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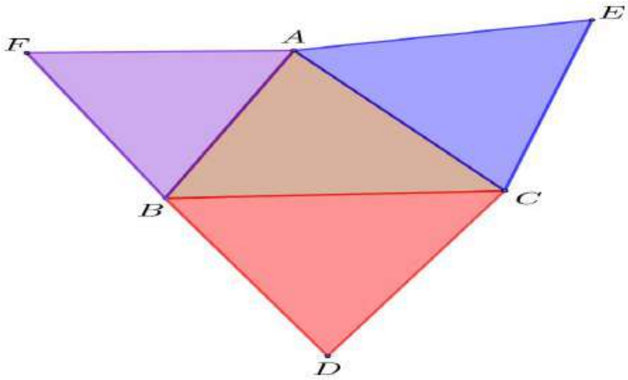
CONTENT

A MUTI KARA AMAZING GEOMETRICAL CONFIGURATION

by Muti Kara-Ankara-Turkiye



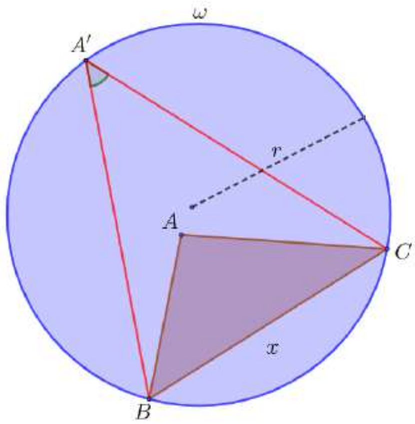
ABC is an acute triangle and BCD, ACE, and AFB are equilateral triangles.

Let's call circumradius of ΔDEF as ρ.

Prove that the following relationship holds:

$$2R \geq \rho$$

Proof.

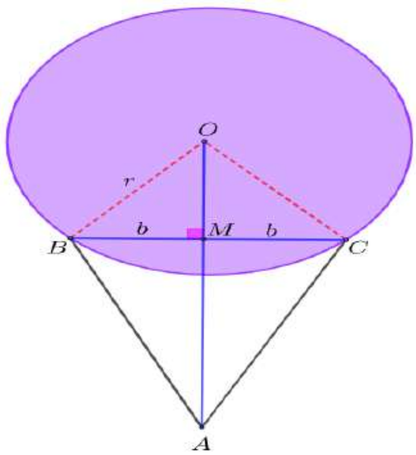


Lemma 1. Circumradius at acute triangle ABC is always smaller or equal than the radius of circles which has A, B and C as interior points.

Proof. $A = A' + \mu(\widehat{A'CA}) + \mu(\widehat{A'BA}) \Rightarrow A \geq A'$

$$2r = \frac{x}{\sin A'} \geq \frac{x}{\sin A} = 2R$$

Since sine is continuously increasing in the interval $[0, 90^\circ]$, the inequality above is correct.



Lemma 2. ω is a circle with center O and $B, C \in \omega$. ΔABC is a equilateral triangle. The following inequality holds: $|AO| \leq 2r$.

Proof. $BC \cap OA = \{M\}$

$$|BM| = |MC| \Rightarrow |OM| = \sqrt{r^2 - b^2} \text{ and } |MA| = b\sqrt{3}$$

We want to prove that:

$$2r \geq \sqrt{r^2 - b^2} + b\sqrt{3} \Leftrightarrow 2r - b\sqrt{3} \geq \sqrt{r^2 - b^2} \Leftrightarrow$$

$$4r^2 + 3b^2 - 4\sqrt{3}rb \geq r^2 - b^2 \Leftrightarrow \left(2b - r\sqrt{3}\right)^2 \geq 0 \text{ which is true.}$$

The equality holds when $2b = r\sqrt{3}$, which means $\mu(\widehat{BC}) = 120°$

Let call ω the image of circumcicle of ΔABC from hhomothehty with center O and factor 2.

According to Lemma 2, D, E and F must be interior points of ω.

If triangle DEF is an acute triangle, from Lemma 1, ρ must be less or equal than $2R$.

