN = \sqrt{\frac{a^2 + b^2 + c^2}{6} + \frac{2F\sqrt{3}}{3}}; F = [ABC]$$

Area of Napoleon's outer triangle:

$$[MNP] = \frac{\sqrt{3}}{4} \cdot MN^2 = \frac{\sqrt{3}}{4} \cdot \left(\frac{b^2 + c^2 + a^2}{6} + \frac{2F\sqrt{3}}{3}\right), \qquad [MNP] = \frac{(a^2 + b^2 + c^2)\sqrt{3}}{4} + \frac{F}{2}$$

Observation 1: If the original triangle ABC is an equilateral one $(a = b = c)$ then:

$$[MNP] = \frac{3a^2\sqrt{3}}{24} + \frac{F}{2} = \frac{a^2\sqrt{3}}{8} + \frac{a^2\sqrt{3}}{8} = \frac{a^2\sqrt{3}}{4} = [ABC]$$

Observation 2: Using Ionescu – Weitzenbock's inequality $a^2+b^2+c^2\geq 4\sqrt{3}F$ the following inequality can be obtained:

$$[MNP]=\frac{(a^2+b^2+c^2)\sqrt{3}}{24}+\frac{F}{2}\geq\frac{4\sqrt{3}F\cdot\sqrt{3}}{24}+\frac{F}{2}=\frac{F}{2}+\frac{F}{2}=F$$

$$[MNP]\geq F$$

Observation 3: Denote $k=[MNP];K=[MNP];s_k$ – semiperimeter of $\Delta MNP;r_kR_k,r_a^k$ – inradii, circumradii, respectively exradii of $\Delta MNP,N_0$ – the center of ΔMNP.

$$k=\sqrt{\frac{a^2+b^2+c^2}{6}+\frac{2F\sqrt{3}}{3}},K=\frac{(a^2+b^2+c^2)\sqrt{3}}{24}+\frac{F}{2},s_k=\frac{3k}{2}$$

$$r_k=\frac{\sqrt{3}}{6}\cdot k=\frac{1}{6}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}},\qquad R_k=\frac{\sqrt{3}}{3}\cdot k=\frac{1}{3}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}$$

$$r_a^k=\frac{\sqrt{3}}{8}\cdot k=\frac{1}{8}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}$$

Observation 4: The trilinear coordinates of N_0 are:

$$\left(\sec\left(A-\frac{\pi}{3}\right);\sec\left(B-\frac{\pi}{3}\right),\sec\left(C-\frac{\pi}{3}\right)\right)$$

Observation 5: The barycentric coordinates of N_0 are:

$$\left(a\csc\left(A+\frac{\pi}{6}\right),b\csc\left(B+\frac{\pi}{6}\right),c\csc\left(C+\frac{\pi}{6}\right)\right)$$

Observation 6: Using Ionescu-Weitzenbock's inequality:

$$r_k=\frac{1}{6}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}\geq\frac{1}{6}\sqrt{\frac{4\sqrt{3}F}{2}+2F\sqrt{3}}=\frac{1}{6}\sqrt{4F\sqrt{3}}=\frac{1}{3}\sqrt{F\sqrt{3}}$$

$$R_k=\frac{1}{3}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}\geq\frac{1}{3}\sqrt{\frac{4\sqrt{3}F}{2}+2F\sqrt{3}}=\frac{1}{3}\sqrt{4F\sqrt{3}}=\frac{2}{3}\sqrt{F\sqrt{3}}$$

$$r_k+R_k\geq\frac{1}{3}\sqrt{F\sqrt{3}}+\frac{2}{3}\sqrt{F\sqrt{3}}=\sqrt{F\sqrt{3}}$$

$$r_a^k=\frac{1}{8}\sqrt{\frac{a^2+b^2+c^2}{2}+2F\sqrt{3}}\geq\frac{1}{8}\sqrt{\frac{4\sqrt{3}F}{2}+2F\sqrt{3}}=\frac{1}{8}\sqrt{4F\sqrt{3}}=\frac{1}{4}\sqrt{F\sqrt{3}}$$

Reference: [1] Romanian Mathematical Magazine – www.ssmrmh.ro

ABOUT CEBYSHEV'S INEQUALITY INTEGRAL FORM-II

By Florică Anastase-Romania

Theorem:(Cebyshev's Inequality):

For $f, g:[a,b]\to R$ continuous function with same monotonicity and $p:[a,b]\to[0,\infty)$ integrable function. Then:

$$\left(\int_a^b p(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right)\ge\left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right)\ (*)$$

In the case f and g different monotonicity:

$$\left(\int_a^b p(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right)\le\left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right)$$

Proof:

If f and g are same monotonicity, $p(x)>0,\forall x\in[a,b]\Rightarrow$

$$p(x)p(y)\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\ge 0,,\forall x\in[a,b]\Rightarrow$$

$$p(x)p(y)f(x)g(x)-p(x)p(y)f(y)g(x)-p(x)p(y)f(x)g(y)+p(x)p(y)f(y)g(y)\ge 0$$

$$p(y)\int_a^b p(x)f(x)g(x)dx-p(y)f(y)\int_a^b p(x)g(x)dx-$$

$$-p(y)g(y)\int_a^b p(x)f(x)dx+p(y)f(y)g(y)\int_a^b p(x)dx\ge 0\leftrightarrow$$

$$\left(\int_a^b p(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right)-\left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right)$$
$$-\left(\int_a^b p(x)g(x)dx\right)\left(\int_a^b p(x)f(x)dx\right)$$
$$+\left(\int_a^b p(x)f(x)g(x)dx\right)\left(\int_a^b p(x)dx\right)\ge 0\Leftrightarrow(*)$$

Application: **If $f:[0,1]\to\mathbb{R}, f$ −continuous and convex function such that**

$\boldsymbol{f(0)=0, f(1)=1}$, **then:**

$$\int_0^1 \sqrt{1+x^2}\cdot \log(1+x)\cdot \left(f'(x)\right)^2 dx \ge \frac{\log(\sqrt{2})}{\log(1+\sqrt{2})}$$

Solution:

$$\int_0^1 \sqrt{1+x^2}\cdot \log(1+x)\cdot \left(f'(x)\right)^2 dx = \int_0^1 \frac{1}{\sqrt{1+x^2}}\cdot \log(1+x)\cdot (1+x^2)\left(f'(x)\right)^2 dx$$

Applying Chebyshev's Inequality:

Let: $p(x)=\frac{1}{\sqrt{1+x^2}}; u(x)=\log(1+x); v(x)=(1+x^2)\left(f'(x)\right)^2,$

u, v *−increasing, we have:*

$$\left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)\left(\int_0^1 \frac{1}{\sqrt{1+x^2}}\cdot \log(1+x)\cdot (1+x^2)\left(f'(x)\right)^2 dx\right) \ge$$

$$\ge \left(\int_0^1 \frac{\log(1+x)}{\sqrt{1+x^2}} dx\right)\left(\int_0^1 \frac{1+x^2}{\sqrt{1+x^2}}\cdot \left(f'(x)\right)^2 dx\right) =$$

$$= \left(\int_0^1 \frac{\log(1+x)}{\sqrt{1+x^2}} dx\right)\left(\int_0^1 \sqrt{1+x^2}\cdot \left(f'(x)\right)^2 dx\right); \quad (1)$$

Now,

$$\int_0^1 \frac{\log(1+x)}{\sqrt{1+x^2}} dx = \int_0^1 \frac{1}{1+x^2}\cdot \sqrt{1+x^2}\cdot \log(1+x)\, dx$$

Let $p_1(x)=\frac{1}{1+x^2}; u_1(x)=\sqrt{1+x^2}, v_1(x)=\log(1+x); u_1, v_1$ *−increasing.*

$$\left(\int_0^1 \frac{dx}{1+x^2}\right)\left(\int_0^1 \frac{1}{1+x^2}\cdot \sqrt{1+x^2}\cdot \log(1+x)\, dx\right)$$

$$\ge \left(\int_0^1 \frac{\sqrt{1+x^2}}{1+x^2} dx\right)\left(\int_0^1 \frac{\log(1+x)}{1+x^2} dx\right) \Leftrightarrow$$

$$\frac{\pi}{4}\int_0^1 \frac{\log(1+x)}{\sqrt{1+x^2}} dx \ge \left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)\left(\int_0^1 \frac{\log(1+x)}{1+x^2} dx\right) \Leftrightarrow$$

$$\frac{\pi}{4}\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx \geq log(1+\sqrt{2})\left(\int_0^1 \frac{log(1+x)}{1+x^2}dx\right); \ (2)$$

$$\int_0^1 \frac{log(1+x)}{1+x^2}dx \overset{x=tan\,u}{=} \int_0^{\frac{\pi}{4}} \frac{log(1+tanu)}{\frac{1}{cos^2u}}\cdot\frac{du}{cos^2u} =$$

$$= \int_0^{\frac{\pi}{4}} log\left(\frac{sinu+cosu}{cosu}\right)du = \int_0^{\frac{\pi}{4}} log\left[\frac{\sqrt{2}cos\left(\frac{\pi}{4}-u\right)}{cosu}\right]du =$$

$$= \int_0^{\frac{\pi}{4}} log\sqrt{2}du + \int_0^{\frac{\pi}{4}} log\left[cos\left(\frac{\pi}{4}-u\right)\right]du - \int_0^{\frac{\pi}{4}} log(cosu)du =;$$

$$\int_0^{\frac{\pi}{4}} log\left[cos\left(\frac{\pi}{4}-u\right)\right]du \overset{\frac{\pi}{4}-u=v}{=} -\int_0^{\frac{\pi}{4}} log(cosv)dv = v$$

$$\int_0^1 \frac{log(1+x)}{1+x^2}dx = \frac{\pi}{8}log2; \ (3)$$

Replacing (3) in (2), we get:

$$\frac{\pi}{4}\cdot\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx \geq log(1+\sqrt{2})\cdot\frac{\pi}{8}log2 \Leftrightarrow$$

$$\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}}dx \geq log(1+\sqrt{2})log(\sqrt{2}); \ (4)$$

Now,

$$1 = f(1)-f(0) = \int_0^1 f'(x)\,dx = \int_0^1 \frac{1}{\sqrt[4]{1+x^2}}\cdot\sqrt[4]{1+x^2}\cdot\left(f'(x)\right)^2 dx \overset{CBS}{\leq}$$

$$\leq \left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)^{\frac{1}{2}}\cdot\left(\int_0^1 \sqrt{1+x^2}\cdot\left(f'(x)\right)^2 dx\right)^{\frac{1}{2}} =$$

$$= \sqrt{log(1+\sqrt{2})} \cdot \left(\int_0^1 \sqrt{1+x^2} \cdot (f'(x))^2 dx\right)^{\frac{1}{2}}$$

Hence,

$$\int_0^1 \sqrt{1+x^2} \cdot (f'(x))^2 dx \geq \frac{1}{log(1+\sqrt{2})}; \quad (5)$$

Now, we get:

$$\left(\int_0^1 \frac{dx}{\sqrt{1+x^2}}\right)\left(\int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot log(1+x) \cdot (1+x^2)(f'(x))^2\, dx\right) \geq$$

$$\geq \left(\int_0^1 \frac{log(1+x)}{\sqrt{1+x^2}} dx\right)\left(\int_0^1 \sqrt{1+x^2} \cdot (f'(x))^2 dx\right) \overset{(4)}{\geq}$$

$$\overset{(4)}{\geq} log(1+\sqrt{2}) log(\sqrt{2}) \left(\int_0^1 \sqrt{1+x^2} \cdot (f'(x))^2 dx\right) \overset{(5)}{\geq}$$

$$\overset{(5)}{\geq} log(1+\sqrt{2}) \cdot log(\sqrt{2}) \cdot \frac{1}{log(1+\sqrt{2})} = log(\sqrt{2}) \Leftrightarrow$$

$$log(1+\sqrt{2}) \cdot \int_0^1 \sqrt{1+x^2} \cdot log(1+x) \cdot (f'(x))^2\, dx \geq log(\sqrt{2})$$

$$\int_0^1 \sqrt{1+x^2} \cdot log(1+x) \cdot (f'(x))^2\, dx \geq \frac{log(\sqrt{2})}{log(1+\sqrt{2})}$$

REFERENCES:

[1]. F. Anastase - ''ABOUT CEBYSHEV's INEQUALITY INTEGRAL FORM''-www.ssmrmh.ro /2019/09/30

[2]. D. Sitaru, M.Ursărescu-''CALCULUS MARATHON''-Publishing House STUDIS-2018

[3]. D. Sitaru, M. Ursărescu-''ICE MATH CONTESTS PROBLEMS''- Publishing House STUDIS-2019

[4]. M. Bencze, D. Sitaru, M. Ursărescu-''OLYMPIC MATHEMATICAL ENERGY''- Publishing House STUDIS-2018

[5]-Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT DOUCET'S INEQUALITY

By Marian Dincă-Romania

In this paper is presented a new demonstration for Doucet's inequality which holds in ΔABC:

$$\boldsymbol{s\sqrt{3} \leq 4R + r};\ (\boldsymbol{Doucet's})$$

It is well-known the following identities: $s = R(\sin A + \sin B + \sin C)$

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1;\ (Carnot\ identity)$$

$$\Leftrightarrow R(\sin A + \sin B + \sin C)\sqrt{3} \leq 4R + r \text{ or } (\sin A + \sin B + \sin C)\sqrt{3} \leq \tfrac{r}{R} + 4$$

$$(\sin A + \sin B + \sin C)\cot\frac{\pi}{6} \leq \cos A + \cos B + \cos C + 3$$

$$\left(\sin A \cos\frac{\pi}{6} - \cos A \sin\frac{\pi}{6}\right) + \left(\sin B \cos\frac{\pi}{6} - \cos B \sin\frac{\pi}{6}\right) + \left(\sin C \cos\frac{\pi}{6} - \cos C \sin\frac{\pi}{6}\right) \leq \frac{3}{2}$$

$$\sin\left(A - \frac{\pi}{6}\right) + \sin\left(B - \frac{\pi}{6}\right) + \sin\left(C - \frac{\pi}{6}\right) \leq \frac{3}{2}$$

Let: $A \geq B \geq C \Rightarrow A \geq \frac{A+B}{2} \geq \frac{A+B+C}{3} \geq C$.

$$\sin\left(A - \frac{\pi}{6}\right) + \sin\left(B - \frac{\pi}{6}\right) = 2\sin\left(\frac{A+B}{2} - \frac{\pi}{6}\right)\cos\left(\frac{A-B}{2}\right) \leq 2\sin\left(\frac{A+B}{2} - \frac{\pi}{6}\right)$$

$$\sin\left(C - \frac{\pi}{6}\right) + \sin\left(\frac{A+B+C}{3} - \frac{\pi}{6}\right) = 2\sin\left(\frac{C + \frac{A+B+C}{3}}{2} - \frac{\pi}{6}\right)\cos\left(\frac{\frac{A+B+C}{3} - C}{2}\right) \leq$$

$$\leq 2\sin\left(\frac{C + \frac{A+B+C}{3}}{2} - \frac{\pi}{6}\right) \text{ and}$$

$$2\sin\left(\frac{A+B}{2} - \frac{\pi}{6}\right) + 2\sin\left(\frac{C + \frac{A+B+C}{3}}{2} - \frac{\pi}{6}\right) =$$

$$= 4\sin\left(\frac{\frac{A+B}{2} + \frac{C+\frac{A+B+C}{3}}{2}}{2} - \frac{\pi}{6}\right)\cos\left(\frac{\frac{A+B}{2} - \frac{C+\frac{A+B+C}{3}}{2}}{2}\right) \leq$$

$$\leq 4\sin\left(\frac{\frac{A+B}{2}+\frac{C+\frac{A+B+C}{3}}{2}}{2}-\frac{\pi}{6}\right)=4\sin\left(\frac{\frac{\pi}{2}+\frac{\pi}{6}}{2}-\frac{\pi}{6}\right)=4\sin\frac{\pi}{6}$$

$$\sin\left(A-\frac{\pi}{6}\right)+\sin\left(B-\frac{\pi}{6}\right)+\sin\left(C-\frac{\pi}{6}\right)+\sin\left(\frac{A+B+C}{3}-\frac{\pi}{6}\right)\leq 4\sin\frac{\pi}{6}$$

$$\sin\left(A-\frac{\pi}{6}\right)+\sin\left(B-\frac{\pi}{6}\right)+\sin\left(C-\frac{\pi}{6}\right)\leq 4\sin\frac{\pi}{6}-\sin\left(\frac{A+B+C}{3}-\frac{\pi}{6}\right)=3\sin\frac{\pi}{6}$$

Reference: ROMANIAN MATHEMATCAL MAGAZINE-www.ssmrmh.ro

ABOUT NAGEL'S AND GERGONNE'S CEVIANS-VIII

By Bogdan Fuștei-Romania

In ΔABC the following relationship holds:

$$n_a g_a \geq r_b r_c,\qquad b^2+c^2=n_a^2+g_a^2+2rr_a,\qquad 2bc=2r_b r_c+2rr_a$$

$$4m_a^2=n_a^2+g_a^2+2r_b r_c$$

$$g_a^2\geq\frac{(r_b r_c)^2}{n_a^2}\Rightarrow n_a^2+g_a^2+2r_b r_c\geq n_a^2+2r_b r_c+\left(\frac{r_b r_c}{n_a}\right)^2$$

$$\Rightarrow \boldsymbol{4m_a^2\geq n_a^2+\left(\frac{r_b r_c}{n_a}\right)^2+2n_a\cdot\frac{r_b r_c}{n_a}\Rightarrow 2m_a\geq n_a+\frac{r_b r_c}{n_a};(1)}$$

Adding these up relations, it follows:

$$\boldsymbol{2(m_a+m_b+m_c)\geq\sum_{cyc}n_a+\sum_{cyc}\frac{r_b r_c}{n_a};(2)}$$

$$\boldsymbol{8m_a m_b m_c\geq\prod_{cyc}\left(n_a+\frac{r_b r_c}{n_a}\right);(3)}$$

$$\boldsymbol{2m_a-n_a\geq\frac{r_b r_c}{n_a}\Rightarrow\prod_{cyc}(2m_a-n_a)\geq\frac{(r_a r_b r_c)^2}{n_a n_c n_c};(4)}$$

$$\boldsymbol{n_a(2m_a-n_a)\geq r_b r_c\Rightarrow 2m_a n_a\geq n_a^2+r_b r_c;(5)}$$

$$\boldsymbol{\because r_a r_b+r_b r_c+r_c r_a=s^2\Rightarrow 2\sum_{cyc}m_a n_a=s^2+\sum_{cyc}n_a^2;(6)}$$

$$\because \boldsymbol{n_a^2+n_b^2+n_c^2 \geq n_an_b+n_bn_c+n_cn_a \Rightarrow 2\sum_{cyc} m_an_a \geq s^2+\sum_{cyc} n_an_b}\ ;(7)$$

But $\boldsymbol{n_an_b+n_bn_c+n_cn_a \geq s^2 \Rightarrow \sum_{cyc} m_an_a \geq s^2}; (\mathbf{8})$

$$\because n_a^2+n_b^2+n_c^2=\frac{s^2(3R-r)-r(4R+r)^2}{R} \Rightarrow$$

$$2\sum_{cyc} m_an_a \geq s^2+\frac{s^2(3R-r)-r(4R+r)^2}{R} \Rightarrow$$

$$\boldsymbol{\sum_{cyc} m_an_a \geq \frac{s^2(4R-r)-r(4R+r)^2}{2R}}\ ;(\mathbf{9})$$

$$8m_an_a \geq 4n_a^2+4r_br_c \Rightarrow 8m_an_a-2r_br_c \geq 4n_a^2+2r_br_c$$

$$\because (b-c)^2=n_a^2+g_a^2-2r_br_c$$

$$4(n_a-m_a)^2=4m_a^2+4n_a^2-8n_am_a=n_a^2+g_a^2+2r_br_c+4n_a^2-8n_am_a$$

$$\Rightarrow 8m_an_a-2r_br_c \geq 4n_a^2+2r_br_c$$

$$\Rightarrow n_a^2+g_a^2+8m_an_a-2r_br_c \geq n_a^2+g_a^2 4n_a^2+2r_br_c \Rightarrow (b-c)^2 \geq 4(n_a-m_a)^2$$

$$\frac{1}{4}(b-c)^2 \geq (n_a-m_a)^2 \Rightarrow \frac{1}{2}|b-c| \geq n_a-m_a$$

So, we get a new inequality: $\boldsymbol{\frac{1}{2}|b-c| \geq n_a-m_a}; (\mathbf{10})$

Adding, it follows $\frac{1}{2}\sum_{cyc}|b-c| \geq \sum_{cyc}(n_a-m_a)$

But $\frac{1}{2}\sum_{cyc}|b-c| = \max\{a,b,c\}-\min\{a,b,c\}$ hence,

$$\boldsymbol{\max\{a,b,c\}-\min\{a,b,c\} \geq \sum_{cyc}(n_a-m_a)}\ ;(\mathbf{11})$$

$$\because \begin{cases} \boldsymbol{\frac{1}{2}|b-c| \geq m_a-s_a} \\ \boldsymbol{\frac{1}{2}|b-c| \geq n_a-m_a} \end{cases} \Rightarrow \boldsymbol{|b-c| \geq n_a-s_a}; (\mathbf{12})$$

$$\Rightarrow \boldsymbol{\sum_{cyc}|b-c| \geq \sum_{cyc}(n_a-s_a)}\ ;(\mathbf{13})$$

$$\Rightarrow \boldsymbol{\max\{a,b,c\}-\min\{a,b,c\} \geq \frac{1}{2}\sum_{cyc}(n_a-s_a)}\ ;(\mathbf{14})$$

$$\because \begin{cases}|b-c|\ge n_a-g_a\\ |b-c|\ge n_a-s_a\end{cases} \Rightarrow 2|b-c|\ge 2n_a-g_a-s_a \Rightarrow |b-c|\ge \frac{1}{2}(2n_a-g_a-s_a);(15)$$

$$\Rightarrow \sum_{cyc}|b-c|\ge \frac{1}{2}\sum_{cyc}(2n_a-s_a-g_a);(16)$$

$$\Rightarrow \max\{a,b,c\}-\min\{a,b,c\}\ge \frac{1}{4}\sum_{cyc}(2n_a-g_a-s_a);(17)$$

$$\because n_a g_a\ge m_a w_a \Rightarrow n_a\ge \frac{m_a w_a}{g_a}\Rightarrow \frac{1}{2}|b-c|\ge \frac{m_a w_a}{g_a}-m_a$$

$$\Rightarrow \frac{1}{2}|b-c|\ge \frac{m_a(w_a-g_a)}{g_a}\Rightarrow \frac{1}{2}\cdot\frac{|b-c|}{m_a}\ge \frac{w_a}{g_a}-1;(18)$$

$$\frac{1}{2}\sum_{cyc}\frac{|b-c|}{m_a}\ge \frac{w_a}{g_a}+\frac{w_b}{g_b}+\frac{w_c}{g_c}-3;(19)$$

$$|b-c|\ge \frac{m_a w_a}{g_a}-s_a;(20)$$

$$\max\{a,b,c\}-\min\{a,b,c\}\ge \frac{1}{2}\sum_{cyc}\left(\frac{m_a w_a}{g_a}-s_a\right);(21)$$

$$\because \frac{n_a^2}{h_a^2}=1+\frac{(b-c)^2}{4R^2}$$

$$\frac{1}{2}|b-c|\ge n_a-m_a\Rightarrow \frac{|b-c|}{2r}\ge \frac{n_a-m_a}{r}\Rightarrow \frac{(b-c)^2}{4r^2}\ge \frac{(n_a-m_a)^2}{r^2}\Rightarrow$$

$$\frac{n_a^2}{h_a^2}\ge \frac{r^2+(n_a-m_a)^2}{r^2}\Rightarrow \frac{n_a}{m_a}\ge \frac{\sqrt{(n_a-m_a)^2+r^2}}{r};(22)$$

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge \sum_{cyc}\frac{\sqrt{(n_a-m_a)^2+r^2}}{r};(23)$$

$$\because \frac{r}{h_a}\ge \frac{\sqrt{(n_a-m_a)^2+r^2}}{n_a^2},\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Rightarrow$$

$$1\ge \sum_{cyc}\frac{\sqrt{(n_a-m_a)^2+r^2}}{n_a^2};(24)$$

From $|b-c|\ge n_a-s_a\Rightarrow (b-c)^2\ge (n_a-s_a)^2=n_a^2+s_a^2-2n_a s_a$

$$n_a^2+g_a^2-2r_b r_c=n_a^2+s_a^2-2n_a s_a\Rightarrow s_a(2n_a-s_a)\ge 2r_b r_c-g_a^2\Rightarrow$$

$$s_a(2n_a-s_a)\ge 2r_b r_c-g_a^2;(25)$$

$$\because s_a \geq \frac{2r_br_c - g_a^2}{2n_a - s_a} \Rightarrow s_a + s_b + s_c \geq \sum_{cyc} \frac{2r_br_c - g_a^2}{2n_a - s_a}; (26)$$

$$\because r_br_c \geq w_a^2 \Leftrightarrow s(s-a) \geq w_a^2 \Rightarrow s_a + s_b + s_c \geq \sum_{cyc} \frac{2w_a^2 - g_a^2}{2n_a - s_a}; (27)$$

$$s_as_bs_c \geq \prod_{cyc} \frac{2r_br_c - g_a^2}{2n_a - s_a}; (28)$$

$$s_as_bs_c \geq \prod_{cyc} \frac{2w_a^2 - g_a^2}{2n_a - s_a}; (29)$$

$$\because 2n_a - s_a \geq \frac{2r_br_c - g_a^2}{s_a} \Rightarrow 2n_a \geq s_a + \frac{2r_br_c - g_a^2}{s_a}; (30)$$

$$\because 2n_a \geq s_a + \frac{2r_br_c - g_a^2}{s_a}, \qquad g_a^2 = (s-a)^2 + 2rh_a$$

$$2r_br_c - g_a^2 = 2s(s-a) - g_a^2 = 2s(s-a) - (s-a)^2 - 2rh_a$$

$$2r_br_c - g_a^2 = (s-a)(2s - s + a) - 2rh_a$$

$$2r_br_c - g_a^2 = (s-a)(s+a) - 2rh_a = s^2 - a^2 - 2rh_a$$

$$s^2 = n_a^2 + 2r_ah_a \Rightarrow 2r_br_c - g_a^2 = n_a^2 - a^2 + 2h_a - a^2 - 2rh_a$$

$$\Rightarrow 2n_a \geq s_a + \frac{n_a^2 - a^2 + 2h_a(r_a - r)}{s_a}; (31)$$

$$2\sum_{cyc} n_a \geq \sum_{cyc} s_a + \sum_{cyc} \frac{n_a^2 - a^2 + 2h_a(r_a - r)}{s_a}; (32)$$

$$\because s^2 = n_a^2 + 2r_ah_a \Rightarrow s^2 - n_a^2 = 2r_ah_a \Rightarrow (s+n_a)(s-n_a) = 2r_ah_a$$

$$\Rightarrow \frac{s - n_a}{h_a} = \frac{2r_a}{s + n_a} \Rightarrow \frac{s}{h_a} = \frac{n_a}{h_a} + \frac{2r_a}{s + n_a}, \frac{s}{h_a} = \frac{a}{2r} \Rightarrow$$

$$\frac{a}{2r} = \frac{n_a}{h_a} + \frac{2r_a}{s + n_a}$$

$$\frac{n_a}{h_a} \geq \frac{\sqrt{(n_a - m_a)^2 + r^2}}{r} \Rightarrow \frac{a - \sqrt{(n_a - m_a)^2 + r^2}}{2r} \geq \frac{2r_a}{s + n_a}$$

$$a - 2\sqrt{(n_a - m_a)^2 + r^2} \geq \frac{4rr_a}{s + n_a} = \frac{4(s-b)(s-c)}{s + n_a}$$

$$\boldsymbol{a \geq 2\sqrt{(n_a - m_a)^2 + r^2} + \frac{4rr_a}{s + n_a} = \frac{4(s-b)(s-c)}{s + n_a}; (33)}$$

$$\boldsymbol{s \geq \sum_{cyc}\left(\sqrt{(n_a - m_a)^2 + r^2} + \frac{2(s-b)(s-c)}{s + n_a}\right); (34)}$$

$$\because \frac{s}{s-a} = \frac{h_a}{h_a - 2r} = \frac{r_a}{r} \Rightarrow \frac{s-a}{h_a - 2r} = \frac{s}{h_a}, \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}, \frac{s-a}{h_a - 2r} = \frac{a}{2r}$$

$$\boldsymbol{\frac{s-a}{h_a - 2r} \geq \frac{\sqrt{(n_a - m_a)^2 + r^2}}{r} + \frac{2r_a}{s + n_a}; (35)}$$

$$\boldsymbol{\Rightarrow \frac{s}{r} = \sum_{cyc}\frac{s-a}{h_a - 2r} \geq \sum_{cyc}\frac{\sqrt{(n_a - m_a)^2 + r^2}}{r} + \frac{2r_a}{s + n_a}; (36)}$$

$$\because \frac{s^2}{h_a^2} = \frac{n_a^2}{h_a^2} + \frac{2r_a}{h_a} \Rightarrow \frac{a^2}{4r^2} = \frac{n_a^2}{h_a^2} + \frac{2r_a}{h_a}$$

$$\frac{r}{2R} \cdot \frac{r_a}{ha} = \frac{r_a - r}{4R} = \sin^2\frac{A}{2} \Rightarrow \frac{r_a}{h_a} = \frac{r_a - r}{2r}$$

$$\frac{a^2}{4r^2} = \frac{n_a^2}{h_a^2} + \frac{r_a - r}{r} \geq \frac{(n_a - m_a)^2 + r^2}{r^2} + \frac{r_a - r}{r}$$

$$\frac{a^2}{4R^2} \geq \frac{(n_a - m_a)^2}{r^2} + 1 - 1 + \frac{r_a}{r} = \frac{(n_a - m_a)^2 + (s-b)(s-c)}{r^2}$$

$$\Rightarrow \frac{a^2}{4} \geq (n_a - m_a)^2 + (s-b)(s-c), rr_a = (s-b)(s-c) \Rightarrow$$

$$\boldsymbol{\frac{a}{2} \geq \sqrt{(n_a - m_a)^2 + (s-b)(s-c)}; (37)}$$

$$\boldsymbol{s \geq \sum_{cyc}\sqrt{(n_a - m_a)^2 + (s-b)(s-c)}; (38)}$$

$$2\sum_{cyc} m_a n_a \geq s^2 + \sum_{cyc} n_a^2$$

$$\boldsymbol{2\sum_{cyc} m_a n_a \geq \left[\sum_{cyc}\sqrt{(n_a - m_a)^2 + (s-b)(s-c)}\right]^2 + \sum_{cyc} n_a^2; (39)}$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY BOGDAN FUȘTEI-V

By Marin Chirciu-Romania

1) In ΔABC: $\sum\frac{m_a}{h_a}\ge\frac{1}{2}\sum\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$

Proposed by Bogdan Fuștei – Romania

Solution: We prove: **Lemma: 2) In ΔABC: $\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$**

Proof: Using $h_a=\frac{2S}{a}=\frac{bc}{2R}$ and Tereshin's inequality $m_a\ge\frac{b^2+c^2}{4R}$ we obtain:

$m_a\ge\frac{b^2+c^2}{4R}=\frac{b^2+c^2}{\frac{2bc}{h_a}}=h_a\cdot\frac{b^2+c^2}{2bc}$, wherefrom $m_a\ge h_a\cdot\frac{b^2+c^2}{2bc}\Leftrightarrow\frac{m_a}{h_a}\ge\frac{b^2+c^2}{2bc}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$.

Let's get back to the main problem.Using the Lemma and the inequality $\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$, (Adil Abdullayev Inequality) we obtain:

$$\left(\frac{m_a}{h_a}\right)^2=\frac{m_a}{h_a}\cdot\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\cdot\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)=\frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$$

wherefrom it follows that: $\left(\frac{m_a}{h_a}\right)^2\ge\frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)\Leftrightarrow\frac{m_a}{h_a}\ge\frac{1}{2}\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$

Adding we deduce the conclusion.Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

3) In ΔABC: $\sum\frac{m_a}{h_a}\ge\frac{27R}{2(4R+r)}$

Marin Chirciu

Solution: We prove **Lemma: 4) In ΔABC: $\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$**

Proof: Using $h_a=\frac{2S}{a}=\frac{bc}{2R}$ and Tereshin's inequality $m_a\ge\frac{b^2+c^2}{4R}$ we obtain:

$m_a\ge\frac{b^2+c^2}{4R}=\frac{b^2+c^2}{\frac{2bc}{h_a}}=h_a\cdot\frac{b^2+c^2}{2bc}$, wherefrom $m_a\ge h_a\cdot\frac{b^2+c^2}{2bc}\Leftrightarrow\frac{m_a}{h_a}\ge\frac{b^2+c^2}{2bc}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$

Let's get back to the main problem.Using the Lemma we obtain:

$$LHS=\sum\frac{m_a}{h_a}\ge\sum\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\overset{(1)}{\ge}$$

$\overset{(1)}{\ge}\frac{1}{2}\cdot\frac{27R}{2(4R+r)}+\frac{1}{2}\cdot\frac{27R}{2(4R+r)}=\frac{27R}{2(4R+r)}=RHS$, where (1) follows from inequality:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{27R}{2(4R+r)}$$

Let's prove the inequality: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{27R}{2(4R+r)}$

5) In ΔABC: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{27R}{2(4R+r)}$

Proof: We use the algebraic inequality:

6) If $a,b,c>0$ then: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{9(a^2+b^2+c^2)}{(a+b+c)^2}$

Indeed: The inequality can be written equivalently: $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(a+b+c)^2\geq 9\sum a^2 \Leftrightarrow$

$$\Leftrightarrow\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(a+b+c)^2=\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\sum a^2+2\sum ab\right)=$$

$$=\sum\frac{a^3}{b}+\sum\frac{ac^2}{b}+2\sum\frac{a^2c}{b}+2\sum a^2+3\sum ab$$

The inequality can be written:

$$\sum\frac{a^3}{b}+\sum\frac{ac^2}{b}+2\sum\frac{a^2c}{b}+2\sum a^2+3\sum ab\geq 9\sum a^2\Leftrightarrow$$

$$\Leftrightarrow\sum\left(\frac{a^3}{b}-\frac{2a^2c}{b}+\frac{ac^2}{b}\right)+\sum\left(\frac{4a^2c}{b}-8ac+4bc\right)\geq 7\sum a^2-7\sum ab\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{a(a-c)^2}{b}+\sum\frac{4c(a-b)^2}{b}\geq\frac{7}{2}\sum(a-b)^2\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{b(b-a)^2}{c}+\sum\frac{4c(a-b)^2}{b}\geq\frac{7}{2}\sum(a-b)^2\Leftrightarrow\sum(a-b)^2\left(\frac{b}{c}+\frac{4c}{b}-\frac{7}{2}\right)\geq 0\Leftrightarrow$$

$\Leftrightarrow\sum(a-b)^2\left[\frac{(b-2c)^2}{bc}+\frac{1}{2}\right]\geq 0$, obviously with equality for $a=b=c$.

Application in triangle:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{9(a^2+b^2+c^2)}{(a+b+c)^2}=\frac{9\cdot 2(s^2-r^2-4Rr)}{4s^2}=\frac{9(s^2-r^2-4Rr)}{2s^2}\overset{Gerretsen}{\geq}\frac{27R}{2(4R+r)}.$$

We obtain $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{27R}{2(4R+r)}$

Equality holds if and only if the triangle is equilateral. **Remark:** The inequality can be strengthened.

7) In ΔABC: $\sum\frac{m_a}{h_a}\geq\sqrt{\frac{3s^2}{r(4R+r)}}$

Marin Chirciu

Solution: We prove **Lemma: 8) In ΔABC:** $\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$

Proof: Using $h_a=\frac{2S}{a}=\frac{bc}{2R}$ and Tereshin's inequality $m_a\ge\frac{b^2+c^2}{4R}$ we obtain:

$m_a\ge\frac{b^2+c^2}{4R}=\frac{b^2+c^2}{\frac{2bc}{h_a}}=h_a\cdot\frac{b^2+c^2}{2bc}$, wherefrom $m_a\ge h_a\cdot\frac{b^2+c^2}{2bc}\Leftrightarrow\frac{m_a}{h_a}\ge\frac{b^2+c^2}{2bc}=\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$

Let's get back to the main problem. Using the Lemma we obtain:

$$LHS=\sum\frac{m_a}{h_a}\ge\sum\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{2}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\overset{(1)}{\ge}$$

$\overset{(1)}{\ge}\frac{1}{2}\sqrt{\frac{3s^2}{r(4R+r)}}+\frac{1}{2}\sqrt{\frac{3s^2}{r(4R+r)}}=\sqrt{\frac{3s^2}{r(4R+r)}}=RHS$, where (1) follows from:

$$\sqrt{\frac{3s^2}{r(4R+r)}}\le\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le\frac{s^2}{r(4r+r)},\ \text{(Mateescu-2016)}$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way:

$$\textbf{9) In } \Delta ABC:\ \sum\frac{h_a}{w_a}\ge 3\left(\frac{2r}{R}\right)^{\frac{2}{3}}$$

Marin Chirciu

Solution: We prove **Lemma: 10) In ΔABC:** $\frac{h_a}{w_a}=\frac{b+c}{a}\sin\frac{A}{2}$

Proof: We have: $\frac{h_a}{w_a}=\cos\frac{B-C}{2}=\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}=$

$$=\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}+\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}=$$

$$=\left(\frac{s}{a}+\frac{s-a}{a}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}=\frac{b+c}{a}\sqrt{\frac{(s-b)(s-c)}{bc}}=\frac{b+c}{a}\sin\frac{A}{2}$$

Let's get back to the main problem. Using the Lemma and the means inequality we obtain:

$$\sum\frac{h_a}{w_a}=\sum\frac{b+c}{a}\sin\frac{A}{2}\ge 3\sqrt[3]{\prod\frac{b+c}{a}\sin\frac{A}{2}}=3\sqrt[3]{\frac{\prod(b+c)\prod\sin\frac{A}{2}}{abc}}=$$

$$=3\sqrt[3]{\frac{2s(s^2+r^2+2Rr)\cdot\frac{r}{4R}}{4Rrs}}=\frac{3}{2}\sqrt[3]{\frac{s^2+r^2+2Rr}{R^2}}\overset{Gerretsen}{\ge}$$

$$\geq \frac{3}{2}\sqrt[3]{\frac{16Rr-5r^2+r^2+2Rr}{R^2}}=\frac{3}{2}\sqrt[3]{\frac{18Rr-4r^2}{R^2}}\overset{Euler}{\geq}\frac{3}{2}\sqrt[3]{\frac{32r^2}{R^2}}=3\sqrt[3]{\frac{4r^2}{R^2}}=3\left(\frac{2r}{R}\right)^{\frac{2}{3}}$$

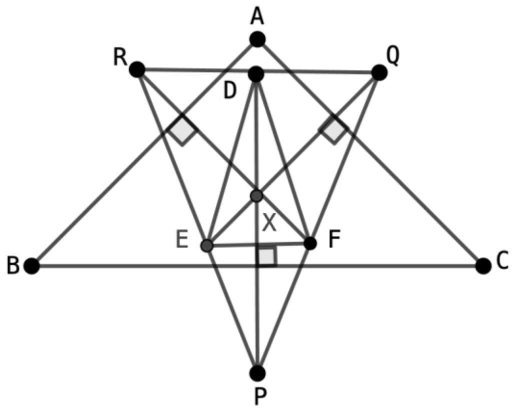
Equality holds if and only if the triangle is equilateral.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

METRIC RELATIONSHIPS IN ȘAHIN'S TRIANGLE (II)

By Daniel Sitaru – Romania

Abstract: This article follows to [1] and prove more metric relationships in a geometrical configuration created by the mathematician **Mehmet Șahin from Ankara – Turkiye.**



Theorem (Mehmet Șahin) Let ΔABC be an acute triangle and $X \in Int\ (\Delta ABC)$ such that $XP \perp BC; XQ \perp AC;\ XR \perp AB; XQ = AC; XP = BC; XR = AB$ (such in above figure) and let ΔDEF be the pedal triangle of X according to ΔPQR.

In these conditions:

1. If r^* is inradii of ΔPQR then:

$$r^* = \frac{3F}{m_a + m_b + m_c}$$

2. If $XD = x; XE = y; XF = z; XD \perp RQ, XE \perp PR, XF \perp QP$ then:

$$x = \frac{F}{m_a}; y = \frac{F}{m_b}; z = \frac{F}{m_c}, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{m_a + m_b + m_c}{F}$$

3. $\cos(\angle RPQ)=\cos(\angle P)=\frac{5a^2-b^2-c^2}{8m_bm_c}, \cos(\angle PQR)=\cos(\angle Q)=\frac{5b^2-c^2-a^2}{8m_cm_a}$

$$\cos(\angle QRP)=\cos(\angle R)=\frac{5c^2-a^2-b^2}{8m_am_b}$$

4. $\sin(\angle RPQ)=\sin(\angle P)=\frac{3F}{2m_bm_c}, \sin(\angle PQR)=\sin(\angle Q)=\frac{3F}{2m_cm_a}$

$$\sin(\angle QRP)=\sin(\angle R)=\frac{3F}{2m_am_b}$$

5. $[DEF]=\frac{9F^3(a^2+b^2+c^2)}{16m_a^2m_b^2m_c^2}$

6. $DE+EF+FD=\frac{3(am_a+bm_b+cm_c)}{2m_am_bm_c}$

7. If R_* is circumradii of ΔDEF then: $R_*=\frac{3abc}{2(a^2+b^2+c^2)}$

Proof (Daniel Sitaru)

1. **According to [1]:** $QR=2m_a, RP=2m_b, PQ=2m_c$ **and** $[PQR]=3F$

$$r^*=\frac{[PQR]}{\frac{QR+RP+PQ}{2}}=\frac{3F}{\frac{2m_a+2m_b+2m_c}{2}}=\frac{3F}{m_a+m_b+m_c}$$

2. **According to [1]:** $[XQR]=F; QR=2m_a$

$$F=\frac{x\cdot 2m_a}{2}\Rightarrow \frac{F}{m_a}$$

Analogous: $y=\frac{F}{m_b}; z=\frac{F}{m_c}, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{m_a}{F}+\frac{m_b}{F}+\frac{m_c}{F}=\frac{m_a+m_b+m_c}{F}$

3. $\cos P=\frac{PR^2+PQ^2-QR^2}{2PR\cdot PQ}=\frac{4m_c^2+4m_b^2-4m_a^2}{2\cdot 2m_c\cdot 2m_b}=\frac{m_b^2+m_c^2-m_a^2}{2m_bm_c}$

$$=\frac{\frac{1}{2}(a^2+c^2)-\frac{1}{4}b^2+\frac{1}{2}(a^2+b^2)-\frac{1}{4}a^2-\frac{1}{2}(b^2+c^2)+\frac{1}{4}a^2}{2m_bm_c}=\frac{5a^2-b^2-c^2}{8m_bm_c}$$

Analogous: $\cos Q=\frac{5b^2-c^2-a^2}{8m_cm_a}; \cos R=\frac{5c^2-a^2-b^2}{8m_am_b}$

4. $\sin P = \frac{QR}{2R^*}$, R^* **- circumradii of** ΔPQR.