Therefore, for acute ΔDEF our statement is true.

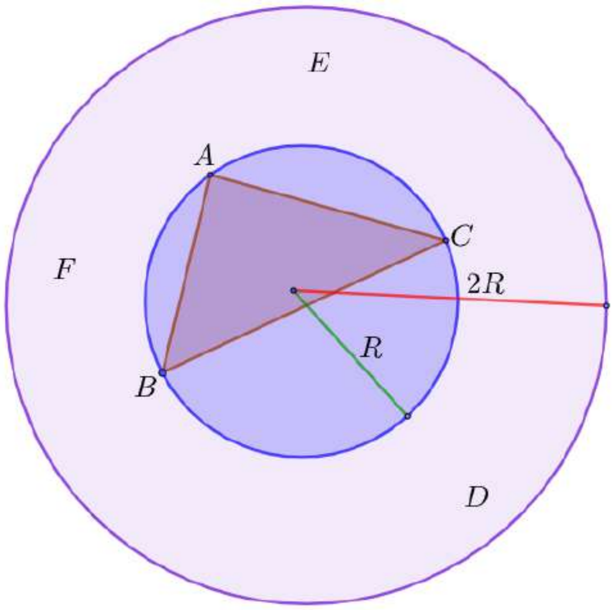
Let assume $\mu(\widehat{FDE}) > 90°$. We distinguish the following cases:

$(i)\ \widehat{B}; \widehat{C} < 60°$

$(ii)\ \widehat{B}; \widehat{C} > 60°$

$(iii) \widehat{B} > 60°$ and $\widehat{C} < 60°$

Because B and C are both less than $60°$, $\widehat{FBD}$ and $\widehat{ECD}$ are less than straight angle.



Case (i)

Because B and C are both less than $60°$, $\widehat{FBD}$ and $\widehat{ECD}$ are less than straight angle, hence

$$60° = \mu(\widehat{BDC}) > \mu(\widehat{FDE}) > 90°$$

Case (ii)

$$\mu(\widehat{FBD}) = 240° - B \text{ and } \mu(\widehat{ECD}) = 240° - C$$

$$\mu(\widehat{BFD}) + \mu(\widehat{BDF}) = B - 60°$$

$$\Rightarrow \mu(\widehat{BDF}) < B - 60°$$

Similarly, we have:

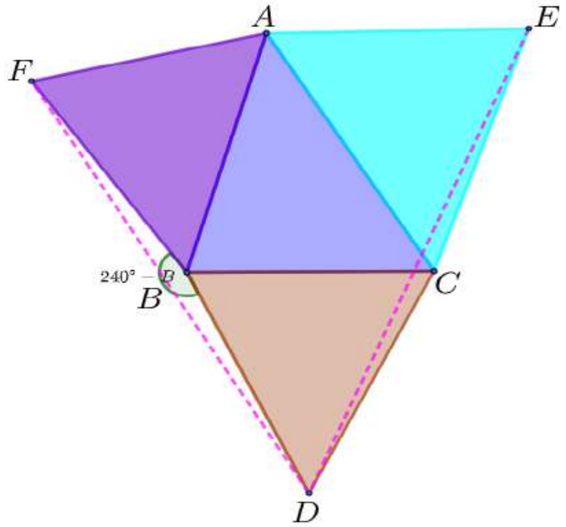
$$\mu(\widehat{CDE}) < C - 60°$$

$$\mu(\widehat{FDE}) = \mu(\widehat{BDF}) + \mu(\widehat{CDE}) + 60° <$$

$$< B + C - 60° \Rightarrow$$

$$\mu(\widehat{FDE}) < B + C - 60° = 120° - A$$

Because $\mu(\widehat{FDE}) > 90°, A < 30°$.



Case (iii)

We know that $\mu(\widehat{BDF}) < B - 60^\circ$ from Case (ii), we get $\mu(\widehat{CDF}) < B$. Because $90^\circ < \mu(\widehat{EDF}) < \mu(\widehat{CDF}) < B, B$ must be obtuse angle, but

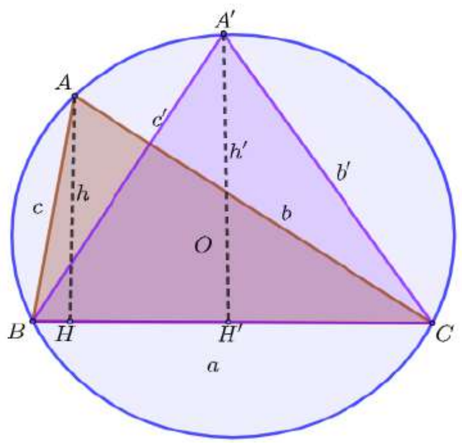
ΔABC must be acute triangle. Therefore, only possible case is case (ii). Let say $|AD| = x$.

Lemma 3. These equalities always holds:

$$|FD|^2 = a^2 + c^2 - b^2 + x^2$$
$$|ED|^2 = a^2 + b^2 - c^2 + x^2$$
$$|FE|^2 = b^2 + c^2 - a^2 + x^2$$

***Proof*. I will prove the first identity. Others have similar proofs. From cosine theorem, we get:** $x^2 =$

$$a^2 + c^2 - 2ac \cdot \cos(B + 60^\circ)$$
$$|FD|^2 = a^2 + c^2 - 2ac \cdot \cos(B + 120^\circ)$$
$$|FD|^2 - x^2 = 2ac(\cos(B + 60^\circ) - \cos(B + 120^\circ)) =$$
$$= 2ac\left(\frac{1}{2}\cos B - \frac{\sqrt{3}}{2}\sin B - \left[-\frac{1}{2}\cos B - \frac{\sqrt{3}}{2}\sin B\right]\right) =$$
$$= 2ac \cdot \cos B = a^2 + c^2 - b^2.$$



Lemma 4. A' is the midpoint of $\widehat{BC}$, then

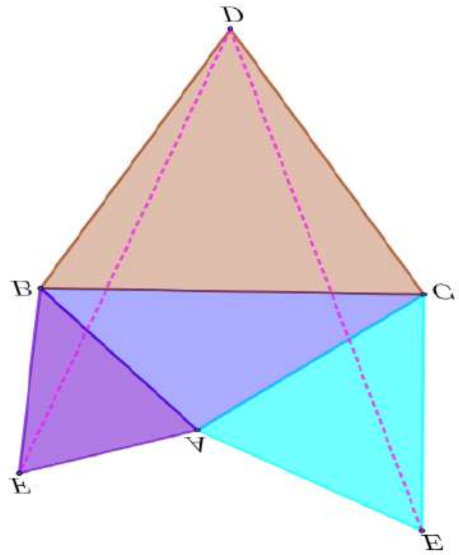
$$b'^2 + c'^2 \geq b^2 + c^2$$

***Proof*. Because $h' \geq h$, we get**

$$[A'BC] \geq [ABC] \Leftrightarrow$$
$$bc \cdot \sin A \leq b'c' \cdot \sin A \Leftrightarrow$$
$$bc \leq b'c'; \text{(o)}$$

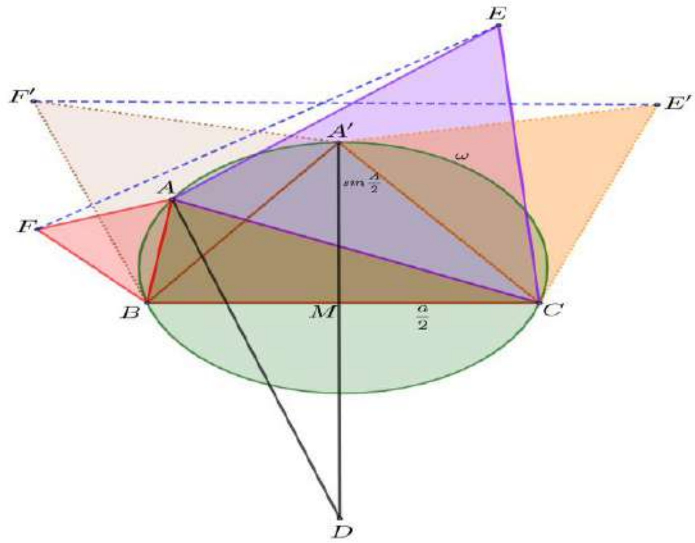
From cosine theorem, we get:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A =$$
$$= b'^2 + c'^2 - 2b'c' \cdot \cos A \Leftrightarrow$$
$$b^2 + c^2 - 2bc \cdot \cos A =$$
$$= b'^2 + c'^2 - 2b'c' \cdot \cos A \Leftrightarrow$$
$$2\cos A\,(b'c' - bc) = b'^2 + c'^2 - b^2 - c^2$$