According to [1]: $R^* = \frac{8}{3}\cdot\frac{m_a m_b m_c R}{abc}$; $QR = 2m_a$

$$\sin P = \frac{2m_a}{2\cdot\frac{8}{3}\cdot\frac{m_a m_b m_c R}{abc}} = \frac{3abc}{8m_b m_c R} = \frac{3\cdot 4RF}{8m_a m_b m_c R} = \frac{3F}{2m_b m_c}$$

Analogous: $\sin Q = \frac{3F}{2m_c m_a}$; $\sin R = \frac{3F}{2m_a m_b}$

5. $[DEF] = [DXE] + [EXF] + [FXD] = \frac{1}{2}xy\sin R + \frac{1}{2}yz\sin P + \frac{1}{2}zx\sin Q =$

$$= \frac{1}{2}\sum_{cyc} xy\sin R = \frac{1}{2}\sum_{cyc}\frac{F}{m_a}\cdot\frac{F}{m_b}\cdot\frac{3F}{2m_a m_b} = \frac{3F^3}{4}\sum_{cyc}\frac{1}{m_a^2 m_b^2} =$$

$$= \frac{3F^3}{4m_a^2 m_b^2 m_c^2}\left(m_a^2 + m_b^2 + m_c^2\right) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{(a^2+b^2+c^2)F^3}{m_a^2 m_b^2 m_c^2} = \frac{9(a^2+b^2+c^2)F^3}{16m_a^2 m_b^2 m_c^2}$$

6. $DE^2 = x^2 + y^2 - 2xy\cos(\angle EXD) = \frac{F^2}{m_a^2} + \frac{F^2}{m_b^2} - 2\cdot\frac{F}{m_a}\cdot\frac{F}{m_b}\cdot\cos(\pi - R) =$

$$= F^2\left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + 2\cdot\frac{F}{m_a}\cdot\frac{F}{m_b}\cdot\cos R\right) = F^2\left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{2}{m_a m_b}\cdot\frac{5c^2 - a^2 - b^2}{8m_a m_b}\right) =$$

$$= \frac{F^2}{4}\cdot\frac{4m_b^2 + 4m_a^2 + 5c^2 - a^2 - b^2}{m_a^2 m_b^2} =$$

$$= \frac{F^2}{4m_a^2 m_b^2}(2a^2 + 2c^2 - b^2 + 2b^2 + 2c^2 - a^2 + 5c^2 - a^2 - b^2) =$$

$$= \frac{9c^2F^2}{4m_a^2 m_b^2} \Rightarrow DE = \frac{3cF}{2m_a m_b}$$

Analogous: $EF = \frac{3aF}{2m_b m_c}$; $FD = \frac{3bF}{2m_c m_a}$

$$DE + EF + FD = \frac{3F}{2}\left(\frac{c}{m_a m_b} + \frac{a}{m_b m_c} + \frac{b}{m_c m_a}\right) = \frac{2F(am_a + bm_b + cm_c)}{2m_a m_b m_c}$$

7. $$R_* = \frac{DE\cdot EF\cdot FD}{4[DEF]} = \frac{\frac{3cF}{2m_am_b}\cdot\frac{3aF}{2m_bm_c}\cdot\frac{3bF}{2m_cm_a}}{4\cdot\frac{9(a^2+b^2+c^2)F^3}{16m_a^2m_b^2m_c^2}} = \frac{27abcF^3}{8m_a^2m_b^2m_c^2}\cdot\frac{4m_a^2m_b^2m_c^2}{9(a^2+b^2+c^2)F^3} =$$

$$= \frac{27abc}{18(a^2+b^2+c^2)} = \frac{3abc}{2(a^2+b^2+c^2)}$$

Reference:

[1] Daniel Sitaru, *Metric relationships in Șahin's triangle,* www.ssmrmh.ro

[2] Romanian Mathematical Magazine -www.ssmrmh.ro

ABOUT AN INEQUALITY BY VASILE MIRCEA POPA-II

Proposed by Marin Chirciu – Romania

1) If $x,y,z>0, x+y+z=\frac{3}{2}$ **then:** $\frac{x}{1+y}+\frac{y}{1+z}+\frac{z}{1+x}\ge 1$

Proposed by Vasile Mircea Popa – Romania

Solution *Using Bergström's inequality, we obtain:*

$\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}=\frac{x^2}{x+xy}+\frac{y^2}{y+yz}+\frac{z^2}{z+zx}\ge\frac{(x+y+z)^2}{x+y+z+xy+yz+zx}=\frac{\frac{9}{4}}{\sum xy+\frac{3}{2}}\ge 1$, *the last inequality is equivalent with* $\frac{9}{4}\ge 3(xy+yz+zx)\Leftrightarrow(x+y+z)^2\ge 3(xy+yz+zx)\Leftrightarrow$

$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2\ge 0$, *obviously, with equality if and only if*

$$x=y=z=\frac{1}{2}.$$

Remark. *The inequality can be developed.*

2) If $\mathrm{x},\mathrm{y},\mathrm{z}>0, x+y+z=\frac{3}{2}$ **and** $\mathrm{n}\ge 0$**, then:**

$$\frac{x}{n+y}+\frac{y}{n+z}+\frac{z}{n+x}\ge\frac{3}{2n+1}$$

Proposed by Marin Chirciu – Romania

Solution *Using Berström's inequality:*

$\frac{x}{n+y}+\frac{y}{n+z}+\frac{z}{n+x}=\frac{x^2}{nx+xy}+\frac{y^2}{ny+yz}+\frac{z^2}{nz+zx}\ge\frac{(x+y+z)^2}{n(x+y+z)+xy+yz+zx}=\frac{\frac{9}{4}}{\sum xy+\frac{3n}{2}}\ge\frac{3}{2n+1}$ *where the last inequality is equivalent with* $\frac{9}{4}\ge 3(xy+yz+zx)\Leftrightarrow$

$$\Leftrightarrow (x+y+z)^2 \geq 3(xy+yz+zx) \Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2 \geq 0, \textit{ obviously,}$$
with equality if and only if $x=y=z=\frac{1}{2}$.

Note: For $n=1$ we obtain problem VIII. 23, from RMM-24, Spring Edition 2020, Vasile Mircea Popa.

3) If $x, y, z > 0, x+y+z=\frac{3}{2}$ and $n \geq 0$, then:

$$\frac{x}{1+ny}+\frac{y}{1+nz}+\frac{z}{1+nx} \geq \frac{3}{n+2}$$

Proposed by Marin Chirciu – Romania

Solution *Using Bergström we obtain:*

$$\frac{x}{1+ny}+\frac{y}{1+nz}+\frac{z}{1+nx}=\frac{x^2}{x+nxy}+\frac{y^2}{y+nyz}+\frac{z^2}{z+nzx} \geq$$

$$\geq \frac{(x+y+z)^2}{x+y+z+n(xy+yz+zx)}=\frac{\frac{9}{4}}{n\sum xy+\frac{3}{2}} \geq \frac{3}{n+2}$$

where the last inequality is equivalent with $\frac{9n}{4} \geq 3n(xy+yz+zx)$

For $n=0$ *is obvious, and for* $n>0$ *is equivalent with* $(x+y+z)^2 \geq 3(xy+yz+zx)$

$\Leftrightarrow (x-y)^2+(y-z)^2+(z-x)^2 \geq 0$, *obviously, with equality if and only if*

$$x=y=z=\frac{1}{2}.$$

Note. For $n=1$, we obtain problem VIII.23, from RMM-24, Spring Edition 2020, Vasile Mircea Popa.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

BEAUTIFUL GENERALIZATION FOR THREE FAMOUS INEQUALITIES IN TRIANGLE

By D.M. Bătinețu-Giurgiu,Daniel Sitaru-Romania

Abstract: *In this paper we prove a theorem which generalize simultaneous Mitrinovic's, Ionescu-Weitzenbock's and Goldner's inequality in triangle*.

Main result: If $m \geq 0$ then in any triangle ABC the following relationship holds:

$$\boldsymbol{a^{m+1}+b^{m+1}+c^{(m+1)}\geq 2^{m+1}\cdot\left(\sqrt[4]{3}\right)^{3-m}\cdot\left(\sqrt{F}\right)^{m+1};(1)}$$

where a,b,c −length sides in triangle and F − area of triangle ABC.

Lemma. ***(Mehmet Șahin identity-Problem 11857-A.M.M.-Vol.1240-Year 2015.)***

Let a,b,c −be length sides in a triangle. The triangle UVW with sides $u=\sqrt{a}, v=\sqrt{b}$,

$w=\sqrt{c}$ has area $\Delta=\frac{1}{2}\sqrt{r(4R+r)}$, Δ −area of ΔUVW.

Proof: $\Delta\overset{Heron}{=}\sqrt{\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}\cdot\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}}{2}\cdot\frac{\sqrt{a}+\sqrt{c}-\sqrt{b}}{2}\cdot\frac{\sqrt{b}+\sqrt{c}-\sqrt{a}}{2}}=$

$$=\frac{1}{4}\sqrt{\left(\left(\sqrt{a}+\sqrt{b}\right)^2-\left(\sqrt{c}\right)^2\right)\left(\left(\sqrt{c}\right)^2-\left(\sqrt{a}-\sqrt{b}\right)^2\right)}=$$

$$=\frac{1}{4}\sqrt{\left(a+b+2\sqrt{ab}-c\right)\left(c-a-b+2\sqrt{ab}\right)}=$$

$$=\frac{1}{4}\sqrt{\left(2\sqrt{ab}+(a+b-c)\right)\left(2\sqrt{ab}-(a+b-c)\right)}=\frac{1}{4}\sqrt{4ab-(a+b+c)^2}=$$

$$=\frac{1}{4}\sqrt{4ab-a^2-b^2-c^2-2ab+2bc+2ca}=\frac{1}{4}\sqrt{2(ab+bc+ca)-(a^2+b^2+c^2)}=$$

$$=\frac{1}{4}\sqrt{2s^2+2r^2+8Rr-2s^2+2R^2+8Rr}=\frac{1}{4}\sqrt{4r^2+16Rr}=\frac{1}{2}\sqrt{r(4R+r)}$$

Observation: $\Delta=\frac{1}{2}\sqrt{r(4R+r)}\overset{Doucet}{\geq}\frac{1}{2}\sqrt{r\cdot s\sqrt{3}}=\frac{\sqrt[4]{3}}{2}\cdot\sqrt{rs}=\frac{\sqrt[4]{3}}{2}\cdot\sqrt{F};(2)$

Back to the main result:

Proof 1: Let's consider ΔABC with sides a,b,c and ΔUVW with sides u,v,w such that $u=\sqrt{a},\ v=\sqrt{b}, w=\sqrt{c}$. Then:

$$a^{m+1}+b^{m+1}+c^{m+1}=(u^2)^{m+1}+(v^2)^{m+1}+(w^2)^{m+1}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}3\cdot\sqrt[3]{(u^2v^2w^2)^{m+1}}=3\left(\sqrt[3]{u^2v^2w^2}\right)^{m+1}\overset{Carlitz}{\geq}$$

$$\overset{Carlitz}{\geq} \frac{1}{3^m}\left(4\sqrt{3}\Delta\right)^{m+1} = 4^{m+1}\cdot\left(\sqrt{3}\right)^{-2m}\cdot\left(\sqrt{3}\right)^{m+1}\cdot\Delta^{m+1} =$$

$$= 2^{2m+2}\cdot\left(\sqrt{3}\right)^{1-m}\cdot\Delta^{m+1} \overset{(2)}{\geq} 2^{2m+2}\cdot\left(\sqrt{3}\right)^{1-m}\cdot\left(\frac{\sqrt[4]{3}}{2}\cdot\sqrt{F}\right)^{m+1} =$$

$$= 2^{m+1}\cdot\left(\sqrt[4]{4}\right)^{2-2m}\cdot\left(\sqrt{F}\right)^{m+1} = 2^{m+1}\cdot\left(\sqrt[4]{3}\right)^{3-m}\cdot\left(\sqrt{F}\right)^{m+1}$$

Proof 2.

$$a^{m+1}+b^{m+1}+c^{m+1} = (u^2)^{m+1}+(v^2)^{m+1}+(w^2)^{m+1} =$$

$$= \frac{(u^2)^{m+1}}{1^m}+\frac{(v^2)^{m+1}}{1^m}+\frac{(w^2)^{m+1}}{1^m} \overset{Radon}{\geq} \frac{(u^2+v^2+w^2)^{m+1}}{(1+1+1)^m} \overset{Ionescu-Weitzenbock}{\geq}$$

$$\geq \frac{1}{3^m}\left(4\sqrt{3}\Delta\right)^{m+1} \overset{(2)}{\geq} \frac{1}{3^m}\left(4\sqrt{3}\cdot\frac{\sqrt[4]{3}}{2}\cdot\sqrt{F}\right)^{m+1} =$$

$$= 4^{m+1}\cdot 3^{-m}\cdot\left(\sqrt{3}\right)^{m+1}\cdot\left(\sqrt[4]{3}\right)^{m+1}\cdot\frac{1}{2^{m+1}}\cdot\left(\sqrt{F}\right)^{m+1} =$$

$$= 2^{m+1}\cdot\left(\sqrt[4]{3}\right)^{-4m+2n+2+m+1}\cdot\left(\sqrt{F}\right)^{m+1} = 2^{m+1}\cdot\left(\sqrt[4]{3}\right)^{3-m}\cdot\left(\sqrt{F}\right)^{m+1}$$

Conclusions: If we take in (1) $m = 0$ then:

$$a+b+c \geq 2\cdot\sqrt[4]{27}\cdot\sqrt{F}, \qquad 2s \geq 2\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

$$s \geq \sqrt[4]{27}\cdot\sqrt{rs} \Rightarrow \sqrt{s} \geq \sqrt[4]{27}\cdot\sqrt{r} \Rightarrow s \geq \sqrt{27}r \Rightarrow s \geq 3\sqrt{3}r\ (\boldsymbol{Mitrinovic}).$$

If we take in (1) $m = 1$ then: $a^2+b^2+c^2 \geq 2^2\cdot\left(\sqrt[4]{3}\right)^2\cdot\left(\sqrt{F}\right)^2 = 4\sqrt{3}F$

$$a^2+b^2+c^2 \geq 4\sqrt{3}\ (\textbf{Ionescu} - \textbf{Weitzenbock's})$$

If we take in (1) $m = 3$ then: $a^4+b^4+c^4 \geq 2^{3+1}\cdot\left(\sqrt[4]{3}\right)^{3-3}\cdot\left(\sqrt{F}\right)^{3+1} = 16F^2$

$$a^4+b^4+c^4 \geq 16F^2\ (\boldsymbol{Goldner}).$$

SOME OF JENSEN'S TYPE INEQUALITIES

By Neculai Stanciu-Romania

We consider the function $f: D \to \mathbb{R}$, convex on $D \subset \mathbb{R}$. For $\forall \lambda_i \in \mathbb{R}_+$ with $\sum_{i=1}^{m} \lambda_i^2 \neq 0$, $a_i \in D, i = \overline{1,m}$ we have the following *Jensen's* inequality:

$$f\left(\frac{\sum_{j=1}^{m} \lambda_j a_j}{\sum_{j=1}^{m} \lambda_j}\right) \leq \frac{\sum_{j=1}^{m} \lambda_j f(a_j)}{\sum_{j=1}^{m} \lambda_j}; (1)$$

Let $\alpha_i, \beta_i \in (0,\infty); p_i, q_i, k_i \in \mathbb{R}$ and the function $u_i: (0,\infty) \to \mathbb{R}$ given by

$$u_i(x) = \left(\alpha_i x_i^{p_i} + \beta_j x_j^{q_j}\right)^{k_j}, i = \overline{1,n}$$

If we denote $f(x) = \prod_{i=1}^{n} u_n(x)$; (2). By induction, we obtain that:

$$f'(x) = \sum_{i=1}^{n} A_i(x) u_i'(x), \text{where } A_i(x) = \prod_{j=1, j\neq i}^{n} u_j(x)$$

$$f''(x) = \sum_{i=1}^{n} A_i(x) u_i''(x) + \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}(x) u_i'(x) u_j'(x), \text{where } B_{ij}(x) = \prod_{k=1, k\neq i,j}^{n} u_k(x)$$

We have:

$$u_i'(x) = k_i(\alpha_i x^{p_i} + \beta_i x^{q_i})^{k_i - 1}(\alpha_i p_i x^{p_i - 1} + \beta_i q_i x^{q_i - 1}), \forall i = \overline{1,n}$$

$$u_i''(x) = k_i(k_i - 1)(\alpha_i x^{p_i} + \beta_i x^{q_i})^{k_i - 2}(\alpha_i p_i x^{p_i - 1} + \beta_i q_i x^{q_i - 1})^2 +$$

$$+k_i(\alpha_i x^{p_i} + \beta_i x^{q_i})^{k_i - 1}(\alpha_i p_i(p_i - 1)x^{p_i - 2} + \beta_i q_i(q_i - 1)x^{q_i - 2}), \forall i = \overline{1,n}$$

If $D = \{x \in (0,\infty) | (\alpha_i p_i x^{p_i - 1} + \beta_i q_i x^{q_i - 1}) > 0, (\alpha_i p_i(p_i - 1)x^{p_i - 2} + \beta_i q_i(q_i - 1)x^{q_i - 2} > 0, \forall i = \overline{1,n}\}$, then $\forall x \in D$ yields that $f''(x) \geq 0$, so f is convex on D.

By (1) we obtain

$$\sum_{j=1}^{m}\prod_{i=1}^{n}\left(\alpha_i a_j^{p_i} + \beta_i a_j^{q_i}\right)^{k_i} \geq m \prod_{i=1}^{n}\left[\alpha_i\left(\frac{a}{m}\right)^{p_i} + \beta_i\left(\frac{a}{m}\right)^{q_i}\right]^{k_i}; (3)$$

where $\sum_{j=1}^{m} a_j = a$ and $\lambda_j = 1, \forall j = \overline{1,m}$ with $\sum_{j=1}^{m} \lambda_j = m$.

If in (3) we take $n = 1$, then $\forall \alpha, \beta \in (0,\infty)$ we have:

$$\sum_{j=1}^{m}\left(\alpha a_j^{p} + \beta a_j^{q}\right)^{k} \geq m\left[\alpha\left(\frac{a}{m}\right)^{p} + \beta\left(\frac{a}{m}\right)^{q}\right]^{k}; (4)$$

Applications.

1. If in (4) we take $\alpha = 1, \beta = 0, k = 1$, then

$$\frac{1}{m}\sum_{j=1}^{m} a_j^p \geq \frac{1}{m^p}\left(\sum_{j=1}^{m} a_j\right)^p ;(5),$$

i.e. a generalization of the problem 8807 from Romanian Mathematical Gazette (G.M.) no 3/1968, proposed by *Iosif Bohler* and problem 8785 from G.M. no. 3/1968, proposed by *N. Pantazi.*

2. If in (5) we take $p = 2$, then

$$\sum_{j=1}^{n} a_j^2 \geq \frac{a^2}{m};(6)$$

i.e. a problem published in 1964 (*Journal de matematiques elementaries)* and in G.M. no. 10/1964, Problem 6579.

3. If in (4) we take $a = 1, \alpha = \beta = 1$ and $q = -1$, then

$$\sum_{j=1}^{m}\left(a_j + \frac{1}{a_j}\right)^k \geq \frac{(1+m^2)^k}{m^{k-1}};(7)$$

i.e. problem 8745 from G.M. no. 2/1968, proposed by *Liviu Pîrșan*, and related to problem

7877, C.d. *Skiliarski, 1965, p.67)*

4) If $\alpha = 0, \beta = k = 1, q = -\frac{1}{s}, s \geq 2, s \in \mathbb{N}$, then:

$$\sum_{i=1}^{m}\frac{1}{\sqrt[5]{a_j}} \geq m\sqrt[5]{\frac{m}{a}};(8)$$

i.e. the problem 8796 from G.M. no. 3/1968 proposed by *Liviu Pîrșan,* and related to problem 6641 from G.M. no.12/1964, proposed by *Cornel Popovici*, and to problem 8358 from G.M. no. 7/1967, proposed by *Dan Stănescu* and to problem 8688 from G.M. no. 1/1968.

New Result.

a) If $a_j > 0, \forall j = \overline{1,m}$ with $\sum_{j=1}^{m} a_j = a$ then:

$$\sum_{j=1}^{m} a_j^q\left(\alpha a_j^r + \beta\right) \geq \frac{a^q(\alpha a^r + \beta m^r)}{m^{q+r-1}};(9)$$

Solution. We take in (4) $k = 1$ and $p = q + r$.

b) If $a_j > 0, \forall j = \overline{1,m}$ with $\sum_{j=1}^{m} a_j = a$, then:

$$\sum_{j=1}^{m} a_j^q\left(a_j^r + 1\right) \geq \frac{a^q(a^r + m^r)}{m^{q+r-1}}; (10)$$

Solution. We take in (9) $\alpha = \beta$.

c) If $a_j > 0, \forall j = \overline{1,m}$ with $\sum_{j=1}^{m} a_j = a$, then:

$$\sum_{j=1}^{m} a_j^q\left(a_j^r + 1\right)(a_1 a_2 \dots a_m)^{-1} \geq \left(\frac{m}{a}\right)^{m-q-r} + \left(\frac{m}{a}\right)^{m-q}; (11)$$

Solution. We take in (9) $\alpha = \beta = (a_1 a_2 \dots a_m)^{-1}$ and taking account by $a_1 a_2 \dots a_m \leq \left(\frac{a}{m}\right)^m$.

d) If $a_j > 0, \forall j = \overline{1,m}$, then:

$$\sum_{j=1}^{m} a_j^p\left(a_j^r + 1\right)(a_1 a_2 \dots a_m)^{-1} \geq m^{m-q-r} + m^{m-q}; (12)$$

(related to G.M. no. 10/1968, problem 9234, author *Liviu Pîrșan*).

Solution. We take in (11) $a = 1$.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY D.M.BĂTINEȚU-GIURGIU-II

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_c h_a} \geq \frac{\sqrt{3}}{S}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution. *We prove the following lemma:* ***Lemma.***

2) In ΔABC the following relationship holds:

$$\frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_c h_a} = \frac{s^2 + r^2 + 4Rr}{4s^2 r^2}$$

Proof. *Using* $h_a = \frac{2S}{a}$ *we obtain* $\sum \frac{1}{h_b h_c} = \sum \frac{1}{\frac{2S}{b}\cdot\frac{2S}{c}} = \sum \frac{bc}{4S^2} = \frac{s^2+r^2+4Rr}{4s^2r^2}$

Let's get back to the main problem. Using the Lemma the inequality from enunciation can be written:

$\frac{s^2+r^2+4Rr}{4s^2r^2} \geq \frac{\sqrt{3}}{sr} \Leftrightarrow s^2 + r^2 + 4Rr \geq 4sr\sqrt{3}$ *which follows from Mitrinovic's inequality:*

$s \leq \frac{3R\sqrt{3}}{2}$*. It remains to prove that:* $s^2 + r^2 + 4Rr \geq 4 \cdot \frac{3R\sqrt{3}}{2} \cdot r\sqrt{3} \Leftrightarrow$

$\Leftrightarrow s^2 + r^2 + 4Rr \geq 18Rr \Leftrightarrow s^2 \geq 14Rr - r^2$*, which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$*. It remains to prove that:*

$16Rr - 5r^2 \geq 14Rr - r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *The inequality can be strengthened:*

3) In ΔABC ***the following relationship holds:***

$$\frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_c h_a} \geq \frac{5R - r}{Ss}$$

Solution. *Using Lemma the inequality can be written:*

$\frac{s^2+r^2+4Rr}{4s^2r^2} \geq \frac{5R-r}{rs^2} \Leftrightarrow s^2 \geq 16Rr - 5r^2$ *(Gerretsen's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *Inequality 3) is stronger than inequality 1).*

4) In ΔABC ***the following relationship holds:***

$$\frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_c h_a} \geq \frac{5R - r}{Ss} \geq \frac{\sqrt{3}}{S}$$

Solution. *See inequality 3) and* $\frac{5R-r}{Ss} \geq \frac{\sqrt{3}}{S} \Leftrightarrow 5R - r \geq s\sqrt{3}$*, which follows from Mitrinovic's inequality* $s \leq \frac{3R\sqrt{3}}{2}$*. It remains to prove that*

$$5R - r \geq \frac{3R\sqrt{3}}{2} \cdot \sqrt{3} \Leftrightarrow R \geq 2r \text{ (Euler)}$$

Remark. *Inequality 3) can be developed.*

5) In ΔABC ***the following relationship holds:***

$$\frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_c h_a} \geq \frac{nR + (9-2n)r}{Ss}, \text{ where } n \leq 5.$$

Solution. *Using Lemma the inequality can be written:*

$\frac{s^2+r^2+4Rr}{4s^2r^2} \geq \frac{nR+(9-2n)r}{rs^2} \Leftrightarrow s^2 \geq Rr(4n-4)+r^2(35-8n)$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2 \geq Rr(4n-4)+r^2(35-8n) \Leftrightarrow R(5-n) \geq 2r(5-n)$, *obviously from Euler's inequality* $R \geq 2r$ *and the condition from hypothesis* $n \leq 5$.

Equality holds if and only if the triangle is equilateral.

Remark. *Let's find an inequality having an opposite sense:*

6) In ΔABC the following inequality holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a} \leq \frac{1}{3r^2}$$

Solution. *Using Lemma the inequality from enunciation can be written:*

$\frac{s^2+r^2+4Rr}{4s^2r^2} \leq \frac{1}{3r^2} \Leftrightarrow s^2 \geq 12Rr+3r^2$, *which follows from Gerretsen's inequality.*

$s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2 \geq 12Rr+3r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *Inequality 6) can be strengthened.*

7) InΔABC the following inequality holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a} \leq \frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right)$$

Solution. *Using Lemma the inequality can be written:*

$\frac{s^2+r^2+4Rr}{4s^2r^2} \leq \frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right) \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *Inequality 7) is stronger than inequality 6)*

8) In ΔABC the following inequality holds:

$$\frac{1}{h_a h_b}+\frac{1}{h_b h_c}+\frac{1}{h_c h_a} \leq \frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right) \leq \frac{1}{3r^2}$$

Solution. *See inequality 7) and* $\frac{1}{4r^2}\left(1+\frac{9Rr}{2s^2}\right) \leq \frac{1}{3r^2} \Leftrightarrow 2s^2 \geq 27Rr$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr-5r^2$. *It remains to prove that:*

$2(16Rr-5r^2) \geq 27Rr \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the double inequality:*

9) In ΔABC the following inequality holds:

$$\frac{5R-r}{Ss} \le \frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_c h_a} \le \frac{1}{4r^2}\left(1 + \frac{9Rr}{2s^2}\right)$$

Proposed by Marin Chirciu – Romania

Solution. *See inequalities 3) and 7). Equality holds if and only if the triangle is equilateral.*

Remark. *We can write the sequence of inequalities.*

10) In ΔABC the following inequality holds:

$$\frac{\sqrt{3}}{S} \le \frac{5R-r}{Ss} \le \frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_c h_a} \le \frac{1}{4r^2}\left(1 + \frac{9Rr}{2s^2}\right) \le \frac{1}{3r^2}$$

Solution. *See inequalities 4) and 8). Equality holds if and only if the triangle is equilateral.*

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY D.M.BĂTINEȚU-GIURGIU-III

BY Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{h_a - r}{h_a + r} + \frac{h_b - r}{h_b - r} + \frac{h_c - r}{h_c - r} \ge \frac{3}{2}$$

Proposed by D.M. Bătinețu-Giurgiu, Romania

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\frac{h_a - r}{h_a + r} + \frac{h_b - r}{h_b + r} + \frac{h_c - r}{h_c + r} = \frac{15s^2 - r^2 - 10Rr}{9s^2 + r^2 + 6Rr}$$

Proof. *Using* $h_a = \frac{2S}{a}$ *and* $r = \frac{S}{s}$ *we obtain* $\sum \frac{h_a - r}{h_a + r} = \sum \frac{\frac{2S}{a} - \frac{S}{s}}{\frac{2S}{a} + \frac{S}{s}} = \sum \frac{2s - a}{2s + a} = \frac{15s^2 - r^2 - 10Rr}{9s^2 + r^2 + 6Rr}$

Let's get to the main problem.Using Lemma the inequality can be written:

$\frac{15s^2-r^2-10Rr}{9s^2+r^2+6Rr} \geq \frac{3}{2} \Leftrightarrow R \geq 2r$ *(Euler's inequality), which follows from Gerretsen's inequality*

$s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$3(16Rr - 5r^2) \geq 38Rr + 5r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark. *Let's find an inequality having an opposite sense:*

3) In ΔABC the following inequality holds:

$$\frac{h_a - r}{h_a + r} + \frac{h_b - r}{h_b + r} + \frac{h_c - r}{h_c + r} \leq \frac{3R}{4r}$$

Proposed by Marin Chirciu – Romania

Solution. *Using Lemma the inequality holds:*

$$\frac{15s^2 - r^2 - 10Rr}{9s^2 + r^2 + 6Rr} \leq \frac{3R}{4r} \Leftrightarrow s^2(27R - 60r) + r(18R^2 + 43Rr + 4r^2) \geq 0$$

We distinguish the following cases:

Case 1) If $(27R - 60r) \geq 0$, *the inequality is obvious.*

Case 2). If $(27R - 60r) < 0$, *the inequality can be written:*

$r(18R^2 + 43Rr + 4r^2) \geq s^2(60r - 27R)$, *which follows from Gerretsen's inequality:*

$s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$r(18R^2 + 43Rr + 4r^2) \geq (4R^2 + 4Rr + 3r^2)(60r - 27R)$$

$$\Leftrightarrow 54R^3 - 57R^2r - 58Rr^2 - 88r^3 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(54R^2 + 51Rr + 44r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the double inequality:*

4) In ΔABC the following inequality can be written:

$$\frac{3}{2} \leq \frac{h_a - r}{h_a + r} + \frac{h_b - r}{h_b + r} + \frac{h_c - r}{h_c + r} \leq \frac{3R}{4r}$$

Solution. *See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.*

Remark. *Switching* $\frac{h_a-r}{h_a+r}$ *we obtain:*

5) In ΔABC the following relationship holds:

$$6 \le \frac{h_a+r}{h_a-r}+\frac{h_b+r}{h_b-r}+\frac{h_c+r}{h_c-r} \le \frac{3R}{r}$$

Proposed by Marin Chirciu – Romania

Solution. *We prove the following lemma:* ***Lemma.***

6) In ΔABC the following inequality holds:

$$\frac{h_a+r}{h_a-r}+\frac{h_b+r}{h_b-r}+\frac{h_c+r}{h_c-r}=\frac{7s^2-r^2+2Rr}{s^2+r^2+2Rr}$$

Proof. *Using $h_a=\frac{2S}{a}$ and $r=\frac{S}{s}$ we obtain $\sum\frac{h_a+r}{h_a-r}=\sum\frac{\frac{2S}{a}+\frac{S}{s}}{\frac{2S}{a}-\frac{S}{s}}=\sum\frac{2s+a}{2s-a}=\frac{7s^2-r^2+2Rr}{s^2+r^2+2Rr}$*

Let's get back to the main problem.LHS Using Lemma the inequality can be written:

$\frac{7s^2-r^2+2Rr}{s^2+r^2+2Rr}\ge 6 \Leftrightarrow s^2\ge 10Rr+7r$^2*, which follows from Gerretsen's inequality*

$s^2\ge 16Rr-5r^2$*. It remains to prove that:*

$16Rr-5r^2\ge 10Rr+7r^2 \Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.RHSUsing the Lemma the inequality can be written:

$$\frac{7s^2-r^2+2Rr}{s^2+r^2+2Rr}\le\frac{3R}{r}\Leftrightarrow s^2(3R-7r)+r(6R^2+Rr+r^2)\ge 0$$

We distinguish the following cases:

Case 1) If $(3R-7r)\ge 0$, the inequality is obvious.

Case 2) If $(3R-7r)<0$, the inequality can be rewritten:

$r(6R^2+Rr+r^2)\ge s^2(7r-3R)$*, which follows from Gerretsen's inequality*

$s^2\le 4R^2+4Rr+3r^2$*. It remains to prove that:*

$$r(6R^2+Rr+r^2)\ge(4R^2+4Rr+3r^2)(7r-3r)\Leftrightarrow 6R^3-5R^2r-9Rr^2-10r^3\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(6R^2+7Rr+5r^2)\ge 0$*, obviously from Euler's inequality $R\ge 2r$.*

Equality holds if and only if the triangle is equilateral.

Remark.*If we replace h_a with r_a we propose:*

7) In ΔABC the following relationship holds:

$$\frac{3}{2}\le\frac{r_a-r}{r_a+r}+\frac{r_b-r}{r_b+r}+\frac{r_c-r}{r_c+r}\le\frac{3R}{4r}$$

Proposed by Marin Chirciu – Romania

Solution.We prove the following lemma:

Lemma.

8) In ΔABC the following relationship holds:

$$\frac{r_a - r}{r_a + r} + \frac{r_b - r}{r_b + r} + \frac{r_c - r}{r_c + r} = \frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr}$$

Proof. *Using* $r_a = \frac{S}{s-a}$ *and* $r = \frac{S}{s}$ *we obtain* $\sum \frac{r_a - r}{r_a + r} = \sum \frac{\frac{S}{s-a} - \frac{S}{s}}{\frac{S}{s-a} + \frac{S}{s}} = \sum \frac{a}{b+c} = \frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr}$

Let's get back to the main problem.LHSUsing the Lemma the inequality can be written:

$\frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} \geq \frac{3}{2} \Leftrightarrow s^2 \geq 10Rr + 7r^2$, *which follows from Gerretsen's inequality*

$s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$16Rr - 5r^2 \geq 10Rr + 7r^2 \Leftrightarrow R \geq 2r$, *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.RHSUsing Lemma the inequality can be written:

$$\frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} \leq \frac{3R}{4r} \Leftrightarrow s^2(3R - 8r) + r(6R^2 + 11Rr + 8r^2) \geq 0$$

We distinguish the following cases:

Case 1) If $(3R - 8r) \geq 0$, *the inequality is obvious.*

Case 2) If $(3R - 8r) < 0$, *the inequality can be rewritten:*

$r(6R^2 + 11Rr + 8r^2) \geq s^2(8r - 3R)$, *which follows from Gerretsen's inequality*

$s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$r(6R^2 + 11Rr + 8r^2) \geq (4R^2 + 4Rr + 3r^2)(8r - 3r) \Leftrightarrow 6R^3 - 7R^2r - 6Rr^2 - 8r^3 \geq 0$

$\Leftrightarrow (R - 2r)(6R^2 + 5Rr + 4r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle equilateral.

9) In ΔABC the following relationship holds:

$$6 \leq \frac{r_a + r}{r_a - r} + \frac{r_b + r}{r_b - r} + \frac{r_c + r}{r_c - r} \leq \frac{3R}{r}$$

Proposed by Marin Chirciu – Romania

Solution.We prove the following lemma:

10) In ΔABC the following relationship holds:

$$\frac{r_a+r}{r_a-r}+\frac{r_b+r}{r_b-r}+\frac{r_c+r}{r_c-r}=\frac{s^2+r^2-2Rr}{2Rr}$$

Solution.*Using* $r_a=\frac{S}{s-a}$ *and* $r=\frac{S}{s}$ *we obtain* $\sum\frac{r_a+r}{r_a-r}=\sum\frac{\frac{S}{s-a}+\frac{S}{s}}{\frac{S}{s-a}-\frac{S}{s}}=\sum\frac{b+c}{a}=\frac{s^2+r^2-2Rr}{2Rr}$

Let's get back to the main problem.LHSUsing Lemma the inequality can be written:

$\frac{s^2+r^2-2Rr}{2Rr}\ge 6\Leftrightarrow s^2\ge 14R-r^2$, *which follows from Gerretsen's inequality*

$s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.RHSUsing Lemma the inequality can be written:

$\frac{s^2+r^2-2Rr}{2Rr}\le\frac{3R}{r}\Leftrightarrow s^2\le 6R^2+2Rr-r^2$, *which follows from Gerretsen's inequality*

$s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$4R^2+4Rr+3r^2\le 6R^2+2Rr-r^2\Leftrightarrow R^2-Rr-2r^2\ge 0\Leftrightarrow (R-2r)(R+r)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only the triangle is equilateral.

Remark.*If in the above sums we replace* r *with* $2r$ *we obtain new inequalities:*

11) In ΔABC ***the following inequality holds:***

$$\frac{3}{5}\le\frac{h_a-2r}{h_a+2r}+\frac{h_b-2r}{h_b+2r}+\frac{h_c-2r}{h_c+2r}\le\frac{3R}{10r}$$

Proposed by Marin Chirciu – Romania

Solution.*We prove the following lemma.*

12) In ΔABC ***the following relationship holds:***

$$\frac{h_a-2r}{h_a+2r}+\frac{h_b-2r}{h_b+2r}+\frac{h_c-2r}{h_c+2r}=\frac{4s^2-r^2-16Rr}{4s^2+r^2+8Rr}$$

Proof.*Using* $h_a=\frac{2S}{a}$ *and* $r=\frac{S}{s}$ *we obtain* $\sum\frac{h_a-2r}{h_a+2r}=\sum\frac{\frac{2S}{a}-\frac{2S}{s}}{\frac{2S}{a}+\frac{2S}{s}}=\sum\frac{s-a}{s+a}=\frac{4s^2-r^2-16Rr}{4s^2+r^2+8Rr}$

Let's get back to the main problem.LHSUsing the Lemma the inequality can be written:

$\frac{4s^2-r^2-16Rr}{4s^2+r^2+8Rr}\ge\frac{3}{5}\Leftrightarrow s^2\ge 13Rr+r^2$, *which follows from Gerretsen's inequality*

$s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$$16Rr-5r^2\ge 13Rr+r^2\Leftrightarrow R\ge 2r\ (Euler's\ inequality)$$

Equality holds if and only if the triangle is equilateral.RHSUsing the Lemma the inequality can be written:

$$\frac{4s^2-r^2-16Rr}{4s^2+r^2+8Rr}\le\frac{3R}{10r}\Leftrightarrow s^2(12R-40r)+r(24R^2+163Rr+10r^2)\ge 0$$

We distinguish the following cases:

Case 1) If $(12R-40r)\ge 0$, *the inequality is obvious.*

Case 2). If $(12R-40r)<0$, *the inequality can be rewritten:*

$r(24R^2+163Rr+10r^2)\ge s^2(40r-12R)$, *which is obvious from Gerretsen's inequality*

$s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$r(24R^2+163Rr+10r^2)\ge(4R^2+4Rr+3r^2)(40r-12R)$$

$$\Leftrightarrow 48R^3-88R^2r-39Rr^2-110r^3\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(48R^2+8Rr+55r^2)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.*Switching* $\frac{h_a-2r}{h_a+2r}$ *we obtain:*

13) In $\boldsymbol{\Delta ABC}$ ***the following relationship holds:***

$$\mathbf{15}\le\frac{h_a+2r}{h_a-2r}+\frac{h_b+2r}{h_b-2r}+\frac{h_c+2r}{h_c-2r}\le\frac{15R^2}{4r^2}$$

Proposed by Marin Chirciu – Romania

Solution.*We prove the following lemma.*

14) In $\boldsymbol{\Delta ABC}$ ***the following relationship holds:***

$$\frac{h_a+2r}{h_a-2r}+\frac{h_b+2r}{h_b-2r}+\frac{h_c+2r}{h_c-2r}=\frac{8R-r}{r}$$

Proof. *Using* $h_a=\frac{2S}{a}$ *and* $r=\frac{S}{s}$ *we obtain* $\sum\frac{h_a+2r}{h_a-2r}=\sum\frac{\frac{2S}{a}+\frac{2S}{s}}{\frac{2S}{a}-\frac{2S}{s}}=\sum\frac{s+a}{s-a}=\frac{8R-r}{r}$

Let's get back to the main problem.LHS Using Lemma the inequality can be written:

$$\frac{8R-r}{r}\ge 15\Leftrightarrow R\ge 2r\ (Euler's\ inequality)$$

Equality holds if and only if the triangle is equilateral.RHS Using Lemma the inequality can be written:

$$\frac{8R-r}{r} \le \frac{15R^2}{4r^2} \Leftrightarrow 15R^2 - 32R + 4r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(15R-2r) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only the triangle is equilateral.

Remark. *If we replace* h_a *with* r_a *we propose:*

15) In ΔABC ***the following relationship holds:***

$$\frac{3}{5} \le \frac{r_a-2r}{r_a+2r} + \frac{r_b-2r}{r_b+2r} + \frac{r_c-2r}{r_c+2r} \le \frac{3R}{10r}$$

Proposed by Marin Chirciu – Romania

Solution.*We prove the following lemma.*

16) In ΔABC ***the following relationship holds:***

$$\frac{r_a-2r}{r_a+2r} + \frac{r_b-2r}{r_b+2r} + \frac{r_c-2r}{r_c+2r} = \frac{5s^2-28r^2-16Rr}{3s^2+12r^2+16Rr}$$

Proof. *Using* $r_a = \frac{S}{s-a}$ *and* $r = \frac{S}{s}$ *we obtain* $\sum \frac{r_a-2r}{r_a+2r} = \sum \frac{\frac{S}{s-a}-\frac{2S}{s}}{\frac{S}{s-a}+\frac{2S}{s}} = \sum \frac{2a-s}{3s-2a} = \frac{5s^2-28R^2-16Rr}{3s^2+12r^2+16Rr}$

Let's get back to the main problem.LHSUsing Lemma the inequality can be written:

$\frac{5s^2-28r^2-16Rr}{3s^2+12r^2+16Rr} \ge \frac{3}{5} \Leftrightarrow 8Rr + 11r^2$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr - 5r^2$. *It remains to prove that:*

$16Rr - 5r^2 \ge 8Rr + 11r^2 \Leftrightarrow R \ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.RHS

Using Lemma the inequality can be written:

$$\frac{5s^2-28r^2-16Rr}{3s^2+12r^2+16Rr} \le \frac{3R}{10r} \Leftrightarrow s^2(9R-50r) + r(48R^2+196Rr+280r^2) \ge 0$$

We distinguish the following cases:

Case 1) If $(9R-50r) \ge 0$, *inequality is obvious.*

Case 2) If $(9R-50r) < 0$, *inequality is rewritten:*

$r(48R^2+196Rr+280r^2) \ge s^2(50r-9R)$, *which follows from Gerretsen's inequality*

$s^2 \le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$r(48R^2+196Rr+280r^2) \ge (4R^2+4Rr+3r^2)(50r-9R) \Leftrightarrow$$

$\Leftrightarrow (R-2r)(36R^2-44Rr-65r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

17) In ΔABC the following relationship holds:

$$\frac{r_a+2r}{r_a-2r}+\frac{r_b+2r}{r_b-2r}+\frac{r_c+2r}{r_c-2r} \geq 15$$

Proposed by Marin Chirciu – Romania

Solution. *We prove the following lemma:*

18) In ΔABC the following relationship holds:

$$\frac{r_a+2r}{r_a-2r}+\frac{r_b+2r}{r_b-2r}+\frac{r_c+2r}{r_c-2r}=\frac{s^2+20r^2-16Rr}{16Rr-4r^2-s^2}$$

Proof. *Using* $r_a=\frac{S}{s-a}$ *and* $r=\frac{S}{s}$ *we obtain* $\sum\frac{r_a+2r}{r_a-2r}=\sum\frac{\frac{S}{s-a}+\frac{2S}{s}}{\frac{S}{s-a}-\frac{2S}{s}}=\sum\frac{3s-2a}{2a-s}=\frac{s^2+20r^2-16Rr}{16Rr-4r^2-s^2}$

Let's get back to the main problem. Using Lemma the inequality can be written:

$\frac{s^2+20r^2-16Rr}{16Rr-4r^2-s^2} \geq 15 \Leftrightarrow s^2 \geq 16Rr-5r^2$ *(Gerretsen's inequality)*

Equality holds if and only if the triangle is equilateral.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

BENCZE'S CRITERION

By Florică Anastase-Romania

Abstract: *In this paper are presented few applications of an outstanding result.*

THEOREM (*Mihàly Bencze)*

Let be $f:[0,1]\to(0,\infty)$ continuous function and $\alpha:\mathbb{R}\to[0,1]$ such that $\lim_{x\to\infty}\alpha(x)=0$. If exists the unique sequence $(x_n)_{n\geq1}$ such that

$$\int_0^{x_n} f(x)\,dx=\alpha(n)\int_0^1 f(x)\,dx.$$

then:

$$\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\cdot\int_0^1 f(x)\,dx$$

Proof 1 by proposer.

$$\text{Let be } g_n(x)=\int_0^{x_n} f(x)\,dx-\alpha(n)\int_0^1 f(x)\,dx$$

The function g_n is differentiable, continuous and:

$$g_n(0)\cdot g_n(1)=-\alpha(n)\big(1-\alpha(n)\big)\left(\int_0^1 f(x)\,dx\right)^2<0,$$

so, from Rolle Theorem, exists $x_n\in(0,1)$ such that $g_n(x_n)=0$. But $g_n'(x)=f(x)>0$.

Hence, g_n is decreasing, then g_n is injective and result the equation $g_n(x)=0$ have unique solution $x_n\in(0,1)$.

Let be: $m=\min\limits_{x\in[0,1]} f(x)\,,M=\max\limits_{x\in[0,1]} f(x)$ such that

$$m\cdot x_n\le\int_0^{x_n} f(x)\,dx=\alpha(n)\int_0^1 f(x)\,dx\le M\cdot\alpha(n) \text{ and } 0\le x_n\le\frac{M}{m}\cdot\alpha(n).$$

Therefore, $0\le\lim\limits_{n\to\infty} x_n\le\frac{M}{m}\lim\limits_{n\to\infty}\alpha(n)=0$ and then, $\lim\limits_{n\to\infty} x_n=0$. But

$$\alpha(n)\int_0^1 f(x)\,dx=\int_0^{x_n} f(x)\,dx=\frac{F(x_n)-F(0)}{x_n}\cdot x_n$$

$$\int_0^1 f(x)\,dx=\frac{F(x_n)-F(0)}{x_n}\cdot\frac{x_n}{\alpha(n)},\ \lim_{n\to\infty}\frac{F(x_n)-F(0)}{x_n}=f(0)$$

$$\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\int_0^1 f(x)\,dx$$

Proof 2 by Marius Olteanu

Because f −continuous function, then f admits primitives $F:[0,1]\to\mathbb{R}$ such that

$$\int_0^x f(t)\,dt=F(x),\forall x\in[0,1]$$

How F is continuous on $[0,1]$, then have Darboux property on $[0,1]$ and have values on the interval $\left[0,\int_0^1 f(x)\,dx\right]$. Because $\alpha(n)\in[0,1]$, then

$$\alpha(n)\cdot\int_0^1 f(x)\,dx\in\left[0,\int_0^1 f(x)\,dx\right]=J.$$

More, $f(x)>0$ and $F'(x)=f(x)>0$ imply that F is strictly increases. Therefore, F bijective function. So, $\forall y_0\in J,\exists!\,x_0\in[0,1]$ such that $F(x_0)=y_0$.

If $y_0=\alpha(n)\cdot\int_0^1 f(x)\,dx$ then $\exists x_0=x_n\in[0,1]$ such that:

$$F(x_n)=\alpha(n)\cdot\int_0^1 f(x)\,dx=\int_0^{x_n} f(x)\,dx$$

$$\lim_{n\to\infty}\int_0^{x_n} f(x)\,dx=\lim_{n\to\infty}\left[\alpha(n)\cdot\int_0^1 f(x)\,dx\right]=\left(\int_0^1 f(x)\,dx\right)\cdot\lim_{n\to\infty}\alpha(n)=$$

$$= \lim_{n\to\infty}[F(x_n)-F(0)] = 0 \Leftrightarrow \lim_{n\to\infty}F(x_n)=F(0) \Leftrightarrow \lim_{n\to\infty}(f(\xi_n)\cdot x_n)=0; (0<\xi_n<x_n)$$

because $F(x_n)-F(0)=(x_n-0)\cdot F'(\xi_n)=x_n\cdot f(\xi_n)$

Because f is continuous function on $[0,1]$ then $f(x)\in[m,M]$ and then $f(\xi_n)\in[m,M], \forall n\in\mathbb{N}^*$. It follows that $\left(f(\xi_n)\right)_{n\ge1}$ is bounded and

$$0<x_n\cdot m\le x_n\cdot f(\xi_n)\le x_n\cdot M \Leftrightarrow$$

$$0\le \lim_{n\to\infty}(m\cdot x_n)\le \lim_{n\to\infty}\left(x_n\cdot f(\xi_n)\right)=0; (1)$$

$$\text{Now, } \int_0^{x_n} f(x)\,dx = x_n\cdot f(\xi_n); (0<\xi_n<x_n) \Rightarrow \lim_{n\to\infty}\xi_n=0$$

$$x_n\cdot f(\xi_n)=\alpha(n)\cdot\int_0^1 f(x)\,dx \Rightarrow \frac{x_n}{\alpha(n)}=\frac{1}{f(\xi_n)}\int_0^1 f(x)\,dx$$

$$\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\lim_{n\to\infty}\left(\frac{1}{f(\xi_n)}\int_0^1 f(x)\,dx\right)=\int_0^1 f(x)\,dx\cdot\lim_{n\to\infty}\frac{1}{f(\xi_n)}=\frac{1}{f(0)}\int_0^1 f(x)\,dx$$

Application 1.

If exists an unique $(x_n)_{n\ge1}$ sequence of real numbers and

$\alpha:\mathbb{R}\to[0,1], \lim_{n\to\infty}\alpha(n)=0$ such that:

$$\int_0^{x_n}\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\alpha(n)\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx$$

$$\textbf{then find}: \Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}$$

Solution.

$$\text{Let: } F(y)=\int_0^1\frac{\tan^{-1}(xy)}{x\sqrt{1-x^2}}dx \text{ then } F'(y)=\int_0^1\frac{dx}{(1+x^2y^2)\sqrt{1-x^2}}=\int_0^{\frac{\pi}{4}}\frac{dt}{1+y^2\cos^2 t}=$$

$$=\frac{1}{\sqrt{1+y^2}}\tan^{-1}\left(\frac{\tan t}{\sqrt{1+y^2}}\right)=\frac{\pi}{2\sqrt{1+y^2}}$$

$$\text{So, } F(y)=\frac{\pi}{2}\log\left(y+\sqrt{1+y^2}\right)+C. \text{ Put } y=0\Rightarrow C=0\Rightarrow$$

$$\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log\left(1+\sqrt{2}\right)$$

Using Bencze's Criterion for $f(x)=\frac{\tan^{-1}x}{x\sqrt{1-x^2}}$, we get:

$$\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\int_0^1\frac{\tan^{-1}x}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log\left(1+\sqrt{2}\right)$$

Application 2: If exists an unique $(x_n)_{n\ge1}$ sequence of real numbers and

$$\boldsymbol{\alpha}:\mathbb{R}\to[\mathbf{0},\mathbf{1}],\lim_{n\to\infty}\boldsymbol{\alpha}(\boldsymbol{n})=\mathbf{0}\text{ such that:}$$

$$\int_0^{x_n}\sqrt{1+\sqrt{x}}\,dx=\alpha(n)\int_0^1\sqrt{1+\sqrt{x}}\,dx$$

$$\textbf{Find}:\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}$$

Solution. Let $f(x)=\sqrt{1+\sqrt{x}}$ continuous function. On the interval $[\varepsilon,1]\subset[0,1],\varepsilon>0$, let

$$g:[\varepsilon,1]\to\mathbb{R},g(x)=\sqrt{1+\sqrt{x}}\left(1+\sqrt{x}\right)'\left(2\sqrt{x}+2-2\right)=$$

$$=2\varphi^{\frac{3}{2}}(x)\varphi'(x)-2\varphi^{\frac{1}{2}}(x)\varphi'(x),\text{where }\varphi(x)=1+\sqrt{x},\varphi'(x)=\frac{1}{2\sqrt{x}}.$$

On $[\varepsilon,1]$ function g is continuous and $G_\varepsilon(x)=\frac{4}{5}\left(1+\sqrt{x}\right)^{\frac{5}{2}}-\frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}}$

$$I_\varepsilon=G_\varepsilon(1)-G_\varepsilon(\varepsilon),\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}}I_\varepsilon=\frac{8\left(\sqrt{2}+1\right)}{15}\text{ then }I=\int_0^1f(x)\,dx=\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}}I_\varepsilon=\frac{8\left(\sqrt{2}+1\right)}{15}$$

Using Bencze's Criterion for $f(x)=\sqrt{1+\sqrt{x}}$, we get:

$$\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\int_0^1f(x)\,dx=\frac{8\left(\sqrt{2}+1\right)}{15}$$

Observation. For $f(x)=\sqrt{1+\sqrt{x}}$, $f:[0,1]\to[1,\sqrt{2}]$ continuous function we take

$$t=1+\sqrt{x}\Rightarrow x=\varphi(t)=(t-1)^2,\text{where }\varphi:[1,2]\to[0,1],\text{with }\varphi^{-1}(0)=1,\varphi^{-1}(1)=2$$

$$\varphi'(t)=2(t-1)\Rightarrow\int_0^1f(x)\,dx=\int_1^2\sqrt{2}(2t-2)\,dt=\frac{8\left(\sqrt{2}+1\right)}{15}.$$

Application 3: If exists an unique $(x_n)_{n\ge1}$ sequence of real numbers and $\alpha:\mathbb{R}\to[0,1]$,

$$\lim_{n\to\infty}\boldsymbol{\alpha}(\boldsymbol{n})=\mathbf{0}\text{ such that:}$$

$$\int_0^{x_n}\frac{\log(1-x^2)}{x}\,dx=\alpha(n)\int_0^1\frac{\log(1-x^2)}{x}\,dx.$$

$$\textbf{then find}:\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}$$

Solution:

$$\text{Let }I=\int_0^1\frac{\log(1-x)}{x}dx+\int_0^1\frac{\log(1+x)}{x}dx=I_1+I_2$$

$$\text{We know: } \log(1+x)=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{n+1};x\in(-1,1]$$

$$\log(1-x)=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1};x\in(-1,1)$$

$$\log\frac{1+x}{1-x}=2\cdot\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1};x\in(-1,1)$$

$$\text{We have: } I_1=\int_0^1\left(\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^n}{n+1}\right)dx=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(n+1)^2}=\frac{\pi^2}{12}$$

$$I_2=\int_0^1\frac{\log(1-x)}{x}dx=\lim_{x\to1}\int_0^x\frac{\log(1-t)}{t}dt=-\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}=-\frac{\pi^2}{6}$$

$$\text{Hence, }\int_0^1\frac{\log(1-x)}{x}dx=I_1+I_2=-\frac{\pi^2}{12}$$

Using Bencze's Criterion for $f(x)=\frac{\log(1-x)}{x}$, we get:

$$\Omega=\lim_{n\to\infty}\frac{x_n}{\alpha(n)}=\frac{1}{f(0)}\int_0^1\frac{\log(1-x)}{x}dx=\frac{\pi^2}{12},\text{where } f(0)=\lim_{t\to0}\frac{\log(1-t)}{t}=-1$$

Application 4:If exists an unique $(x_n)_{n\geq1}$ sequence of real numbers, $\alpha:\mathbb{R}\to[0,1]$, $\lim_{n\to\infty}\alpha(n)=0$ such that:

$$\int_0^{x_n}\frac{dx}{1+\sqrt[n]{x}+\sqrt[n]{x^2}+\dots+\sqrt[n]{x^n}}=\alpha(n)\cdot\int_0^1\frac{dx}{1+\sqrt[n]{x}+\sqrt[n]{x^2}+\dots+\sqrt[n]{x^n}}$$

$$\textbf{then find}:\Omega=\lim_{n\to\infty}\frac{n\cdot x_n}{\alpha(n)}$$

Solution. Let $\sqrt[n]{x}=t\Rightarrow x=t^n,dx=nt^{n-1}dt$ then

$$I_n=\int_0^1\frac{dx}{1+\sqrt[n]{x}+\sqrt[n]{x^2}+\dots+\sqrt[n]{x^n}}=n\int_0^1\frac{t^{n-1}\,dt}{1+t+t^2+\dots+t^n}$$

$$\because (1-t^{n+1})\left(\sum_{i=0}^{p}t^{i(n+1)}\right)+t^{(p+1)(n+1)}=1\Rightarrow$$

$$\frac{1}{1+t+\dots+t^n}=(1-t)\left(\sum_{i=0}^{p}t^{i(n+1)}\right)+\frac{t^{(p+1)(n+1)}}{1+t+\dots+t^n}\Leftrightarrow$$

$$\frac{t^{n-1}}{1+t+t^2+\cdots+t^n}=\sum_{i=0}^{p}\left[t^{i(n+1)+n-1}-t^{i(n+1)+n}\right]+\frac{t^{(p+1)(n+1)+n-1}}{1++t+\cdots+t^n}\Rightarrow$$

$$\int_0^1\frac{t^{n-1}\,dt}{1+t+t^2+\cdots+t^n}=\sum_{i=0}^{p}\left[\frac{1}{i(n+1)+n}-\frac{1}{i(n+1)+n+1}\right]+\int_0^1\frac{t^{(p+1)(n+1)+n-1}}{1++t+\cdots+t^n}dt$$

$$I_n=a_{pn}+b_{pn},\forall n\in\mathbb{N},\forall p\in\mathbb{N};(1),\text{where}$$

$$a_{np}=\frac{n}{n+1}\sum_{i=0}^{p}\frac{1}{(i+1)[i(n+1)-n]}$$

$$b_{pn}=n\int_0^1\frac{t^{(p+1)(n+1)+n-1}}{1++t+\cdots+t^n}dt\le n\int_0^1 t^{(p+1)(n+1)}\,dt=\frac{n}{(p+1)(n+1)+1}$$

How, $\lim_{n\to\infty}b_{pn}=0,\forall n\in\mathbb{N}$ from (1) it follows that:

$$I_n=\lim_{p\to\infty}a_{pn}=\frac{n}{n+1}\sum_{i=0}^{\infty}\frac{1}{(i+1)[i(n+1)+n]}$$

Let: $g_i:[0,\infty)\to\mathbb{R},g_i(x)=\frac{1}{(i+1)(i+1-x)}$, we have: $I_n=\frac{n}{(n+1)^2}\sum_{i=0}^{\infty}g_i\left(\frac{1}{n+1}\right);(2)$

But $0\le g_i(x)\le\frac{1}{(i+1)^2};\forall x\ge 0$ and $\sum_{i=0}^{\infty}\frac{1}{(i+1)^2}<\infty$ from $Weiestrass$, we get that

$$\sum_{i=0}^{\infty}g_i\ \text{converges uniform on }[0,\infty)$$

Hence, $nI_n=\left(\frac{n}{n+1}\right)^2 f\left(\frac{1}{n+1}\right)$ and using Bencze's Criterion, we get:

$$\Omega=\lim_{n\to\infty}\frac{n\cdot x_n}{\alpha(n)}=\lim_{n\to\infty}n\cdot I_n=\lim_{n\to\infty}g\left(\frac{1}{n+1}\right)=g(0)=\sum_{i=0}^{\infty}\frac{1}{(i+1)^2}=\frac{\pi^2}{6}$$

Application 5: For $x_n\in(0,1)$ let $\lim_{n\to\infty}n\int_0^{x_n}x^nf(x)\,dx=0$, where

$f:[0,1]\to\mathbb{R}$ integrable on $[0,1]$
and continuous in $x=1$. Prove that:

$$\lim_{n\to\infty}n\int_0^1 x^nf(x)\,dx=f(1)$$

Solution. How f integrable function, then f bounded function. So, $\exists M>0$ such that

$|f(x)|\le M,\forall x\in[0,1]$. We have:

$$\left|n\int_0^{x_n} x^n f(x)\,dx\right| \le n\int_0^{x_n} x^n|f(x)|\,dx \le nM\cdot\frac{x_n^{n+1}}{n+1}$$

$$\text{But } x_n \in (0,1) \Rightarrow \lim_{n\to\infty} x_n^n = 0, \text{so } \lim_{n\to\infty}\left(nM\cdot\frac{x_n^{n+1}}{n+1}\right) = 0$$

$$\text{Hence, } \lim_{n\to\infty} n\int_0^{x_n} x^n f(x)\,dx = 0$$

Function f continuous at point $x = 1$, then we have:

$$n\int_0^1 x^n f(x)\,dx - f(1) = n\int_0^1 x^{n-1}[f(x) - f(1)]\,dx =$$

$$= n\int_0^{x_n} x^{n-1}[xf(x) - f(1)]\,dx + n\int_{x_n}^1 x^{n-1}[xf(x) - f(1)]\,dx; (1)$$

Now, f –continous at point $x = 1$ then $\exists x_n \in (0,1)$ such that

$$|xf(x) - f(1)| < \varepsilon, \forall x \in [x_n, 1], \varepsilon > 0\ (fixed); (2)$$

Applying up these strategy for function $x \to xf(x) - f(1), x \in [0,1], \exists N_\varepsilon \ge 1$ such that

$$\left|n\int_0^{x_n} x^{n-1}[xf(x) - f(1)]\,dx\right| < \varepsilon, \forall n \ge N_\varepsilon; (3)$$

Hence, we obtain:

$$\left|n\int_{x_n}^1 x^{n-1}[xf(x) - f(1)]\,dx\right| \le n\int_{x_n}^1 \varepsilon x^{n-1}\,dx = \varepsilon(1 - a^n) < \varepsilon; (4)$$

From (1),(3),(4) we get:

$$\left|n\int_0^1 x^n f(x)\,dx - f(1)\right| \le 2\varepsilon \Rightarrow \lim_{n\to\infty} n\int_0^1 x^n f(x)\,dx = f(1)$$

REFERENCES:

[1] ROMANIAN MATHEMATICAL MAGAZINE CHALLENGES 1-500-D. Sitaru, M. Ursărescu-Studis 2021

[2] QUANTUM MATHEMATICAL POWER- M. Bencze, D. Sitaru-Studis, 2018

[3] OLYMPIC MATHEMATICAL ENERGY- M. Bencze, D. Sitaru-Studis, 2018

[4] 699 OLYMPIC MATHEMATICAL CHALLENGES- M. Bencze, D. Sitaru-Studis, 2017

[5] ANALYTIC PHENOMENON- D. Sitaru -Studis,2018

[6] OLYMPIC MATHEMATICAL POWER-M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[7] ANALIZĂ MATEMATICA –D.M.Bătineţu-Giurgiu & Co. –MatrixRom,2003

[8] OLIMPIC MATHEMATICAL BEAUTIES- M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[9] ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

[10] OCTOGON MATHEMATICAL MAGAZINE-www.uni-miskolk.hu/~mastefi/Octogon/

SPECIAL TECHNIQUES FOR PRIMITIVES

By Florică Anastase-Romania

Abstract: *I this paper are presented some special techniques for determining the primitives of a function.*

Theorem 1. Let I −interval, $f: I \to \mathbb{R}, x_0 \in I$. Then function $F: I \to \mathbb{R}$,

$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

is differentiable on I and $F'(x) = f(x), \forall x \in I$.

More, we can generalize that result.

Theorem 2. Let $f: I \to \mathbb{R}$ continuous function, $\varphi: J \to I$ differentiable function and $x_0 \in I$. Then function $G: J \to \mathbb{R}$,

$$G(x) = \int_{x_0}^{\varphi(x)} f(t)\,dt$$

is differentiable on J and $G'(x) = f(\varphi(x)) \cdot \varphi'(x), \forall x \in J$.

Proof.

If $G(x) = F(\varphi(x))$ and using theorem of differentiable compound function, we have:

$$G'(x) = F'(\varphi(x))\varphi'(x).$$

On the other hand, we observe that if

$$G(x) = \int_{\psi(x)}^{\varphi(x)} f(t)\,dt$$

where φ and ψ are differentiable functions, then

$$G'(x) = f(\varphi(x))\varphi'(x) - f(\psi(x))\psi'(x), \forall x \in J.$$

Let $x_0 \in I$, then

$$G(x)=\int_{\psi(x)}^{x_0} f(t)\,dt+\int_{x_0}^{\varphi(x)} f(t)\,dt=F(\varphi(x))-F(\psi(x))$$

Application 1: Prove that function $f:\mathbb{R}\to\mathbb{R}, f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ admits primitives and find these primitives.

Solution 1. Let be the function $h:\mathbb{R}\to\mathbb{R}, h(x)=\frac{2x}{1+x^2}$ and $g:[-1,1]\to\mathbb{R}, g(y=\sin^{-1} y$ continuous on $[-1,1]$ and observe that $\left|\frac{2x}{1+x^2}\right|\le 1, \forall x\in\mathbb{R}$.

How $f=g\circ h$ then f —is continuous on $\mathbb{R}$. We use I.B.P. method:

$$\int f(x)\,dx=\int x'\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx=x\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)-\int x\left(\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right)'dx$$

The problem is when the function $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is not differentiable, really

$$f'(x)=\frac{2(1-x^2)}{(1+x^2)|1-x^2|}, \forall x\in\mathbb{R}-\{-1,1\}$$

$$\lim_{\substack{x\to-1\\x<-1}} f'(x)=-1;\ \lim_{\substack{x\to-1\\x>-1}} f'(x)=1$$

$$\lim_{\substack{x\to1\\x<1}} f'(x)=1;\ \lim_{\substack{x\to1\\x>1}} f'(x)=-1$$

Then $f_l'(-1)=-1; f_r'(-1)=1; f_l'(1)=1; f_r'(1)=-1$ which means that f is not differentiable on $\{-1,1\}$. So, we cannot to apply I.B.P. method on $\mathbb{R}$ but we can apply on $(-\infty,-1),(-1,1),(1,\infty)$.

Case 1) For $x\in(-\infty,-1)$ we have:

$$\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx\overset{IBP}{=}x\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\int x\cdot\frac{2}{1+x^2}dx=$$

$$=x\cdot\sin^{-1}\frac{2x}{1+x^2}+\log(1+x^2)+C_1$$

Case 2) For $x\in(-1,1)$ we have:

$$\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx\overset{IBP}{=}x\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)-\int x\cdot\frac{2}{1+x^2}dx=$$

$$=x\cdot\sin^{-1}\frac{2x}{1+x^2}-\log(1+x^2)+C_2$$

Case 3) For $x\in(1,\infty)$ we have:

$$\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx\overset{IBP}{=}x\cdot\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\int x\cdot\frac{2}{1+x^2}dx=$$

$$= x \cdot \sin^{-1} \frac{2x}{1+x^2} + \log(1+x^2) + C_3$$

If denote with $F: \mathbb{R} \to \mathbb{R}$ a primitive of f then we have:

$$F(x) = \begin{cases} x \cdot \sin^{-1} \frac{2x}{1+x^2} + \log(1+x^2) + C_1; x \in (-\infty, -1) \\ x \cdot \sin^{-1} \frac{2x}{1+x^2} - \log(1+x^2) + C_2; x \in (-1,1) \\ x \cdot \sin^{-1} \frac{2x}{1+x^2} + \log(1+x^2) + C_3; x \in (1, \infty) \end{cases}$$

Because F is continuous on $\{-1,1\}$ because is differentiable we have:

$$\lim_{\substack{x \to -1 \\ x < -1}} F(x) = \lim_{\substack{x \to -1 \\ x > -1}} F(x) \text{ then } C_2 = C_1 + 2\log 2 \text{ and}$$

$$\lim_{\substack{x \to -1 \\ x < -1}} F(x) = \lim_{\substack{x \to 1 \\ x > 1}} F(x) \text{ then } C_1 = C_3$$

Therefore,

$$F(x) = \begin{cases} x \cdot \sin^{-1} \frac{2x}{1+x^2} + \log(1+x^2) + C; x \in (-\infty, -1) \\ x \cdot \sin^{-1} \frac{2x}{1+x^2} - \log(1+x^2) + C + 2\log 2\,; x \in (-1,1) \\ x \cdot \sin^{-1} \frac{2x}{1+x^2} + \log(1+x^2) + C; x \in (1, \infty) \end{cases}$$

Solution 2. Using **Theorem 2**, we fix it the point $x_0 = 0$ the we get that $F(x) = \int_0^x f(t)\,dt$ is primitive for f.

If $x \in [-1,1]$, using I.B.P. we find that:

$$\int_0^x f(t)\,dt = x \cdot \sin^{-1}\left(\frac{2x}{1+x^2}\right) - \log(1+x^2)$$

If $x \in (-\infty, -1)$, using I.B.P. we find that:

$$\int_0^x f(t)\,dt = \int_0^{-1} f(t)\,dt + \int_{-1}^x f(t)\,dt = x \cdot \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \log(1+x^2) - 2\log 2$$

If $x \in (1, \infty)$, using I.B.P. we find that:

$$\int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = x \cdot \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \log(1+x^2) + 2\log 2.$$

Application 2: Let be the function $f: [0.1] \to \mathbb{R}$,

$$\boldsymbol{f(x)=\begin{cases}0, x=0\\ 0, x\in[0,1]-\mathbb{Q}\\ \frac{1}{q}, x=\frac{p}{q}\,(p,q\in\mathbb{N}^*;(p,q)=1)\end{cases}; F(x)=\int_0^x f(t)\,dt, \forall x\in[0,1]}$$

Prove that F is differentiable on $[0,1]$ but $F'(x)\neq f(x)$ in any rational point.

Solution: Let $x_0=\frac{p}{q}; p,q\in\mathbb{N}^*,(p,q)=1$ and $(x_n)_{n\geq1}$ sequence of irrational numbers such that $x_n\subset[0,1], x_n\to x$, then $f(x_n)\to0\neq f(x_0)$. So, f is not continuous on x_0.

Now, let $x_0\in\mathbb{R}-\mathbb{Q}, x_n\subset[0,1], x_n\to x$. We want to prove that for any $q\in\mathbb{N}^*$ exists $p\in\mathbb{N}^*$ such that $p<x_0q<q+1$, and then $\frac{p}{q}<x_0<\frac{p+1}{q}$ which means the interval (a_q,b_q) not contain any number with denominator q. For any $k\in\mathbb{N}^*$ denote

$a_k=\max\{a_q\}$ and $b_k=\min\{b_q\}$ for $q\in[1,k]$. Because $x_n\to x_0$ and $x_0\in(a_k,b_k)$ we get that for all $k\in\mathbb{N}^*,\exists n_k\in\mathbb{N}$ such that $x_n\in(a_k,b_k),\forall n\geq n_k$.

If $x_n\in\mathbb{Q}, n\geq n_k$ then x_n is an irreducible fraction with denominator greater than k and we have $f(x_n)<\frac{1}{k}$. If $x_n\in\mathbb{R}-\mathbb{Q}, n\geq n_k$ then $f(x_n)=0$ hence, $\forall k\in\mathbb{N}^*,\exists n_k\in\mathbb{N}$ such that

$$0\leq f(x_n)\leq\frac{1}{k},\forall n\geq n_k$$

So, $f(x_n)\to0$ which , means $\forall\varepsilon>0,\exists k(\varepsilon)\in\mathbb{N}^*$ such that $\frac{1}{k(\varepsilon)}<\varepsilon$. Thus, f is continuous in any irrational point. Now, using Darboux Criterion for integrable function:

$$I=[a,b],\Delta: a=x_0<x_1<x_2<\dots<x_n=b; S_\Delta(f)-s_\Delta(f)<\varepsilon,\forall\varepsilon>0$$

$$S_\Delta=\sum_{i=1}^n M_i(x_i-x_{i-1}), s_\Delta(f)\sum_{i=1}^n m_i(x_i-x_{i-1}), m_i=\inf_{x\in[x_{i-1},x_i]}f(x), M_i=\sup_{x\in[x_{i-1},x_i]}f(x)$$

$$\text{Thus, }\int_a^b f(x)\,dx=\inf_\Delta S_\Delta(f)=\sup_\Delta s_\Delta(f)$$

Because $f(t)\geq0,\forall t\in[0,1]$ we have:

$$0\leq\int_0^x f(t)\,dt\leq\int_0^1 f(t)\,dt=0,\forall x\in[0,1]$$

So, $F(x)=0,\forall x\in[0,1]$ and so $F'(x)=0,\forall x\in[0,1]$ hence, $F'(x)\neq f(x), x\in\mathbb{Q}^*$.

Application 3: For $y>0$ let

$$\boldsymbol{F(y)=\int_0^{\frac{\pi}{2}}\frac{dx}{y+\cos x}dx}$$

Prove that f is continuous at point $y=1$.

Solution:Let $\tan\frac{x}{2}=t \Leftrightarrow x=2\tan^{-1}t\,, dx=\frac{1}{1+t^2}$ and thus, $F(y)=2\int_0^1\frac{dt}{(y-1)t^2+1+y}$

For $y>1$ we get:

$$F(y)=\frac{2}{y-1}\int_0^1\frac{dt}{t^2+\left(\sqrt{\frac{1+y}{y-1}}\right)^2}=\frac{2}{y-1}\cdot\sqrt{\frac{y-1}{y+1}}\cdot\tan^{-1}\sqrt{\frac{y-1}{y+1}}\Bigg|_0^1=$$

$$=\frac{2}{\sqrt{y^2-1}}\tan^{-1}\sqrt{\frac{y-1}{y+1}}$$

For $y<1$ we get: $F(y)=\frac{1}{y-1}\sqrt{\frac{1-y}{1+y}}\int_0^1\left(\frac{1}{t-\sqrt{\frac{1+y}{1-y}}}-\frac{1}{t+\sqrt{\frac{1+y}{1-y}}}\right)dt=$

$$=\frac{1}{\sqrt{1-y^2}}\log\left|\frac{t+\sqrt{\frac{1+y}{1-y}}}{t-\sqrt{\frac{1+y}{1-y}}}\right|\Bigg|_0^1=\frac{1}{\sqrt{1-y^2}}\log\frac{\left(\sqrt{1-y}+\sqrt{1+y}\right)^2}{2|y|}$$

$$\lim_{\substack{y\to 1\\ y<1}}F(y)=\frac{1}{\sqrt{2}}\cdot\lim_{\substack{y\to 1\\ y<1}}\frac{2\log\left(\sqrt{1-y}+\sqrt{1+y}\right)-\log 2-\log y}{\sqrt{1-y}}=$$

$$=-\sqrt{2}\cdot\lim_{\substack{y\to 1\\ y<1}}\left(\frac{1}{\sqrt{1+y}}\cdot\frac{\sqrt{1-y}-\sqrt{1+y}}{\sqrt{1-y}+\sqrt{1+y}}-\frac{\sqrt{1-y}}{y}\right)=1$$

$$\lim_{\substack{y\to 1\\ y>1}}F(y)=\frac{1}{\sqrt{2}}\cdot\lim_{\substack{y\to 1\\ y>1}}\frac{2}{\sqrt{y^2-1}}\tan^{-1}\sqrt{\frac{y-1}{y+1}}=\sqrt{2}\cdot\lim_{\substack{y\to 1\\ y>1}}\frac{1}{y\sqrt{y+1}}=1$$

Because $F(1)=1$ it follows that F is continuous at point $y=1$.

Application 4: If $f:\mathbb{R}\to\mathbb{R}$ is continuous function and

$$\boldsymbol{F(x)=\int_0^x f(x)\,dx}$$

Prove that F is ω −periodic if and only if f is ω −periodic and $\int_0^{\omega}f(t)\,dt=0$.

Solution:$G(x)\overset{def.}{=}F(x+\omega)-F(x)=\int_0^{x+\omega}f(t)\,dt-\int_0^x f(t)\,dt$

$$G'(x)=f(x+\omega)-f(x)$$

Suppose that, f is ω −periodic and $\int_0^{\omega}f(t)\,dt=0$ we get $G'(x)=0$. So, G −constant function. Then:

$$G(x)=G(0)=\int_0^{\omega} f(t)\,dt=0\Rightarrow F(x+\omega)-F(x)=0\Leftrightarrow F(x+\omega)=F(x).$$

If $F(x+\omega)=F(x),\forall x\in\mathbb{R}\Rightarrow F'(x+\omega)=F'(x)\Leftrightarrow f(x+\omega)=f(x)$.

Application 5: Let $f_n: J\to\mathbb{R}, f_n(x)=\frac{1}{\sin^n x}, n\in\mathbb{N}^*$ and F_n −primitive of function f_n.

Prove that:

$$F_n(x)=\frac{n-2}{n-1}\cdot F_{n-2}(x)-\frac{1}{n-1}\cdot\frac{\cos x}{\sin^{n-1}x}$$

Solution. It is suffices to prove that $F_n'(x)=\frac{1}{\sin^n x}$. We have:

$$F_n'(x)=\frac{n-2}{n-4}\cdot\frac{1}{\sin^{n-2}x}+\frac{\sin^n x+(n-1)\cdot\sin^{n-2}x-(n-1)\cdot\sin^n x}{(n-1)\cdot\sin^{2n-2}x}=$$

$$=\frac{(n-1)\cdot\sin^{n-2}x}{(n-1)\cdot\sin^{2n-2}x}=\frac{1}{\sin^n x}.$$

Application 6: Let $f:\mathbb{R}\to\mathbb{R}_+^*$ differentiable function and

$$F(a)=\int_0^a\frac{f(x)\cdot f'(a-x)}{\left(f(x)+f(a-x)\right)^2}dx$$

Prove that F is differentiable and find $F'(0)$.

Solution: $F(a)=\int_0^a\frac{f(x)\cdot f'(a-x)}{\left(f(x)+f(a-x)\right)^2}dx\overset{a-x=y}{=}-\int_0^a\frac{f(a-y)\cdot f'(y)}{\left(f(a-y)+f(y)\right)^2}dx$

$$2F(a)=\int_0^a\frac{f(x)\cdot f'(a-x)+f'(x)\cdot f(a-x)}{\left(f(a-x)+f(x)\right)^2}dx=$$

$$=\int_0^a\left(\frac{f(x)}{f(x)+f(a-x)}\right)'dx=\frac{f(x)}{f(x)+f(a-x)}\Bigg|_0^a=\frac{f(a)-f(0)}{f(a)+f(0)}$$

So,

$$F(a)=\frac{1}{2}\cdot\frac{f(a)-f(0)}{f(a)+f(0)},F'(a)=\frac{f'(a)\cdot f(0)}{\left(f(a)+f(0)\right)^2},F'(0)=\frac{f'(0)}{4f(0)}$$

Application 7: Let $f:\left[\frac{1}{a},a\right]\to\mathbb{R}$ continuous function and $a>1$. Find:

$$F(a)=\int_{\frac{1}{a}}^{a}f(x^p+x^{-p})\cdot\frac{\log x}{x}dx\text{, where } p\in\mathbb{R}.$$

Solution: $F(a)=\int_{\frac{1}{a}}^{a}f(x^p+x^{-p})\cdot\frac{\log x}{x}dx=$

$$=\int_{\frac{1}{a}}^{1} f(x^p+x^{-p})\cdot\frac{\log x}{x}dx+\int_1^a f(x^p+x^{-p})\cdot\frac{\log x}{x}dx=A+B=0,\text{where}$$

$$B=\int_1^a f(x^p+x^{-p})\cdot\frac{\log x}{x}dx\overset{x=\frac{1}{y}}{=}\int_1^{\frac{1}{a}} f(y^{-p}+y^p)\cdot\frac{\log y}{y}dy$$
$$=-\int_{\frac{1}{a}}^{1} f(x^p+x^{-p})\cdot\frac{\log x}{x}dx=-A$$

Application 10: Let $f:[0,\pi]\to\mathbb{R}$ integrable function such that

$$\boldsymbol{f(x)=\begin{cases}\frac{\sin 2x}{1+\cos x};x\in\left[0,\frac{\pi}{2}\right]\cap\mathbb{Q}\\ -\frac{2+\log x}{\pi};x\in\left[\frac{\pi}{2},\pi\right]\end{cases}}$$ **. Find: $F(x)=\int_0^\pi f(x)\,dx$.**

Solution. Because f –integrable function, then by Riemann, from $\overline{\mathbb{Q}}\cap\left[\frac{\pi}{2},\pi\right]=\left[\frac{\pi}{2},\pi\right]$ and $\overline{\left[0,\frac{\pi}{2}\right]-\mathbb{Q}}=[0,\frac{\pi}{2}]$ it follows that: $\int_0^\pi f(x)\,dx=\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+\cos x}dx-\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(2+\log x)dx$

$$I_1=\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+\cos x}dx\overset{\cos x=t}{=}-2\int_1^0\frac{t}{1+t}dt=2(1-\log(1+t)|_0^1)=2\cdot\log\left(\frac{e}{2}\right)$$

$$I_2=\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(2+\log x)dx\,1+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\log x\,dx\overset{IBP}{=}1+\frac{1}{\pi}(x\cdot\log x-x)\Big|_{\frac{\pi}{2}}^{\pi}=$$

$$=1+\left(\frac{1}{2}\cdot\log\pi+\frac{1}{2}\log 2-\frac{1}{2}\right)=\frac{1}{2}(1+\log(2\pi))$$

Therefore: $F(x)=\int_0^\pi f(x)\,dx=2\log\left(\frac{e}{2}\right)-\frac{1}{2}(1+\log(2\pi))=\log\sqrt{\frac{e^3}{2^5\cdot\pi}}$

Application 11: Find:

$$\boldsymbol{I=\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2x)+\cos^2(\cos 2x))\cdot\log\left(1+\tan\frac{x}{2}\right)dx}$$

Marian Ursărescu

Soution: $I=\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2x)+\cos^2(\cos 2x))\cdot\log\left(1+\tan\frac{x}{2}\right)dx\overset{x=\frac{\pi}{2}-t}{=}$

$$=\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\cdot\log\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt=$$

$$=\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\log\left(1+\frac{1-\tan t}{1+\tan t}\right)dt$$

Hence: $I=\frac{\log 2}{2}\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt$

$$J=\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt=$$

$$=\int_0^{\frac{\pi}{4}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt=J_1+J_2$$

$$x=\frac{\pi}{2}-t\Rightarrow dx=-dt\Rightarrow J=\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt=$$

$$=2\int_0^{\frac{\pi}{4}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt=2J_1$$

$$x=\frac{\pi}{4}-t\Rightarrow dx=-dt\Rightarrow J=\int_0^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt=$$

$$=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin^2(\sin 2t)+\cos^2(\cos 2t))\,dt=2J_2$$

Therefore: $J=J_1+J_2=\int_0^{\frac{\pi}{4}}2\,dt=\frac{\pi}{2}\Rightarrow J=\pi\Rightarrow I=\frac{\pi}{4}\log 2.$

Application 12: Find:

$$\boldsymbol{\Omega=\int\frac{x(\tan x+2\tan 2x+4\tan 4x)}{\cot x-8\cot 8x}dx}$$

Daniel Sitaru

Solution. We prove that: $\tan x+2\tan 2x+4\tan 4x=\cot x-8\cot 8x$

$$\cot x-\tan x-2\tan 2x-4\tan 4x=\frac{1}{\tan x}-\tan x-2\tan 2x-4\tan 4x=$$

$$=\frac{1-\tan^2 x}{\tan x}-2\tan 2x-4\tan 4x=2\left(\frac{1-\tan^2 x}{2\tan x}-\tan 2x\right)-4\tan 4x=$$

$$=2(\cot 2x-\tan 2x)-4\tan 4x=2\left(\frac{1}{\tan 2x}-\tan 2x\right)-4\tan 4x=$$

$$=2\left(\frac{1-\tan^2 2x}{\tan 2x}\right)-4\tan 4x=4\left(\frac{1-\tan^2 2x}{2\tan 2x}-\tan 4x\right)=$$

$$=4(\cot 4x-\tan 4x)=4\left(\frac{1}{\tan 4x}-\tan 4x\right)=4\cdot\frac{1-\tan^2 4x}{2\tan 4x}=8\tan 8x$$

Therefore,

$$\Omega = \int \frac{x(\tan x + 2\tan 2x + 4\tan 4x)}{\cot x - 8\cot 8x} dx = \frac{x}{2} + C$$

Application 13: **For $0 < x < \frac{\pi}{2(a+b)}$ find:**

$$\boldsymbol{\Omega(a,b) = \int (\tan(ax)\tan(bx)\tan((a+b)x)\,dx, a, b} > 0,$$

Daniel Sitaru

Solution: $\tan(ax+bx) = \frac{\tan(ax)+\tan(bx)}{1-\tan(ax)\tan(bx)}$

$$\tan(a+b)x - \tan(ax)\tan(bx)\tan(a+b)x = \tan(ax) + \tan(bx)$$

$$\tan(ax)\tan(bx)\tan(a+b)x = \tan(a+b)x - \tan(ax) - \tan(bx)$$

$$\Omega(a,b) = \int (\tan(a+b)x - \tan(ax) - \tan(bx))\,dx =$$

$$= \int \tan(a+b)x\,dx - \int \tan(ax)\,dx - \int \tan(bx)\,dx =$$

$$= \frac{1}{a+b}\log|\sec(a+b)x| + \frac{1}{a}\log|\cos(ax)| + \frac{1}{b}\log|\cos(bx)| + C$$

Application 14: If $a > 0$ find:

$$\boldsymbol{\Omega = \int \frac{x^n dx}{a^x + \sum_{k=0}^{n} \frac{(x\log a)^k}{k!}}}$$

Mihàly Bencze

Solution: $f(x) = a^x + \sum_{k=0}^{n} \frac{(x\log a)^k}{k!} \Rightarrow f'(x) = a^x\log a + \sum_{k=0}^{n} \frac{x^{k-1}\log^k a}{(k-1)!}$

$$f(x)\log a - f'(x) = \frac{x^n \log^{n+1} a}{n!}$$

Hence,

$$\Omega = \int \frac{x^n dx}{a^x + \sum_{k=0}^{n} \frac{(x\log a)^k}{k!}} = \frac{n!}{\log^{n+1} a} \int \frac{\frac{x^n \log^{n+1} a}{n!}}{f(x)} dx =$$

$$= \frac{n!}{\log^{n+1} a} \int \frac{f(x)\log a - f'(x)}{f(x)} dx = x \cdot \frac{n!}{\log^n a} - \frac{n!}{\log^{n+1} a}\log|f(x)| + C, \text{for } a \neq 1$$

$$\text{For } a = 1 \Rightarrow \Omega = \int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Application 15: Let be $f:\mathbb{R}\to\mathbb{R}$ differentiable, with the continuous derivative. If the graph of f' admits $x=a$, the symmetry axis, then:

$$\int_0^{2a} f(x)\,dx = a\big(f(2a)+f(0)\big)$$

Marian Ursărescu

Solution: f' admits $x=a$ symmetry axis, then $f'(a-x)=f'(a+x), \forall x\in\mathbb{R}$

$$f'(x)=f'(2a-x)$$

$$\int_0^{2a} f(x)\,dx=\int_0^{2a} x'\cdot f(x)\,dx = xf(x)\big|_0^{2a}-\int_0^{2a} x\cdot f'(x)\,dx=$$

$$=2af(a)-\int_0^{2a} x\cdot f'(x)\,dx;\ (1)$$

$$I=\int_0^{2a} x\cdot f'(x)\,dx \overset{x=2a-t}{=} \int_0^{2a}(2a-t)f'(2a-t)\,dt=\int_0^{2a}(2a-t)f'(t)\,dt=$$

$$=2a\int_0^{2a} f'(t)\,dt-\int_0^{2a} tf'(t)\,dt=2af(t)\big|_0^{2a}-I=(2af(a)-2af(0))I$$

Hence, $I=af(2a)-af(0);\ (2)$

From (1),(2) it follows that: $\int_0^{2a} f(x)\,dx = a\big(f(2a)+f(0)\big)$

REFERENCES:

[1] ROMANIAN MATHEMATICAL MAGAZINE CHALLENGES 1-500-D. Sitaru, M. Ursărescu-Studis 2021

[2] QUANTUM MATHEMATICAL POWER- M. Bencze, D. Sitaru-Studis, 2018

[3] OLYMPIC MATHEMATICAL ENERGY- M. Bencze, D. Sitaru-Studis, 2018

[4] 699 OLYMPIC MATHEMATICAL CHALLENGES- M. Bencze, D. Sitaru-Studis, 2017

[5] ANALYTIC PHENOMENON- D. Sitaru -Studis,2018

[6] OLYMPIC MATHEMATICAL POWER-M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[7] ANALIZĂ MATEMATICA –D.M.Bătineţu-Giurgiu & Co. –MatrixRom,2003

[8] OLIMPIC MATHEMATICAL BEAUTIES- M. Bencze, D. Sitaru, M. Ursărescu-Studis, 2018

[9] ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

[10] OCTOGON MATHEMATICAL MAGAZINE-www.uni-miskolk.hu/~mastefi/Octogon/

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-XIV

By Marin Chirciu-Romania

1) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\geq\frac{6R}{r}$$

Proposed by Marian Ursărescu-Romania

Solution We prove: **Lemma.**

2) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\sum\left(\frac{BC}{B_1C_1}\right)^2=\frac{2R(2R-r)}{r^2}$$

Proof: Using $B_1C_1=2r\cos\frac{A}{2}$, (which follows from: In cyclic quadrilateral AB_1IC_1, the angles A and I are sum $180°$, M – the middle of B_1C_1, and $\sin(B_1IM)=\frac{MB_1}{r}=\frac{\frac{B_1C_1}{2}}{r}$ and $\sin(B_1IM)=\sin\left(\frac{1}{2}B_1IC_1\right)=\sin\left(\frac{\pi-A}{2}\right)=\cos\frac{A}{2}$) we obtain:

$$\sum\left(\frac{BC}{B_1C_1}\right)^2=\sum\left(\frac{a}{2r\cos\frac{A}{2}}\right)^2=\frac{1}{4r^2}\sum\frac{a^2}{\cos^2\frac{A}{2}}=\frac{1}{4r^2}\cdot 8R(2R-r)=\frac{2R(2R-r)}{r^2}$$

which follows from $\sum\frac{a^2}{\cos^2\frac{A}{2}}=8R(2R-r)$.

Let's get back to the main problem. Using the Lemma we obtain:

$$LHS=\sum\left(\frac{BC}{B_1C_1}\right)^2=\frac{2R(2R-r)}{r^2}\overset{Euler}{\geq}\frac{6R}{r}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: The inequality can be strengthened.

3) In $\Delta ABC, A_1B_1C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\geq 4\left(\frac{2R}{r}-1\right)$$

Marin Chirciu

Solution: Using the lemma we obtain:

$$LHS=\sum\left(\frac{BC}{B_1C_1}\right)^2=\frac{2R(2R-r)}{r^2}\overset{Euler}{\geq}\frac{4(2R-r)}{r}=4\left(\frac{2R}{r}-1\right)=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: Inequality 3) is stronger than inequality 1).

4) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\geq 4\left(\frac{2R}{r}-1\right)\geq\frac{6R}{r}$$

Solution: See inequality 3) and Euler's inequality $R\geq 2r$. Equality holds if and only if the triangle is equilateral.**Remark:** Also, inequality 3) can be strengthened.

5) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\geq 3\left(\frac{R}{r}\right)^2$$

Marin Chirciu

Solution: Using the Lemma we obtain:

$$LHS=\sum\left(\frac{BC}{B_1C_1}\right)^2=\frac{2R(2R-r)}{r^2}\overset{Euler}{\geq}\frac{2R\left(2R-\frac{R}{2}\right)}{r^2}=3\frac{R^2}{r^2}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: Inequality 5) is stronger then inequality 3)

6) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\geq 3\left(\frac{R}{r}\right)^2\geq 4\left(\frac{2R}{r}-1\right)$$

Solution: See inequality 3) and Euler's inequality $R\geq 2r$.Equality holds if and only if the triangle is equilateral.**Remark:** We can write the inequalities:

7) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\geq 3\left(\frac{R}{r}\right)^2\geq 4\left(\frac{2R}{r}-1\right)\geq\frac{6R}{r}$$

Solution: See above.Equality holds if and only if the triangle is equilateral.