Because of (o) LHS is greater or equal than 0, RHS must be $\geq 0 \Rightarrow b'^2 + c'^2 \geq b^2 + c^2$.

Lemma 5. $|EF| \le \frac{a}{\sin\frac{A}{2}}\sqrt{\frac{1-\cos(A+120^\circ)}{2}}$



Proof. A' is the midpoint of $\widehat{BC}$. From Lemma 3, we have: $|EF|^2 = b^2 + c^2 - a^2 + |AD|^2$
$|E'F'|^2 = b'^2 + c'^2 - a^2 + |A'D|^2$. From geometric location of A, A', ω and D, we can say that

$$|A'D| \ge |AD|.$$

From Lemma 4, we can say that : $b'^2 + c'^2 \ge b^2 + c^2$, (1): $|E'F'| \ge |EF|$

From cosines theorem: (2): $|E'F'|^2 = b'^2 + c'^2 - 2b'c'\cos(A+120^\circ)$

$$(3): b' = c' = \frac{a}{2\sin\frac{A}{2}}$$

From (1),(2) and (3), we get: $|EF| \le \frac{a}{\sin\frac{A}{2}}\sqrt{\frac{1-\cos(A+120^\circ)}{2}}$. Now, we know that: $|EF| \le \frac{a}{\sin\frac{A}{2}}\sqrt{\frac{1-\cos(A+120^\circ)}{2}}$ $(Lemma\ 5)$; $0 < A < 30^\circ$ and $90^\circ \le \mu(\widehat{FDE}) < 120^\circ - A$ assumption and case (ii). Because of sine is continuously decreasing in interval $(90^\circ, 180^\circ)$, we can say that:

$$2\rho = \frac{|EF|}{\sin(\widehat{FOE})} < \frac{\frac{a}{\sin\frac{A}{2}}\sqrt{\frac{1-\cos(A+120^\circ)}{2}}}{\sin(120^\circ - A)} < \frac{2a}{\sin A} = 4R \Leftrightarrow$$

$$\sin A\sqrt{\frac{1-\cos(A+120^\circ)}{2}} < 2\sin\frac{A}{2}\cdot\sin(120^\circ - A) \Leftrightarrow$$

$$\cos\frac{A}{2}\sqrt{\frac{1-\cos(A+120^\circ)}{2}} < \frac{\sqrt{3}\cos A + \sin A}{2} \Leftrightarrow$$

$$\cos\frac{A}{2}\sqrt{2+\cos A + \sqrt{3}\sin A} < \sqrt{3}\cos A + \sin A \Leftrightarrow$$

$$\cos^2\frac{A}{2}\left(2+\cos A+\sqrt{3}\sin A\right)<3\cos^2 A+2\sqrt{3}\cos A\sin A+\sin^2 A\Leftrightarrow$$

$$\left(\frac{1+\cos A}{2}\right)\left(2+\cos A+\sqrt{3}\sin A\right)<2\cos^2 A+2\sqrt{3}\sin A\cos A+1\Leftrightarrow$$

$$3\cos A+\sqrt{3}\sin A<3\cos A\left(\cos A+\sqrt{3}\sin A\right)\Leftrightarrow \cos(A+30^\circ)<3\cos A\cdot\sin(A+30^\circ)$$