Remark: Let's find an inequality with opposite sense.

8) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\le\frac{2R}{r}\left(\frac{2R}{r}-1\right)$$

Marin Chirciu

Solution: Using the Lemma we obtain:

$LHS=\sum\left(\frac{BC}{B_1C_1}\right)^2=\frac{2R(2R-r)}{r^2}\le\frac{2R}{r}\left(\frac{2R}{r}-1\right)=RHS$, obviously with equality.

Remark: We can write the inequalities:

9) In $\Delta ABC, A_1, B_1, C_1$ are contact points with incircle. Prove that:

$$\frac{6R}{r}\le 4\left(\frac{2R}{r}-1\right)\le 3\left(\frac{R}{r}\right)^2\le\left(\frac{AB}{A_1B_1}\right)^2+\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2\le\frac{2R}{r}\left(\frac{2R}{r}-1\right)$$

Solution: See above.Equality holds if and only if the triangle is equilateral.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

VECTORIAL GEOMETRY-(II)

COLLINEAR POINTS

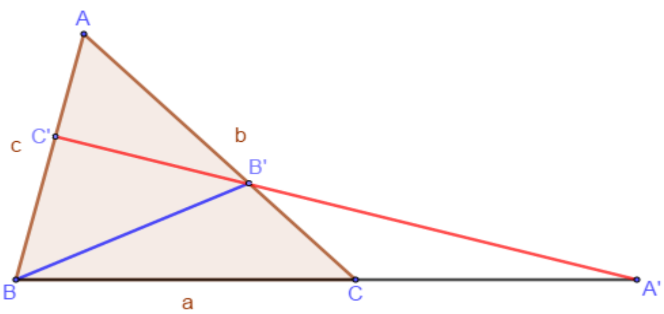
By Florică Anastase-Romania

Abstract: *In this paper I was to present some applications about collinear points using vectorial geometry. This paper is dedicated to students who participate to Olympics and math competitions as well as young people passionate about geometry.*

Theorem (Menelaus): In $\Delta ABC, A' \in BC, B' \in CA, C' \in AB$.If A', B', C' are collinear then,

$$\frac{\overline{A'B}}{\overline{A'C}}\cdot\frac{\overline{B'C}}{\overline{B'A}}\cdot\frac{\overline{C'A}}{\overline{C'B}}=1$$

Proof.



Let us denote: $\frac{\overline{A'B}}{\overline{A'C}}=m,\ \frac{\overline{B'C}}{\overline{B'A}}=n,\ \frac{\overline{C'A}}{\overline{C'B}}=p$ then, $\overrightarrow{A'B}=m\overrightarrow{A'C},\ \overrightarrow{B'C}=n\overrightarrow{B'A},\overrightarrow{C'A}=p\overrightarrow{C'B}$

Now, the points A',B',C' are collinear if and only if exists $x,y\in\mathbb{R}$, with $x+y=1$ such that

$$\overrightarrow{BB'}=x\overrightarrow{BA'}+y\overrightarrow{BC'};(1)$$

Other, $\overrightarrow{B'C}=n\overrightarrow{B'A}$ then, $\overrightarrow{BB'}=\frac{1}{1-n}\overrightarrow{BC}-\frac{n}{1-n}\overrightarrow{BA};(2)$.

$$\overrightarrow{BC}=\overrightarrow{BA'}+\overrightarrow{A'C}=\overrightarrow{BA'}+\frac{1}{m}\overrightarrow{A'B}=\left(1-\frac{1}{m}\right)\overrightarrow{BA'}$$

$$\overrightarrow{BA}=\overrightarrow{BC'}+\overrightarrow{C'A}=\overrightarrow{BC'}+p\overrightarrow{C'B}=(1-p)\overrightarrow{BC'}$$

Hence, relation (2) becomes as:

$$\overrightarrow{BB'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{n(1-p)}{1-n}\overrightarrow{BC'};(3)$$

From (1),(3) it follows that:

$$x\overrightarrow{BA'}+y\overrightarrow{BC'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{n(1-p)}{1-n}\overrightarrow{BC'}$$

How, vectors $\overrightarrow{BA'}$ and $\overrightarrow{BC'}$ are not collinear, we get: $x=\frac{m-1}{m(1-n)},\ y=-\frac{n(1-p)}{1-n}$ and because $x+y=1$ it follows that $mnp=1$. Therefore,

$$\frac{\overline{A'B}}{\overline{A'C}}\cdot\frac{\overline{B'C}}{\overline{B'A}}\cdot\frac{\overline{C'A}}{\overline{C'B}}=1$$

Theorem (Reciprocal Menelaus)

In $\Delta ABC, A'\in BC, B'\in CA, C'\in AB$. If $\frac{\overline{A'B}}{\overline{A'C}}\cdot\frac{\overline{B'C}}{\overline{B'A}}\cdot\frac{\overline{C'A}}{\overline{C'B}}=1$ then A',B',C' are collinear.

Proof.

Let us denote: $\frac{\overline{A'B}}{\overline{A'C}}=m,\ \frac{\overline{B'C}}{\overline{B'A}}=n,\ \frac{\overline{C'A}}{\overline{C'B}}=p$ then, $\overrightarrow{A'B}=m\overrightarrow{A'C},\ \overrightarrow{B'C}=n\overrightarrow{B'A},\overrightarrow{C'A}=p\overrightarrow{C'B}$

How $\overrightarrow{B'C}=n\overrightarrow{B'A}$ then, $\overrightarrow{BB'}=\frac{1}{1-n}\overrightarrow{BC}-\frac{n}{1-n}\overrightarrow{BA};(1)$.

$$\overrightarrow{BC}=\overrightarrow{BA'}+\overrightarrow{A'C}=\overrightarrow{BA'}+\frac{1}{m}\overrightarrow{A'B}=\left(1-\frac{1}{m}\right)\overrightarrow{BA'}$$

$$\overrightarrow{BA}=\overrightarrow{BC'}++\overrightarrow{C'A}=\overrightarrow{BC'}+p\overrightarrow{C'B}=(1-p)\overrightarrow{BC'}$$

So, (1) becomes as: $\overrightarrow{BB'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{n(1-p)}{1-n}\overrightarrow{BC'};(2)$ and how $mnp=1$, we get $p=\frac{1}{mn}$ and $\overrightarrow{BB'}=\frac{m-1}{m(1-n)}\overrightarrow{BA'}-\frac{mn-1}{m(1-n)}\overrightarrow{BC'};(3)$.

If $x=\frac{m-1}{m(1-n)},\ y=-\frac{mn-1}{m(1-n)}$ then $x+y=\frac{m-1}{m(1-n)}-\frac{mn-1}{m(1-n)}=\frac{m(1-n)}{m(1-n)}=1$.

So, exists $x, y \in \mathbb{R}$, with $x + y = 1$ such that $\overrightarrow{BB'} = x\overrightarrow{BA'} + y\overrightarrow{BC'}$ and then the points A', B', C' are collinear.

Application 1: ***In $ABCD$ parallelogram, poits E, F are such that $2\overrightarrow{BE} = \overrightarrow{AB}$ and $\overrightarrow{AF} = 3\overrightarrow{AD}$. Prove that E, F and C are collinear.***

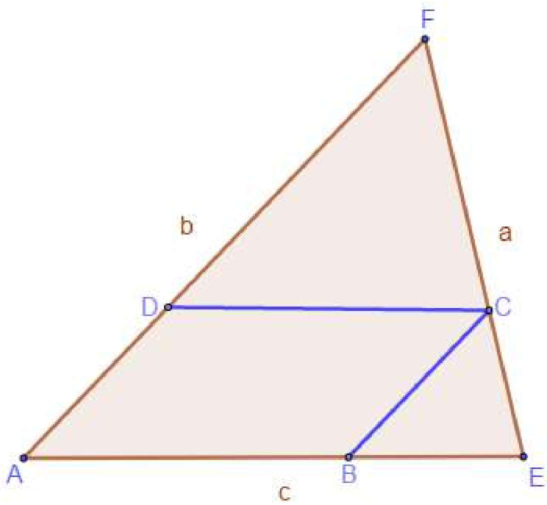
Solution.

How $2\overrightarrow{BE} = \overrightarrow{AB}$ and $\overrightarrow{AF} = 3\overrightarrow{AD}$ then $\overrightarrow{CE} = \overrightarrow{CB} + \overrightarrow{BE} = \overrightarrow{DA} + \overrightarrow{BE} = -\frac{1}{3}\overrightarrow{AF} + \frac{1}{2}\overrightarrow{AB}$. Hence,

$$\overrightarrow{CE} = -\frac{1}{3}\overrightarrow{AF} + \frac{1}{2}\overrightarrow{AB}; (1)$$

Other, $\overrightarrow{FC} = \overrightarrow{FA} + \overrightarrow{AC} = \overrightarrow{FA} + \overrightarrow{AD} + \overrightarrow{AB} = -\overrightarrow{AF} + \frac{1}{3}\overrightarrow{AF} + \overrightarrow{AB} = -\frac{2}{3}\overrightarrow{AF} + \overrightarrow{AB}$.

Thus, $\overrightarrow{FC} = -\frac{2}{3}\overrightarrow{AF} + \overrightarrow{AB}$; (2). From (1),(2) we have $\overrightarrow{FC} = 2\overrightarrow{CE}$ and then, the points E, F and C are collinear.



Application 2: ***In $\Delta ABC, E \in AB, F \in AC$ such that $EF \parallel BC, M \in EF, N \in BC$ such that***

$$\frac{ME}{MF} = \frac{NB}{NC} = \lambda, \lambda > 0$$

Prove that M, N and A are collinear.

Solution.

How $\frac{ME}{MF} = \frac{NB}{NC} = \lambda, \lambda > 0$, we have:

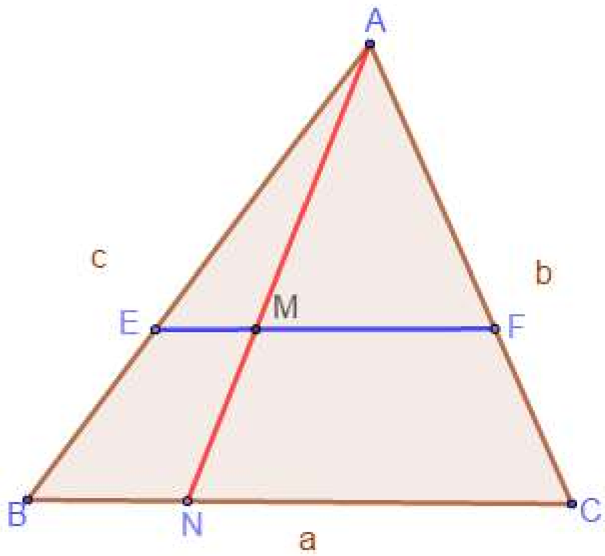
$$\overrightarrow{ME} = -\lambda\overrightarrow{MF}, \qquad \overrightarrow{NB} = -\lambda\overrightarrow{NC} \Rightarrow$$

$$\overrightarrow{AM} = \frac{1}{1+\lambda}\overrightarrow{AE} + \frac{\lambda}{1+\lambda}\overrightarrow{AF}; (1)$$

$$\overrightarrow{AN} = \frac{1}{1+\lambda}\overrightarrow{AB} + \frac{\lambda}{1+\lambda}\overrightarrow{AC}; (2)$$

But $\Delta AEF \sim \Delta ABC$ then, $\frac{AE}{AB} = \frac{AF}{AC} = k$.

Thus, $\overrightarrow{AE} = k\overrightarrow{AB}, \overrightarrow{AF} = k\overrightarrow{AC}; (3)$.



From (2),(3) relation (1) becomes as:

$$\overrightarrow{AM} = \frac{k}{1+\lambda}\overrightarrow{AB} + \frac{\lambda k}{1+\lambda}\overrightarrow{AC} = k\left(\frac{1}{1+\lambda}\overrightarrow{AB} + \frac{\lambda}{1+\lambda}\overrightarrow{AC}\right) = k\overrightarrow{AN}$$

Therefore, A, M and N are collinear.

Application 3: ***In $\Delta ABC, BF, CE$ −symmedians from B and C respectively. If points E, F and I are collinear if and only if $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$.***

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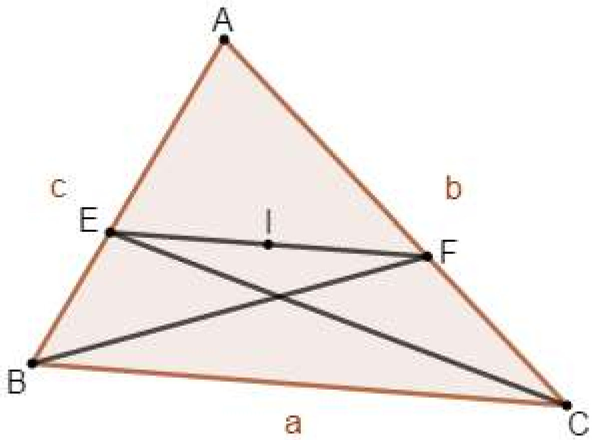
Solution.

From transversals theorem:

$$I \in EF \Leftrightarrow b \cdot \frac{EB}{EA} + c \cdot \frac{FC}{FA} = a; (1)$$



But, from Steiner;s theorem, we have:

$$\begin{cases} \frac{EB}{EA} = \frac{a^2}{b^2} \\ \frac{FC}{FA} = \frac{a^2}{c^2} \end{cases}; (2)$$

From (1),(2) it follows that: $b \cdot \frac{a^2}{b^2} + c \cdot \frac{a^2}{c^2} = a$

$$\Leftrightarrow \frac{a^2}{b} + \frac{a^2}{c} = a \Leftrightarrow \frac{a}{b} + \frac{a}{c} = 1 \Leftrightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a}$$

Application 3: ***In $\Delta ABC, I \in Int(\Delta ABC)$. Prove that I −incentre if and only if***

$$a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \vec{0}.$$

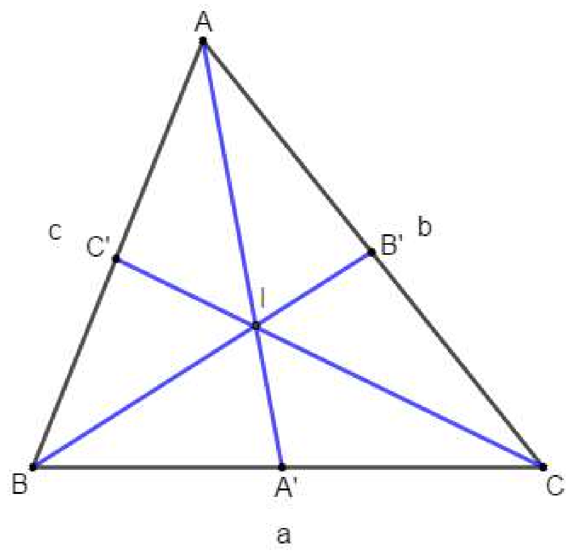
Solution.

Let $A' \in (BC), B' \in (CA), C' \in (AB)$.

Applying bisector theorem, we get:

$$\overrightarrow{BA'} = \frac{c}{b+c}\overrightarrow{BC},$$

$$\overrightarrow{AB'} = \frac{c}{a+c}\overrightarrow{AC}.$$



Thus,

$$\overrightarrow{AA'} = \frac{b}{b+c}\overrightarrow{AB} + \frac{c}{b+c}\overrightarrow{AC}$$

$$\overrightarrow{BB'} = -\overrightarrow{AB} + \frac{c}{a+c}\overrightarrow{AC}$$

How $I \in (AA')$, then exist $x \in (0,1)$ such that $\overrightarrow{AI} = x\overrightarrow{AA'}$. It follows that: $\overrightarrow{BI} = \left(\frac{xb}{b+c} - 1\right)\overrightarrow{AB} + \frac{xc}{b+c}\overrightarrow{AC}$.

How $\overrightarrow{BI}$ and $\overrightarrow{BB'}$ are collinear, we get: $\frac{\frac{xb}{b+c}-1}{-1}=\frac{\frac{xc}{b+c}}{\frac{c}{a+c}}$ and then $x=\frac{b+c}{a+b+c}$.

So, we have: $\overrightarrow{AI}=\frac{b}{a+b+c}\overrightarrow{AB}+\frac{c}{a+b+c}\overrightarrow{AC}\ (and\ analogs)$. Adding, it follows that:

$$a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=(a+b+c)\overrightarrow{IA}+b\overrightarrow{AB}+c\overrightarrow{AC}=\left(-b\overrightarrow{AB}-c\overrightarrow{AC}\right)+b\overrightarrow{AB}+c\overrightarrow{AC}=0.$$

Therefore, I −incenter.

Reverse, let $I'\in Int(\Delta ABC)$ who verify relation $a\overrightarrow{I'A}+b\overrightarrow{I'B}+c\overrightarrow{I'C}=\vec{0}$ and from $a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=\vec{0}$, we obtain: $(a+b+c)\overrightarrow{II'}=\vec{0}$ and know that $a+b+c\neq 0$, it follows that $I=I'$.

Application 4.

In ΔABC, BF, CE −symmedians from B and C respectively. If points E, F and O are collinear if and only if $\cot B+\cot C=\cot A$.

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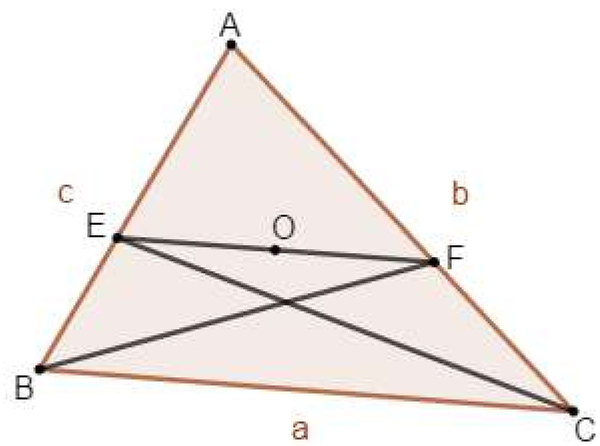
Solution.

From transversals theorem: $O\in EF\Leftrightarrow$

$\frac{EB}{EA}\cdot\sin 2B+\frac{FC}{FA}\cdot\sin 2C=\sin 2A$; (1)



From Steiner's theorem, we have:

$$\begin{cases}\dfrac{EB}{EA}=\dfrac{a^2}{b^2}\\ \dfrac{FC}{FA}=\dfrac{a^2}{c^2}\end{cases};(2)$$

From (1),(2) it follows that: $O\in EF\Leftrightarrow$

$$\frac{a^2}{b^2}\cdot\sin 2B+\frac{a^2}{c^2}\cdot\sin 2C=\sin 2A\Leftrightarrow$$

$$\frac{\sin^2 A}{\sin^2 B}\cdot 2\sin B\cos B+\frac{\sin^2 A}{\sin^2 C}\cdot 2\sin C\cos C=2\sin A\cos A\Leftrightarrow$$

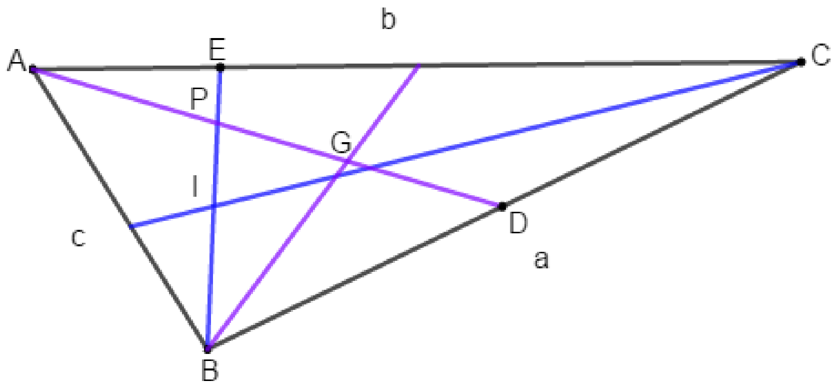
$$\frac{\cos B}{\sin B}\cdot\sin A+\frac{\cos C}{\sin C}\cdot\sin A=\cos A\Leftrightarrow\cot B+\cot C=\cot A.$$

Application 5.

In ΔABC, D −middle point of (BC), G −centroid, BE −internal bisector, $\{P\}=AD\cap BE$. Prove that $\overrightarrow{PG}=\overrightarrow{GD}$ if and only if $|\overrightarrow{BC}|=4|\overrightarrow{AB}|$.

Solution.

Let us denote:



$$\alpha=\frac{AB}{BC}=\frac{AE}{EC},\beta=\frac{AP}{PD}$$

We have:

$$\overrightarrow{BE}=\frac{\overrightarrow{BA}+\alpha\overrightarrow{BC}}{1+\alpha}=$$

$$=\frac{1}{1+\alpha}\overrightarrow{BA}+\frac{\alpha}{1+\alpha}\overrightarrow{BC}$$

$$\overrightarrow{BP}=\frac{\overrightarrow{BA}+\beta\overrightarrow{BD}}{1+\beta}=\frac{1}{1+\beta}\overrightarrow{BA}+\frac{1}{2}\cdot\frac{\beta}{1+\beta}\overrightarrow{BC}$$

How $\overrightarrow{BP}$ and $\overrightarrow{BE}$ are collinear, then $\frac{\frac{1}{1+\alpha}}{\frac{1}{1+\beta}}=\frac{\frac{\alpha}{1+\alpha}}{\frac{\beta}{1+\beta}}$. Therefore, $\beta=2\alpha\Leftrightarrow\overrightarrow{BC}=4\overrightarrow{AB}$.

Application 6.

In ΔABC, AD −internal bisector and $M\in AB, N\in AC$.

a) ***Find $y,z\in\mathbb{R}$ such that $\overrightarrow{AD}=y\cdot\overrightarrow{AB}+z\cdot\overrightarrow{AC}$.***

b) ***If $P_i\in(ABC)$ and $(x_i,y_i,z_i)\in\mathbb{R}^3 i=\overline{1,3}$ such that $x_i+y_i+z_i=1,\forall i=\overline{1,3}$ and $\overrightarrow{OP_i}=x_i\cdot\overrightarrow{OA}+y_i\cdot\overrightarrow{OB}+z_i\cdot\overrightarrow{OC},\forall O\in(ABC)$, then P_1,P_2,P_3 are collinear if and only if exists $u,v,w\in\mathbb{R}$ with property $ux_i+vy_i+wz_i=0,\forall i=\overline{1,3}$.***

c) ***Prove that the points M,N,D are collinear if and only if $b\cdot\frac{\overline{BM}}{\overline{AM}}+c\cdot\frac{\overline{CN}}{\overline{AN}}=\frac{a^2}{b+c}$.***

Solution.

a) It is easy to prove that D middle point of $[II_a]$; $(usual\ notations)$ then,

$\overrightarrow{AD}=\frac{1}{2}\overrightarrow{AI}+\frac{1}{2}\overrightarrow{AI_a}$. We know the following relations:

$$\overrightarrow{AI}=\frac{b}{a+b+c}\cdot\overrightarrow{AB}+\frac{c}{a+b+c}\cdot\overrightarrow{AC};\ \overrightarrow{AI_a}=\frac{b}{-a+b+c}\cdot\overrightarrow{AB}+\frac{c}{-a+b+c}\cdot\overrightarrow{AC}$$

Hence,

$$\overrightarrow{AD}=\frac{b}{-a^2+(b+c)^2}\cdot\overrightarrow{AB}+\frac{c}{-a^2+(b+c)^2}\cdot\overrightarrow{AC}$$

Therefore, $y=\frac{b}{-a^2+(b+c)^2}$ and $z=\frac{c}{-a^2+(b+c)^2}$.

b) Let the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ then P_1,P_2,P_3 are collinear if and only if

$$\frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} = \frac{z_1 - z_2}{z_1 - z_3}; (1)$$

For "⇒" , we get $u = y_1z_2 - y_2z_1, v = z_1x_2 - z_2x_1, w = x_1y_2 - x_2y_1$.

For "⇐", if $ux_i + vy_i + wz_i = 0, \forall i = \overline{1,3}$ then $\frac{x_1-x_2}{y_1-y_2} = -\frac{v-w}{u-w} = \frac{x_1-x_3}{y_1-y_3}$ hence,

$$\frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} = \frac{z_1 - z_2}{z_1 - z_3}$$

c) Let $M(x_m, y_M, 0)$ hence, $\frac{x_M}{y_M} = \frac{\overrightarrow{BM}}{\overrightarrow{MA}}$ and for $N(x_N, 0, y_N)$ we have $\frac{x_N}{z_N} = \frac{\overrightarrow{CN}}{\overrightarrow{NA}}$.

$d_{MN}: ux + vy + wz = 0$, then $\begin{cases} \frac{v}{u} = -\frac{x_M}{y_M} = \frac{\overrightarrow{BM}}{\overrightarrow{AM}} \\ \frac{w}{u} = -\frac{x_n}{y_n} = \frac{\overrightarrow{CN}}{\overrightarrow{AN}} \end{cases}$ and using point a) it follows that

$$D\left(\frac{-a^2}{-a^2 + (b+c)^2}, \frac{b(b+c)}{-a^2 + (b+c)^2}, \frac{c(b+c)}{-a^2 + (b+c)^2}\right)$$

So, $D \in MN$ if and only if $b \cdot \frac{\overrightarrow{BM}}{\overrightarrow{AM}} + c \cdot \frac{\overrightarrow{CN}}{\overrightarrow{AN}} = \frac{a^2}{b+c}$.

Application 7: ***In ΔABC, N −Nagel's point, BF, CE −symmedians from B and C respectively. Prove that the points E, F and N are collinear if and only if***

$$\mathbf{\frac{1}{b^2 r_b} + \frac{1}{c^2 r_c} += \frac{1}{a^2 r_a}}$$

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Solution: From transversal's theorem: $\frac{PB}{PA} \cdot (s-b) + \frac{QC}{QA} \cdot (s-c) = s-a; (1)$

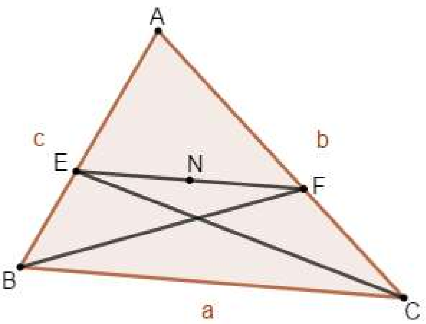
From Steiner's theorem, we have: $\begin{cases} \frac{PB}{PA} = \left(\frac{BC}{AC}\right)^2 = \frac{a^2}{b^2} \\ \frac{QC}{QA} = \left(\frac{BC}{AB}\right)^2 = \frac{a^2}{c^2} \end{cases}; (2)$

From (1),(2) it follows that: $\frac{a^2}{b^2} \cdot (s-b) + \frac{a^2}{c^2} \cdot (s-c) = s-a$

$$\frac{s-b}{b^2} + \frac{s-c}{c^2} = \frac{s-a}{a^2}$$

But, $r_a = \frac{F}{s-a} \Rightarrow s-a = \frac{F}{r_a} \Rightarrow$

$$\frac{1}{b^2 r_b} + \frac{1}{c^2 r_c} = \frac{1}{a^2 r_a}$$



Application 8: ***In $\Delta ABC, BE, CF$ –internal bisectors and O –circumcenter. Prove that the points E, O and F are collinear if and only if $\cos A = \cos B + \cos C$.***

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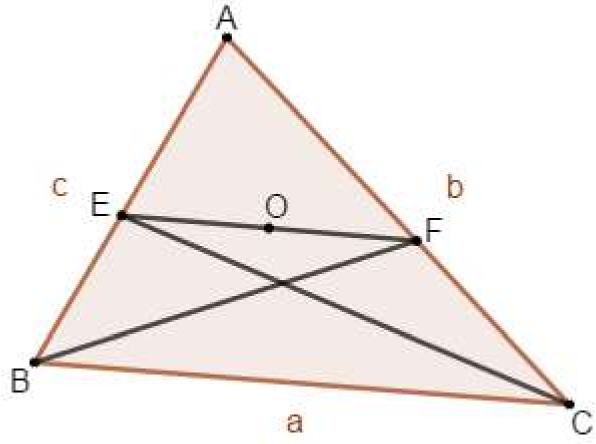
Solution.

Applying transversals theorem, we have:

$O \in EF \Leftrightarrow$

$$\frac{EB}{EA} \cdot \sin 2B + \frac{FC}{FA} \cdot \sin 2C = \sin 2A;\ (1)$$



From bisector theorem, we have:

$$\begin{cases} \frac{FB}{FA} = \frac{a}{b} \\ \frac{EC}{EA} = \frac{a}{c} \end{cases};(2)$$

From (1),(2) it follows that: $\frac{a}{b} \cdot \sin 2B + \frac{a}{c} \cdot \sin 2C = \sin 2A \Leftrightarrow$

$$2\sin B\cos B \cdot \frac{\sin A}{\sin B} + 2\sin C\cos C \cdot \frac{\sin A}{\sin C} = 2\sin A\cos A \Leftrightarrow$$

$$2\sin A\cos B + 2\sin A\cos C = 2\sin A\cos A \Leftrightarrow \cos B + \cos C = \cos A$$

Application 9.

In $ABCD$ parallelogram, I –middle point of AB and $E \in ID$ such that $3\overrightarrow{IE} = \overrightarrow{ID}$. Prove that the points A, E and C are collinear.

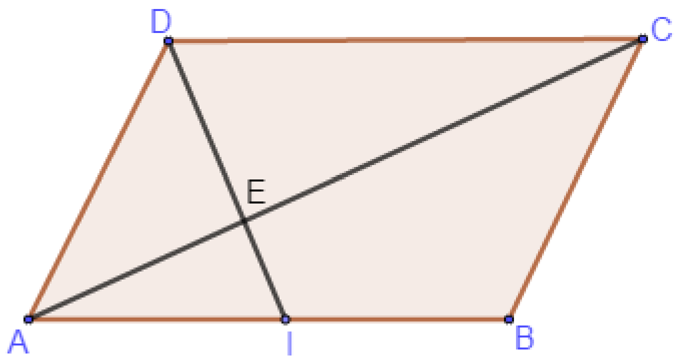
Solution.

Because $3\overrightarrow{IE} = \overrightarrow{ID} \Rightarrow \frac{IE}{ED} = \frac{1}{2} \Rightarrow$

$$\overrightarrow{ED} = -2\overrightarrow{EI}$$

$$\overrightarrow{AE} = \frac{1}{1—2}\overrightarrow{AD}—21—2\overrightarrow{AI} =$$

$$= \frac{1}{3}\overrightarrow{AD} + \frac{2}{3}\overrightarrow{AI}$$



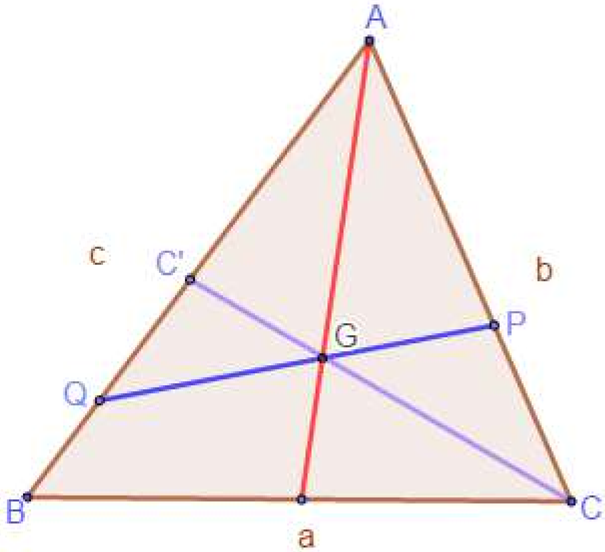
How $2\overrightarrow{AI} = \overrightarrow{AB}$ it follows that:

$\overrightarrow{AE} = \frac{1}{3}\overrightarrow{AD} + \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\overrightarrow{AD} + \overrightarrow{AB}) = \frac{1}{3}\overrightarrow{AC} \Rightarrow A, E$ and C are collinear.

Application 10.

In ΔABC, G –centroid and $P \in AC, Q \in BC$ such that $\frac{CP}{PA} + \frac{BQ}{QA} = 1$. Then prove that the points P, Q and G are collinear.

Solution.



Let us denote $\frac{CP}{PA} = m$, $\frac{BQ}{QA} = n$ and let C' middle point of AB. Because $\overrightarrow{GC} = -2\overrightarrow{GC'}$ we have:

$$\overrightarrow{AG} = \frac{1}{1-2}\overrightarrow{AC} + \frac{-2}{1-2}\overrightarrow{AC'} =$$

$$= \frac{1}{3}\overrightarrow{AC} + \frac{2}{3}\overrightarrow{AC'}; (1)$$

From $\frac{CP}{PA} = m$, $\frac{BQ}{QA} = n$ it follows that

$$\overrightarrow{CP} = m\overrightarrow{PA}, \qquad \overrightarrow{BQ} = n\overrightarrow{QA} \Rightarrow$$

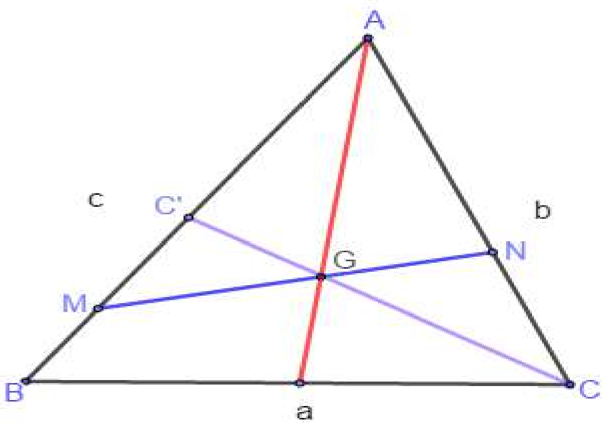
$\overrightarrow{AC} = (m+1)\overrightarrow{AP}$, $\overrightarrow{AC'} = \frac{n+1}{2}\overrightarrow{AQ}$ and relation (1) becomes as:

$$\overrightarrow{AG} = \frac{m+1}{3}\overrightarrow{AP} + \frac{n+1}{3}\overrightarrow{AQ}; (2)$$

Let $x = \frac{m+1}{3}, y = \frac{n+1}{3}$ and from $m + n = 1$ we get $x + y = 1$. So, exists $x, y \in \mathbb{R}$ such that $x + y = 1$ and $\overrightarrow{AG} = x\overrightarrow{AP} + y\overrightarrow{AQ}$, namely the points P, Q and G are collinear.

Application 11: ***In ΔABC, G –centroid and $M \in AB, N \in AC$ such that $\frac{MB}{MA} + \frac{NC}{NA} = k$. Prove that the points M, N and G are collinear if and only if $k = 1$.***

Solution.



Let us denote $\frac{MB}{MA} = \alpha, \frac{NC}{NA} = \beta$ then,

$\overrightarrow{MA} = -\frac{1}{\alpha+1}\overrightarrow{AB}$, $\overrightarrow{AN} = \frac{1}{\beta+1}\overrightarrow{AC}$. We have:

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = -\frac{1}{\alpha+1}\overrightarrow{AB} + \frac{1}{\beta+1}\overrightarrow{AC}$$

$$\overrightarrow{MG} = \overrightarrow{MA} + \overrightarrow{AG} = -\frac{1}{\alpha+1}\overrightarrow{AB} + \frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC})$$

$$= \left(\frac{1}{3} - \frac{1}{\alpha+1}\right)\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$$

We observe that the vectors $\overrightarrow{MG}$ and $\overrightarrow{MN}$ have same direction then,

$$\frac{\frac{1}{3}-\frac{1}{\alpha+1}}{-\frac{1}{\alpha+1}}=\frac{\frac{1}{3}}{\frac{1}{\beta+1}}\Leftrightarrow \alpha+\beta=1$$

Application 12.

In plane of ΔABC let be the points D, M, S and T such that $5\overrightarrow{AT}=3\overrightarrow{AB}, 2\overrightarrow{SA}+\overrightarrow{SC}=\vec{0}$,

$35\overrightarrow{AD}=18\overrightarrow{AB}$ and $34\overrightarrow{MA}+36\overrightarrow{MB}+5\overrightarrow{MC}=\vec{0}$.

a) Find $x, y\in\mathbb{R}$ such that $x\overrightarrow{MT}+y\overrightarrow{MS}=\vec{0}$.

b) Prove that the points C, M and D are collinear.

Solution.

a) From $5\overrightarrow{AT}=3\overrightarrow{AB}\Rightarrow\overrightarrow{AT}=\frac{3}{2}\overrightarrow{TB}\Rightarrow\overrightarrow{MT}=\frac{2}{5}\overrightarrow{MA}+\frac{3}{5}\overrightarrow{MB}$ and from $\overrightarrow{AS}=\frac{1}{2}\overrightarrow{SC}$ we get:

$$\overrightarrow{MS}=\frac{2}{3}\overrightarrow{MA}+\frac{1}{3}\overrightarrow{MC}=\frac{2}{3}\overrightarrow{MA}+\frac{1}{3}\left(-\frac{34}{5}\overrightarrow{MA}-\frac{36}{5}\overrightarrow{MB}\right)=$$

$$=-\frac{8}{5}\overrightarrow{MA}-\frac{12}{5}\overrightarrow{MB}=-4\overrightarrow{MT}$$

So, $\overrightarrow{MS}+4\overrightarrow{MT}=\vec{0}$ and we can choose $x=4, y=1$.

b) $\overrightarrow{AD}=\frac{18}{35}\overrightarrow{AB}\Rightarrow\overrightarrow{AD}=\frac{18}{17}\overrightarrow{DB}\Rightarrow\overrightarrow{MD}=\frac{17}{35}\overrightarrow{MA}+\frac{18}{35}\overrightarrow{MB}$ and from

$34\overrightarrow{MA}+36\overrightarrow{MB}+5\overrightarrow{MC}=\vec{0}$ we get $-\frac{1}{14}\overrightarrow{MC}=\frac{17}{35}\overrightarrow{MA}+\frac{18}{35}\overrightarrow{MB}$.

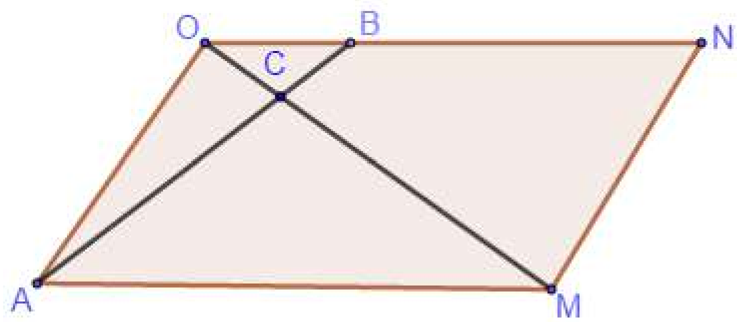
So, it follows $\overrightarrow{MD}=-\frac{1}{14}\overrightarrow{MC}$ and then, the points M, D, C are collinear.

Application 13.

In $AMNO$ parallelogram the points B, C are such that $\overrightarrow{OB}=\frac{1}{n}\overrightarrow{ON}, \overrightarrow{OC}=\frac{1}{n+1}\overrightarrow{OM}$, where $n\in\mathbb{N}^*, n\geq 2$. Prove that the points A, B, C are collinear.

Solution.

We must to prove that exist $\alpha\in\mathbb{R}$ such that $\overrightarrow{AC}=\alpha\overrightarrow{AB}$.



How, $\overrightarrow{OM}=(n+1)\overrightarrow{OC}$ we have

$\overrightarrow{CM}=-n\overrightarrow{CO}$.

It follows that:

$$\overrightarrow{AC}=\frac{1}{1-n}\overrightarrow{AM}-n1-n\overrightarrow{AO}=\frac{1}{n+1}\overrightarrow{AM}+\frac{n}{n+1}\overrightarrow{AO}$$

Because $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{AO}+\frac{1}{n}\overrightarrow{ON}=\overrightarrow{AO}+\frac{1}{n}\overrightarrow{AM}$ then, $\overrightarrow{AC}=\alpha\overrightarrow{AB}\Leftrightarrow$

$$\frac{n}{n+1}\overrightarrow{AO}+\frac{1}{n+1}\overrightarrow{AM}=\alpha\overrightarrow{AO}+\frac{\alpha}{n}\overrightarrow{AM}$$

How the vectors $\overrightarrow{AO}$ and $\overrightarrow{AM}$ are not collinear, we have $\frac{n}{n+1}=\alpha,\frac{1}{n+1}=\frac{\alpha}{n}$.

So, $\alpha=\frac{n}{n+1},\overrightarrow{AC}=\frac{n}{n+1}\overrightarrow{AB}$ and then the points A,B,C are collinear.

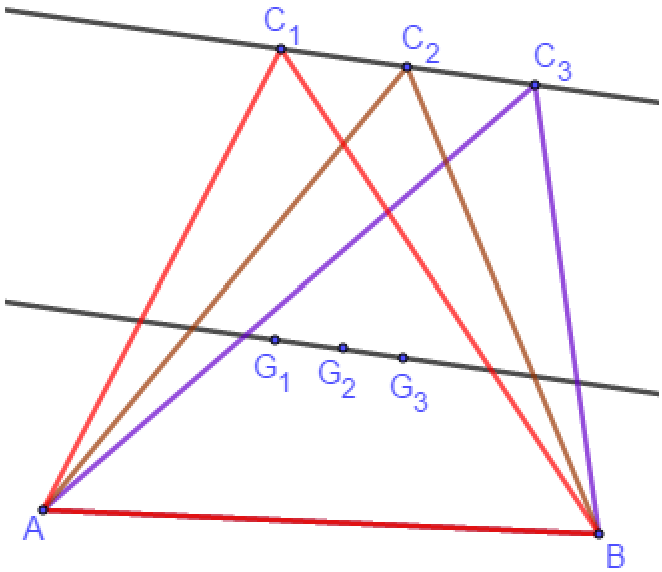
Application.

In $\Delta ABC_1,\Delta ABC_2,\Delta ABC_3,G_1,G_2,G_3$ –centroids. Prove that the points G_1,G_2,G_3 are collinear if and only if the points C_1,C_2,C_3 are collinear.

Solution.

Let O in plane of that triangles.

From Leinbiz relation, we have:



$$\overrightarrow{OG_1}=\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)$$

$$\overrightarrow{OG_2}=\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC_2}\right)$$

$$\overrightarrow{OG_3}=\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC_3}\right)$$

It follows that $\overrightarrow{G_1G_2}=\overrightarrow{OG_2}-\overrightarrow{OG_1}=\frac{1}{3}\left(\overrightarrow{OC_2}-\overrightarrow{OC_1}\right)=\frac{1}{3}\overrightarrow{C_1C_2}$ and similarly, $\overrightarrow{G_1G_3}=\frac{1}{3}\overrightarrow{C_1C_3}$.

The points G_1,G_2,G_3 are collinear if and only if exist $\alpha\in\mathbb{R}$ such that $\overrightarrow{G_1G_2}=\alpha\overrightarrow{G_1G_3}\Leftrightarrow$

$\frac{1}{3}\overrightarrow{C_1C_2}=\frac{\alpha}{3}\overrightarrow{C_1C_3}\Leftrightarrow\overrightarrow{C_1C_2}=\alpha\overrightarrow{C_1C_3}\Leftrightarrow C_1,C_2,C_3$ are collinear.

References:

[1]. ROMANIAN MATHEMATICAL MAGAZINE-*Interactive Journal-www.ssmrmh.ro.*

[2]. **OCTOGON MATHEMATICAL MAGAZINE**.

[3]. Daniel Sitaru, George Apostolopoulos-***Olympic Mathematical Marathon***,*C.R.2018*.

[4]. Daniel Sitaru, Marian Ursărescu-***Geometry math problems-Vol.I-*** *Studis 2020.*

[5]. Mihaly Bencze, Daniel Sitaru-***699 Olympic Mathematical Challenges***, *Studis, 2017.*

[7]. Daniel Sitaru-***Affine and euclidiane geometry problems***, *Ecko Print 2012.*

[8].Dana Heuberger &Co. -***Matematică pentru grupele de excelenţă***.*Paralela 45.*

[9]. Camelia Magdaş &Co-***Geometrie pentru clasele IX-X***, *Studia 2010*

[10]. Octavian Marica- ***Metode de rezolvarea problemelor de coliniaritate și concurență.*** *ROVIMED PUBLISHERS-2013.*

NEW REFINEMENT FOR RADON'S INEQUALITY

By D.M. Bătineţu-Giurgiu,Mihaly Bencze,Daniel Sitaru-Romania

Theorem.(Radon's Inequality)

If $x_k, y_k \in (0,\infty), \forall k=\overline{1,n}, n\geq 2$ and $t\geq 0$, then:

$$\frac{x_1^{t+1}}{y_1^t}+\frac{x_2^{t+1}}{y_2^t}+\cdots+\frac{x_n^{t+1}}{y_n^t}\geq\frac{(x_1+x_2+\cdots+x_n)^{t+1}}{(y_1+y_2+\cdots+y_n)^t};(R)$$

Equality holds if and only if $\frac{x_1}{y_1}=\frac{x_2}{y_2}=\cdots\frac{x_n}{y_n}$.

Theorem. (Bergstrom's Inequality)

If $x_k, y_k \in (0,\infty), \forall k=\overline{1,n}, n\geq 2$, then:

$$\sum_{k=1}^{n}\frac{x_k^2}{y_k}\geq\frac{(x_1+x_2+\cdots+x_n)^2}{y_1+y_2+\cdots+y_n};(B)$$

Equality holds if and only if $\exists u\in\mathbb{R}_+^*$ such that $|x_k|=u\cdot y_k;\forall k=\overline{1,n}$.

We observe that inequality (B) cannot be a consequence of inequality (R) because inequality (R) is not possible for $x_k\in\mathbb{R}-\mathbb{R}_+^*;\forall k=\overline{1,n}$.

Theorem.

If $u\geq 0, v>0$ and $t\geq 0, x_k, y_k\in\mathbb{R}_+^*, \forall k=\overline{1,n}, x_{n+1}=x_1, y_{n+1}=y_1$, then:

$$\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}\geq\frac{1}{2u+v}\left(u\cdot\sum_{k=1}^{n}\frac{(x_k+x_{k+1})^{t+1}}{(y_k+y_{k+1})^t}+v\cdot\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^{t+1}}\right)\geq\frac{(x_1+x_2+\cdots+x_n)^{t+1}}{(y_1+y_2+\cdots+y_n)^t};(*)$$

Proof. We have:

$$(2u+v)\cdot\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}=2u\cdot\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}+v\cdot\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}=$$

$$=u\cdot\sum_{k=1}^{n}\left(\frac{x_k^{t+1}}{y_k^t}+\frac{x_{k+1}^{t+1}}{y_k^t}\right)+v\cdot\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}\overset{(R)}{\geq} u\cdot\sum_{k=1}^{n}\frac{(x_k+x_{k+1})^{t+1}}{(y_k+y_{k+1})^t}+v\cdot\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}\overset{(R)}{\geq}$$

$$\geq u\cdot\frac{\left(\sum_{k=1}^{n}(x_k+x_{k+1})\right)^{t+1}}{\left(\sum_{k=1}^{n}(y_k+y_{k+1})\right)^t}+v\cdot\frac{\left(\sum_{k=1}^{n}x_k\right)^{t+1}}{\left(\sum_{k=1}^{n}y_k\right)^t}\geq$$

$$\geq u\cdot\frac{\left(2\sum_{k=1}^{n}x_k\right)^{t+1}}{\left(2\sum_{k=1}^{n}y_k\right)^t}+v\cdot\frac{\left(\sum_{k=1}^{n}x_k\right)^{t+1}}{\left(\sum_{k=1}^{n}y_k\right)^t}=$$

$$=\frac{2^{t+1}\cdot u}{2^t}\cdot\frac{\left(\sum_{k=1}^{n}x_k\right)^{t+1}}{\left(\sum_{k=1}^{n}y_k\right)^t}+v\cdot\frac{\left(\sum_{k=1}^{n}x_k\right)^{t+1}}{\left(\sum_{k=1}^{n}y_k\right)^t}=(2u+v)\cdot\frac{\left(\sum_{k=1}^{n}x_k\right)^{t+1}}{\left(\sum_{k=1}^{n}y_k\right)^t}\Leftrightarrow$$

$$\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}\geq\frac{1}{2u+v}\left(u\cdot\sum_{k=1}^{n}\frac{(x_k+x_{k+1})^{t+1}}{(y_k+y_{k+1})^t}+v\cdot\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^{t+1}}\right)\geq\frac{(x_1+x_2+\cdots+x_n)^{t+1}}{(y_1+y_2+\cdots+y_n)^t}$$

If $u=0$, inequality $(*)$ becomes as:

$$\frac{x_1^{t+1}}{y_1^t}+\frac{x_2^{t+1}}{y_2^t}+\cdots+\frac{x_n^{t+1}}{y_n^t}\geq\frac{(x_1+x_2+\cdots+x_n)^{t+1}}{(y_1+y_2+\cdots+y_n)^t};(R)$$

If $u=v$, inequality $(*)$ becomes as:

$$\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}\geq\frac{1}{3}\left(\sum_{k=1}^{n}\frac{(x_k+x_{k+1})^{t+1}}{(y_k+y_{k+1})^t}+\sum_{k=1}^{n}\frac{x_k^{t+1}}{y_k^t}\right)\geq\frac{(x_1+x_2+\cdots+x_n)^{t+1}}{(y_1+y_2+\cdots+y_n)^t}$$

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

A SIMPLE PROOF FOR POPOVICIU'S INEQUALITY INTEGRAL FORM

By Daniel Sitaru-Romania

Abstract: In this paper is gived a simple proof for Popoviciu's inequality and an application.

Theorem: If $a,b,c>0, f:(0,\infty)\to\mathbb{R}; f$ −integrable and convexe function then:

$$\frac{1}{a}\int_0^a f(x)\,dx+\frac{1}{b}\int_0^b f(x)\,dx+\frac{1}{c}\int_0^c f(x)\,dx+\frac{9}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)\,dx\geq$$

$$\geq\frac{4}{a+b}\int_0^{\frac{a+b}{2}} f(x)\,dx+\frac{4}{b+c}\int_0^{\frac{b+c}{2}} f(x)\,dx+\frac{4}{c+a}\int_0^{\frac{c+a}{2}} f(x)\,dx$$

Lemma: If $a>0$; $f:(0,\infty)\to\mathbb{R}, f$ −integrable and convexe then:

$$\int_0^1 f(ax)\,dx=\frac{1}{a}\int_0^a f(x)\,dx \quad (1)$$

Proof. For $ax=y\Rightarrow\begin{cases}x=\frac{1}{a}y\\ dx=\frac{1}{a}dy\end{cases}$ and $\begin{cases}x=0\\x=1\end{cases}\Rightarrow\begin{cases}y=0\\y=a\end{cases}$.

$$\int_0^1 f(ax)\,dx=\int_0^a f(y)\cdot\frac{1}{a}dy=\frac{1}{a}\int_0^a f(y)\,dy=\frac{1}{a}\int_0^a f(x)\,dx$$

Analogous with (1):

$$\int_0^1 f(bx)\,dx=\frac{1}{b}\int_0^b f(x)\,dx;\ \ (2)$$

$$\int_0^1 f(cx)\,dx=\frac{1}{c}\int_0^c f(x)\,dx;\ \ (3)$$

$$\int_0^1 f\left(\frac{a+b+c}{3}\cdot x\right)dx=\frac{3}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)\,dx;\ \ (4)$$

$$\int_0^1 f\left(\frac{a+b}{2}\cdot x\right)dx=\frac{2}{a+b}\int_0^{\frac{a+b}{2}} f(x)\,dx;(5)$$

$$\int_0^1 f\left(\frac{b+c}{2}\cdot x\right)dx=\frac{2}{b+c}\int_0^{\frac{b+c}{2}} f(x)\,dx;\ \ (6)$$

$$\int_0^1 f\left(\frac{c+a}{2}\cdot x\right)dx=\frac{2}{c+a}\int_0^{\frac{c+a}{2}} f(x)\,dx;\ \ (7)$$

By classical Popoviciu’s inequality:

$$f(ax)+f(bx)+f(cx)+3f\left(\frac{a+b+c}{3}\cdot x\right)\geq$$

$$\geq 2f\left(\frac{a+b}{2}\cdot x\right)+2f\left(\frac{b+c}{2}\cdot x\right)+2f\left(\frac{c+a}{2}\cdot x\right);(8)$$

Integrating (8), it follows:

$$\int_0^1 f(ax)\,dx+\int_0^1 f(bx)\,dx+\int_0^1 f(cx)\,dx+3\int_0^1 f\left(\frac{a+b+c}{3}\cdot x\right)dx\ge$$

$$\ge 2\int_0^1 f\left(\frac{a+b}{2}\cdot x\right)dx+2\int_0^2 f\left(\frac{b+c}{2}\cdot x\right)dx+2\int_0^1 f\left(\frac{c+a}{2}\cdot x\right)dx$$

By (1),(2),…,(7) we get:

$$\frac{1}{a}\int_0^a f(x)\,dx+\frac{1}{b}\int_0^b f(x)\,dx+\frac{1}{c}\int_0^c f(x)\,dx+\frac{9}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)\,dx\ge$$

$$\ge 2\cdot\frac{2}{a+b}\int_0^{\frac{a+b}{2}} f(x)\,dx+2\cdot\frac{2}{b+c}\int_0^{\frac{b+c}{2}} f(x)\,dx+2\cdot\frac{2}{c+a}\int_0^{\frac{c+a}{2}} f(x)\,dx$$

Hence,

$$\frac{1}{a}\int_0^a f(x)\,dx+\frac{1}{b}\int_0^b f(x)\,dx+\frac{1}{c}\int_0^c f(x)\,dx+\frac{9}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)\,dx\ge$$

$$\ge\frac{4}{a+b}\int_0^{\frac{a+b}{2}} f(x)\,dx+\frac{4}{b+c}\int_0^{\frac{b+c}{2}} f(x)\,dx+\frac{4}{c+a}\int_0^{\frac{c+a}{2}} f(x)\,dx$$

If $a=b=c$:

$$LHS=\frac{3}{a}\int_0^a f(x)\,dx+\frac{9}{3a}\int_0^a f(x)\,dx=\frac{6}{a}\int_0^a f(x)\,dx$$

$$RHS=3\cdot\frac{4}{2a}\int_0^{\frac{a+a}{2}} f(x)\,dx=\frac{6}{a}\int_0^a f(x)\,dx$$

$$LHS=RHS$$

Application: If $n\in\mathbb{N}; n\ge 2; a,b,c>0$ then:

$$a^n+b^n+c^n+\frac{(a+b+c)^n}{3^{n-2}}\ge\frac{(a+b)^n}{2^{n-1}}+\frac{(b+c)^n}{2^{n-1}}+\frac{(c+a)^n}{2^{n-1}}$$

Proof. We take in (8): $f(x)=x^n$, then:

$$\frac{1}{a}\int_0^a x^n\,dx+\frac{1}{b}\int_0^b x^n\,dx+\frac{1}{c}\int_0^c x^n\,dx+\frac{9}{a+b+c}\int_0^{\frac{a+b+c}{3}} x^n\,dx\ge$$

$$\ge\frac{4}{a+b}\int_0^{\frac{a+b}{2}} x^n\,dx+\frac{4}{b+c}\int_0^{\frac{b+c}{2}} x^n\,dx+\frac{4}{c+a}\int_0^{\frac{c+a}{2}} x^n\,dx$$

$$\frac{1}{a}\cdot\frac{a^{n+1}}{n+1}+\frac{1}{b}\cdot\frac{b^{n+1}}{n+1}+\frac{1}{c}\cdot\frac{c^{n+1}}{n+1}+\frac{9}{a+b+c}\cdot\frac{\left(\frac{a+b+c}{3}\right)^{n+1}}{n+1}\geq$$

$$\geq\frac{4}{a+b}\cdot\left(\frac{a+b}{2}\right)^{n+1}\cdot\frac{1}{n+1}+\frac{4}{b+c}\cdot\left(\frac{b+c}{2}\right)^{n+1}\cdot\frac{1}{n+1}+\frac{4}{c+a}\cdot\left(\frac{c+a}{2}\right)^{n+2}\cdot\frac{1}{n+1}$$

Therefore,

$$a^n+b^n+c^n+\frac{(a+b+c)^n}{3^{n-2}}\geq\frac{(a+b)^n}{2^{n-1}}+\frac{(b+c)^n}{2^{n-1}}+\frac{(c+a)^n}{2^{n-1}}$$

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

APPLICATIONS OF GIREAUX'S THEOREM

By Alexander Bogomolny-USA,Daniel Sitaru-Romania

Abstract: If a continuous function of several variables is defined on a hyperbrick and is convex in each of the variables, it attains its maximum at one of the corners. More formally:

Assume $I_k=[a_k,b_k]\subset\mathbb{R}, k=\overline{1,n}$ and $f: I_1\times I_2\times\ldots\times I_n\to\mathbb{R}$ is a continuous function convex separately in each of the variables in the domain of definition. Then it attains its maximum at point $C=(c_1,\ldots,c_n)$ where $c_k\in\{a_k,b_k\}, k\in\overline{1,n}$.

The statement of the theorem is a specification of a theorem of Weierstrass (the Extreme Values Theorem) that states that a continuous function defined on a compact set attains its extremes in the set. Assumme now that the function is convex in each of its variables (i.e., as a function of one argument, with other arguments fixed.) A continuous function of one variable, convex on a closed interval, attains its maximum at one of the endpoints of the interval. This means that the maximum of the given function is attained at either, say, $a_1\times I_2\times\ldots\times I_n$ or $b_1\times I_2\times\ldots.\times I_n$, which reduces the dimension of the search for the maximum by 1. Doing this recursively proves the statement.

Application 1-USA 1980: Prove that, for $a,b,c\in[0,1]$,

$$\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c)\leq 1$$

The function $f(a,b,c)=\sum_{cycl}\frac{a}{b+c+1}+\prod_{cycl}(1-a)$ is convex in each of the three variables a,b,c, so that f takes its maximum value in one of either vertices of the cube

$0\leq a\leq 1, 0\leq b\leq 1, 0\leq c\leq 1$. Since $f(a,b,c)$ takes value 1 in each of these points, the required inequality is proven.

Application 2: Prove that, for $a,b,c,d\in[0,2]$,

$$\frac{9a}{1+bcd}+\frac{9b}{1+cda}+\frac{9c}{1+dab}+\frac{9d}{1+abc}+9e^{abcd}\le 8+9e^{16}$$

Daniel Sitaru

$$f:[0,2]^4\to\mathbb{R}, f(a,b,c,d)=9\sum\frac{a}{1+bcd}+9e^{abcd}.$$

$$f'_a=\frac{9}{1+bcd}-\frac{9bcd}{(1+cda)^2}-\frac{9cdb}{(1+dab)^2}+\frac{9dbc}{(1+abc)^2}+9bcde^{abcd},$$

$$f''_{aa}=\frac{18bc^2d^2}{(1+cda)^3}+\frac{18cd^2b^2}{(1+dab)^3}+\frac{18db^2c^2}{(1+abc)^3}+9b^2c^2d^2e^{abcd}>0$$

f strictly convex in variable a and, similarly, in the rest of the variables. f defined on a compact set $[0,2]^4$, hence, by Gireaux's theorem f attains it maximum at the vertices of the hypercube $[0,1]^4$. It is easy to check that the maximum is attained for

$f(2,2,2,2)=4\cdot\frac{18}{1+8}+9e^{16}=8+9e^{16}$, thus proving the inequality.

Application 3: Prove that, for $x,y,z\in[0,1]$,

$$\frac{x}{y+z+2016}+\frac{y^2}{z+x+2016}+\frac{z^3}{x+y+2016}+(1-x)(1-y)(1-z)\le 1.$$

Daniel Sitaru

$$f:[0,2]^3\to\mathbb{R},$$

$$f(x,y,z)=\frac{x}{y+z+2016}+\frac{y^2}{z+x+2016}+\frac{z^3}{x+y+2016}+(1-x)(1-y)(1-z)$$

We easily check that

$$f'_x x=\frac{2y^2}{(x+z+2016)^3}+\frac{2z^3}{(x+y+2016)^3}>0$$

f strictly convex in variable a and, similarly, in the rest of the variables. f defined on a compact set $[0,2]^4$, hence, by Gireaux's theorem f attains it maximum at the vertices of the hypercube $[0,1]^3$. It is easy to check that the maximum is attained for $f(0,0,0)=1$, thus proving the inequality.

References:

1. **Israel Meireles Chrisostomo, *Trigonometria Pura e Aplicacoes e um pouco alem: problemas de Olimpiadas,* 3 de Julho, 2015**
2. **M.S. Klamkin, USA Mathematical Olympiads 1972-1986, MAA, 1988**
3. **Romanian Mathematical Magazine-www.ssmrmh.ro**
4. **https://www.cut-the-knot.org/**

ABOUT D.M.BĂTINEȚU'S SEQUENCE

By Mihaly Bencze,Claudia Nănuți,Florică Anastase,Daniel Sitaru

The problem of studying of the Traian Lalescu sequence was put, first, by Mihail Ghermănescu and later by Tiberiu Popoviciu. Tiberiu Popoviciu request to establish correctness of calculus for that limit and the solution was given by Traian Lalescu. In that solution, Traian Lalescu find tha limit of the sequence $(L_n)_{n\geq 2}, L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$ used Cesaro-Stolz theorem.

First who give correct solution of that limit was Jozsef Kobeniban, because that solution was not in plane of Tiberiu Popoviciu because he used Stirling's formulae, was required an elementary solution for calculus that limit.

Following that request by Tiberiu Popoviciu, the first elementary solution for calculus limit for the sequence $(L_n)_{n\geq 2}$ was given by Alexandru Lupaș and then follows another elementary solutions among which solution by D.M.Bătinețu-Giurgiu:

Let be $(a_n)_{n\geq 1}$ a sequence of real numbers strictly positive. We say that the sequence $(a_n)_{n\geq 1}$ has Lalescu's property or is L –sequence is exist $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a \in \mathbb{R}_+^* = (0,\infty)$.

The sequence $(a_n)_{n\geq 1}$ with Lalescu's property (L –sequence $(a_n)_{n\geq 1}$) defined the sequence $(L_n)_{n\geq 2}, L_n = \sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}$ and named Lalescu's sequence defined by $(a_n)_{n\geq 1}$.

Theorem 1: Let be L –sequence $(a_n)_{n\geq 1}$ with $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0$, then

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{a}{e};\ (1)$$

Proof. First, we observe that we cannot apply Stirling formulae for find that limit. So, we have:

$$L_n = \sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \sqrt[n]{a_n}\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1\right) = \sqrt[n]{a_n}(u_n - 1) =$$

$$= \sqrt[n]{a_n}\cdot\frac{u_n - 1}{\log u_n}\cdot\log u_n = \frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n - 1}{\log u_n}\cdot\log u_n^n;\quad \forall n \geq 2;\quad (2)$$

But,

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} \overset{C-D'A}{=} \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}\cdot\left(\frac{n}{n+1}\right)^{n+1} = \frac{a}{e}.$$

Because $u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}, \forall n \geq 2$, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \cdot \frac{n}{\sqrt[n]{a_n}} \cdot \frac{n+1}{n} = \frac{a}{e} \cdot \frac{e}{a} \cdot 1 = 1$$

$$\lim_{n\to\infty} \frac{u_n - 1}{\log u_n} = 1, \lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n}{n+1} = a \cdot \frac{e}{a} \cdot 1 = e$$

Finally, applying limit as $n \to \infty$, it follows: $\lim_{n\to\infty} L_n = \frac{a}{e} \cdot 1 \cdot \log e = \frac{a}{e}$.

If $a_n = n!$, then $\frac{a_{n+1}}{n \cdot a_n} = \frac{(n+1)!}{n! \cdot n} = \frac{n+1}{n}; \forall n \ge 2$, so $\lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = 1$, then $(n!)_{n\ge1}$ is Lalescu's sequence with $a = 1$ and from Theorem 1, we obtain: $\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{1}{e}$ and we get the solution by D.M.Bătințu-Giurgiu of Lalescu sequence.

Now, we say that the sequence $(b_n)_{n\ge1}$ has Bătinețu-Giurgiu property or (B-G)-sequence if exist $t \in [0,\infty)$ such that $\lim_{n\to\infty} \frac{n^t \cdot b_{n+1}}{b_n} = b > 0$.

If $(b_n)_{n\ge1}$ is a (B-G)-sequence, then that can be defined by: $(B_n)_{n\ge2}$,

$$B_n = n^{t+1} \cdot \left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right); \forall n \ge 2; \quad (3)$$

Theorem 2: If $(b_n)_{n\ge1}$ is a (B-G)-sequence $\exists t \in \mathbb{R}_+$ such that $\lim_{n\to\infty} \frac{n^t \cdot b_{n+1}}{b_n} = b \in \mathbb{R}_+^*$, then:

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} n^{t+1}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right) = -b \cdot t \cdot e^t; \quad (4)$$

Proof. We have:

$$B_n = n^{t+1} \cdot \sqrt[n]{b_n}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} - 1\right) = n^{t+1} \cdot \sqrt[n]{b_n} \cdot (v_n - 1) =$$

$$= n^{t+1} \cdot \sqrt[n]{b_n} \cdot \frac{v_n - 1}{\log v_n} \cdot \log v_n = n^t \cdot \sqrt[n]{b_n} \cdot \frac{v_n - 1}{\log v_n} \cdot \log v_n^n; \forall n \ge 2; (5)$$

But,

$$\lim_{n\to\infty} n^t \cdot \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{n^{nt} \cdot b_n} \overset{C-D'A}{=} \lim_{n\to\infty} \frac{(n+1)^{(n+1)t} \cdot b_{n+1}}{n^{nt} \cdot b_n} =$$

$$= \lim_{n\to\infty} \frac{n^t \cdot b_{n+1}}{b_n}\left(\frac{n+1}{n}\right)^{(n+1)t} = b \cdot e^t$$

$$\lim_{n\to\infty} v_n = \frac{\lim_{n\to\infty} \sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}} = \lim_{n\to\infty} \frac{(n+1)^t \cdot \sqrt[n+1]{b_{n+1}}}{n^t \cdot \sqrt[n]{b_n}} \cdot \left(\frac{n}{n+1}\right)^t = \frac{b \cdot e^t}{b \cdot e^t} \cdot 1 = 1$$

$$\lim_{n\to\infty}\frac{v_n-1}{\log v_n}=1$$

$$\lim_{n\to\infty}v_n^n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{b_{n+1}}}{\sqrt[n]{b_n}}\right)^n=\lim_{n\to\infty}\frac{b_{n+1}}{b_n}\cdot\frac{1}{\sqrt[n+1]{b_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{n^t\cdot b_{n+1}}{b_n}\cdot\frac{1}{(n+1)^t\cdot\sqrt[n+1]{b_{n+1}}}\cdot\left(\frac{n+1}{n}\right)^t=b\cdot\frac{1}{b\cdot e^t}\cdot 1=\frac{1}{e^t}=e^{-t}$$

So, we get:

$$\lim_{n\to\infty}B_n=b\cdot e^t\cdot\log\left(\lim_{n\to\infty}v_n^n\right)=b\cdot e^t\cdot\log(e^{-t})=-b\cdot t\cdot e^t$$

So, using Theorem 2, we take $b_n=\frac{1}{n!};\forall n\geq 1$ and we get:

$$\lim_{n\to\infty}\frac{n\cdot b_{n+1}}{b_n}=\lim_{n\to\infty}\frac{n!\cdot n}{(n+1)!}=\lim_{n\to\infty}\frac{n}{n+1}=1$$

Therefore, the sequence of Bătinețu-Giurgiu defined by (B-G)-sequence $\left(\frac{1}{n!}\right)_{n\geq 1}$ is the sequence

$$(B_n)_{n\geq 2},B_n=n^2\left(\frac{1}{\sqrt[n+1]{(n+1)!}}-\frac{1}{\sqrt[n]{n!}}\right);\forall n\geq 2$$

Using Theorem 2, we obtain that $\lim\limits_{n\to\infty}B_n$ can be find in this way:

$$\lim_{n\to\infty}B_n=\lim_{n\to\infty}n^2\left(\frac{1}{\sqrt[n+1]{(n+1)!}}-\frac{1}{\sqrt[n]{n!}}\right)=\lim_{n\to\infty}\frac{n^2}{\sqrt[n+1]{(n+1)!}\cdot\sqrt[n]{n!}}\left(\sqrt[n]{n!}-\sqrt[n+1]{(n+1)!}\right)=$$

$$=\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n}{n+1}\cdot L_n=-e\cdot e\cdot 1\cdot\frac{1}{e}=-e$$

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT FINSLER-HADWIGER'S INEQUALITY

By D.M. Bătineţu-Giurgiu,Mihaly Bencze,Daniel Sitaru

In any ΔABC we denote with F –area, s –semiperimeter.

Theorem 1: If $m\in[0,\infty)$ and $x,y\in(0,\infty)$, then in ΔABC the following relationship holds:

$$\boldsymbol{(x^2+y^2)(a^{2m+2}+b^{2m+2}+c^{2m+2})}$$
$$\boldsymbol{\geq 2^{2m+3}\cdot xy\left(\sqrt{3}\right)^{1-m}F^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2};\ (*)$$

Proof. We have:

$$\sum_{cyc}(xa^{m+1}-yb^{m+1})^2=(x^2+y^2)(a^{2m+2}+b^{2m+2}+c^{2m+2})-2xy\sum_{cyc}(ab)^{m+1}\Leftrightarrow$$

$$(x^2+y^2)(a^{2m+2}+b^{2m+2}+c^{2m+2})=2xy\sum_{cyc}(ab)^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2\overset{Radon}{\geq}$$

$$\geq\frac{2xy}{3^m}\left(\sum_{cyc}ab\right)^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2\overset{Gordon}{\geq}$$

$$\geq\frac{2xy}{3^m}\left(4\sqrt{3}\cdot F\right)^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2=$$

$$=\frac{2^{2m+3}xy}{3^m}\left(\sqrt{3}\right)^{m+1}\cdot F^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2=$$

$$=2^{2m+3}xy\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2$$

Theorem 2.

If $\boldsymbol{m\in[0,\infty)},\boldsymbol{\Delta A_1B_1C_1}$ and in $\boldsymbol{\Delta A_2B_2C_2},\boldsymbol{\mu(\widehat{A_2})=90^\circ}$, then:

$$\boldsymbol{a_2^2(a_1^{2m+2}+b_1^{2m+2}+c_1^{2m+2})}$$
$$\boldsymbol{\geq 4^{m+2}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F_1^{m+1}\cdot F_2+\sum_{cyc}\left(a_1^{m+1}b_1-b_1^{m+1}c_2\right)^2};\ (**)$$

Proof. We have: $b_2^2+c_2^2=a_2^2$ and $2F_2=b_2\cdot c_2$. So,

$$\sum_{cyc}(a_1^{m+1}b_1-b_1^{m+1}c_2)^2=(b_2^2+c_2^2)(a_1^{2m+2}+b_1^{2m+2}+c_1^{2m+2})-2b_2c_2\sum_{cyc}(a_1b_1)^{m+1}$$

$$\Leftrightarrow(b_2^2+c_2^2)(a_1^{2m+2}+b_1^{2m+2}+c_1^{2m+2})=$$

$$=2b_2c_2\sum_{cyc}(a_1b_1)^{m+1}+\sum_{cyc}(a_1^{m+1}b_1-b_1^{m+1}c_2)^2\overset{Radon}{\geq}$$

$$\geq\frac{2b_2c_2}{3^m}\left(\sum_{cyc}a_1b_1\right)^{m+1}+\sum_{cyc}(a_1^{m+1}b_1-b_1^{m+1}c_2)^2\overset{Gordon}{\geq}$$

$$\geq \frac{2b_2c_2}{3^m}\left(4\sqrt{3}\cdot F_1\right)^{m+1}+\sum_{cyc}(a_1^{m+1}b_1-b_1^{m+1}c_2)^2=$$

$$=4\text{\textasciicircum}(m+20\left(\sqrt{3}\right)^{1-m}\cdot F_1^{m+1}\cdot F_2+\sum_{cyc}(a_1^{m+1}b_1-b_1^{m+1}c_2)^2$$

If in $(*)$ we take $x=y$, then:

$$a^{2m+2}+b^{2m+2}+c^{2m+2}\geq 4^{m+1}\left(\sqrt{3}\right)^{1-m}\cdot F^{m-1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2\,;(***)$$

and for $m=0$, we get:

$$a^2+b^2+c^2\geq 4\sqrt{3}\cdot F+\frac{1}{2}\sum_{cyc}(a-b)^2\,;(F-H)$$

If in $(***)$ we take $m=\frac{1}{2}$, we get:

$$a^3+b^3+c^3\geq 8\cdot\sqrt[4]{3}\cdot\left(\sqrt{F}\right)^3+\frac{1}{2}\sum_{cyc}\left(a\sqrt{a}-b\sqrt{b}\right)^2\,;(1)$$

If in $(*)$ we take $x=2$ and $y=3$, we get:

$$25(a^{2m+2}+b^{2m+2}+c^{2m+2})\geq 4^{m+2}\left(\sqrt{3}\right)^{3-m}\cdot F^{m+1}+\sum_{cyc}(2a^{m+1}-3b^{m+1})^2\,;(2)$$

and for $m=0$, we find:

$$25(a^2+b^2+c^2)\geq 48\sqrt{3}\cdot F+\sum_{cyc}(2a-3b)^2\,;(3)$$

If in $(**)$, we take $b_2=2$, $c_2=3$, then $a_2=5$ and we get:

$$25(a_1^{2m+2}+b_1^{2m+2}+c_1^{2m+2})\geq 4^{m+2}\left(\sqrt{3}\right)^{3-m}\cdot F_1^{m+1}+\sum_{cyc}(2a_1^{m+1}-3b_1^{m+1})^2\,;(4)$$

and for $m=0$, it follows that:

$$25(a_1^2+b_1^2+c_1^2)\geq 48\sqrt{3}\cdot F_1+\sum_{cyc}(2a_1-3b_1)^2\,;(5)$$

If in $(**)$ triangle $A_2B_2C_2$ is rectangular isosceles $(b_2=c_2=t)$, then $a_2=t\sqrt{2}$ and

$$2t^2(a_1^{2m+2}+b_1^{2m+2}+c_1^{2m+2})\geq 4^{m+2}\left(\sqrt{3}\right)^{1-m}\cdot F_1^{m+1}\cdot F_2+t^2\sum_{cyc}(a_1^{m+1}-b_1^{m+1})^2=$$

$$=4^{m+2}\left(\sqrt{3}\right)^{1-m}\cdot F_1^{m+1}\cdot\frac{t^2}{2}+t^2\sum_{cyc}(a_1^{m+1}-b_1^{m+1})^2$$

$$a_1^{2m+2}+b_1^{2m+2}+c_1^{2m+2}\geq 4^{m+1}\left(\sqrt{3}\right)^{1-m}F_1^{m+1}+\frac{1}{2}\sum_{cyc}(a_1^{m+1}-b_1^{m+1})^2\ ;(6)$$

If in (6) we take $m=0$, it follows:

$$a_1^2+b_1^2+c_1^2\geq 4\sqrt{3}\cdot F_1+\frac{1}{2}\sum_{cyc}(a_1-b_1)^2\ ;(F-H)$$

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT GORDON'S INEQUALITY

By D.M. Bătineţu-Giurgiu, Mihaly Bencze,Daniel Sitaru-Romania

In ΔABC the following relationship holds: $ab+bc+ca\geq 4\sqrt{3}\cdot F$; $(Gordon)$

where F −area of ΔABC.

Theorem 1: If $x,y,z>0$, then in ΔABC the following relationship holds:

$$(x+y)ab+(y+z)bc+(z+x)ca\geq 8\sqrt{xy+yz+zx}\cdot F;\ (*)$$

Proof. Applying V. Jiglău inequality:

$$xa^2+yb^2+zc^2\geq 2F\cdot\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}}\ ;(J_1)\text{ and}$$

$$(x+y)ab+(y+z)bc+(z+x)ca+8F\sqrt{xy\sin^2\frac{C}{2}+yz\sin^2\frac{A}{2}+zx\sin^2\frac{B}{2}}\ ;(J_2)$$

Let us denote: $B=\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}},H=xy\sin^2\frac{C}{2}+yz\sin^2\frac{A}{2}+zx\sin^2\frac{B}{2}$

So, we must to prove the following inequality:

$$(x+y)ab+(y+z)bc+(z+x)ca\geq 8\sqrt{xy+yz+zx}\cdot F;\ (*)$$

Using (J_1) and (J_2), we have:

$$(x+y)ab+(y+z)bc+(z+x)ca\geq 2F\cdot\sqrt{B}+8F\cdot\sqrt{H}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq}2\cdot\sqrt{2F\sqrt{B}\cdot 8F\sqrt{H}}=2\cdot 4\cdot F\sqrt[4]{BH}=$$

$$= 8F \cdot \sqrt[4]{\left(\frac{xy}{\sin^2\frac{C}{2}} + \frac{yz}{\sin^2\frac{A}{2}} + \frac{zx}{\sin^2\frac{B}{2}}\right)\left(xy\sin^2\frac{C}{2} + yz\sin^2\frac{A}{2} + zx\sin^2\frac{B}{2}\right)} \overset{CBS}{\geq}$$

$$\geq 8F \cdot \sqrt[4]{(xy + yz + zx)^2} = 8F \cdot \sqrt{xy + yz + zx}$$

Let $x = y = z$ in inequality $(*)$, then we get:

$$2x(ab + bc + ca) \geq 8\sqrt{3x^2} \cdot F \Leftrightarrow ab + bc + ca \geq 4\sqrt{3} \cdot F; (Gordon)$$

Theorem 2: If $\boldsymbol{m \geq 0}; \boldsymbol{x, y, z} > 0, xyz = 1$, then in $\boldsymbol{\Delta ABC}$ the following relationship holds:

$$\boldsymbol{(x + y)(ab)^{m+1} + (y + z)(bc)^{m+1} + (z + x)(ca)^{m+1} \geq 2^{2m+3}\left(\sqrt{3}\right)^{1-m} \cdot F^{m+1}}; (**)$$

Proof. We have:

$$\sum_{cyc}(x + y)(ab)^{m+1} \geq 2\sum_{cyc}\sqrt{xy} \cdot (ab)^{m+1} \overset{AM-GM}{\geq} 2 \cdot 3 \cdot \sqrt[3]{\prod_{cyc}\left(\sqrt{xy}(ab)^{m+1}\right)} =$$

$$= 6 \cdot \sqrt[3]{xyz(a^2b^2c^2)^{m+1}} = 6 \cdot \sqrt[3]{(abc)^{2m+2}} = 2 \cdot \frac{3^{m+1}}{3^m} \cdot \sqrt[3]{(a^2b^2c^2)^{m+1}} =$$

$$= \frac{2}{3^m}\left(3 \cdot \sqrt[3]{a^2b^2c^2}\right)^{m+1} \overset{Carliz}{\geq} \frac{2}{3^m} \cdot \left(4\sqrt{3}\right)^{m+1} = \frac{2^{2m+3} \cdot \left(\sqrt{3}\right)^{m+1}}{3^m} \cdot F^{m+1} =$$

$$= 2^{2m+3}\left(\sqrt{3}\right)^{1-m} \cdot F^{m+1}$$

If in $(**)$ we take $m = 0$, then it follows Gordon's inequality.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

PROBLEMS FOR JUNIORS

J.1222 If $m \geq 0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{a^{m+2} + b^{m+2}}{a^m + b^n} + \frac{b^{m+2} + c^{m+2}}{b^m + c^m} + \frac{c^{m+2} + a^{m+2}}{c^m + a^m} \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihály Bencze – Romania

J.1223 If $x, y, z > 0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{(y+z)(h_b+h_c)a^3}{z}+\frac{(z+x)(h_c+h_a)b^3}{y}+\frac{(x+y)(h_a+h_b)c^3}{z}\geq 48\sqrt{3}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihály Bencze – Romania

J.1224 If $x, y, z \geq 0, x^2+y^2+z^2=3$ then:

$$\frac{x^3+1}{\sqrt{x^2-x+1}}+\frac{y^3+1}{\sqrt{y^2-y+1}}+\frac{z^3+1}{\sqrt{z^2-z+1}}\geq 6$$

Proposed by Daniel Sitaru – Romania

J.1225 If $a, b, c, d > 0, a+b+c+d=1$ then:

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}}\geq 4\sqrt[3]{\frac{4}{5}}$$

Proposed by Daniel Sitaru – Romania

J.1226 Solve for real numbers:

$$\begin{cases} x+\dfrac{9}{[x]}=\dfrac{6}{1+x-[x]} \\ z+2^z+\log_2 z=x+y, [*]-GIF \\ y+\dfrac{16}{[y]}=\dfrac{8}{1+y-[y]} \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.1227 If $x, y, z > 0, xyz=1$ then:

$$(x-y)^4+(y-z)^4+(z-x)^4\geq 2\left(3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)^2$$

Proposed by Daniel Sitaru – Romania

J.1228 If $x, y, z > 0$ then:

$$\frac{(x+y)^4}{x^4+x^2y^2+y^4}+\frac{(y+z)^4}{y^4+y^2z^2+z^4}+\frac{(z+x)^4}{z^4+z^2x^2+x^4}\leq 16$$

Proposed by Daniel Sitaru – Romania

J.1229 If $x, y, z, t > 0$ then:

$$\frac{75x+36(y+z)}{y+z+t}+\frac{75y+36(z+t)}{z+t+x}+\frac{75z+36(t+x)}{t+x+y}+\frac{75t+36(x+y)}{x+y+z}\geq 196$$

Proposed by Daniel Sitaru – Romania

J.1230 Solve for real numbers:

$$\begin{cases}(x+y)(\sqrt{6}-\sqrt{x})=\sqrt{x}\\ (x+y)(1+\sqrt{y})=\sqrt{y}\end{cases}$$

Proposed by Florică Anastase-Romania

J.1231 If $x,y,z>0$ that $xy+yz+zx=1$. Prove that:

$$\frac{1}{9x^2+1}+\frac{1}{9y^2+1}+\frac{1}{9z^2+1}\geq\frac{3}{4}$$

Proposed by Marin Chirciu – Romania

J.1232 Solve for real numbers:

$$5+\log_{12}\frac{x}{x^3+16}=x+\frac{2}{\sqrt{x-1}}$$

Proposed by Marin Chirciu – Romania

J.1233 In acute ΔABC the following relationship holds:

$$\frac{6}{Rp^2}\leq\sum\frac{1}{s_a^3}\leq\frac{2R-r}{3S^2}\sum\left(\frac{b^2+c^2}{2bc}\right)^3$$

Proposed by Marin Chirciu – Romania

J.1234 In ΔABC the following relationship holds:

$$\frac{1}{9r^3}\leq\sum\frac{1}{h_a^3}\leq\frac{R}{18r^4}$$

Proposed by Marin Chirciu – Romania

J.1235 In ΔABC the following relationship holds:

$$\frac{1}{9r^3}\leq\sum\frac{1}{r_a^3}\leq\frac{1}{r^3}\left(1-\frac{16r}{9R}\right)$$

Proposed by Marin Chirciu – Romania

J.1236 GENERALIZATION FOR OPPENHEIMER INEQUALITY

If $x,y,z>0$ then in ΔABC the following relationship holds:

$$(x+y+z)^2\geq 2\sqrt{4-\frac{2r}{R}}(xy\sin C+yz\sin A+zx\sin B)$$

Proposed by Bogdan Fuștei – Romania

J.1237 NEW BLUNDON TYPE INEQUALITIES

In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$2\left(\frac{R^2}{r^2}-\frac{R}{r}-\frac{h_a}{r_a}\right)-\frac{2R\sqrt{R^2-2Rr}}{r^2}\le\left(\frac{n_a}{r_a}\right)^2$$

$$\left(\frac{n_a}{r_a}\right)^2\le 2\left(\frac{R^2}{r^2}-\frac{R}{r}-\frac{h_a}{r_a}\right)+\frac{2R\sqrt{R^2-2Rr}}{r^2}$$

Proposed by Bogdan Fuștei – Romania

J.1238 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$s\sqrt{2}\sum_{cyc}\frac{1}{w_a}\ge 2\left(\sqrt{\frac{R}{r}}+\sqrt{\frac{r}{R}}\right)+\sum_{cyc}\frac{n_a}{w_a}$$

Proposed by Bogdan Fuștei – Romania

J.1239 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$2\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}-1\right)+\frac{r}{R}\ge\sum_{cyc}\frac{n_a^2+g_a^2}{bc}$$

Proposed by Bogdan Fuștei – Romania

J.1240 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\prod_{cyc}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\ge\frac{1}{8}\prod_{cyc}\frac{n_a+g_a+\sqrt{2r_br_c}+2\sqrt{rr_a}}{h_a-r}$$

Proposed by Bogdan Fuștei – Romania

J.1241 In ΔABC the following relationship holds:

$$2\sqrt{3}(R-2r)\ge\frac{|b-c|\cdot|m_b-m_c|}{a}$$

Proposed by Bogdan Fuștei – Romania

J.1242 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{a}\ge\sqrt{\left(\frac{2r}{R}-\frac{r^2}{R^2}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

Proposed by Bogdan Fuștei – Romania

J.1243 In ΔABC the following relationship holds:

$$\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{h_b}{r_b}}+\sqrt{\frac{h_c}{r_c}}\ge\frac{s+3(2-\sqrt{3})r}{\sqrt{2Rr}}$$

Proposed by Bogdan Fuștei – Romania

J.1244 If $m,x,y,z,t\geq 0,x+y,z+t>0$ then in any ABC triangle the following inequality holds:

$$\frac{(bx+cy)^{2m+2}}{(zw_b+tw_c)w_a}+\frac{(cx+ay)^{2m+2}}{(zw_c+tw_a)w_b}+\frac{(ax+by)^{2m+2}}{(zw_a+tw_b)w_c}\geq\frac{4^{m+1}\cdot 3^m(x+y)^{2m+2}\cdot r^{2m}}{t+z}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1245 If $m\geq 0$, then in ABC triangle having the area F the following inequality holds:

$$\frac{a^{m+1}b}{h_b^m}+\frac{b^{m+1}c}{h_c^m}+\frac{c^{m+1}a}{h_a^m}\geq 2^{m+2}\left(\sqrt{3}\right)^{1-m}F$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1246 If $m\geq 0,x,y>0$, then in ABC triangle with the area F the following inequality holds:

$$\frac{a^{m+1}b}{(ax+by)^m}+\frac{b^{m+1}c}{(bx+ay)^m}+\frac{c^{m+1}a}{(cx+by)^m}\geq\frac{4\sqrt{3}F}{(x+y)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1247 If $m\geq 0,x,y,z\in(0,1)$ and ABC is a triangle with the area F, then:

$$\frac{x^m\cdot a^{2m+2}}{(y+z)^{m+1}(1-x)^2}+\frac{y^m b^{2m+2}}{(z+x)^{m+1}(1-y^2)}+\frac{z^m c^{2m+2}}{(x+y)^m(1-z^2)}\geq 2^m\left(\sqrt{3}\right)^{4-m}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1248 If $x,y,z>0$ then in ABC triangle with the semiperimeter s, the following inequality holds:

$$\frac{x\cdot r_a^2}{y+z}+\frac{y\cdot r_b^2}{z+x}+\frac{z\cdot c^2}{x+y}\geq\frac{1}{2}(4s^2-(4R+r)^2)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1249 In any ABC triangle with the area F the following inequality holds:

$$\frac{a^3+b}{\sqrt{a^3-a\sqrt{ab}+b}}+\frac{b^3+c}{\sqrt{b^3-b\sqrt{bc}+c}}+\frac{c^2+a}{\sqrt{c^3-c\sqrt{ca}+a}}\geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1250 If $x,y\geq 0,x+y>0$ and $ABC,A_1B_1C_1$ are two triangles having the circumradii R, respectively R_1, then:

$$\frac{1}{xa+y\sigma(a_1)}+\frac{1}{xb+y\sigma(b_1)}+\frac{1}{xc+y\sigma(c_1)}\geq\frac{\sqrt{3}}{xR+yR_1}$$

where σ is a permutation of the set $\{a_1,b_2,c_1\}$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1251 Solve for real numbers:

$$\begin{cases}12x^3+12y^3+9z=2\\12y^3+12z^3+9x=2\\12z^3+12x^3+9y=2\end{cases}$$

Proposed by Asmat Qatea-Afghanistan

J.1252 If $a,b,c>0, a+b+c=1$ then:

$$\frac{1}{\sqrt{a+bc}}+\frac{1}{\sqrt{b+ca}}+\frac{1}{\sqrt{c+ab}}\ge\frac{9}{2}$$

Proposed by Rajeev Rastogi-India

J.1253 Prove that in triangle ABC, the following relationship holds:

$$\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}}\ge\frac{2s}{r}$$

Proposed by Daniel Sitaru – Romania

J.1254 If $a,b,c>0, (a+b)^3+(b+c)^3+(c+a)^3=24$ then:

$$(a+b)(a^2+b^2)+(b+c)(b^2+c^2)+(c+a)(c^2+a^2)\ge 12$$

Proposed by Daniel Sitaru – Romania

J.1255 Solve for real numbers:

$$\begin{cases}x,y,z>0\\x^3(z+y^2)+y^3(x+z^2)+z^3(y+x^2)=6\\xyz(1+xyz)=2\end{cases}$$

Proposed by Daniel Sitaru – Romania

J.1256 If in $\Delta ABC, a\le b\le c$ then:

$$\frac{\frac{a(x+y)}{2}+b\sqrt[3]{\frac{x^3+y^3}{2}}+\frac{c(x^2+y^2)}{x+y}}{\frac{x+y}{2}+\sqrt[3]{\frac{x^3+y^3}{2}}+\frac{x^2+y^2}{x+y}}\ge 2\sqrt{3}r, \forall x,y,z>0$$

Proposed by Daniel Sitaru – Romania

J.1257 If $a,b,c>0$ then:

$$\sum_{cyc}\frac{(a^6+b^6)(a^8+b^8)}{(a^5+b^5)(a^{11}+b^{11})}\le\sum_{cyc}\frac{1}{a^2-ab+b^2}$$

Proposed by Daniel Sitaru – Romania

J.1258 In ΔABC the following relationship holds:

$$16\sqrt{RF}\sum_{cyc}\frac{a}{b+c}\le 24\sqrt{RF}+\sum_{cyc}\frac{(a-b)^2}{\sqrt{c}}$$

Proposed by Daniel Sitaru – Romania

J.1259 Solve for real numbers:

$$\begin{cases} \frac{1}{\log x}+\frac{1}{\log y}+\frac{1}{\log z}=\frac{3}{\log 3} \\ xyz=27 \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.1260 $ABCD$ – tetrahedron, $AB=CD=a, BC=DA=b, CA=BD=c$. Prove that:

$$Volume[ABCD]\le \frac{\sqrt{6}}{108}(a^2+b^2+c^2)\sqrt{a^2+b^2+c^2}$$

When equality holds?