$1<3\cos A\cdot\tan(A+30^\circ)$ **which is clearly true.**

FEW AMAZING PROPERTIES OF MEANS

By Daniel Sitaru-Romania

Abstract: ***In this paper are presented few amazing properties of means.***

Notations:

$$A_n=\frac{a_1+a_2+\cdots+a_n}{n};G_n=\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}$$

Property 1: If $0<a_1\le a_2\le\cdots\le a_n$ then:

$$A_{k-1}\le A_k\le a_n;\forall k\in\overline{2,n},n\in\mathbb{N},n\ge 2;(1)$$

Proof: $A_{k-1}\le A_k\Leftrightarrow\frac{a_1+a_2+\cdots+a_{k-1}}{k-1}\le\frac{a_1+a_2+\cdots+a_k}{k}\Leftrightarrow$

$$k(a_1+a_2+\cdots+a_{k-1})\le(k-1)(a_1+a_2+\cdots+a_{k-1}+a_k)$$

$$k(a_1+a_2+\cdots+a_{k-1})\le k(a_1+a_2+\cdots+a_{k-1})+ka_k-(a_1+a_2+\cdots+a_{k-1}+a_k)$$

$$a_1+a_2+\cdots+a_{k-1}+a_k\le ka_k$$

which result by adding all relationships $a_i\le a_k;\forall i\in\overline{1,k}$

$$A_k\le a_n\Leftrightarrow\frac{a_1+a_2+\cdots+a_k}{k}\le a_n\Leftrightarrow a_1+a_2+\cdots+a_k\le ka_n$$

which result by adding all relationships $a_i\le a_n;\forall i\in\overline{1,k}$.

Corollary 1: If $0<a\le b\le c$ then:

$$\frac{a+b}{2}\le\frac{a+b+c}{3}\le c;(2)$$

Proof. We take in (1): $n=3,a_1=a,a_2=b,a_3=c$.

Corollary 2: If $0<a\le b\le c\le d$ then:

$$\frac{a+b+c}{3}\le\frac{a+b+c+d}{4}\le d;(3)$$

Proof. We take in (1): $n=3,a_1=a,a_2=b,a_3=c,a_4=d$.

Corollary 3: If $x,y>0$ then:

$$\frac{\sqrt{xy}+\frac{x+y}{2}}{2}\le\frac{\sqrt{xy}+\frac{x+y}{2}+\sqrt{\frac{x^2+y^2}{2}}}{3}\le\sqrt{\frac{x^2+y^2}{2}}$$

Proof. We take in (2): $a=\sqrt{xy};b=\frac{x+y}{2};c=\sqrt{\frac{x^2+y^2}{2}}$

Corollary 4: If $x,y>0$ then:

$$\frac{\frac{2xy}{x+y}+\sqrt{xy}+\frac{x+y}{2}}{3}\le\frac{\frac{2xy}{x+y}+\sqrt{xy}+\frac{x+y}{2}+\sqrt{\frac{x^2+y^2}{2}}}{4}\le\sqrt{\frac{x^2+y^2}{2}}$$

Proof. We take in (3): $a=\frac{2xy}{x+y};b=\sqrt{xy};c=\frac{x+y}{2};d=\sqrt{\frac{x^2+y^2}{2}}$

Corollary 5: If $x,y,z>0$ then:

$$\frac{\sqrt[3]{xyz}+\frac{x+y+z}{3}}{2}\le\frac{\sqrt[3]{xyz}+\frac{x+y+z}{3}+\sqrt{\frac{x^2+y^2+z^2}{3}}}{3}\le\sqrt{\frac{x^2+y^2+z^2}{3}}$$

Proof. We take in (2): $a=\sqrt[3]{xyz};b=\frac{x+y+z}{3};c=\sqrt{\frac{x^2+y^2+z^2}{3}}$

Property 2: If $0<a_1\le a_2\le\cdots\le a_n$ then:

$$a_k\cdot A_{k-1}^{k-1}\le A_k^k;k\in\overline{2,n},n\in\mathbb{N},n\ge 2;(4)$$

Proof. By (1): $A_{k-1}\le A_k\Rightarrow\frac{A_k}{A_{k-1}}\ge 2;\forall k\in\overline{2,n}$

$$\frac{A_k^k}{A_{k-1}^{k-1}}=A_{k-1}\cdot\left(\frac{A_k}{A_{k-1}}\right)^k\overset{Bernoulli}{\ge}A_{k-1}\left(1+k\left(\frac{A_k}{A_{k-1}}-1\right)\right)=$$

$$=A_{k-1}+kA_k-kA_{k-1}=kA_k-(k-1)A_{k-1}=$$

$$=k\cdot\frac{a_1+a_2+\cdots+a_k}{k}-(k-1)\cdot\frac{a_1+a_2+\cdots+a_{k-1}}{k-1}$$

$$\frac{A_k^k}{A_{k-1}^{k-1}}\ge a_k\Rightarrow a_k\cdot A_{k-1}^{k-1}\le A_k^k$$

Corollary 6: If $0<a\le b\le c$ then:

$$c\left(\frac{a+b}{2}\right)^2\le\left(\frac{a+b+c}{3}\right)^3;(5)$$

Proof. We take in (4): $n=3,a_1=a,a_2=b,a_3=c$.

Corollary 7: If $0<a\le b\le c\le d$ then:

$$d\left(\frac{a+b+c}{3}\right)^3\le\left(\frac{a+b+c+d}{4}\right)^4;(6)$$

Proof. We take in (4): $n=4, a_1=a, a_2=b, a_3=c, a_4=d$.

Corollary 8: If $x, y > 0$ then:

$$\frac{x+y}{2}\left(\frac{\frac{2xy}{x+y}+\sqrt{xy}}{2}\right)^2 \le \left(\frac{\frac{2xy}{x+y}+\sqrt{xy}+\frac{x+y}{2}}{3}\right)^3$$

Proof. We take in (5): $a=\frac{2xy}{x+y}, b=\sqrt{xy}, c=\frac{x+y}{2}$.

Corollary 9: If $x, y > 0$ then:

$$\sqrt{\frac{x^2+y^2+z^2}{3}}\left(\frac{\frac{3xyz}{xy+yz+zx}+\sqrt[3]{xyz}+\frac{x+y+z}{3}}{3}\right)^3 \le$$

$$\le \left(\frac{\frac{3xyz}{xy+yz+zx}+\sqrt[3]{xyz}+\frac{x+y+z}{3}+\sqrt{\frac{x^2+y^2+z^2}{3}}}{4}\right)^4$$

Proof. We take in (6):

$$a=\frac{3xyz}{xy+yz+zx}, b=\sqrt[3]{xyz}, c=\frac{x+y+z}{3}, d=\sqrt{\frac{x^2+y^2+z^2}{3}}$$

Property 3: If $0 < a_1 \le a_2 \le \dots \le a_n, n \in \mathbb{N}, n \ge 2$ then:

$$G_{k-1} \le G_k \le a_n; \forall k \in \overline{2,n}; (7)$$

Proof.

$$\left(\frac{G_k}{G_{k-1}}\right)^{k(k-1)} = \left(\frac{\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k}}{\sqrt[k-1]{a_1a_2\cdot\ldots\cdot a_{k-1}}}\right)^{k(k-1)} = \frac{(a_1a_2\cdot\ldots\cdot a_k)^{k-1}}{(a_1a_2\cdot\ldots\cdot a_{k-1})^k} =$$

$$= \frac{a_k^{k-1}}{a_1a_2\cdot\ldots\cdot a_{k-1}} = \frac{a_k}{a_1}\cdot\frac{a_k}{a_2}\cdot\ldots\cdot\frac{a_{k-1}}{a_k} \ge 1 \text{ because } a_k \ge a_i; \forall i \in \overline{1,k}.$$

$$\left(\frac{G_k}{G_{k-1}}\right)^{k(k-1)} \ge 1 \Rightarrow \frac{G_k}{G_{k-1}} \ge 1 \Rightarrow G_k \ge G_{k-1}$$