Proposed by Daniel Sitaru – Romania

J.1261 If $a,b,c>0$ then:

$$\frac{a}{3b+\sqrt[7]{ab^6}}+\frac{b}{3c+\sqrt[7]{bc^6}}+\frac{c}{3a+\sqrt[7]{ca^6}}\ge \frac{3}{8}$$

Proposed by Daniel Sitaru – Romania

J.1262 If $a,b,c>0$ then:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+a^2)(c^2+b^2)}}{c^2+ab}\ge 3$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.908 Let be $f:\mathbb{R}_+^*\times\mathbb{R}_+^*\to\mathbb{R}_+^*=(0,\infty), f(u,v)=\frac{u^{16}+v^4}{\sqrt{u^{16}-u^8v^2+v^4}}$ and ABC a triangle with the area F, then:

$$\frac{f(a,y+z)}{x}+\frac{f(b,z+x)}{y}+\frac{f(c,x+y)}{z}\ge 64F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.909 Let be $n \in \mathbb{N}^* \setminus \{1\}, x_k \in [1, \infty), \forall k = \overline{1, n}$ and a the arithmetic mean of these numbers, then:

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{n} x_k^{x_j}\right) \geq n^n \cdot a^{na}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.910 If $t \in \mathbb{R}_+ = [0, \infty); x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{y+z+3t}{x}a^2 + \frac{z+x+2t}{y+t}b^2 + \frac{x+y+t}{z+2t}c^2 \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

S.911 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{y+z}{x}\left(a+b-\sqrt{ab}\right)^2 + \frac{z+x}{y}\left(b+c-\sqrt{bc}\right)^2 + \frac{x+y}{z}\left(c+a-\sqrt{ca}\right)^2 \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

S.912 Let $m \in \left[\frac{1}{2}, \infty\right), n \in \mathbb{N}^*$ and $a, b, c \in \mathbb{R}_+^* = (0, \infty)$, then:

$$(a^{2m}+b^{2m})\sqrt[2n]{a^{2n}+b^{2n}-a^n b^n} + (b^{2m}+c^{2m})\sqrt[2n]{b^{2n}+c^{2n}-b^n c^n} +$$
$$+(c^{2m}+a^{2m})\sqrt[2n]{c^{2n}+a^{2n}-c^n a^n} \geq 2 \cdot 3^{\frac{1-2m}{2}}(ab+bc+ca)^{\frac{2m+1}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

S.913 If $m \geq 0$ and $t, x, y, z > 0$, then in ABC triangle with the area F the following inequality holds:

$$\frac{a^{m+1}}{b^m(ta+xc)^m \cdot (yb+zc)^{m+1}} + \frac{b^{m+1}}{c^m(tb+xa)^m(yc+za)^{m+1}} +$$
$$+\frac{c^{m+1}}{a^m(tc+xb)^m(ya+zb)^{m+1}} \geq \frac{3}{(t+x)^m \cdot (y+z)^{m+1}(ab+bc+ca)^m}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

S.914 If $x, y \in \mathbb{R}_+^* = (0, \infty)$, then in any ABC triangle the following inequality holds:

$$\sum_{cyc}\frac{(xr_a+yr_b)(xr_a+yr_c)}{r_b r_c} \geq 12xy$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.915 Let $x, y \in \mathbb{R}_+^* = (0, \infty)$, then in ABC triangle with the semiperimeter s the following inequality holds:

$$\frac{m_a}{xb + yc} + \frac{m_b}{xc + ya} + \frac{m_c}{xa + yb} \geq \frac{s}{(x + y)R}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.916 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{(y + z)a}{x \cdot h_b} + \frac{(z + x)b}{y \cdot h_c} + \frac{(x + y)c}{z \cdot h_a} \geq 4\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.917 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ABC triangle with the area F the following inequality holds:

$$x \cdot m_a + y \cdot m_b + z \cdot m_c \geq \sqrt{xy + yz + zx} \cdot \frac{2F}{R}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.918 If $m \geq 0$, then in any ABC triangle the following inequality holds:

$$\frac{a^{2m+2}}{w_b \cdot w_c} + \frac{b^{2m+2}}{w_c \cdot w_a} + \frac{c^{2m+2}}{w_a \cdot w_b} \geq 4^{m+1} \cdot 3^n \cdot r^{2m}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.919 If $x, y, z \in (0,1)$ and ABC is a triangle with the area F, then:

$$\frac{ab}{x^2(1 - x)} + \frac{bc}{y^2(1 - y)} + \frac{ca}{z^2(1 - z)} \geq 27\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.920 In any ABC triangle with the area F the following inequality holds:

$$\frac{1}{\sqrt{3}}\sqrt{(a + b)^2 + (b + c)^2 + (c + a)^2} + \frac{3abc}{ab + bc + ca} \geq 2 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.921 If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then in any ABC triangle with the area F, the following inequality holds:

$$\frac{a^2}{(\sin y + \sin z)\cos^2 x} + \frac{b^2}{(\sin z + \sin x)\cos^2 y} + \frac{c^2}{(\sin x + \sin y)\cos^2 z} \geq 9F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.922 If $x, y, z > 0, ABC$ is a triangle with the area F and s_a, s_b, s_c are the symmedians lengths from A, B respectively C, then:

$$\frac{xs_a + ys_b}{z}c^3 + \frac{ys_b + zs_c}{x}a^3 + \frac{zs_c + xs_a}{y}b^3 \geq 16\sqrt{3}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.923 If $a, b, c, d \in \mathbb{R}_+^* = (0, \infty)$ and $\frac{y+z}{x}a + \frac{z+x}{y}b + \frac{x+y}{z}c \geq d, \forall x, y, z \in \mathbb{R}_+^*$, then:

$$\frac{y+z}{x}a^2 + \frac{z+x}{y}b^2 + \frac{x+y}{z}c^2 \geq \frac{1}{6}d^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.924 Let be $x, y, z > 0$, triangle ABC, and the cevians AD, BE, CF concurrent in P and K, L, M the intersections between AP, BP and CP with EF, FD respectively DE, then:

$$\frac{(y+z)AK}{x \cdot MF} + \frac{(z+x)BL}{y \cdot LE} + \frac{(x+y)CM}{KD} \geq 2$$

Proposed by D.M. Bătinețu – Giurgiu, Gabriel Tică – Romania

S.925 Let $x_1, x_2, \dots, x_n > 0, k \in \mathbb{N}$. Prove that:

$$\frac{x_1^{k+2} + x_2^{k+2}}{x_1^k + x_2^k} + \frac{x_2^{k+2} + x_3^{k+2}}{x_1^k + x_2^k} + \cdots + \frac{x_n^{k+2} + x_1^{k+2}}{x_n^k + x_1^k} + \frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{2}{x_n} \geq 3n$$

Proposed by Nicolai Găitan-Romania

S.926 If $a, b, c, d \geq 1, abcd = 3$ then:

$$a^{-\frac{1}{\sqrt{a}}} + b^{-\frac{1}{\sqrt{b}}} + c^{-\frac{1}{\sqrt{c}}} + d^{-\frac{1}{\sqrt{d}}} > 3$$

Proposed by Seyran Ibrahimov-Azerbaijan

S.927 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\tan^2\frac{A}{2} + \tan^2\frac{B}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}} \leq \frac{3\left(\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}\right)}{\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}}$$

When equality holds?

Proposed by Nguyen Van Canh-Vietnam

S.928 If $m, n > 1, mn = m + n$ then in ΔABC the following relationship holds:

$$n \cdot R^m \cdot r^n + m \geq 2mnr^n$$

Proposed by Seyran Ibrahimov-Azerbaijan

S.929 In ΔABC the following relationship holds:

$$\frac{9}{8(r+4R)}\cdot\frac{(a+b)(b+c)(c+a)}{abc}\le\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\le\frac{1}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

S.930 If $\Omega_n(1)=\underbrace{1111\dots111}_{n-times}$then:

$$\frac{1}{3(2n-2)}\sum_{k=1}^{n}(-1)^{n+k}\cdot\Omega_k(1)\cdot\binom{n}{k}=1$$

Proposed by Mohammed Bouras-Morocco

S.931 If in $\Delta ABC, abc=1$ then:

$$\sum_{cyc}\left(2\sqrt{a}+\frac{1}{a}\right)+\sqrt{\sum_{cyc}\frac{\cos A}{a^3}}\ge 9+\frac{\sqrt{6}}{2}$$

Proposed by Radu Diaconu-Romania

S.932 If in $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian then:

$$\sqrt{\frac{n_a g_a h_a}{h_a-2r}}+\sqrt{\frac{n_b g_b h_b}{h_b-2r}}+\sqrt{\frac{n_c g_c h_c}{h_c-2r}}\ge 3s$$

Proposed by Bogdan Fuştei-Romania

S.933 V –Bevan's point in $\Delta ABC, I_a, I_b, I_c$ –excenters, $R_a, R_b, R-c$ –circumradii of ΔVI_bI_c, $\Delta VI_cI_a, \Delta VI_aI_b$. Prove that:

$$\frac{1}{R_a^2}+\frac{1}{R_b^2}+\frac{1}{R_c^2}=\frac{2R-r}{2R^3}$$

Proposed by Mehmet Şahin-Turkiye

S.934 For $x,y,z\ge 1$ prove that:

$$\sum_{cyc}\frac{z}{\lambda x+\lambda y+z}\ge\frac{3}{1+(n-1)\lambda}$$

Proposed by Amrit Awasthi-India

S.935 Solve for real numbers:

$$\begin{cases}x^{\sqrt{y}}+y^{\sqrt{x}}=145\\ \sqrt{x}+\sqrt{y}=5\end{cases}$$

Proposed by Ghuiam Shah Naseri-Afghanistan

S.936 Prove that:

$$\sum_{n=0}^{\infty}\frac{3072n^2+3072n+832}{4096n^6+12288n^5+14592n^4+8704n^3+2736n^2+432n+27}=\pi^3$$

Proposed by Naren Bhandari-Nepal

S.937 For $a,b,c,m,n>0$ prove that:

$$i)\ n>m:\ \sum_{cyc}\frac{a^n+b^n}{a^m+b^m}\ge(ab)^{\frac{n-m}{2}}+(bc)^{\frac{n-m}{2}}+(ca)^{\frac{n-m}{2}}$$

$$ii)\ n<m:\ \sum_{cyc}\frac{a^n+b^n}{a^m+b^m}\le(ab)^{\frac{n-m}{2}}+(bc)^{\frac{n-m}{2}}+(ca)^{\frac{n-m}{2}}$$

Proposed by Pavlos Trifon-Greece

S.938 If $x,y,z>0$ then prove:

$$\frac{x^3+y^3+z^3+x+y+z}{x^2+y^2+z^2}+\frac{x^5+y^5+z^5+x^3+y^3+z^3}{x^4+y^4+z^4}\ge 4$$

Proposed by Jay Jay Oweifa-Nigeria

S.939 If $z=x+iy\in\mathbb{C},x,y\in\mathbb{R}$ then:

$$|\sin z|^2+|\sinh z|^2+|\cos z|^2+|\cosh z|^2\ge\sinh(2x)+\cosh(2y)$$

Proposed by Daniel Sitaru – Romania

S.940 If $0\le a\le\frac{\pi}{12}$ then:

$$\int_0^a\sin x\cdot\cos(6x)\cdot\cos^6(4x)\cdot\cos^{15}(2x)\,dx\le\frac{1}{193}(1-\cos^{193}a)$$

Proposed by Daniel Sitaru – Romania

S.941 If $u,v,w\in\mathbb{C},|u|=3,|v|=5,|w|=7$ then:

$$|u+v+w|+15\ge\left|\frac{5u}{3}+\frac{3v}{5}\right|+\left|\frac{7v}{5}+\frac{5w}{7}\right|+\left|\frac{3w}{7}+\frac{7u}{3}\right|$$

Proposed by Daniel Sitaru – Romania

S.942 If $0<a\le b$ then:

$$3\int_a^b\sqrt{x^4+x^2+1}dx\ge(b-a)\sqrt{(a^2+ab+b^2)^2+3(a^2+ab+b^2)+9}$$

Proposed by Daniel Sitaru – Romania

S.943 If $x,y\ge 0,n\in\mathbb{N}$ then:

$$(x^{n+1}+y^{n+1})^{n-1}\cdot(x+y)^{n+1}\le 2^{n-1}\cdot(x^n+y^n)^{n+1}$$

Proposed by Daniel Sitaru – Romania

S.944 If $0 < a \leq b$ then:

$$\int_a^b \sinh x\left(e^{\sinh^2 x} + e^{\cosh^2 x}\right)dx \geq \frac{289}{105}\left(\sqrt[17]{(\cosh b)^{18}} - \sqrt[17]{(\cosh a)^{18}}\right)$$

Proposed by Daniel Sitaru – Romania

S.945 Solve for real numbers:

$$(x - \sin x)(x^2 - \cos^2 x) + x^2(\sin x - 1)(\sin^2 x + 1 - \tan^2 x) =$$
$$= \sin^2 x\,(1 - x)(1 + x^2 - \sin^2 x)$$

Proposed by Daniel Sitaru – Romania

S.946 $z \in \mathbb{C} - \{-i, i\}, |z| = 1, Imz > 0, z -$ fixed. Solve for real numbers:

$$(x+1)^2 + \left(\frac{|z+1| + |z-1|}{|z+i|}\right)^2 x + \left(\frac{|z+1| - |z-1|}{|z-i|}\right)^2 = 0$$

Proposed by Daniel Sitaru – Romania

S.947 If $A, B, C \in M_2(\mathbb{R}), \det A > 0, \det B > 0, \det C > 0, \det(ABC) = 8$ then:

$$\det(A + B + C) + \det(-A + B + C) + \det(A - B + C) + \det(A + B - C) \geq 24$$

Proposed by Daniel Sitaru – Romania

S.948 If $a, b, c > 0, a + b + c = 3$ then:

$$e^{a^2} + e^{b^2} + e^{c^2} + 2\sqrt[4]{e^{(a+b)^2}} + 2\sqrt[4]{e^{(b+c)^2}} + 2\sqrt[4]{e^{(c+a)^2}} \geq 9$$

Proposed by Daniel Sitaru – Romania

S.949 Solve for real numbers:

$$\int_1^x \frac{t \cdot \log t}{t^4 + x^2}dt = 0$$

Proposed by Daniel Sitaru – Romania

S.950 Find:

$$\Omega = \lim_{n\to\infty}(n-1)!\sum_{k=0}^{n}\frac{1}{(k+1)^k(n-k+1)^{n-k}}$$

Proposed by Daniel Sitaru – Romania

S.951 Solve for natural numbers:

$$\sum_{i=0}^{n}\sum_{j=0}^{n} 3^{i+j-60}\binom{3n-i-j}{2n-i-j}\binom{2n-i-j}{n-j} = 1$$

Proposed by Daniel Sitaru – Romania

S.952 Find:

$$\Omega = \int \frac{x^4}{x^4 \ln^4 4 + 4(x^3 \ln^3 4 + 3x^2 \ln^2 4 + 6x \ln 4 + 6 + 6 \cdot 4^x)}dx$$

Proposed by Daniel Sitaru – Romania

S.953 If $x, y, z > 0, xy + yz + zx = 3$ then in ΔABC the following relationship holds:

$$\frac{\tan^4 A \cdot \tan^4 B}{x^3 y^3} + \frac{\tan^4 B \cdot \tan^4 C}{y^3 z^3} + \frac{\tan^4 C \cdot \tan^4 A}{z^3 x^3} \geq 243$$

Proposed by Daniel Sitaru – Romania

S.954 K – Lemoine's point in ΔABC. Prove that:

$$\frac{m_a}{AK \cdot \sin A} + \frac{m_b}{BK \cdot \sin B} + \frac{m_c}{CK \cdot \sin C} \geq 3\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

S.955 Solve for real numbers:

$$\cos x \cdot \sqrt{\tan x} = \sin^3 x + \cos^3 x$$

Proposed by Daniel Sitaru – Romania

S.956 Find:

$$\Omega(n) = \int \frac{x^{2n-1}(1-x^2)}{e^{nx^2}} dx, n \in \mathbb{N}, n \geq 1$$

Proposed by Daniel Sitaru – Romania

S.957 If $a, b \geq e\sqrt{e}$ then:

$$\left(\left(\frac{a+2b}{3}\right)^{2a+b} \cdot \left(\frac{2a+b}{3}\right)^{a+2b}\right)^{3ab} \leq (a^b \cdot b^a)^{(a+2b)(2a+b)}$$

Proposed by Daniel Sitaru – Romania

S.958 Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\sum_{i=1}^{k} i\left(i+\frac{1}{2}\right)(i+1)\right]^{-1}$$

Proposed by Vasile Mircea Popa – Romania

S.959 Prove that the number:

$$\frac{\sqrt[3]{5+\sqrt[3]{3}} - \sqrt[3]{-1+\sqrt[3]{3}}}{\sqrt[3]{5+\sqrt[3]{3}} - 2\sqrt[3]{-1+\sqrt[3]{3}}}$$

is a solution of the equation: $2x^3 - 9x^2 + 9x - 3 = 0$

Proposed by Vasile Mircea Popa – Romania

S.960 $A \in M_2(\mathbb{C}), \det A = 1$. Prove that:

$$A^2 B - BA^2 = BA^{-2} - A^{-2} B, \forall B \in M_2(B).$$

Proposed by Marian Ursărescu-Romania

S.961 In ΔABC the following relationship holds:

$$9 \le \sum \frac{2\cot\frac{A}{2}\cot^2\frac{B}{2}}{\cot\frac{A}{2}+\cot\frac{B}{2}} \le \left(\frac{2R}{r}-1\right)^2$$

Proposed by Marian Ursărescu-Romania

S.962 In ΔABC the following relationship holds:

$$\frac{27r^3}{R} \le \sum \frac{m_a m_b^2}{m_a+m_b} \le \frac{27R^2}{8}$$

Proposed by Marian Ursărescu-Romania

S.963 $a \in M_2(\mathbb{R}), \det(A^{4042}+2021I_2)=0$. Find: $\Omega=\det A$.

Proposed by Marian Ursărescu-Romania

S.964 Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\log\left(1+\frac{1}{k}\right)\left(\tan^{-1}\left(\frac{1}{\sqrt{k}}\right)\right)^2$$

Proposed by Florică Anastase-Romania

S.965 $n \in \mathbb{N}^*, n \ge 2$. For $a, x_i, t \in \mathbb{R}, i \in \overline{1,n}, t \ne k\pi, k \in \mathbb{Z}, a \ge 2$ Prove that:

$$\left(\sum_{k=1}^{n} k(n-k)\cos tk - \sum_{k=1}^{n} k(n-k)\sin kt\right)\left(\sum_{i=1}^{n}\frac{\cos x_i}{a^{i-1}} - \sum_{i=1}^{n}\frac{\sin x_i}{a^{i-1}}\right) = 0 \Leftrightarrow \tan\frac{nt}{2} = n\tan\frac{t}{2}.$$

Proposed by Florică Anastase-Romania

S.966 In ΔABC the following relationship holds:

$$\sum \frac{h_b+h_c}{b+c} \le \sum \frac{r_b+r_c}{b+c}$$

Proposed by Marin Chirciu – Romania

S.967 In ΔABC the following relationship holds:

$$3\sum b^3c^3\tan\frac{A}{2} \le \sum b^3c^3\tan\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.968 If $a, b, c > 0$ and $\lambda \ge 0$ then:

$$\sum \frac{a^3}{(b+\lambda c)(b^2+\lambda c^2)} \ge \frac{3}{(\lambda+1)^2}$$

Proposed by Marin Chirciu – Romania

S.969 If $a,b,c>0$ and $\lambda\ge 0,\mu\ge 0$ then:

$$\sum\frac{a^3}{(b+\lambda c)(b^2+\mu c^2)}\ge\frac{3}{(\lambda+1)(\mu+1)}$$

Proposed by Marin Chirciu – Romania

S.970 In ΔABC the following relationship holds:

$$r^2(8R-7r)\le\sum(p-a)^3\tan\frac{A}{2}\le r(2R-r)^2$$

Proposed by Marin Chirciu – Romania

S.971 Solve for real numbers:

$$2x\sqrt{2x-1}=x^2(x+1)-x+1$$

Proposed by Marin Chirciu – Romania

S.972 In ΔABC the following relationship holds:

$$48r^2\le\frac{a^4}{r_br_c}+\frac{b^4}{r_cr_a}+\frac{c^4}{r_ar_b}\le\frac{16}{r}(R^3-5r^3)$$

Proposed by Marin Chirciu – Romania

S.973 In ΔABC the following relationship holds:

$$\frac{16}{9}\sum m_bm_c\le\frac{\sum a^4+3\sum b^2c^2}{\sum a^2}$$

Proposed by Marin Chirciu – Romania

S.974 In ΔABC the following relationship holds:

$$\sum\frac{(r_b+r_c)^2}{b^2+c^2}\le\frac{9R}{4r}$$

Proposed by Marin Chirciu – Romania

S.975 In ΔABC the following relationship holds:

$$m_a\sqrt{s_a}+m_b\sqrt{s_b}+m_c\sqrt{s_c}\le\frac{9\sqrt{6}}{4}R^{\frac{3}{2}}$$

Proposed by Marin Chirciu – Romania

S.976 In ΔABC the following relationship holds:

$$p\le\sum\frac{a^3}{b^2+c^2}\le\frac{R^2(4R+r)^2}{6r^2p}$$

Proposed by Marin Chirciu – Romania

S.977 In ΔABC the following relationship holds:

$$\frac{(m_a^2+m_b^2)^2+(m_b^2+m_c^2)^2+(m_c^2+m_a^2)^2}{m_a^2+m_b^2+m_c^2}\le 9R^2$$

Proposed by Marin Chirciu – Romania

S.978 In ΔABC the following relationship holds:

$$3\left(5-\frac{2r}{R}\right)\le\sum\frac{(a+b)(a+c)}{bc}\le 4\left(\frac{R}{r}+1\right)$$

Proposed by Marin Chirciu – Romania

S.979 In ΔABC the following relationship holds:

$$\frac{4}{9R^2r}\le\sum\frac{1}{w_a^3}\le\frac{2R^2-Rr}{54r^3}$$

Proposed by Marin Chirciu – Romania

S.980 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s\sqrt{3}-m_a}{a}\ge 8\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$$

Proposed by Bogdan Fuștei – Romania

S.981 In ΔABC the following relationship holds:

$$\frac{8R}{s}\cdot\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\ge\sqrt{\frac{r_a+r_c}{r_a+r_b}}+\sqrt{\frac{r_a+r_b}{r_a+r_c}}$$

Proposed by Bogdan Fuștei – Romania

S.982 In $\Delta ABC, A\ge B\ge\frac{\pi}{3}\ge C$, the following relationship holds:

$$n_a+n_b+n_c\ge 3(R+r)$$

Proposed by Bogdan Fuștei – Romania

S.983 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{|b-c|}{b+c}\ge\frac{m_a-s_a}{a}+\frac{m_b-s_b}{b}+\frac{m_c-s_c}{c}$$

Proposed by Bogdan Fuștei – Romania

S.984 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\ge\frac{1}{4}\left(\sum_{cyc}\frac{b+c}{a}+\sum_{cyc}\frac{m_b+m_c}{m_a}\right)$$

Proposed by Bogdan Fuștei – Romania

S.985 In ΔABC the following relationship holds:

$$\max\{a,b,c\}-\min\{a,b,c\}\ge\sum_{cyc}(m_a-s_a)$$

Proposed by Bogdan Fuștei – Romania

S.986 In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\frac{AI+BI+CI}{r}\ge\sum_{cyc}\sqrt{\frac{2(n_a+h_a)}{r_a}}$$

Proposed by Bogdan Fuștei – Romania

S.987 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{R}{r}\ge 1+\frac{\sqrt{\sum n_a n_b}}{s}\ge 2$$

Proposed by Bogdan Fuștei – Romania

S.988 In ΔABC the following relationship holds:

$$|b-c|\ge\frac{1}{2}(n_a+m_a-g_a-s_a)$$

Proposed by Bogdan Fuștei – Romania

S.989 In ΔABC the following relationship holds:

$$\frac{\sqrt{r_a}+\sqrt{r_b}+\sqrt{r_c}}{r}\ge\sum_{cyc}\sqrt{\frac{2(n_a+h_a)}{(r_b-r)(r_c-r)}}$$

Proposed by Bogdan Fuștei – Romania

S.990 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\frac{g_a r_a+g_b r_b+g_c r_c}{r}\ge\sum_{cyc}\left(n_a+\frac{2r_a h_a}{n_a}\right)$$

Proposed by Bogdan Fuștei – Romania

S.991 If $0<a<b$ and $c\in\mathbb{R}_+^*=(0,\infty)$, find:

$$\int_a^b\frac{x(x+2)}{(x+2)^4+c^2x^4}dx$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.992 If $x\in[1,\infty)\setminus\mathbb{N}^*$ then in any ABC triangle with the area F the following inequality holds:

$$a^4+\frac{[x]b^4}{x+\{x\}}+\frac{\{x\}c^4}{x+[x]}\geq 8F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.993 If $x,y,z>0$ and ABC is a triangle with the area F, then:

$$\frac{x\cdot w_a+y\cdot w_b}{z}c^3+\frac{y\cdot w_b+z\cdot w_c}{x}a^3+\frac{z\cdot w_c+x\cdot w_a}{y}b^3\geq 16\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.994 If $x,y,z\in(0,1)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{x\cdot a^8}{(y+z)^2(1-x^2)}+\frac{y\cdot b^8}{(z+x)^2(1-y^2)}+\frac{z\cdot c^8}{(x+y)^2(1-z^2)}\geq 32\sqrt{3}F^4$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.995 In any ABC triangle the following inequality holds:

$$\sum_{cyc}(n_a^2+g_a^2+2r\cdot r)\geq 6\sqrt{3}\cdot\frac{r}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.996 If $m\geq 0$ and $x,y>0$, then in any ABC triangle the following inequality holds:

$$\frac{(xb+yc)^{2m+2}}{w_b\cdot w_c}+\frac{(xc+ya)^{2m+2}}{w_c\cdot w_a}+\frac{(xa+yb)^{2m+2}}{w_a\cdot w_b}\geq 4^{m+1}\cdot 3^m\cdot(x+y)^{2m+2}\cdot r^{2m}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.997 If $n\in\mathbb{N}^*-\{1,2\}$ and $x_k\in(0,\infty),\forall k=\overline{1,n}$ and $\sum_{k=1}^n x_k^2=A,\sum_{k=1}^n x_k=X_n$, then

$$\sum_{k=1}^n\sqrt{A-x_k^2}\geq\sqrt{n-1}X_n$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.998 If $m,n,p\geq 0;x,y,z>0$ and at least one of m,n,p is non-zero, then in any ABC triangle with the area F the following inequality holds:

$$\frac{mx+ny+pz}{m(y+z)+n(z+x)+p(x+y)}a^2+\frac{my+nz+px}{m(z+x)+n(x+y)+p(y+z)}b^2+$$
$$+\frac{mz+nx+py}{m(x+y)+n(y+z)+p(z+x)}c^2\geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.999 Let be $m \in \mathbb{N}$, the in any ABC triangle the following inequality holds:

$$m + 3^m\left(\cot^{2m+2}\frac{A}{2} + \cot^{2m+2}\frac{B}{2} + \cot^{2m+2}\frac{C}{2}\right) \geq$$

$$\geq (m+1)\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)(\cot A + \cot B + \cot C)$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1000 Let be $m, n \in \mathbb{R}_+ = [0,\infty); m + n \in \mathbb{R}_+^* = (0,\infty)$, then in any ABC triangle with the area F the following inequality holds:

$$(m^2 + n^2)(a^2 + b^2 + c^2) \geq 8mn\sqrt{3}F + (ma - nb)^2 + (mb - na)^2 + (mc - na)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1001 Let be a triangle ABC and the points $M \in (BC), N \in (CA), P \in (AB)$. If the cevians AM, BN, CP are concurrent, then:

$$\frac{MB \cdot a^2}{MC} + \frac{NC \cdot b^2}{NA} + \frac{PA \cdot c^2}{PB} \geq 4\sqrt{3}F$$

where F is the area of the triangle.

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1002 Let be $x, y, z \in \mathbb{R}_+^* = (0,\infty)$, the triangle ABC and the points $M \in (BC), N \in (CA)$, $P \in (AB)$. If the cevians AM, BN, CP are concurrent, then:

$$MB^3 \cdot a^2 + NC^3 \cdot b^2 + PA^3 \cdot c^2 \geq 4\sqrt{3} \cdot MC \cdot NA \cdot PB \cdot F$$

where F is the area of the triangle.

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1003 If $A_1A_2 \dots A_n, n \geq 3$ is a convexe polygon having the area F and the lengths sides $a_k = A_kA_{k+1}, \forall k = \overline{1,n}, A_{n+1} = A_1$, then:

$$\sum_{k=1}^{n} \frac{a_k^8 + 1}{\sqrt{a_k^8 - a_k^4 + 1}} \geq 8F \cdot \tan\frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1004 $ABCD$ – cyclic quadrilateral, $E \in Int\ (ABCD), EI \perp AB, EG \perp BD, EH \perp CD, EF \perp AC$,

$EI = a, EG = b, EH = c, EF = d, AI = IB, BG = GD, DH = HC, FC = FA.$

Find circumradii of $[ABCD]$ in terms of a, b, c, d.

Proposed by Amerul Hassan-Myanmar

S.1005 $a_1=4, a_2=2, a_n=a_{n+1}^{\frac{3n+3}{7n}}\cdot a_{n+2}^{\frac{4n+8}{7n}}$. Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\frac{a_k}{a_{k+1}}\right)^k\right)\left(\frac{1}{n!}\sum_{k=1}^{n+1}k\cdot(k+1)\right)^{-1}$$

Proposed by Ruxandra Daniela Tonilă – Romania

S.1006 In $\Delta ABC, N_a$ – Nagel's point, I – incentre, $S_1=[AIN_a], S_3=[BIN_a], S_3=[CIN_a]$.

Prove that:

$$S_1=S_2+S_3 \vee S_2=S_3+S_1 \vee S_3=S_1+S_2$$

Proposed by Adil Abdullayev-Azerbaijan

S.1007 If $0<a\le b$ then:

$$\int_a^b \ln\left(\frac{x+b}{x+a}\right)dx \ge \frac{(b-a)^2}{b+a}$$

Proposed by Asmat Qatea-Afghanistan

S.1008 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i^3+j^3}{i^4+j^4}-\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{k^3-l^3}{k^4-l^4}\right)\right)$$

Proposed by Mikael Bernardo-Nigeria

S.1009 In ΔABC the following relationship holds:

$$\left(\sqrt{\frac{as_ah_a}{3}}+\sqrt{\frac{bs_bh_b}{3}}+\sqrt{\frac{cs_ch_c}{3}}\right)^2\le\frac{\sqrt{3(a^2+b^2+c^2)^3}}{3}$$

Proposed by Radu Diaconu – Romania

S.1010 If $0\le x,y\le\frac{\pi}{2}$ then:

$$(\sin x)^{2\cos x}+(\cos x)^{2\sin x}\le 1+\sin x\cos x$$

Proposed by Seyran Ibrahimov-Azerbaijan

S.1011 We define a progressive sequence as:

$$\begin{cases} q_1=\dfrac{3}{2} \\ q_{n+1}=\dfrac{(q_n)^2+2}{2q_n}; n\ge 1\end{cases}$$

And let's put: $P_n=\sum_{k=1}^{n}\frac{1}{\sqrt{2k+\sqrt{4k^2-1}}}$

Then prove that: $P_n = O(\sqrt{n}q_n)$. Or in other words: $P_n \sim \sqrt{n}q_n$

Proposed by Samir HajAli-Syria

S.1012 If m and n are natural numbers such that $m > 1$, then prove that:

$$\binom{mn}{n} = \left(\frac{m^m}{(m-1)^{m-1}}\right)^n \frac{\left(\frac{m-1}{m}\right)^{(n)}}{n!} \prod_{k=1}^{m-2} \frac{\left(\frac{k}{m}\right)^{(n)}}{\left(\frac{k}{m-1}\right)^{(n)}}$$

Where $x^{(n)} = x(x+1)(x+2)\dots(x+n-1)$ and $x^{(0)} = 1$.

Proposed by Angad Singh – India

S.1013 Prove without softs:

$$\sum_{n=1}^{2020} \frac{1}{\sqrt{n+k}} < 4040, k > 0, k - fixed$$

Proposed by Nikos Ntorvas-Greece

S.1014 In ΔABC the following relationship holds:

$$ab\cos A + bc\cos B + ca\cos C \le \frac{9R^2}{2}$$

Proposed by Ionuț Florin Voinea – Romania

S.1015 Prove that:

$$\frac{1}{2}\csc\left(\frac{\pi}{16}\right) = \cos\left(\frac{\pi}{16}\right) + \cos\left(\frac{3\pi}{16}\right) + \sin\left(\frac{\pi}{16}\right) + \sin\left(\frac{3\pi}{16}\right)$$

Proposed by Mohammed Bouras-Morocco

S.1016 If $0 < a \le b$ then:

$$(\sqrt{a} + \sqrt{b})(\arctan b - \arctan\sqrt{ab}) \le \sqrt{b}(\arctan b - \arctan a)$$

Proposed by Daniel Sitaru – Romania

S.1017 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\sqrt{5\sqrt{8\sqrt{11\sqrt{\dots\sqrt{3n-1}}}}}}{\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \dots + (-1)^{n-1}\frac{1}{n}\binom{n}{n}}\right)$$

Proposed by Daniel Sitaru – Romania

S.1018 Find:

$$\Omega = \lim_{n\to\infty} \int_{e^{H_n}}^{e^{H_{n+1}}} \frac{n^7}{(x+n)^7 - x^7 - n^7}\,dx$$

Proposed by Daniel Sitaru – Romania

S.1019 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \ldots \cdot \sqrt[2^n]{2^n}}{n} \right)$$

Proposed by Daniel Sitaru – Romania

S.1020 If $a, b, c \in \mathbb{C}, |a| + |b| + |c| = \frac{1}{4}$ then:

$$\left| (a+b+c)^3 + 8abc + \prod_{cyc} (a+b-c) \right| \le |ab| + |bc| + |ca|$$

Proposed by Daniel Sitaru – Romania

S.1021 If $0 < a \le b < \frac{\pi}{2}$ then:

$$\log\left|\frac{\pi^2 - 4a^2}{\pi^2 - 4b^2}\right| \le \log\left|\frac{\cos a}{\cos b}\right| \le \frac{\pi^2}{8} \log\left|\frac{\pi^2 - 4a^2}{\pi^2 - 4b^2}\right|$$

Proposed by Daniel Sitaru – Romania

S.1022 If $x, y > 0, x + y = \sqrt{\tan^{-1}\left(\frac{1}{5}\right)}$ then:

$$\frac{4x^2}{\pi} + \frac{y^2}{\tan^{-1}\left(\frac{1}{239}\right)} \ge \frac{1}{4}$$

Proposed by Daniel Sitaru – Romania

S.1023 Find:

$$\Omega = \lim_{n\to\infty} \left(\binom{2n}{n}^{-\frac{1}{n}} \cdot \int_0^\infty (x-1)^n e^x dx \cdot \left((n-2)!\right)^{-1} \right)$$

Proposed by Daniel Sitaru – Romania

S.1024 If $0 < x, y, z < 1$ then:

$$\left(\frac{1-y}{x}\right)^{1-y} \cdot \left(\frac{1-z}{y}\right)^{1-z} \cdot \left(\frac{1-x}{z}\right)^{1-x} \ge \frac{1}{(x+y)(y+z)(z+x)}$$

Proposed by Daniel Sitaru – Romania

S.1025 If $a, b, c > 0$ then:

$$\sum_{cyc} a^{11} \cdot \left(\sum_{cyc} a\right)^2 \geq \sum_{cyc} a^7 \cdot \sum_{cyc} a^4 \cdot \sum_{cyc} a^2$$

Proposed by Daniel Sitaru – Romania

S.1026 If $0 < a \leq b < \frac{\pi^3}{8}$ then:

$$\int_{\sqrt[3]{a}}^{\sqrt[3]{b}} \sin x \cdot \sinh x \leq \frac{b-a}{3}$$

Proposed by Daniel Sitaru – Romania

S.1027 If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\frac{8}{\pi^2} \int_a^b \log\left(\sec\left(\frac{\pi \sin x}{2}\right)\right) dx \leq b - a + \tan b - \tan a$$

Proposed by Daniel Sitaru – Romania

S.1028 If $0 < a \leq b < \frac{\pi}{2}$ then:

$$3(b-a) + 3(\sin b - \sin a) \leq \int_a^b \frac{\sin x}{x} dx \leq 4(b-a) + 2(\sin b - \sin a)$$

Proposed by Daniel Sitaru – Romania

S.1029 Solve for real numbers:

$$\frac{(2x^4 + 5x^2 - 4x + 1)(x^4 + 9x^2 - 8x + 2)(x^4 + 6x^2 - 4x + 1)}{x^2(x+1)^2(3x-1)^2(x^2 + 2x - 1)^2} = 1$$

Proposed by Daniel Sitaru – Romania

S.1030 Find:

$$\Omega = \lim_{n \to \infty} \frac{\sqrt{1! + \sqrt{2! + \sqrt{3! + \cdots + \sqrt{n!}}}}}{H_n}$$

Proposed by Daniel Sitaru,Dan Nănuți – Romania

S.1031 If $0 < a \leq b$ then:

$$(b-a) \int_a^b \left(\sqrt{\frac{b^2 + x^2}{a^2 + x^2}} - \sqrt{\frac{a^2 + x^2}{b^2 + x^2}}\right) dx \leq \int_a^b \frac{1}{x}\left(\tan^{-1}\frac{b}{x} - \tan^{-1}\frac{a}{x}\right)$$

Proposed by Daniel Sitaru,Dan Nănuți – Romania

S.1032 If $0 < a \leq b$ then:

$$\int_a^b 2^{\frac{1}{\sqrt{x}}} dx \geq 3(b-a) + \frac{2^a - 2^b}{\log 4}$$

Proposed by Daniel Sitaru,Claudia Nănuți – Romania

S.1033 If $n \in \mathbb{N}, n \geq 2$ then:

$$\frac{4}{n\pi} \sum_{k=2}^{n} \left(\int_{\frac{1}{k}}^{k} \frac{1}{x} \cdot \tan^{-1} x \, dx \right) \geq \log n$$

Proposed by Daniel Sitaru,Claudia Nănuți – Romania

S.1034 Find without any software:

$$\Omega = \int \frac{\log x}{\log^2 x + (2 - 2ex)\log x + 2e^2x^2 - 2ex + 1} dx$$

Proposed by Daniel Sitaru – Romania

S.1035 In ΔABC the following relationship holds:

$$\frac{a^4(\pi - \mu(A))}{\sin A} + \frac{b^4(\pi - \mu(B))}{\sin B} + \frac{c^4(\pi - \mu(C))}{\sin C} > \frac{16\sqrt{3}\pi F}{3}$$

Proposed by Daniel Sitaru – Romania

S.1036 Find:

$$\Omega = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k+1} \cdot \sqrt{\left(\int_0^1 \frac{x^k dx}{1+x^2} \right)\left(\int_0^1 \frac{x^{k+2} dx}{1+x^2} \right)} \right)$$

Proposed by Daniel Sitaru – Romania

S.1037 Find:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x - \cos x}{1 + \sin x \cos x}$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

S.1038 If $\alpha \in \mathbb{N}, \alpha \geq 1$ then in ΔABC the following relationship holds:

$$\max\left(\frac{n_a g_a}{\hat{A}}, \frac{n_b g_b}{\hat{B}}, \frac{n_c g_c}{\hat{C}} \right) \geq \frac{3 \cdot p^2 \cdot \left(\sqrt{3} \cdot S\right)^\alpha}{\pi \cdot (n_a^{2\alpha} + n_b^{2\alpha} + n_c^{2\alpha})}$$

Proposed by Radu Diaconu – Romania

S.1039 Show that there are two numbers of a and b in the range $\left(0, \frac{\pi}{4}\right)$ that does satisfy the following equation:

$$\cos a + \sin b = \frac{4}{\pi}$$

Proposed by Ata Marangoz-Turkiye

S.1040 In $\Delta ABC, r_L$ – inradii of pedal triangle of Lemoine's point in ΔABC. Prove that:

$$r_L \le \frac{s}{6\sqrt{3}}$$

Proposed by Mehmet Șahin – Turkey

S.1041 Solve for natural numbers:

$$\begin{cases} 4xyz = (z+y)^3 \\ \dfrac{y+z}{x} + \dfrac{x+z}{y} + \dfrac{x+y}{z} = 7 \\ \dfrac{1}{xy} + \dfrac{1}{yz} + \dfrac{1}{xz} = 2 \end{cases}$$

Proposed by Mokhtar Khassani-Algerie

S.1042 In ΔABC the following relationship holds:

$$\frac{\sqrt{m_a^2 + m_b^2 + m_c^2}}{a\cos A + b\cos B + c\cos C} \ge \frac{R}{2r}$$

Proposed by Haxverdiyev Taverdi-Azerbaijan

S.1043 Without softwares: $a^x = bx, a, b \in \mathbb{N}; a \neq b$.Find the value of X

Proposed by Hussain Reza Zadah-Afghanistan

S.1044 Find without softwares:

$$\Omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2020x) \cdot \cos^{2019} x \, dx$$

Proposed by Kafunda Tuesday-Nigeria

S.1045 Find without any software:

$$\Omega(n) = \int \frac{\sqrt[n]{\sin x} - \sqrt[n]{\cos x}}{\left(\sqrt[n]{\sin x} + \sqrt[n]{\cos x}\right)^{2n}\left(\sqrt[n]{\sin x} + \sqrt[n]{\cos x}\right)} dx, n \in \mathbb{N}, n \ge 2$$

Proposed by Serlea Kabay – Liberia

S.1046 Solve for real numbers:

$$\sqrt[a]{b^2c + bc^2 + x(b+c)} - \sqrt[a]{x^2 + bc^2 + xc(1+b)} = \sqrt[a]{b^2c - x^2 + bx(1-c)}$$