$$\left(\frac{a_n}{G_n}\right)^n = \left(\frac{a_n}{\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}}\right)^n = \frac{a_n^n}{a_1a_2\cdot\ldots\cdot a_n} = \frac{a_n}{a_1}\cdot\frac{a_n}{a_2}\cdot\ldots\cdot\frac{a_n}{a_{n-1}} \ge 1 \text{ because}$$

$$a_n \ge a_i, \forall i \in \overline{1,n-1}$$

$$\left(\frac{a_n}{G_n}\right)^n \ge 1 \Rightarrow \frac{a_n}{G_n} \ge 1 \Rightarrow G_n \le a_n$$

$$G_2 \le G_3 \le G_4 \le \dots \le G_n \le a_n.$$

Property 4: If $0 < a_1 \leq a_2 \leq \cdots \leq a_n, n \in \mathbb{N}, n \geq 2$ then:

$$k \cdot G_k - (k-1)G_{k-1} \leq a_n; \forall k \in \overline{2,n}; (8)$$

Proof. By (7): $G_k \geq G_{k-1} \Rightarrow \frac{G_k}{G_{k-1}} \geq 1; \forall k \in \overline{2,n}$

$$a_k = \frac{a_1 a_2 \cdot \ldots \cdot a_{k-1} a_k}{a_1 a_2 \cdot \ldots \cdot a_{k-1}} = \frac{\left(\sqrt[k]{a_1 a_2 \cdot \ldots \cdot a_k}\right)^k}{\left(\sqrt[k-1]{a_1 a_2 \cdot \ldots \cdot a_{k-1}}\right)^{k-1}} = \frac{G_k^k}{G_{k-1}^{k-1}} =$$

$$= G_{k-1} \cdot \left(\frac{G_k}{G_{k-1}}\right)^k \overset{Bernoulli}{\geq} G_{k-1} \cdot \left(1 + k\left(\frac{G_k}{G_{k-1}} - 1\right)\right) =$$

$$= G_{k-1} + kG_k - kG_{k-1} = (1-k)G_{k-1} + kG_k = kG_k - (k-1)G_{k-1}$$

$$a_n \geq a_k \geq kG_k - (k-1)G_{k-1}; \forall k \in \overline{2,n}$$

Corollary 10: If $0 < a \leq b \leq c$ then:

$$3 \cdot \sqrt[3]{abc} - 2\sqrt{ab} \leq c; (9)$$

Proof. We take in (8): $k = 3, n = 3, a_1 = a, a_2 = b, a_3 = c$.

Corollary 11: If $0 < a \leq b \leq c \leq d$ then:

$$4\sqrt[4]{abc} - 3\sqrt[3]{abc} \leq d$$

Proof. We take in (8): $k = n = 3, a_1 = a, a_2 = b, a_3 = c, a_4 = d$.

Corollary 12: If $x, y > 0$ then:

$$3\sqrt{xy} - 2\sqrt{\frac{2xy\sqrt{xy}}{x+y}} \leq \frac{x+y}{2}$$

Proof. We take in (9):

$$a = \frac{2xy}{x+y}, b = \sqrt{xy}, c = \frac{x+y}{2}$$

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HORNICH-HLAWKA-POPOVICIU INEQUALITIES REVISITED

By Florică Anastase-Romania

ABSTRACT: *In this paper are presented proofs, generalizations, equivalent forms and connection between famous Hornich-Hlawka and Popoviciu's inequalities and applications.*

By [7] if X is a complex inner product space with norm $|\cdot|$, and $a, b, c \in X$ arbitrary vectors holds the inequality $|a+b+c| + |a| + |b| + |c| \geq |a+b| + |b+c| + |c+a|$

It seems to have appeared first in a paper of Hornich (1942) (who credits the proof to Hlawka). Several proofs are known, see e.g. [Niculescu and Persson,2006,p.100] or [Mitrinovic, 1970, pp.171-172]. Fechner himself considers the functional Hlawka inequality

$$f(a+b+c)+f(a)+f(b)+f(c)\geq f(a+b)+f(b+c)+f(c+a)$$

and studies real valued functions f on an abelian group $(A,+)$ that satisfy this.