Where $a \le b + c, a \in \mathbb{N} - \{0,1\}, b, c \in \mathbb{R}^2$

Proposed by Serlea Kabay – Liberia

S.1047 Let $\in (0,\infty), n \in \mathbb{N}^*, x_k \in (0,\infty), \forall k \in \overline{1,n}, X_n = \sum_{k=1}^{n} x_k$, then:

$$\sum_{k=1}^{n} \frac{ax_k + b(X_n - x_k)}{x_k} \ge n(a - b + bn)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1048 In any ABC triangle with the area F the following inequality holds:

$$\left(\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}\right)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\geq\frac{9}{16F^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1049 If $x,y,z>0$, then in any ABC triangle with the area F the following inequality holds:

$$\sum_{cyc}\left(\frac{x+y}{z}ab+\frac{z}{x+y}c^2\right)^{m+1}\geq 10^{m+1}\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1050 If $m\geq 0,t,u,v,x,y,z>0$ then in any ABC triangle having the area F the following inequality holds:

$$\sum_{cyc}\left(\frac{t+u}{v}ab+\frac{z}{x+y}c^2\right)^{m+1}\geq 10^{m+1}\left(\sqrt{3}\right)^{1-m}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1051 If $x,y,z>0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{y+z}{xh_bh_c}+\frac{x}{(y+z)h_a^2}\right)^2+\left(\frac{z+x}{yh_ch_a}+\frac{y}{(z+x)h_b^2}\right)^2+\left(\frac{x+y}{zh_ah_b}+\frac{z}{(x+y)h_c^2}\right)^2\geq\frac{25}{4F^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1052 If $x,y,z>0$, then in any ABC triangle with the area F the following inequality holds:

$$\left(\frac{x+y}{z}ab+\frac{z}{x+y}c^2\right)^2+\left(\frac{y+z}{x}bc+\frac{x}{y+z}a^2\right)^2+\left(\frac{z+x}{y}ca+\frac{y}{z+x}b^2\right)^2\geq 64F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1053 In any ABC triangle with the area F the following inequality holds:

$$m_a^2w_a^2+m_b^2w_b^2+m_c^2w_c^2\geq 9F^2+F^2\sum_{cyc}\left(\frac{b-c}{b+c}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1054 In any ABC triangle with the area F the following inequality holds:

$$m_a^2w_a^2+m_b^2w_b^2+m_c^2w_c^2\geq 243r^4$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1055 In any ABC triangle with the area F the following inequality holds:

$$\sum_{cyc}\left(n_a^2+g_a^2+2r_br_c\right)\geq 12\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1056 If $m \geq 0$, then in any ABC triangle the following inequality holds:

$$\frac{a^{m+1}}{h_b^{m+1}}+\frac{b^{m+1}}{h_c^{m+1}}+\frac{c^{m+1}}{h_a^{m+1}} \geq 2^{m+1}\left(\sqrt{3}\right)^{1-m}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1057 If $m,n > 0$ and $x \geq 0$ then:

$$e^{mx}+e^{n[x]}+e^{n\{x\}} \geq 3+(m+n)x$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1058 In any ABC triangle with the area F the following inequality holds:

$$\frac{1}{2}(a^2+b^2+c^2)+\frac{a^2b^2}{a^2+b^2}+\frac{b^2c^2}{b^2+c^2}+\frac{c^2a^2}{c^2+a^2} \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1059 In any ABC triangle with the area F the following inequality holds:

$$\sum_{cyc} a^3b^2c+\sum_{cyc}\frac{ab^2}{c} \geq 32F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1060 If $x,y,z \in \left(0,\frac{\pi}{2}\right)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{(\sin y+\sin z)a^2}{\sin^2 2x}+\frac{(\sin z+\sin x)b^2}{\sin^2 2y}+\frac{(\sin x+\sin y)c^2}{\sin^2 2z} \geq 9F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1061 Solve for real numbers:

$$\frac{16}{(x^4+y^2)(x^6+y^4)(x^2+y^6)}=\frac{1}{x^{12}}+\frac{1}{y^{12}}$$

Proposed by Daniel Sitaru – Romania

S.1062 If $n \in \mathbb{N}, n \geq 2$ then:

$$\sum_{k=2}^{n}\left(H_k+\sqrt[k]{k^{k-1}}\right) < \frac{(n-1)(n+4)}{2}$$

Proposed by Daniel Sitaru – Romania

S.1063 Find without any software:

$$\Omega=\begin{vmatrix} \sin\frac{2\pi}{19} & \sin\frac{3\pi}{19} & \sin\frac{4\pi}{19} & \sin\frac{5\pi}{19} \\ \sin\frac{3\pi}{19} & \sin\frac{4\pi}{19} & \sin\frac{5\pi}{19} & \sin\frac{6\pi}{19} \\ \sin\frac{4\pi}{19} & \sin\frac{5\pi}{19} & \sin\frac{6\pi}{19} & \sin\frac{7\pi}{19} \\ \sin\frac{5\pi}{19} & \sin\frac{6\pi}{19} & \sin\frac{7\pi}{19} & \sin\frac{8\pi}{19} \end{vmatrix}$$

Proposed by Daniel Sitaru – Romania

S.1064 Solve for real numbers:

$$\sin x + \cos x \cdot \sin y + \cos x \cdot \cos y = 1$$

Proposed by Daniel Sitaru – Romania

S.1065 $x_0 = 1, x_1 = 0, x_n = (n-1)(x_{n-1} + x_{n-2}), n \geq 2, n \in \mathbb{N}$. Find:

$$\Omega = \lim_{n\to\infty} \frac{x_n}{n!}$$

Proposed by Daniel Sitaru – Romania

S.1066 If $x \geq y \geq z > 0, x + y + z = 3$ then in ΔABC holds:

$$(x-1) \cdot \frac{m_a}{w_a} + (y-1) \cdot \sqrt{\frac{b^2+c^2}{2bc}} + (z-1) \cdot \frac{b+c}{2\sqrt{bc}} \geq 0$$

Proposed by Daniel Sitaru – Romania

S.1067 If $x, y, z, t \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{\sin^4 t}{\cos x \cdot \cos y \cdot \cos z} + \frac{\cos^4 t}{\sin x \cdot \sin y \cdot \sin z} > 1$$

Proposed by Daniel Sitaru – Romania

S.1068 Solve for real numbers:

$$\log\left(\frac{yz}{x}\right)\left(\log^2 x - \log\left(\frac{zx}{y}\right)\log\left(\frac{xy}{z}\right)\right) = \log^2 y \cdot \log\left(\frac{y}{zx}\right) + \log^2 z \cdot \log\left(\frac{z}{xy}\right)$$

Proposed by Daniel Sitaru – Romania

S.1069 If $a, b, c > 0$ are such that $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{3}{4}$ then:

$$\sum_{cyc} \frac{a+2b}{a^2+2b^2} + \sum_{cyc} \frac{b+2a}{b^2+2a^2} \leq 3$$

Proposed by Daniel Sitaru – Romania

S.1070 $x_0 = 1, x_1 = \sqrt{2}, x_{n+1} + x_{n-1} = \sqrt{2}x_n, n \in \mathbb{N}^*$. Find:

$$\Omega(n) = \sum_{k=1}^{n} \sum_{i=1}^{8} (a_{2k+i} + a_{3k+i} + a_{5k+i})$$

Proposed by Daniel Sitaru – Romania

S.1071 Solve for real numbers:

$$\begin{cases} x^3 + y^3 = 516 - \sqrt[3]{x} - \sqrt[3]{y} \\ y^3 + z^3 = 20200 - \sqrt[3]{y} - \sqrt[3]{z} \\ z^3 + x^3 = 19688 - \sqrt[3]{z} - \sqrt[3]{x} \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.1072 Find without any software:

$$\Omega=\int_1^2 \frac{\log(9x-4)}{3x^2+2}\,dx$$

Proposed by Daniel Sitaru – Romania

S.1073 If $a,b,c>0,[*]$ - GIF, then:

$$3+([a])^a+([b])^b+([c])^c\ge a^{[a]}+b^{[b]}+c^{[c]}$$

Proposed by Daniel Sitaru – Romania

S.1074 Solve for real numbers:

$$\begin{cases}\dfrac{x^2+y^2}{2xy}\left(\dfrac{3(x+y)^2}{2xy}-1\right)=\dfrac{(x+y)^2}{xy}+\dfrac{2xy}{x^2+y^2}\\ \sin x\,(\sin^2 x+3\cos^2 y)=1+3\sin^2 x\cos y\end{cases}$$

Proposed by Daniel Sitaru – Romania

S.1075 If $a,b,c>0$ then:

$$\frac{5}{a}+\frac{8}{b}+\frac{9}{c}\ge\frac{8}{a+b}+\frac{24}{b+c}+\frac{12}{c+a}$$

Proposed by Daniel Sitaru – Romania

S.1076 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^n}$$

Proposed by Daniel Sitaru – Romania

S.1077 Solve for real numbers:

$$\begin{cases}x+y+3=3xy\\ \dfrac{x^2-y^2}{xy-1}+\dfrac{x^2-9}{3x-1}+\dfrac{y^2-9}{3y-1}=0\end{cases}$$

Proposed by Daniel Sitaru – Romania

S.1078 If $m\ge 0,x,y,z>0$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{a^2}{(xa+yb+zc)^m\cdot h_a^m}+\frac{b^2}{(xb+yc+za)^m h_b^m}+\frac{c^2}{(xc+ya+zb)^m\cdot h_c^m}\ge$$

$$\ge\frac{2^{2-m}\sqrt{3}}{(x+y+z)^m\cdot F^{m-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.1079 If $x, y, z > 0$ and ABC is a triangle with the area $F, M \in (BC), N \in (CA), P \in (AB)$ and $c_a = AM, c_b = BN, c_c = CP$, then:

$$\frac{y+z}{x} \cdot (c_b + c_c)a^3 + \frac{z+x}{y} \cdot (c_c + c_a) \cdot b^3 + \frac{x+y}{z} \cdot (c_a + c_b)c^3 \geq 32\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.1080 If $m \geq 0, n \geq 1$ then in any ABC triangle with the area F the following inequality holds:

$$\sum_{cyc} a^{2m+3} \cdot b^{2n-1} + \sum_{cyc} a^{2n-1} \cdot b^{2m+3} \geq 2^{2m+2n+3}\left(\sqrt{3}\right)^{1-m-n} F^{m+n+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.1081 If $m \geq 0$ then in any ABC triangle with the area F the following inequality holds:

$$a^{m+1}b + b^{m+1}c + c^{m+1}a \geq 2^{m+2} \cdot \left(\sqrt{3}\right)^{m+4} \cdot r^{m+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.1082 Let be $x, y > 0$ and ABC triangle with the area F, then there are two triangles MNP and UVW with the sides m, n, p, respectively u, v, w such that:

$$mu + nv + pw = \frac{4\sqrt{3}}{xy} \cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.1083 If $x, y, z > 0, ABC$ a triangle with the area F and the points $M \in (BC), N \in (CA), P \in (AB)$ such that the cevians AM, BN, CP are concurrent, then:

$$\frac{y+z}{x} \cdot \frac{a \cdot MB}{MC} + \frac{z+x}{y} \cdot \frac{b \cdot NC}{NA} + \frac{x+y}{z} \cdot \frac{c \cdot PA}{PB} \geq 4\sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1084 If $t \geq 0, ABC$ a triangle with the area F and M is an interior point in triangle. If x, y, z are the distances of point M to the apices A, B, C respectively u, v, w the distances of point M to the sides BC, CA, AB, respectively, then:

$$\frac{x \cdot a^{2t+2}}{v+w} + \frac{y \cdot b^{2t+2}}{w+u} + \frac{z \cdot c^{2t+2}}{u+v} \geq 4^{t+1} \cdot \left(\sqrt{3}\right)^{1-t} \cdot F^{t+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1085 If $m \in [1, \infty)$ and $x, y \in \mathbb{R}_+^* = (0, \infty)$, then in any ABC triangle the following inequality holds:

$$\frac{a}{(bx+cy)^{m+1}}+\frac{b}{(cx+ay)^{m+1}}+\frac{c}{(ax+by)^{m+1}}\geq\frac{\left(\sqrt{3}\right)^{2-m}}{(x+y)^{m+1}\cdot R^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1086 Let be $m\in\mathbb{R}_+=[0,\infty)$ and ABC is a triangle with the area F and the points $M\in(BC), N\in(CA), P\in(AB)$. If the cevians AM, BN, CP are concurrent, then:

$$\frac{MB\cdot a^{m+1}}{MC}+\frac{NC\cdot b^{m+1}}{NA}+\frac{PA\cdot c^{m+1}}{PB}\geq 2^{m+1}\cdot\left(\sqrt[4]{3}\right)^{3-m}\left(\sqrt{F}\right)^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1087 Let be $m\in\mathbb{N}, a,b,c$ the lengths sides of ABC triangle with the semiperimeter s, then:

$$3m+\left(bc(b+c)\right)^{m+1}+\left(ca(c+a)\right)^{m+1}+\left(ab(a+b)\right)^{m+1}\geq$$
$$\geq 48(m+1)(s-a)(s-b)(s-c)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1088 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ such that $xyz\geq d^3$ the in any ABC triangle with the area F the following inequality holds:

$$x\cdot m_a+y\cdot m_b+z\cdot m_c\geq 2d\cdot\frac{F}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1089 Let M be an interior point in ABC triangle and x,y,z the distances of M to the apices A,B,C and u,v,w the distances of M to the sides BC,CA,AB, then:

$$\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\geq 6$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1090 If $t\in\left(0,\sqrt{3}\right)$, then:

$$\left(\arctan^2 t+\arctan^2\left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right)+\frac{\pi^2}{9}\right)^2=2\left(\arctan^4 t+\arctan^4\left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right)+\frac{\pi^4}{81}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1091 Let be $x,y,z\in\mathbb{R}_+^*=(0,\infty), m,n,p\in\mathbb{R}_+=[0,\infty), m+n=2$ and ABC a triangle with the area F, then:

$$\frac{4x+3y+z+2p}{y+3z+p}\cdot\frac{a^m}{h_a^n}+\frac{x+4y+3z+2p}{z+3x+p}\cdot\frac{b^m}{h_b^n}+\frac{3x+y+4z+2p}{x+3y+p}\cdot\frac{c^m}{h_c^m}\geq$$
$$\geq 2^{3-n}\sqrt{3}\cdot F^{1-n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1092 If $x, y > 0$, then in any ABC triangle having the area F the following inequality holds:

$$\frac{x^2a^2}{y(x+y)}+\frac{y^2b^2}{x(x+y)}+\frac{xyc^2}{x^2+y^2}\geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1093 If $u \in \left(0, \frac{\pi}{2}\right)$, then in any ABC triangle the following inequality holds:

$$\frac{a}{h_a}\cdot\sin u+\frac{b}{h_b}\cos u+\frac{c}{h_c}\sin 2u\geq\sqrt{2}\cdot\sqrt{\sin u}\cdot\sqrt{1+2(\sin u+\cos u)}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1094 Let M be an interior point in ABC triangle. $x = MA, y = MB, z = MC$ and u, v, w are the distances from point M to the sides BC, CA, AM, then:

$$(x^2+1)(y^2+1)(z^2+1)\geq 9(uv+vw+wu)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1095 If $m \in \mathbb{N}$ and $a, b, x, y, z, t \in \mathbb{R}_+^* = (0, \infty)$, then:

$$4m+\frac{x^{7m+7}}{(ax^3+byzt)^{m+1}}+\frac{y^{7m+7}}{(ay^3+bxyt)^{m+1}}+\frac{z^{7m+7}}{(az^3+bxyt)^{m+1}}+\frac{t^{7m+7}}{(at^3+bxyz)^{m+1}}\geq$$

$$\geq\frac{4(m+1)xyzt}{a+b}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1096 If $x, y > 0$ then in any ABC triangle the following inequality holds:

$$\frac{x^2a^2}{y(x+y)h_c^2}+\frac{y^2b^2}{x(x+y)h_b^2}+\frac{xyc^2}{(x^2+y^2)h_c^2}\geq 2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1097 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and a, b, c are the lengths sides of ABC triangle with the area F, then:

$$\left((x+y)^2\left(\frac{a^2}{z}\right)^2+1\right)\left((y+z)^2\left(\frac{b^2}{x}\right)^2+1\right)\left((z+x)\left(\frac{c^2}{y}\right)^2+1\right)\geq 144F^2$$

where F is the triangle's area.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1098 If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{1}{\cos^2 x} - \frac{2}{1+\sin^2 x} + \frac{1}{1-\cos^2 x \sin^2 y} - \frac{2}{1+\cos^2 x \sin^2 y} + \frac{1}{1-\cos^2 x \cos^2 y} -$$
$$-\frac{2}{1+\cos^2 x \cos^2 y} \geq 0$$

Proposed by Daniel Sitaru – Romania

S.1099 In $\Delta ABC, K$ – Lemoine's point, holds:

$$AK \cdot BK \cdot m_a \cdot s_b + BK \cdot CK \cdot m_b \cdot s_c + CK \cdot AK \cdot m_c \cdot s_a \geq 4F^2$$

Proposed by Daniel Sitaru – Romania

S.1100 If $a, b, c > 0, a + b + c = 1$ then:

$$\left(\frac{1-\cos a}{1+\cos a}\right)^{a^2+2bc} \cdot \left(\frac{1-\cos b}{1+\cos b}\right)^{b^2+2ca} \cdot \left(\frac{1-\cos c}{1+\cos c}\right)^{c^2+2ab} \leq$$
$$\leq \frac{1-\cos(a^3+b^3+c^3+6abc)}{1+\cos(a^3+b^3+c^3+6abc)}$$

Proposed by Daniel Sitaru – Romania

S.1101 In $\Delta ABC, O$ – circumcircle, AM, BN, CP – internal bisectors. Prove that:

$$OM^2 + ON^2 + OP^2 + \frac{27abc\sqrt[3]{abc}}{16s^2} \leq 3R^2$$

Proposed by Daniel Sitaru – Romania

S.1102 If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\int_a^b \log\left(\frac{1-\sin x}{1+\sin x}\right) dx \geq (b-a) \log\left(\frac{1-\sin\left(\frac{a+b}{2}\right)}{1+\sin\left(\frac{a+b}{2}\right)}\right)$$

Proposed by Daniel Sitaru – Romania

S.1103 If $1 \leq a \leq b \leq c$ then:

$$a^a \cdot \left(\sqrt{ab}\right)^{b-a} \cdot \left(\sqrt{bc}\right)^{c-b} \cdot e^{c-a} \leq c^c$$

Proposed by Daniel Sitaru – Romania

S.1104 R_1, R_2, R_3 – radii of circles, each one simultaneous externally tangent to circumcircle and tangent at two sides of ΔABC. Prove that:

$$\frac{r_a r_b r_c}{R_1^2 R_2^2 R_3^2} \geq \frac{27r}{256R^4}$$

Proposed by Daniel Sitaru – Romania

S.1105 In ΔABC holds:

$$\begin{cases} \frac{(r_a+r_b)(r_a+r_c)}{r_br_c}+\frac{(r_b+r_c)(r_b+r_a)}{r_cr_a}+\frac{(r_c+r_a)(r_c+r_b)}{r_ar_b}=12 \\ m_a+w_a+h_a+s_a=4\sqrt{3} \end{cases}$$

Find the area of orthic triangle.

Proposed by Daniel Sitaru – Romania

S.1106 If $a,b,c>0, a+b+c=1$ then:

$$\left(\frac{1-\sin a}{1+\sin a}\right)^a\cdot\left(\frac{1-\sin b}{1+\sin b}\right)^b\cdot\left(\frac{1-\sin c}{1+\sin c}\right)^c\le\frac{1-\sin(a^2+b^2+c^2)}{1+\sin(a^2+b^2+c^2)}$$

Proposed by Daniel Sitaru – Romania

S.1107 Prove that for $\frac{\sqrt{3}}{3}\le a,b,c\le 1$, we have:

$$\sqrt[3]{abc}\cdot\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)\le\sqrt{\frac{ab+bc+ca}{3}}\cdot\tan^{-1}\left(\sqrt[3]{abc}\right)$$

S.1108 When does equality occur?

Proposed by Daniel Sitaru – Romania

S.1109 If in $\Delta ABC, m(\sphericalangle B)=60^\circ$ then:

$$\frac{8}{3}\tan^6\frac{A}{2}+\frac{216}{35}\tan^6\frac{C}{2}>\frac{\sqrt{3}}{6}-\frac{1}{8}$$

Proposed by Daniel Sitaru – Romania

S.1110 If $0<x<1$ then:

$$4\sin 2x\cdot\sin^2(1-x)\le 27x(1-x)^2\cdot\sin^3\left(\frac{2}{3}\right)$$

Proposed by Daniel Sitaru – Romania

S.1111 In any scalene ΔABC holds:

$$\frac{bc(1+a^2)}{a(b-a)(c-a)}+\frac{ca(1+b^2)}{b(a-b)(c-b)}+\frac{ab(1+c^2)}{c(c-a)(c-b)}>\frac{\sqrt{3}}{R}$$

Proposed by Daniel Sitaru – Romania

S.1112 $a,b,c,d>0, \frac{ab+bc+cd+da}{a+b+c+d}=\frac{ab}{a+b}+\frac{cd}{c+d}=\frac{ad}{a+d}+\frac{bc}{b+c}$

Prove that: $(bd+a^2)(bd+c^2)=(ac+b^2)(ac+d^2)$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.434 Let $a, b > 0$ and the sequence $(x_n)_{n\geq 1}$ such that $x_n = a + (n-1)b, \forall n \in \mathbb{N}^*$. Find:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[2n]{n!\cdot(2n-1)!!}}\cdot\sum_{k=1}^{n}\sqrt[3]{\frac{1}{b^3}+\frac{1}{x_k^3}+\frac{1}{x_{k+1}^3}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

U.435 Prove that:

$$\lim_{n\to\infty}\frac{1}{n^{20}}\sum_{k_{10}=1}^{n}\sum_{k_9=1}^{k_{10}}\dots\sum_{k_2=1}^{k_3}\sum_{k_1=1}^{k_2}(k_{10}k_9\dots k_3k_2k_1)=\frac{1}{3715891200}$$

Proposed by Naren Bhandari-Nepal

U.436 Prove that:

$$\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{\left[\frac{p}{2}\right]-1}\left(\cos\left(\frac{\pi}{p}(2n+1)(2k+1)\right)\right)}{(2n+1)^s}\right)=\left(\frac{1}{2}-\frac{1}{2p^s}-\frac{\left[\frac{p}{2}\right]}{p^s}\right)(1-2^{-s})\zeta(s)$$

, where p is a prime number and $s \in \mathbb{R}, s > 1$.

Proposed by Rohan Shinde-India

U.437 Find:

$$\Omega=\sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{2n-1}{k}}$$

Proposed by Abdul Mukhtar-Nigeria

U.438 If $I(z)=\int_0^1\int_0^1\frac{dxdy}{(1-xyz)(1+x)(1+y)}, |z|\geq 1$, then prove that:

$$I(z)=\frac{1}{1-z}\left[Li_2(z)-2Li_2\left(\frac{1+z}{2}\right)+\zeta(2)\right]$$

Proposed by Ngulmun George Baite-India

U.439 Prove that:

$$I_2(k)=\int_0^{\frac{\pi}{2}}\frac{x\sin x\cos x}{\sqrt{1-k^2\sin^2 x}}dx=-\frac{\pi k'}{2k^2}+\frac{E(k)}{k^2}$$

Proposed by Onikoyi Adeboye-Nigeria

U.440 For $0\le p<e^a$ prove that:

$$\int_0^{\infty}\frac{1-p\cos x}{(1+p^2-2p\cos x)(a^2+x^2)}dx=\frac{\pi e^a}{2a(e^a-p)}$$

Proposed by Precious Itsuokor-Nigeria

U.441 Prove that:

$$\int_0^1\frac{x^2}{x+x^3}\log\left(\frac{x}{\sqrt{x^2+1}}\right)dx=-\frac{1}{8}(\zeta(2)+\log^2 2)$$

Proposed by Abdul Mukhtar-Nigeria

U.442 Let α and β be positive integers with $\alpha>\beta$ such that $p^{\alpha}-p^{\beta}\equiv 0mod\ (7!)$ and p is any prime with g.c.d. $(p,7!)=1$. If M is the smallest values of $\alpha+\beta$, then show that for

$$k=\phi(M)+\lambda(M)$$

$$\left(\int_0^{\infty}\frac{x^M dx}{x^M+x^{Mk}}\right)\left(\int_{-\infty}^{\infty}\frac{dy}{\left(y^{Mk^{-1}}+1\right)^{n+1}}\right)=\frac{\pi^2\binom{2n}{n}}{30.4^n\sqrt{2-\sqrt{6-2\phi}-2^{-1}\sqrt{3}\phi}}$$

where $\phi(n),\lambda(n),\phi$ are Euler Phi function, Carmichael function for $n\in\mathbb{Z}^+$ and Golden ratio respectively.

Proposed by Naren Bhandari – Nepal

U.443 Prove or disprove:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{((m+3)^2n^2+(m+3)^2n+m+2)^{-1}}{(q+3)^2(n-k+1)^4+(q+3)^2(n-k+1)^2+q+2}$$

$$=\frac{\pi}{4(m+3)}\tan\left(\frac{\pi(m+1)}{m+3}\right)\left\{\frac{1}{(q+2)}-\frac{\pi\sqrt{q+3}}{(q+1)(q+3)}\coth\left(\frac{\pi}{\sqrt{q+3}}\right)+\frac{\pi\sqrt{q+3}}{\sqrt{q+2}(q+1)(q+3)}\coth\left(\pi\sqrt{\frac{q+2}{q+3}}\right)\right\}$$

where m is even positive integer and $q\in\mathbb{Z}^+$.

Proposed by Naren Bhandari - Nepal

U.444 Let $Re(k) > -1$ and if

$$\sum_{1\le n\le m\le\infty} \frac{1}{n^2m+m^2n+kmn} = \frac{(H_{k+1})^2-\psi^1(k+\alpha)}{k+\beta} + \frac{\pi^\alpha}{\lambda(k+\beta)}$$

and

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n} \frac{x^n}{(q+b)\sqrt{q+b+1}+(q+b+1)\sqrt{q+b}} = \frac{1}{\sqrt{b}(1-x)} - \Psi\left(x,\frac{1}{\alpha},b+1\right)$$

where $b\in\mathbb{N}$ and $x\in\mathbb{R}\setminus\{1\}$ then prove that $\Psi(\beta-\alpha,\beta,(\alpha+2\beta+\lambda)^{-1})$

$$= \frac{2\pi}{(\sqrt{5}-1)} + 20\sqrt{\phi+1}\log\left(\theta - \frac{8\theta}{4+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}\right)$$

$$+20\sqrt{10-2\sqrt{5}}\log\left(\frac{\sqrt{3}\theta-1}{\theta+\sqrt{3}}\right)$$

and $\theta = \sqrt{\frac{8+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}{8-\sqrt{10-2\sqrt{5}}-\sqrt{15}-\sqrt{3}}}$

Proposed by Naren Bhandari - Nepal

U.445 For all $k\ge 1$, if

$$J(k) = \lim_{u\to\infty}\lim_{m\to\infty}\left(\sum_{n=1}^{u}\sum_{k=1}^{n}\left(\left(1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{m^k}\right)\frac{1}{\ln m}\right)^{-1} + \ln\left(\frac{n-1}{n}\right)\right)\frac{1}{u}$$

and

$$R = \lim_{k\to\infty}\lim_{N\to\infty}\sum_{K=0}^{N}\sum_{l=0}^{K}\sin\left(\frac{\pi^{2J(k)}}{N+l}\right)\sin^{-1}\left(\frac{\pi^{-J(k)}}{N+K-l+1}\right)\frac{1}{\ln^2 2}$$

then prove or disprove

$$\frac{1}{\zeta(h,2020)}\sum_{U=0}^{\infty}\sum_{V=0}^{U}\frac{1}{2020^V}\binom{2V+\frac{R}{e}}{V}\frac{1}{(U-V+2020)^h} = \frac{2^{\frac{\pi^\gamma}{e}}}{3\sqrt{8}}\sqrt{\frac{505}{7}}\left(1+\sqrt{\frac{14}{505}}\right)^{1-2^1+\frac{\pi^\gamma}{e}}$$

where $h>1, \zeta(s,a)$ is generalized Riemann zeta function and γ is Euler – Mascheroni constant.

Proposed by Naren Bhandari - Nepal

U.446 A Pythagorean Triples is set of positive integers (a,b,c) such that $a^2+b^2=c^2$. For example

$$3^2+4^2=5^2$$

$$5^2+12^2=13^2$$

$$7^2+24^2=25^2$$

If a and b and are of distinct parity. Is it true that a, b and c are always co-prime integers?

Proposed by Naren Bhandari - Nepal

U.447 Prove:

$$\sum_{n}^{\infty}\int_{-1}^{1}(1-x^2)\arccos x\frac{dx}{\sqrt{(1-x^2)}2^n n!}=\frac{\pi^2}{2}\left(\sqrt[4]{e}I_0\left(\frac{1}{4}\right)-1\right)$$

where $I_n(z)$ is modified Bessel function of the first kind.

Proposed by Naren Bhandari - Nepal

U.448 Prove the following result:

$$\lim_{n\to\infty}\frac{n^k}{\sqrt[n]{(n^k+1^k)(n^k+2^k)(n^k+3^k)\dots(2n^k)}}=\frac{e^k}{2\exp\left(\Phi\left(-1,1,\frac{1}{k}\right)\right)}$$

and hence for $k=6$

$$\lim_{n\to\infty}\frac{n^6}{\sqrt[n]{(n^6+1^6)(n^6+2^6)(n^6+3^6)\dots(2n^6)}}=\frac{e^6}{2\left(2+\sqrt{3}\right)^{\sqrt{3}}e^{\pi}}$$

where $k\in\mathbb{N}$ and $\Phi(z,a,b)$ is Lerch transcendent.

Proposed by Naren Bhandari - Nepal

U.449 Show that $1+\sqrt{5}+\sqrt{5+2\sqrt{5}}$ is the zero of the quartic equation $x^4-4x^3-14x^2-4x+1=0$ and also the following relation holds:

$$\frac{8+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}{\sqrt{36-4\sqrt{5}-4\sqrt{30+6\sqrt{5}}}}\left(1-\frac{8}{4+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}\right)=1+\sqrt{5}+\sqrt{5+2\sqrt{5}}$$

Proposed by Naren Bhandari - Nepal

U.450 Prove that:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{0}^{\infty}\frac{dx}{x}\frac{\sin x}{(n^4\sin^2 x+k^4\cos^2 x)}=\frac{7\pi^5}{720}$$

Proposed by Naren Bhandari - Nepal

U.451 Prove that without software

$$\sum_{n=0}^{\infty}\frac{1}{16^{2n}}\binom{2n}{n}\binom{4n}{2n}=\frac{2}{\pi}\sqrt{\frac{2}{3}}K\left(\frac{2}{3}\right)$$

where $K(m)$ is called Eliptical integral of the first kind.

Proposed by Naren Bhandari - Nepal

U.452 If $S=\{(-1)^{k-1}(2k-1)|k\in\mathbb{Z}^+\}$, then prove that:

$$\lim_{n\to\infty^+}\sum_{v=1}^{\infty}\left(\sum_{m\in S}^{v}\int_0^n\frac{n^n dx}{(n+mx)^n\sqrt[n]{x+m}}-\frac{\pi}{4}\right)=\frac{\pi}{8}-\frac{1}{4}$$

Proposed by Naren Bhandari - Nepal

U.453 If n,p are positive integers such that $4n^4-8080n^2-4080375-4p=0$ and $\phi(p)=\phi(a)\phi(b)=72$ with distinct prime factors a and b, then solve the following

$$\sum_{k=1}^{i-1}\left\lfloor\frac{kq}{i}\right\rfloor=\sum_{j=1}^{6}\underbrace{\phi(\phi\dots(\phi(p)\dots))}_{j},i\geq 1$$

for gcd $(q,i)=1$ where $\phi(x)$ is Euler's Totient function, $\lfloor.\rfloor$ denotes Floor function and

$$\underbrace{\phi(\phi(\dots\phi(p)\dots))}_{3}=\phi\left(\phi(\phi\,(p))\right)$$

Proposed by Naren Bhandari - Nepal

U.454 For all $n,k\geq 1$ evaluate in closed form:

$$\int_0^1\frac{dx}{\sqrt{1-\sqrt[n]{1+x+x^2+\dots+x^k}}}$$

Proposed by Naren Bhandari - Nepal

U.455 Prove that:

$$\sqrt[3]{2021}=12+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{16+\cfrac{1}{1+\cfrac{1}{21+\cfrac{1}{\dots}}}}}}}}$$

Proposed by Naren Bhandari - Nepal

U.456 Let α and β be positive integers with $\alpha>\beta$ such that $p^\alpha-p^\beta\equiv 0\ mod(7!)$ where p is any prime with gcd $(p,7!)=1$. If M is the smallest value of $\alpha+\beta$, then show that for

$$k=\phi(M)+\lambda(M)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\int\int_R\frac{x^M dxdy}{(x^M+x^{Mk})\left(y^{Mk^{-1}}+1\right)^{n+1}}=\frac{\pi^2}{15}\frac{\zeta(2)-2\log^2 2}{\sqrt{2-\sqrt{6-2\phi}-2^{-1}\sqrt{3}\phi}}$$

where $\phi(n),\phi,\lambda(n)$ are Euler's phi function, Golden ratio and Carmichael function for

$n \in \mathbb{N}$ repectively.

Proposed by Naren Bhandari - Nepal

U.457 Find the limit

$$\lim_{x\to 0^+}\left(3\log\left(\left|\sqrt{2}-\sqrt{1+\sqrt{1+16x}}\right|\right)\right)+\log\left(\left|1-\sqrt{1+16x}\right|-4\log x\right)$$

Notation: $|.|$ denotes absolute value.

Proposed by Naren Bhandari - Nepal

U.458 Prove that:

$$\int_0^1 \frac{\ln x}{1+x^2}\left(-x+\frac{x^2}{2^2}-\frac{x^3}{3^2}+\cdots\right)dx=\frac{G\zeta(2)}{8}$$

where G is Catalan's constant.

Proposed by Naren Bhandari - Nepal

U.459 Prove:

$$\lim_{m\to\infty^+}\lim_{n\to\infty^+}\prod_{k=1}^{m}\left(1+\int_0^\infty \tanh\left(\frac{x}{n}\right)\frac{e^{-x}dx}{x}\right)^{\frac{n}{k}}\frac{1}{m}=e^{\gamma}$$

Notation: e is called Euler's number and γ is Euler-Mascheroni constant.

Proposed by Naren Bhandari – Nepal

U.460 For all $|z|\le\frac{1}{16}$ prove the following generating function:

$$\sum_{n=1}^{\infty}\frac{sgn^n(z)}{n}\binom{4n}{2n}z^n$$

$$=4\ln 2-\ln\left(1+\sqrt{1-sgn(z)16z}\right)-2\ln\left(\sqrt{2}+\sqrt{1+\sqrt{1-sgn(z)16z}}\right)$$

sgn(z) denotes signum function.

Proposed by Naren Bhandari - Nepal

U.461 Prove the relation:

$$\int_0^1 \frac{Li_5\left(\sqrt[5]{x}\right)}{\sqrt[5]{x}}dx=\frac{5}{4}\left(\frac{25}{3072}-\frac{\zeta(2)}{2^6}+\frac{\zeta(3)}{2^4}+-\frac{\zeta(4)}{2^2}+\zeta(5)\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.462 Evaluate the integral:

$$\int_0^\infty \log^3(1-e^{-\pi x})\tanh(\pi x)\,dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.463 Let

$$\sum_{n=0}^{\infty}\frac{\binom{2n+1}{n}}{2^{3n}}\left(\sqrt{2}n+(-1)^n\right)=\beta\sum_{n=0}^{\infty}\frac{\binom{2n+1}{n}}{2^{3n}}\left(\sqrt{2}n-(-1)^n\right)$$

then find the value of the expression:

$$2\beta^4-4\beta^3-6\beta^2+8\beta$$

Proposed by Srinivasa Raghava-AIRMC-India

U.464 Let for $n>0$

$$S(n)=\int_{\infty}^{\infty}e^{-\pi(x^2+x)}\sin(2\pi x)\cosh(\pi n x)\,dx$$

then show that

$$\left(\int_{\infty}^{\infty}S(n)e^{-\pi n^2}dn\right)\left(\int_{-\infty}^{\infty}S(n+1)\,e^{-\pi n^2}dn\right)=\int_{-\infty}^{\infty}S(n+2)e^{-\pi n^2}\,dn$$

Proposed by Srinivasa Raghava-AIRMC-India

U.465 Prove the relation:

$$\left(\sum_{n=0}^{\infty}\frac{1}{F_{2n}-i}\right)\left(\sum_{n=0}^{\infty}\frac{1}{F_{2n}+i}\right)=1+\phi^2$$

F_n – Fibonacci number; ϕ – Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.466

$$\Omega(n)=\int_0^{\infty}\frac{x^{n-1}(x-n)\log x}{e^x}dx, n\geq 1$$

Find:

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{\Omega(n)}$$

Proposed by Daniel Sitaru – Romania

U.467 Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{x^n(1-x)\log x}{e^x}dx\right)^{-1}$$

Proposed by Daniel Sitaru – Romania

U.468

$$\Omega_n(x) = \int \frac{dx}{x(1+x^n)}, n \in \mathbb{N}^*, \Omega_n(1) = \log 2$$

Find:

$$\Omega(x) = \lim_{n\to\infty}\left(n\Omega_n(x)\right), x > 0$$

Proposed by Daniel Sitaru – Romania

U.469 Find:

$$\Omega(a) = \int_0^\infty \frac{x}{(1+x^4)(1+ax)}dx, a > 0$$

Proposed by Vasile Mircea Popa – Romania

U.470 Find:

$$\Omega = \int_0^\infty \frac{x\ln(1+x^2)}{1+x^2+x^4}dx$$

Proposed by Vasile Mircea Popa – Romania

U.471 Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_\varepsilon^1 \frac{x\sqrt{x}\ln(x)}{x^3+x\sqrt{x}+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.472 Find:

$$\Omega = \lim_{\varepsilon\to 0+} \int_\varepsilon^\infty \frac{x^2\log x}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

U.473 Prove that:

$\psi\left(\frac{7}{8}\right) - \psi\left(\frac{3}{8}\right) = \pi\sqrt{2} - 2\sqrt{2}\log(1+\sqrt{2})$, where $\psi(x) -$ is the digamma function.

Proposed by Vasile Mircea Popa-Romania

U.474 Find a closed form:

$$\Omega = \int_0^\infty \frac{\sqrt{x}\log(1+x)}{x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

U.475 Find:

$$\omega = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k^3}{n^4}\right), \Omega = \lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k^3}{n^4}\right)}.$$

Proposed by Vasile Mircea Popa-Romania

U.476 For all $n > 1$, prove that:

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\int_0^{\frac{\pi}{k^3}}\frac{(1-\sin^{2n}x)(\psi_2(k+2)-\psi_2(2))}{2(1+\sin^{2n}x)\sqrt[n]{1+\sin^{4n}x}}dx = \pi\left(\frac{\pi^6}{1890}+\mathcal{N}+\frac{\zeta^2(3)}{2}-\zeta(3)\right)$$

Here $\mathcal{N}$ is some constant and using $\mathcal{N}$ prove that $3 < \pi < 4$ where $\zeta(z)$ is Riemann zeta function and $\psi_n(x)$ is polygamma function.

Proposed by Narendra Bhandari – Nepal

U.477 Find a closed form:

$$\Omega(n) = \int_0^e \frac{x^n}{\sqrt{1-\log x}}dx, n > 0$$

Proposed by Abdul Mukhtar-Nigeria

U.478 Prove:

$$\int_0^{\infty}\frac{dx}{(4\ln^2(x)+\pi^2)^2(x^2+1)} = \frac{\ln(2)}{4\pi^3}+\frac{1}{96\pi}$$

Proposed by Ty Halpen-USA

U.479 Find:

$$\Omega = \int_0^{\frac{\pi}{2}}\frac{\ln(\sin x)\cdot\ln(\cos x)}{\tan x}dx$$

Proposed by Ghuiam Shah Naseri-Afghanistan

U.480 Find a closed form (without residue theorem):

$$\Omega = \sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{\left(n+\frac{1}{4}\right)^2}$$

Proposed by Lucas Paes Barreto-Brazil

U.481 If $0 < a \le b, f:[a,b] \to (0,\infty), f$ – continuous, then:

$$\int_a^b\int_a^b \frac{f^3(x)dxdy}{f^2(x)+f(x)f(y)+f^2(y)} \ge \frac{b-a}{3}\int_a^b f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

U.482 If $0 < a \le b, f:[a,b] \to (0,\infty), f$ – continuous, then:

$$\int_a^b\int_a^b\int_a^b \frac{f^3(x)dxdydz}{f(y)f(z)+f^2(x)} \ge \frac{(b-a)^2}{2}\int_a^b f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

U.483 If $0 < a \le b$ then:

$$\int_a^b\int_a^b\int_a^b \frac{e^{2x^2+y^2}+e^{2y^2+z^2}}{e^{2(x^2+y^2)}}dxdydz \ge 2(b-a)^2\int_a^b e^{-x^2}\,dx$$

Proposed by Daniel Sitaru – Romania

U.484 If $0 < a \le b, f:[a,b] \to (0,\infty), f-$continuous, then:

$$\int_a^b\int_a^b \frac{dxydy}{\sqrt{\left(f(x)+f(y)\right)f(x)f(y)}} \le \frac{b-a}{2}\int_a^b \frac{dx}{f(x)}+\frac{1}{4}\left(\int_a^b \frac{dx}{f(x)}\right)^2$$

Proposed by Daniel Sitaru – Romania

U.485 Find without any software:

$$\Omega = \sum_{n=1}^{\infty} \frac{3n-1}{\sqrt{2}+\sqrt{5}+\sqrt{8}+\cdots+\sqrt{3n-1}}$$

Proposed by Daniel Sitaru – Romania

U.486 If $0 < a \le b < \frac{\pi}{2}$ then:

$$10\int_a^b\int_a^b \cos(x-y)\cos(x+y)\,dx\,dy \ge 20\int_a^b\int_a^b \log(\cos x\cdot\cos y)\,dxdy+(b-a)(b^5-a^5)$$

Proposed by Daniel Sitaru – Romania

U.487

$$\Omega(n,k) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\cdot\ldots\cdot(n+k)}$$

If $a,b,c > 0, abc = 1$ then:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} > \frac{3}{2\Omega(n,k)}$$

Proposed by Daniel Sitaru – Romania

U.488 Find:

$$I = \int_0^1 \frac{\sqrt[3]{x}\ln^2\left(\sqrt[3]{x}\right)dx}{x^3+x\sqrt{x}+1}$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

U.489 Find:

$$\int_0^{\infty} \frac{x^2\tan^{-1}(x)}{x^4-x^2+1}dx$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

U.490

$$\sum_{k=0}^{\infty}\frac{2^{2k}}{(2k+1)^2({}^{2k}C_k)}\left(\frac{\pi}{2}-\frac{(2k)!!}{(2k+1)!!}\right)=2\pi G-\frac{7}{2}\zeta(3)$$

Here G is the Catalan Constant

Proposed by Kaushik Mahanta – India

U.491 Find:

$$\Omega=\lim_{x\to\infty}\left(\frac{x^{x^2}\cdot(x+2)^{(x+1)^2}}{(x+1)^{2x^2+2x+1}}\right)^{\sum_{n=1}^{\infty}\frac{n(n+1)}{2^{n+1}}}$$

Proposed by Mohammad Hamed Nasery-Afghanistan

U.492 Show that:

$$\int_0^{\infty}\frac{x\tan^{-1}(x)}{1+x^2+x^4}dx=\int_0^{\infty}\frac{x\tan^{-1}(x^2)}{1+x^2+x^4}dx=\frac{\pi^2}{12\sqrt{3}}$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

U.493 Evaluate

$$e^{\left[\sum_{n=1}^{n=\infty}\sum_{k=1}^{k=\infty}\frac{(-1)^{k+1}n^k}{k}-\sum_{n=1}^{n=\infty}\log_e(2+n)\right]}$$

Proposed by Ankush Kumar Parcha-India

U.494 Find:

$$\int_0^{\infty}\frac{\ln^3 x\,dx}{x^2+2x+2}$$

Proposed by Kaushik Mahanta – India

U.495 Check the sum:

$$\sum_{n=1}^{\infty}(-1)^n\left(\operatorname{csch}^2\left(\frac{\pi n}{2}\right)+\operatorname{sech}^2\left(\frac{\pi n}{2}\right)\right)=-\frac{1}{3}$$

Proposed by Lucas Paes Barreto – Brazil

U.496 Find a closed form:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left[\frac{1}{e^{s+t}}-\frac{1-e^{-u}}{e^{s+t}-(1-e^{-u})}\right]dsdtdu$$

Proposed by Abdul Mukhtar-Nigeria

U.497 Prove that:

$$\int_0^{\infty}\frac{\log(1+x^{10}+x^4+x^6)}{x^3+1+3x^2+3x}dx=\pi+\frac{\pi\sqrt{2}}{2}-\frac{5}{2}-\frac{\pi\sqrt{3}}{3}$$

Proposed by Abdul Mukhtar-Nigeria

U.498 Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{x^{2n}}-\frac{1}{x^{2n-1}}=\frac{1}{x}$$

And iff, $y=\frac{e-2}{\sqrt{2}+1}$ and, $\frac{e^{i\pi}-x^{i^2}}{x^2+e^{i\pi}}+1=0$

Resolve for $t, \sum_{n=1}^{\infty} y^{x^n}+y^{e^n}+y^{\pi^n}=\frac{x^{i^2}-e^{i\pi}}{t(x^2+e^{i\pi})}$

Proposed by Jeremie Rioux – Toth-Canada

U.499 Find:

$$\Omega=\lim_{n\to\infty}\left(5n\cdot\sqrt[n]{\frac{\sqrt{n}}{n!}}-\sqrt[n]{\frac{n}{\sqrt[n]{n!}}}\right)$$

Proposed by Jay Jay Oweifa-Ngeria

U.500 Find in a closed form:

$$\omega(n)=\int_0^{\infty}\frac{\cos(\pi x)}{(x^2+1^2)(x^2+2^2)\dots(x^2+n^2)}dx, \forall n\geq 1$$

Proposed by Serlea Kabay – Liberia

U.501 Prove that:

$$\left(\sum_{P=Prime}\left(\frac{P\log(P)}{P-1}\right)+1\right)^2=48\left(\frac{1}{12}-\log A+3\log^2 A\right)$$

Where A denotes Glaisher Kinkelin-constant

Proposed by Serlea Kabay – Liberia

U.502 If $0<a\leq b<1$ then:

$$\int_a^b x^{x-1}\cdot(1-x)^{1-x}dx\geq\log\sqrt{\frac{b}{a}}$$

Proposed by Serlea Kabay – Liberia

U.503 Let $\Delta_{n_1}=\begin{pmatrix}\gamma & \cdots & \gamma^{n-1} & \gamma^n\\ \gamma & \cdots & \gamma & \vdots\\ \vdots & \ddots & \gamma & \gamma^2\\ \gamma & \cdots & \gamma & \gamma\end{pmatrix}$ and

$$\Delta_{n_2}=\begin{pmatrix}\gamma-\gamma^2 & \gamma^2-2 & 0 & \cdots & 0\\ \gamma^2 & \gamma & \gamma^2 & \ddots & \gamma^2\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \gamma^2 & \ddots & \ddots & \gamma & \gamma^2\\ 0 & \cdots & 0 & \gamma^2-\gamma & \gamma-\gamma^2\end{pmatrix}$$

Show that $\lim_{n\to\infty}\frac{(n-1)\det(\Delta_{n_1})}{\det(\Delta_{n_2})}=\frac{1}{\gamma}$

Proposed by Serlea Kabay – Liberia

U.504 If $0 \le x_i \le 1, i \in [1,2022]$ and $y_n = \prod_{k=1,k\neq n}^{2022} x_k$

Prove: $\sum_{n=1}^{2022} \frac{x_n}{1+y_n} \le 2021$

Proposed by Serlea Kabay – Liberia

U.505 Prove that:

$$G = \frac{\partial}{\partial n}\sum_{k=0}^{\infty}\left(n - \frac{(2k)!!}{(2k+1)!!}\right)\frac{2^{2k-1}}{(2k+1)^2}\frac{1}{\binom{2k}{k}},$$ where G is Catalan's constant.

Srinivasa Raghava-AIRMC-India

U.506 For $a, b > 0$ prove that:

$$\int_{-\infty}^{\infty} \frac{x^2 - a}{x^2 + b}\sin\left(\frac{x}{\sqrt{b}}\log\left(\frac{a+b}{a}\right)\right)\frac{dx}{x} = 0$$

Srinivasa Raghava-AIRMC-India

U.507 Evaluate the sum:

$$\Omega = \sum_{n=-\infty}^{\infty}\left(\frac{1}{5n+3} + \frac{1}{5n+2}\right)^2\left(\frac{1}{5n+4} + \frac{1}{5n+1}\right)^2 (3n+1)$$

Srinivasa Raghava-AIRMC-India

U.508 Evaluate the sum:

$\Omega = \sum_{n=1}^{\infty} \frac{L_{2n}+L_{4n}+L_{6n}}{\varphi^{2n}+\varphi^{4n}+\varphi^{6n}+\varphi^{8n}}$, where L_m −Lucas numbers and φ −golden ratio.

Srinivasa Raghava-AIRMC-India

U.509 Prove that:

$$\int_0^{\sqrt[2020]{\tan 1}} \frac{x^{2019}}{1 + x^{2\cdot 2020}}\,dx = 2020$$

Srinivasa Raghava-AIRMC-India

U.510 Prove that:

$$101\sum_{n=1}^{\infty}\frac{16n}{\phi^{12n}} = 2020$$

Srinivasa Raghava-AIRMC-India

U.511 Prove that

$$\int_0^{\frac{\pi}{2}} \sin 2x \cos^3(\log(\tan x))\,dx = 1 + \sum_{k=1}^{\infty}\frac{6(-1)^k(4k^2+3)}{16k^4+40k^2+9} = \sum_{k=0}^{\infty}\frac{3\pi\left(1+e^{2k\pi+\pi}\right)}{4e^{\frac{3}{2}(2k\pi+\pi)}} =$$

$$= \frac{3\pi}{8}\sum_{n=-\infty}^{\infty}\frac{8(-1)^k(4k^2+3)}{\pi(4k^2+1)(4k^2+9)} = \frac{3\pi}{8}\left(\operatorname{csch}\left(\frac{\pi}{2}\right) + \operatorname{csch}\left(\frac{3\pi}{2}\right)\right)$$

Srinivasa Raghava-AIRMC-India

U.512 Let for any positive integer $n \geq 1, F(n) = \int_{-\pi}^{\pi} \frac{\cos^n x}{1+e^{x^3}} dx$ then prove that

$$\sum_{n=1}^{\infty} \frac{F(n)}{n} = \log 2\,; \ \sum_{n=1}^{\infty} \frac{F(n)}{n^2} = \frac{\pi^2}{24} - \frac{\log^2 2}{2}$$

Srinivasa Raghava-AIRMC-India

U.513 If $H(m) = \begin{pmatrix} e^{-\pi m} & e^{-2\pi m} & e^{-3\pi m} & e^{-4\pi m} \\ e^{-2\pi m} & e^{-\pi m} & e^{-2\pi m} & e^{-3\pi m} \\ e^{-3\pi m} & e^{-2\pi m} & e^{-\pi m} & e^{-2\pi m} \\ e^{-4\pi m} & e^{-3\pi m} & e^{-2\pi m} & e^{-\pi m} \end{pmatrix}$ then find $\Omega = \int_0^{\infty} \frac{dm}{|H^{-1}(m)|}$, where

$|H|$ −matrix determinant and H^{-1} −matrix inverse.

Srinivasa Raghava-AIRMC-India

U.514 If $\sum_{n=0}^{\infty} A(n) z^n = \frac{2(5z-2z^2-1)}{z^3-7z^2+7z-1}$ then find the integral in closed-form

$$\Omega = \int_{-\infty}^{\infty} \frac{1}{A(n)} dn$$

Srinivasa Raghava-AIRMC-India

U.515 Prove that:

$$\int_0^{\infty} \frac{Li_{-3}(-x)}{\sqrt{x}} \frac{\log(2+x)}{2+x} dx = \pi\left(\frac{111}{4} - \frac{95}{2\sqrt{2}} + \frac{3}{4}(52\sqrt{2}\log 2 - 49\log(1+\sqrt{2})\right)$$

Srinivasa Raghava-AIRMC-India

U.516. For any complex numbers a, b, c with $Re[a, b, c, n] > 0, Re[a+b+n] > 0$, define

$f_n(a, b, c) = \int_0^{\infty} \frac{(1-e^{-ax})(1-e^{-bx})e^{-nx}}{1-e^{-cx}} dx$ then prove that

$$e^{\int f_n(a,b,c) dn} = \frac{\Gamma\left(\frac{a+n}{c}\right)\Gamma\left(\frac{b+n}{c}\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma\left(\frac{a+b+n}{c}\right)}$$

Srinivasa Raghava-AIRMC-India

U.517 Prove that:

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial e^{-x}} \left(e^{x-x^2} \sin\left(\frac{\pi x}{2}\right)\right) dx = \sqrt{\pi} e^{1-\frac{\pi^2}{16}}$$

Srinivasa Raghava-AIRMC-India

U.518 $G(n)$ – Barnes G – function, $K(n)$ – K function. Find:

$$\Omega = \sum_{n=2}^{\infty} \sqrt[n]{\frac{n!}{K(n+1) \cdot G(n+2)}}$$

Proposed by Daniel Sitaru – Romania

U.519 Prove that:

$$\int_0^{\pi} \cos x\left(\sin\left(\frac{x}{3}\right)+\cos\left(\frac{x}{3}\right)\right)\tanh^{-1}(\sin x)\,dx=\frac{9}{16}\left(2\sqrt{3}-2\log\left(\left(26-15\sqrt{3}\right)\left(\sqrt{3}+2\right)^{\sqrt{3}}\right)\right).$$

Proposed by Srinivasa Raghava-AIRMC-India

U.520 If $\int_0^{\infty}\frac{x+\tanh x+\tanh\left(\frac{x}{8}\right)}{\cosh x}\frac{dx}{x}=\frac{4G}{\pi}+\frac{\pi}{2}+\log a$ then prove that:

$$4096a^4-38912a^3+43392a^2-2656a+1=0.$$

Proposed by Srinivasa Raghava-AIRMC-India

U.521 If $\int_0^{\infty}\frac{\sin(2x)(2\cos x+\phi)\left(2\cos x+\frac{1}{\phi}\right)}{xe^x}dx=\frac{\pi b}{a+c}+a\cdot\tan^{-1}3+c\cdot\tan^{-1}4$ then prove $b^2=a+2c$.

Proposed by Srinivasa Raghava-AIRMC-India

U.522 If $\int_0^{\infty}\log\frac{\cosh(3x)}{\sinh^2\left(\frac{x}{3}\right)}dx=\pi+3\log 2$ then prove

$$a^6-432a^4+13824a^2-110592=0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.523 For $n>0$, prove that

$$\sum_{m=0}^{n}\frac{1}{3+2\sqrt{2}\cos\left(\frac{\pi m}{n}\right)}\ge n+3$$

Proposed by Srinivasa Raghava-AIRMC-India

U.524 Find:

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\sqrt{\tan x+\tan^2 x}}{\sqrt{\tan x-\tan^2 x}}\sin x\,dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.525 Find:

$$\Omega=\int_0^{\frac{\pi}{3}}\left(\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{2}\right)}{2\sin\left(\frac{3x}{4}\right)+1}+\frac{16\sin x}{4\cos x+1}\right)dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.526 Find the value of α, if

$$\int_0^{\frac{\pi}{2}}\sin\left(\frac{x}{2}\right)\tanh^{-1}(\sin(2x))\,dx=\log\alpha$$

Proposed by Srinivasa Raghava-AIRMC-India

U.527 Prove that:

$$\frac{\pi}{12}+\int_0^{\frac{\pi}{3}}\sin^3\left(\frac{x}{3}\right)\tanh^{-1}(\sin x)\,dx=\log\left(\left(\sqrt{3}+2\right)^{\frac{1}{8}-\frac{9}{4}\cos\left(\frac{\pi}{9}\right)}\left(1+2\sqrt{3}\cos\left(\frac{\pi}{18}\right)\right)^{\frac{9\sqrt{3}}{8}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.528 If $A(n-2)+A(n-1)+A(n)=(-1)^n, A(0)=-1, A(1)=1$ then

$$\sum_{n=1}^{\infty}\frac{A(n^4)}{n^4}=\frac{697\pi^4}{58320}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.529 Prove that:

$$\int_0^{\infty}\int_0^{\infty}(xy)^2\operatorname{sech} y\tan^{-1}(\operatorname{sech} x)\,dy\,dx=\frac{1}{48}\pi^4\log^3(1+\sqrt{2})+\frac{1}{64}\pi^6\log\left(1+\sqrt{2}\right).$$

Proposed by Srinivasa Raghava-AIRMC-India

U.530 Prove that:

$$\frac{4m-1}{2m^2}<\psi^{(1)}(m+1)+\psi^{(1)}\left(m+\frac{1}{2}\right)\forall m\in\mathbb{R}^+$$

Notations: $\psi^{(n)}(z)$ is Polygamma function

Proposed by Surjeet Singhania and Kaushik Mahanta – India

U.531 Find:

$$\int_0^{\frac{\pi}{2}}x\cot x\log^2(\cos x)\,dx$$

Proposed by Surjeet Singhania – India

U.532 Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^{\infty}\frac{x}{(1+x^2)^2}\left(\frac{e^{2nx\pi}+1}{e^{2nx\pi}-1}\right)dx=\frac{\pi^4}{144}$$

Proposed by Surjeet Singhania – India

U.533 Consider

$$\phi_n=\frac{4}{\pi}\int_0^{\infty}\frac{\coth(nx^{-1})-xn^{-1}}{n(1+x^2)^2}dx$$

And

$$\Phi_n=\frac{\cos(n\pi)}{n}\forall n\in\mathbb{N}$$

Then show that

$$\lim_{m\to\infty}\lim_{n\to\infty}\prod_{k=1}^{n}\prod_{r=2}^{m}(1+\phi_n)^{nk^{-r}\Phi_r}=e^{\gamma}$$

Notations: γ is Euler – Mascheroni constant

Proposed by Surjeet Singhania – India

U.534 Consider $G=\left\{0,\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n}\right\};x*y=x+y-\lfloor x+y\rfloor$

Then find an isomorphism and prove that $Aut(G)\approx U(n)$

Proposed by Surjeet Singhania – India

U.535 Prove that

$$\int_0^1\frac{\sin(\pi x)}{x^x(1-x)^{1-x}(x-2)^2}dx=\frac{\pi}{4}\ln(2)$$

Proposed by Surjeet Singhania – India

U.536 Assume $X=\left\lfloor\sum_{k=1}^{1729}\frac{1}{k}\right\rfloor, Y=\left\lfloor\sum_{k=1}^{1729}\frac{H_k}{k}\right\rfloor$ and Z is total number of digits of $(X+Y)!$.

Find $m,n\in\mathbb{Z}_+$ such that $X+Y+Z=(m^2-n^2)(m+n^2)$

Notations: Where $\lfloor*\rfloor$ floor function and H_n is nth harmonic number.

Proposed by Surjeet Singhania – India

U.537 Prove that for $n\geq 1$

$$\int_0^1\frac{x}{x^2+1}\sqrt[n]{\frac{x}{1-x}}dx=\pi\csc\left(\frac{\pi}{n}\right)\left\{1-\frac{\pi}{\sqrt[2n]{2}}\cos\left(\frac{\pi}{4n}\right)\right\}$$

Proposed by Surjeet Singhania – India

U.538 Let $f(z),g(z)$ be two entire functions such that $f(z)\notin|\omega-2|<1,\forall z\in\mathbb{C}$ and $\mathcal{I}\{g(z)\}\leq\mathcal{R}\{f(z)\}$. Then show that $g(z)$ is constant.

Proposed by Surjeet Singhania – India

U.539 Find a closed form

$$\sum_{n=1}^{\infty}\frac{\sinh(\sqrt{2}\pi n)+\sin(\sqrt{2}\pi n)}{n^7\left(\cosh(\sqrt{2}\pi n)-\cos(\sqrt{2}\pi n)\right)}$$

Proposed by Surjeet Singhania – India

U.540 Prove that:

$$\int_0^{\infty}\frac{x\tanh(\pi nx)}{(x^2+1)^2}dx=\frac{\pi^2}{4}n\sec^2(n\pi)-\frac{n}{4}\left(\psi^{(1)}\left(\frac{1}{2}-n\right)-\psi^{(1)}\left(\frac{1}{2}+n\right)\right)$$

Notations: $\psi^{(1)}(z)$ is Trigamma function

Proposed by Surjeet Singhania – India

U.541 Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{\left(\left[\sqrt[3]{n}\right]\right)^{10}}$$

$[*]$ - great integer function.

Proposed by Surjeet Singhania – India

U.542 Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^4n^2(m^2+n^2)}$$

Proposed by Surjeet Singhania – India

U.543 Assume for $n>j$

$$\theta_j(n)=\sum_{j\le k\le n}\binom{n}{k}\binom{k}{j}$$

Then prove:

$$\lim_{n\to\infty}\sum_{1\le j\le n}\frac{(-1)^{j+1}\theta_j(n)2^{j-n}H_j}{j}=\zeta(2)$$

Proposed by Surjeet Singhania – India

U.544 If

$$I=\int_0^{\frac{\pi}{2}}\left(\left[e^{e^{\ln\ln\left(\frac{\left(2\sqrt{2}+2\sqrt{2}\sin^2x\right)^2}{8+\sin^2x}\right)}}\right]-1\right)dx$$

Find the possible closed form for I without using any form of standard series. $[\,.\,]$ is floor function.

Proposed by Tobi Joshua – Nigeria

U.545 If

$$\lim_{n\to\infty}\frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=0}^{n}\sin\left(\frac{\pi k}{4}\right)=[x]\cot\left(\frac{[y]}{8}\right)$$

Then find $\Omega=2024-(x+y)$, $[.]$ represent floor function

Proposed by Tobi Joshua - Nigeria

U.546 Show that:

$$\lim_{s\to 1}\int_0^\infty\left(\sum_{n=0}^{\infty}\frac{t^n\sin\left(\frac{\pi n}{4}\right)}{n!}\right)dt=\frac{(i-1)}{\left(\sqrt{2}+(i-1)\right)^2}$$

Proposed by Tobi Joshua - Nigeria

U.547 Show that:

$$\log\left(\frac{8(1+\sin^2 x)^2}{8+\sin^2 x}\right)=\log\left(\frac{2\left(4+\pi x^2\left(\sum_{j=0}^{\infty} Re\ s_{s=-j}\frac{4^s(x^2)^{-s}\Gamma(s)}{\Gamma\left(\frac{3}{2}-s\right)}\right)^2\right)^2}{32+\pi x^2\left(\sum_{j=0}^{\infty} Re\ s_{s=-j}\frac{4^s(x^2)^{-s}\Gamma(s)}{\Gamma\left(\frac{3}{2}-s\right)}\right)^2}\right)$$

Proposed by Tobi Joshua - Nigeria

U.548 If $y'+\alpha y=|x^2|-\left[e^{|x-\beta|}\right]+1, y(0)=0$ then find the possible expression for $y(x)$ where $[.]$ is the floor function, and $\alpha,\beta>0$.

You are not to use Laplace transform

Proposed by Tobi Joshua - Nigeria

U.549 Given that:

$$\int_a^\infty x^2e^{-ax^2}\,dx=\frac{1}{4}\sqrt{\frac{\pi}{a}}$$

for $a>0$. Then show by converting the integral below to a partial differential equation (PDE) that:

$$\int_0^\infty xe^{-ax^2}\sinh(3x)\,dx=\frac{3}{4a}\sqrt{\frac{\pi}{a}}e^{\left(\frac{9}{4a}\right)}, Re\ a>0$$

(Restriction: No special functions are allowed)

Proposed by Tobi Joshua - Nigeria

U.550 Show that:

$$\int_0^\infty\frac{([e]-2\cos^{2n}x)}{2x^2}dx=\left(\frac{n\left[e^{\frac{\pi}{4}}-1\right]}{2^{2n-2}}\right)\frac{\Gamma(2n+1)}{\Gamma^2(n+1)}; n\in\mathbb{N}$$

Where $[.]$ is the greatest integer function.

Proposed by Tobi Joshua - Nigeria

U.551 If

$$\lim_{\beta\uparrow 1}\int_0^{\infty}\frac{\cos ax\cosh(\beta x)}{\cosh x+\cos(\eta)}dx=-[T]\cot(\eta)\frac{\sinh(\alpha\eta)}{\sinh(\alpha[T])}$$

for α,β and $\eta>0$, then find the value of $\{T\}$.

Where $[.]$ and $\{.\}$ are the floor and fractional function.

Proposed by Tobi Joshua - Nigeria

U.552 Show that:

$$\sum_{n=1}^{\infty}\frac{1}{n^3\sin^2 n}=3\zeta(5)-\frac{2}{\pi^2}\zeta(2)\zeta(3)-2\left(\frac{\cot 1}{1^4}+\frac{\cot 2}{2^4}+\frac{\cot 4}{3^4}+\cdots\right)$$

Proposed by Tobi Joshua - Nigeria

U.553 Let $\frac{\partial T}{\partial t}=3\frac{\partial^2 T}{\partial x^2}; 0<x<\infty, \forall t>0$ knowing that $T(x,0)=0, T(0,t)=4; t>0$

then, show that $T(x,t)=4\left(1-\operatorname{erf}\left(\frac{x}{2\sqrt{3t}}\right)\right), \forall\, t>0$

Proposed by Tobi Joshua - Nigeria

U.554 Prove that:

$$e^{\lim\limits_{k\to\infty}\left(\prod_{n=0}^{k}C_n^k\right)^{\frac{1}{k(k+1)}}}=e^{\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j}{j!}}=e^{\sqrt{e}}$$

Proposed by Tobi Joshua - Nigeria

U.555 If

$$\prod_{k=0}^{\infty}\exp\left(\frac{\left[\prod_{m=1}^{k}m\sum_{n=1}^{\infty}\frac{1}{n!}\right]}{\prod_{m=1}^{k}m\,3^k}\right)=e^{\left(1+be^{\frac{a}{b}}\right)}$$

find a and b, such that $b>a$ where $[.]$ is the greatest integer function.

Proposed by Tobi Joshua - Nigeria

U.556 Suppose $f(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$, every $a_i\in\mathbb{R}\setminus\{0\}$

$a_i\neq a_j$ for $i\neq j$ and $1\le i,j\le n$.Define $\mathcal{A}=\{\alpha: f(\alpha)=0\}, |\mathcal{A}|=n$. Take $p\in\mathbb{N}$ such that $p\le n-1$. Evaluate

$$\sum_{x\in\mathcal{A}}\frac{x^p+2x+1}{(n+1)x^n+n(a_1-1)+\sum_{k=2}^{n}(n-k+1)(a_k-a_{k-1})x^{n-k}}$$

Proposed by Surjeet Singhania – India

U.557 Suppose α,β,γ be the roots of x^3-5x+7, then evaluate the sum:

$$\sum_{\alpha,\beta,\gamma} \frac{x^3}{4x^3 - 18x^2 - 10x + 37}$$

Proposed by Surjeet Singhania - India

U.558 Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous functions at $x = 0$ such that

$$f(x) + f\left(\frac{x}{x+1}\right) = x^2; \forall x \in \mathbb{R}$$

Then find the value of $f(1)$.

Proposed by Surjeet Singhania - India

U.559 Let $y(x)$ be solutions of differentiable equation $\frac{d^2y}{dx^2} - \alpha^2 y = 0$.Such that $y(0) = 2$ and $y'(0) = 2\beta$. Here $0 \le \alpha \le \beta$. Find all $\alpha, \beta \in \mathbb{N}$. So that $y(\ln(\alpha)) = 1$

Proposed by Surjeet Singhania - India

U.560 Prove that

$$\lim_{n\to\infty}\left\{2n\left(1 - \sqrt[n]{\mathcal{A}_n}\right) + H_n\right\} = \gamma + \ln\left(\frac{1}{4\pi}\right)$$

Where

$$\mathcal{A}_n = \sum_{k=1}^{n} \frac{4^k}{k\binom{2k}{k}}, H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and γ is Euler Mascheroni Constant

Proposed by Surjeet Singhania - India

U.561 Let $\varphi(z)$ be an entire function and $\{z_k\}_{k=1}^{\infty}$ be sequence of zeros of $\varphi(z)$. Such that $|z_{n+k} - z_k| < \epsilon$ for all $n, k \ge m$ where $n, k, m \in \mathbb{N}$. Then find all $\varphi(z)$

Proposed by Surjeet Singhania - India

U.562 Prove or disprove

$$\gamma = -1 + \sum_{n=1}^{\infty}\left\{\frac{\mathcal{A}_n}{n^2} - \ln\left(\frac{n+1}{n}\right)\right\}$$

Where

$$\mathcal{A}_n = \sum_{k=1}^{n} \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}}$$

and γ is Euler Mascheroni Constant.

Proposed by Surjeet Singhania - India

U.563 Prove that for $m \in \mathbb{N}$

$$\int_0^1 \frac{\ln(2) - x^m \ln(1+x)}{1-x} dx = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2} + \{H_m + \overline{H}_m\}\ln(2) - \frac{\left(\overline{H}_m\right)^2 + H_m^{(2)}}{2}$$

Where

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \overline{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

and

$$H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}$$

Proposed by Surjeet Singhania - India

U.564 Suppose $f:[0,1] \to \mathbb{R}$ be a differentiable function, then prove that

$$\lim_{n\to\infty} \int_0^1 e^{-nx} f'(x)dx = 2f(0)$$

Proposed by Surjeet Singhania - India

U.565 Let

$$\phi(n) = \int_0^1 \frac{\sqrt[n]{x^x} \sin\left(\frac{\pi x}{n}\right)}{\sqrt[n]{(1-x)^x}} dx, \forall n \in \mathbb{N}$$

Then evaluate

$$\lim_{m\to\infty} \left(\frac{2}{\pi}\right)^m \frac{1}{m} \prod_{n=1}^{m} n\phi(n)$$

Proposed by Surjeet Singhania - India

U.566 Let α, β, γ be roots of $125x^3 - 171x^2 - 105x - 25$ then show that the sum

$$\sum_{\alpha,\beta,\gamma} \frac{x^2}{500x^3 - 2388x^2 + 1500x + 500}$$

is Rational.

Proposed by Surjeet Singhania - India

U.567

$$x_n = \sum_{k=1}^{n-1} \frac{1}{1+\omega^k}, y_n = \sum_{k=1}^{n-1} \frac{\omega^{2k}}{1+\omega^k}, \omega^n = 1, \omega \neq 1, n \in \mathbb{N}, n \geq 3$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{e^{H_n}}{x_n - y_n}$$

Proposed by Surjeet Singhania - India

U.568 If $f(n) = \prod_{k=0}^{n} \left(\frac{(10+12k)^4+324}{(4+12k)^4+324}\right)$ then find:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{f(n)}{n^3}$$

Proposed by Surjeet Singhania - India

U.569 Let x_1, x_2, x_3 and x_4 be roots of polynomial $ax^4 - x^3 + bx^2 + cx + d$ such that

$$\sum_{cyc} \frac{x_1}{x_2 + x_3 + x_4} = \frac{4}{3}$$

here each $x_i > 0$ hence find the values of a, b, c and d

Proposed by Surjeet Singhania - India

U.570 Prove that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)!_{(r)}} = r$$

Where $n!_{(r)}$ is multifactorial function

Proposed by Surjeet Singhania - India

U.571 Find without softs:

$$\Omega = \int_0^{2\pi} \left(\ln(5 + 4\cos x) + \arctan\left(\frac{\sin x}{2 + \cos x}\right)\right) dx$$

Proposed by Surjeet Singhania - India

U.572 If $0 < a \leq b$ then find:

$$\Omega(a,b) = \int_a^b \int_a^b \frac{x - \cos y}{x^2 - 2x\cos y + 1} dx\, dy$$

Proposed by Daniel Sitaru-Romania

U.573 If $1 < a \leq b$ then find:

$$\Omega(a,b) = \int_a^b \int_a^b \int_a^b \tan^{-1}\left(\frac{x + y + z - xyz}{1 - xy - yz - zx}\right) dx\, dydz$$

Proposed by Daniel Sitaru-Romania

U.574 If $0 < a \leq b$. Find a closed form:

$$\Omega(a,b) = \int_a^b \left(\frac{x}{1 + \frac{x^2}{3 + \frac{x^2}{5 + \frac{x^2}{7 + \cdots}}}} \right) dx$$

Proposed by Daniel Sitaru-Romania

U.575 If $0 < a \leq b < \frac{\pi}{2}$ then find:

$$\Omega(a,b) = \int_a^b \frac{3 + \cos 4x}{1 - \cos 4x} dx$$

Proposed by Daniel Sitaru-Romania

U.576 If $-1 < a \leq b < 1, n \in \mathbb{N}^*, P_n$ —Legendre's polynomials. Find:

$$\Omega(a,b) = \lim_{n \to \infty} \frac{1}{n} \int_a^b \frac{P_n'(x)}{P_{n-1}(x) - xP_n(x)} dx$$

Proposed by Daniel Sitaru-Romania

U.577 Solve for complex numbers:

$$4x^4 + 5x^2 + 4 = x\left(\tan\frac{\pi}{24}\tan\frac{11\pi}{24} + \tan\frac{5\pi}{24}\tan\frac{7\pi}{24}\right)$$

Proposed by Daniel Sitaru-Romania

U.578 If $5 < a \leq b$ then find:

$$\Omega(a,b) = \int_a^b \tan^{-1}\left(\frac{4x - 4x^3}{x^4 - 6x^2 + 1}\right) dx$$

Proposed by Daniel Sitaru-Romania

U.579 Solve for real numbers: $7\sin 6x + 35\sin x = \sin 7x + 21\sin 3x$

Proposed by Daniel Sitaru-Romania

U.580 Find:

$$\Omega = \lim_{n \to \infty} \frac{1}{n^2}\left(1 + 2\sum_{k=1}^{n} \frac{1}{2k+5}\right)^n$$

Proposed by Daniel Sitaru-Romania

U.581 In ΔABC let R_A —be the radii of circle tangent simultaneous to AB, AC and external tangent to circumcircle of ΔABC. Prove that:

$$\frac{R_A R_B}{r_a r_b} + \frac{R_B R_C}{r_b r_c} + \frac{R_C R_A}{r_c r_a} \geq \frac{64r^2}{3R^2}$$

Proposed by Daniel Sitaru-Romania

U.582 Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\cos^{2n}\frac{\pi}{7} - 2^{1-2n} \sum_{0\le i<j\le n} \binom{n}{i}\binom{n}{j} \cos\frac{2(j-i)\pi}{7}}$$

Proposed by Daniel Sitaru-Romania

U.583 If $x_i, y_i > -, i \in \overline{0,7}, 512\sum_{i=0}^{7}(x_i + y_i) = 1225$. Prove that:

$$\sum_{i=0}^{7} \frac{\sin^6\left(\frac{i\pi}{8}\right)}{x_i} + \sum_{i=0}^{7} \frac{\cos^6\left(\frac{i\pi}{8}\right)}{y_i} \ge 1$$

Proposed by Daniel Sitaru-Romania

U.584 If $\sec\frac{\pi}{7} < a \le b$ then find:

$$\Omega(a,b) = \int_a^b \left(\tan^{-1}\left(\frac{x}{\sec\frac{\pi}{7} - x\tan\frac{\pi}{7}}\right) - \tan^{-1}\left(x\sec\frac{\pi}{7} - \tan\frac{\pi}{7}\right)\right)dx$$

Proposed by Daniel Sitaru-Romania

U.585 Solve for real numbers:

$$\sin 5x + 10\sin x = 5\sin 3x$$

Proposed by Daniel Sitaru-Romania

U.586 If $x \ge 0$ then:

$$\begin{vmatrix} \sin^3 4x & \sin^2 4x\cos 4x & \sin 4x\cos^2 4x & \cos^2 4x \\ \sin^3 3x & \sin^2 3x\cos 3x & \sin 3x\cos^2 3x & \cos^3 3x \\ \sin^3 2x & \sin^2 2x\cos 2x & \sin 2x\cos^2 2x & \cos^3 2x \\ \sin^3 x & \sin^2 x\cos x & \sin x\cos^2 x & \cos^3 x \end{vmatrix} \le 12x^6$$

Proposed by Daniel Sitaru-Romania

U.587 If $x, y, z, t \ge 0$ then:

$$x^2\cot\frac{\pi}{19} + y^2\cot\frac{2\pi}{19} + z^2\cot\frac{4\pi}{19} + t^2\tan\frac{8\pi}{19} \ge \left(x + y\sqrt{2} + 2z + 2t\sqrt{2}\right)^2\tan\frac{\pi}{19}$$

Proposed by Daniel Sitaru-Romania

U.588 If $\frac{1}{\sqrt{31}} < a \le b$ then find:

$$\Omega(a,b) = \int_a^b \tan^{-1}\left(\frac{30x^3 - 10x}{31x^2 - 1}\right)dx$$

Proposed by Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-SUMMER 2023

PROBLEMS FOR JUNIORS

JP.421 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2+ab+bc+ca}{2s+a}\le 3\sqrt{3}R$$

Proposed by Daniel Sitaru-Romania

JP.422 If $a,b,c>0$, then:

$$\frac{(a^2+c^2)(b^2+c^2)}{(ab+c^2)(ac+bc)}+\frac{(b^2+a^2)(c^2+a^2)}{(bc+a^2)(ba+ca)}+\frac{(c^2+b^2)(a^2+b^2)}{(ca+b^2)(cb+ab)}\ge 3$$

Proposed by Daniel Sitaru-Romania

JP.423 If $a,b,c>0$, then:

$$\left(\frac{a}{c}\right)^2 e^{\left(\frac{c}{a}\right)^2}+\left(\frac{b}{a}\right)^2 e^{\left(\frac{a}{b}\right)^2}+\left(\frac{c}{b}\right)^2 e^{\left(\frac{b}{c}\right)^2}\ge 3e$$

Proposed by Daniel Sitaru-Romania

JP.424 Solve for real numbers:

$$(x+1)(x-1)\begin{vmatrix}\overline{x111} & \overline{1x11} & \overline{11x1} & \overline{111x}\\ \overline{1x11} & \overline{11x1} & \overline{111x} & \overline{x111}\\ \overline{11x1} & \overline{111x} & \overline{x111} & \overline{1x11}\\ \overline{111x} & \overline{x111} & \overline{1x11} & \overline{11x1}\end{vmatrix}+(y+3)(y-1)\begin{vmatrix}\overline{y111} & \overline{1y11} & \overline{11y1} & \overline{111y}\\ \overline{1y11} & \overline{11y1} & \overline{111y} & \overline{y111}\\ \overline{11y1} & \overline{111y} & \overline{y111} & \overline{1y11}\\ \overline{111y} & \overline{y111} & \overline{1y11} & \overline{11y1}\end{vmatrix}=0$$

Proposed by Daniel Sitaru-Romania

JP.425 If $x,y,z>0, x^2+y^2+z^2=3$, then:

$$\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2}+xy+yz+zx\ge 6$$

Proposed by Daniel Sitaru-Romania

JP.426 If $a,b,c\in\mathbb{C}$ then:

$$\frac{|a+1|}{|b+1|+|b+c|+|c|}+\frac{|b+1|}{|c+1|+|c+a|+|a|}+\frac{|c+1|}{|a+1|+|a+b|+|b|}\geq$$

$$\geq 3+|a|+|b|+|c|$$

Proposed by Daniel Sitaru-Romania

JP.427 If $a,b>1$ then:

$$\left(a^x\cdot e^{a^{2x}}+b^x\cdot e^{b^{2x}}\right)\cdot e^{a^x\cdot b^x}\geq (a^x+b^x)\cdot e^{a^{2x}+b^{2x}};\forall x\in\mathbb{R}$$

Proposed by Daniel Sitaru-Romania

JP.428 Let be $A=\{a,b,c|a,b,c\in\mathbb{R}^*\}$ and $B=\{u,v,w,t|u,v,w,t\in\mathbb{R}^*\}$ such that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}=3$$

$$\frac{u^2}{v^2}+\frac{v^2}{w^2}+\frac{w^2}{t^2}+\frac{t^2}{u^2}=\frac{v^2}{u^2}+\frac{w^2}{v^2}+\frac{t^2}{w^2}+\frac{u^2}{t^2}=4$$

Find:

$$\Omega=\sum_{a,y\in A}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in A}\left|\frac{x}{y}\right|+\sum_{a,y\in B}\left|\frac{x}{y}\right|\cdot\prod_{x,y\in B}\left|\frac{x}{y}\right|$$

Proposed by Daniel Sitaru-Romania

JP.429 Let $x\in\mathbb{R}$, then in ΔABC holds:

$$\frac{2abc}{R}\leq\frac{a^x}{r_a}+\frac{b^x}{r_b}+\frac{c^x}{r_c}\leq\frac{abc}{r}$$

Proposed by Alex Szoros-Romania

JP.430 If $x,y,z>0$, then prove that:

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\geq\frac{3x+y+z}{y+z}+\frac{x+3y+z}{z+x}+\frac{x+y+3z}{x+y}$$

Proposed by Neculai Stanciu-Romania

JP.431 If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$3+\sqrt[3]{\prod_{cyc}(2+\tan^6 x)}\geq\sec^2 x+\sec^2 y+\sec^2 z$$

Proposed by Daniel Sitaru-Romania

JP.432 In ΔABC the following relationship holds:

$$\left(\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\right)\left(\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\right)\geq\frac{4R}{r}+1$$

Proposed by Marian Ursărescu-Romania

JP.433 In $\Delta ABC, AA', BB', CC'$ −internal bisectors and A'', B'', C'' −contact points with circumcircle of ΔABC. Prove that:

$$\frac{1}{3}\left(7-\frac{2r}{R}\right)^2\leq\frac{AA''}{A'A''}+\frac{BB''}{B'B''}+\frac{CC''}{C'C''}\leq 6\left(\left(\frac{R}{r}\right)^2-2\right)$$

Proposed by Marian Ursărescu-Romania

JP.434 If $x,y,z\in(0,1); 4(x^2+y^2+z^2)=3$ then:

$$x^2y^2(1-x^2)^3+y^2z^2(1-y^2)^3+z^2x^2(1-z^2)^3\leq\frac{243}{1024}$$

Proposed by Daniel Sitaru – Romania

JP.435 If $x,y,z>0; x^2+y^2+z^2=1$ then:

$$(x^6+y^6+z^6)^3\geq\left(x^5+y^5+z^5\right)^4$$

Proposed by Daniel Sitaru-Romania

PROBLEMS FOR SENIORS

SP.421 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}{\cos^4\frac{A}{2}+\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^4\frac{B}{2}}\geq\frac{8}{3}$$

Proposed by Marian Ursărescu-Romania

SP.422 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_b+r_c}{r_b^2+r_br_c+r_c^2}\geq\frac{2}{2R-r}$$

Proposed by Marian Ursărescu-Romania

SP.423 If $z_1,z_2,z_3\in\mathbb{C}^*$ different in pairs such that $|z_1|=|z_2|=|z_3|=1$, $A(z_1),B(z_2),C(z_3)$. Prove that:

$$\sum_{cyc} \frac{z_2 z_3}{3z_2 z_3 - z_2^2 - z_3^2} = \frac{3}{4} \Leftrightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

SP.424 If $x, y, z > 0, 27(x^3y + y^3z + z^3x) = 1$ then

$$45(x^2y + y^2z + z^2x) + 6(xy + yz + zx) \leq 4 + 3(x + y + z)$$

Proposed by Daniel Sitaru-Romania

SP.425 If $x \in [0, 1]$, then

$$1 + x^2 \leq \int_0^1 e^{t^2}\, dt + \int_0^x 2t^2 e^{t^2}\, dt + \int_x^1 (2t^2 - 2t)e^{t^2}\, dt \leq e^x$$

Proposed by Alex Szoros-Romania

SP.426 Let R_1, R_2, R_3 be circumradii of $\Delta A_1B_1C_1, \Delta A_2B_2C_2, \Delta A_3B_3C_3$ with sides a_1, a_2, a_3 respectively b_1, b_2, b_3 and c_1, c_2, c_3. Prove that:

$$\frac{1}{a_1a_2a_3} + \frac{1}{b_1b_2b_3} + \frac{1}{c_1c_2c_3} \geq \frac{9\sqrt{3}}{(R_1 + R_2 + R_3)^3}$$

Proposed by D.M. Bătinețu-Giurgiu –Romania

SP.427 If $f: [0, n] \to \left[0, \frac{1}{n-1}\right]$ continuous function, $n \in \mathbb{N}, n \geq 3$ then:

$$\int_0^n x^\alpha f(x) \cdot \sqrt[n-1]{1 - (n-1)f(x)}\, dx \leq \frac{1}{\alpha + 1} \cdot \sqrt[n-1]{n^{\alpha(n-1)-1}}; \alpha > 0$$

Proposed by Florică Anastase-Romania

SP.428 Solve for real numbers:

$$\sum_{k=1}^{n} \frac{1}{\cos x - \cos(2k+1)x} = \frac{\sin nx}{\sin(n+1)x} \cdot \cot x$$

Proposed by Florică Anastase-Romania

SP.429 Let $(x_n)_{n\geq 1}$ is a sequence of real numbers such that

$$x_n = \int_0^1 x^n \cdot \log(1 + x)\, dx. \text{ Find:}$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} (-1)^{k-1} x_k$$

Proposed by Florică Anastase-Romania

SP.430 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{\cos\frac{A}{2}}\le\frac{3}{2s}\sum_{cyc}\frac{a}{\cos\frac{A}{2}}\le\frac{6R}{s}\sqrt{2+\frac{r}{2R}}\le\sqrt{2+\frac{5R}{r}}$$

Proposed by Alex Szoros-Romania

SP.431 In ΔABC the following relationship holds:

$$1\ge\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s^2(s^2+r^2+2Rr)}\ge\frac{2r}{R}$$

Proposed by Alex Szoros-Romania

SP.432 If $a,b,c>0$ then:

$$\frac{a^6+15a^4+15a^2+1}{3b^5+10b^3+3b}+\frac{b^6+15b^4+15b^2+1}{3c^5+10c^3+3c}+\frac{c^6+15c^4+15c^2+1}{3a^5+10a^3+3a}\ge 6$$

Proposed by Daniel Sitaru-Romania

SP.433 Let be $f:\mathbb{R}\to\mathbb{R}$, $f(x+2)+10f(x)=7f(x+1);\forall x\in\mathbb{R}$. If $f(0)=2,f(1)=7$ then find:

$$\Omega=\log 2\cdot\log 5\cdot\int_0^1 f(x)\,dx$$

Proposed by Daniel Sitaru-Romania

SP.434 Let be $f:\mathbb{R}\to\mathbb{R}, f(0)=3, f(1)=10, f(2)=38$

$$f(x+3)+31f(x+1)=10f(x+2)+30f(x)$$

Solve for real numbers: $f(x)=10$.

Proposed by Daniel Sitaru-Romania

SP.435 Let ΔABC with inradius r, circumradius R, and exradii r_a,r_b,r_c. Prove that:

$$\frac{R}{2r}\ge\frac{1}{3}\sqrt{\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+6}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

UNDERGRADUATE PROBLEMS

UP.421 Let $P_{n-1}(x)=a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1}$ ($n\ge 2, n\in\mathbb{N}$) such that: $\sqrt{1-x^2}.|P_{n-1}(x)|\le 1,\forall x\in[-1,1]$. Prove that: $|a_0|\le 2^{n-1}$

Proposed by Nguyen Van Canh-Vietnam

UP.422 Find:

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n}\left(\sqrt[n]{\frac{(a+1)(k+n)-a}{(a+1)n}}\right)^{\frac{(a+1)(k+n)-a}{(a+1)n}};\ a\in\mathbb{N}^*$$

Proposed by Neculai Stanciu-Romania

UP.423 Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n^3}\int_0^n \frac{x^2+n^2}{2^{-x}+1}dx$$

Proposed by Neculai Stanciu-Romania

UP.424 Solve for integers:

$$\sqrt[3]{(1+x)^x\cdot(2x-4)^{2x-5}\cdot(3x-9)^{3x-10}} = \left(1+\sqrt[3]{6x^3-35x^2+50x}\right)^{1+\sqrt[3]{6x^3-35x^2+50x}}$$

Proposed by Daniel Sitaru-Romania

UP.425 If $0 < a \le b$ then:

$$a^{a+1}\cdot\exp\big(2(b-a)\big)\le b^{b+1}$$

Proposed by Daniel Sitaru-Romania

UP.426 Let $(a_n)_{n\ge1}$ be a sequence of real numbers such that $a_n \le n;\forall n\ge 1$ and

$$\sum_{k=1}^{n-1}\cos\frac{\pi a_k}{n} = 0;\forall n\ge 2.\ \text{Find}:$$

$$\Omega = \lim_{n\to\infty}\left(a_n\cdot\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}\right)$$

Proposed by Florică Anastase-Romania

UP.427 Let $(a_n)_{n\ge1}$ be a sequence of real numbers such that $a_n \le n;\forall n\ge 1$ and

$$\sum_{k=1}^{n-1}\cos\frac{\pi a_k}{n} = 0;\forall n\ge 2.\ \text{Find}:$$

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2}\cdot\sum_{k=0}^{2n}\frac{\binom{2n}{k}}{\binom{4n}{2k}}\right)^{a_{2n+1}}$$

Proposed by Florică Anastase-Romania

UP.428 If $(a_n)_{n\geq 1}$ is a positive real sequence, such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(n!)^2}}=a;\ a\in\mathbb{R}_+^*$

then find:

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n]{(2n-1)!!}}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP.429 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\pi^2}{16}-\left(\sum_{k=2}^{n+1}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)\right)^2\right)\cdot\sqrt[n]{n!}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.430 If $a>0;t\in\mathbb{N};a,t$ –fixed then find:

$$\Omega(a,t)=\lim_{n\to\infty}\left(\sqrt[nt]{a}-1\right)\cdot\sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.431 Find:

$$\Omega(a)=\lim_{t\to\infty}e^{H_n}\cdot\sqrt[n]{n!}\left(\sqrt[n^2]{a}-1\right);a>0;a-\text{fixed}.$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.432 Let $(b_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^{t+1}}=b>0;a>0;t\geq 0;a,t-\text{fixed}.\ \text{Find}:$$

$$\Omega(a,b,t)=\lim_{n\to\infty}\frac{\left(\sqrt[n]{a}-1\right)\cdot\sqrt[n]{b_n}}{n^t}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.433 If $m\geq 0;m$ –fixed, $u,v>0,u+v=3$ then

$$\frac{1}{u^m}\left(\int_0^1 e^{x^3}\,dx\right)^{m+1}+\frac{1}{v^m}\left(\int_0^1\sqrt[3]{\log x}\,dx\right)^{m+1}\geq\frac{1}{3^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.434 If $a,b>0;a,b$ –fixed, find: $\Omega(a,b)=\int_{-a}^{a}\frac{x^{2022}}{b+\sqrt{b^2+x^2}+x}dx$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.435 If $\Omega(n)=\lim\limits_{t\to\infty}\int_0^t \frac{dx}{(x^2+n^2)(x^2+n^4)(x^2+n^6)}; n\in\mathbb{N}, n\geq 2$

Find:

$$\Omega=\lim_{n\to\infty}\frac{n^{\alpha}}{\Omega(n)}\cdot\int_0^1 \sqrt[n]{1+x+x^n}\,dx\,;\alpha\in\mathbb{R}$$

Proposed by Florică Anastase-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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