Observe also the resemblance between Fechner and Popoviciu's inequality, which states for a convex function f on a real interval I and $a,b,c\in I$ holds

$$3f\left(\frac{a+b+c}{3}\right)+f(a)+f(b)+f(c)\geq 2\left(f\left(\frac{a+b}{2}\right)+f\left(\frac{b+c}{2}\right)+f\left(\frac{c+a}{2}\right)\right)$$

THEOREM 1. (HORNICH-HLAWKA)

If $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{C}$ then holds:

$$|\boldsymbol{u}|+|\boldsymbol{v}|+|\boldsymbol{w}|+|\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}|\geq|\boldsymbol{u}+\boldsymbol{v}|+|\boldsymbol{v}+\boldsymbol{w}|+|\boldsymbol{w}+\boldsymbol{u}|$$

PROOF 1. Using the identity

$|u|^2+|v|^2+|w|^2+|v+|u+v+w|^2=|u+v|^2+|v+w|^2+|w+u|^2$, then from

$(|u|+|v|+|w|+|u+v+w|)^2\geq(|u+v|+|v+w|+|w+u|)^2$ holds

$$|uv|+|vw|+|wu|+|u+v+w|(|u|+|v|+|w|)\geq$$
$$\geq|(u+v)(v+w)|+|(v+w)(w+u)|+|(w+u)(u+v)|;(*)$$

But: $|(u+v)(v+w)|=|v(u+v+w)+uw|\leq|v||u+v+w|+|uw|$.

Similarly,

$|(v+w)(w+u)|\leq|w||u+v+w|+|vu|$ and

$$|(w+u)(u+v)|\leq|u||u+v+w|+|wu|$$

Adding these three inequality holds inequality $(*)$.

PROOF 2. We have:

$$(|u+v+w|+|u|+|v|+|w|-|u+v|-|v+w|-|w+u|)$$
$$\cdot(|u+v+w|+|u|+|v|+|w|)=$$
$$=(|v|+|w|-|v+w|)(|u|-|v+w|+|u+v+w|)+$$
$$+(|w|+|u|-|w+u|)(|v|-|w+u|+|w+v+u|)+$$
$$+(|u|+|v|-|u+v|)(|w|-|u+v|+|u+v+w|)\geq 0$$

APPLICATION 1. If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\mathbb{R}$ then

$$\sum_{cyc}\sqrt{\boldsymbol{a}^2+\boldsymbol{b}^2}+\sqrt{2}|\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|\geq\sum_{cyc}\sqrt{(\boldsymbol{a}+\boldsymbol{b})^2+(\boldsymbol{b}+\boldsymbol{c})^2}$$

PROOF. In Theorem 1 we take $u = a + bi, v = b + ic, c = c + ia$ and the given inequality is equivalent to:

$$\sum_{cyc} \frac{(b+c)^2 + 2(a+c)b}{\sqrt{a^2+b^2} + \sqrt{(a+b)^2 + (b+c)^2}} \leq \sqrt{2}|a+b+c|$$

APPLICATION 2. If $a, b, c \in \mathbb{R}$ then

$$\sum_{cyc} \sqrt{a^2b^2 + c^2} + \sqrt{\left(\sum_{cyc} a\right)^2 + \left(\sum_{cyc} ab\right)^2} \geq \sum_{cyc} |a+c|\sqrt{b^2+1}$$

PROOF. In Theorem 1 we take $u = ab + ic, v = bc + ia, w = ca + ib$.

APPLICATION 3. If $a, b, c \in \mathbb{R}$ then

$$\sum_{cyc} \sqrt{a^4 + b^2c^2} + \sqrt{\left(\sum_{cyc} a^2\right)^2 + \left(\sum_{cyc} ab\right)^2} \geq \sum_{cyc} \sqrt{(a^2+b^2) + c^2(a+b)^2}$$

PROOF. In Theorem 1 we take $u = a^2 + ibc, v = b^2 + ica, w = c^2 + iab$.

APPLICATION 4. If $a, b, c \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ then

$$\sum_{cyc} \sqrt{a^2 + \lambda^2} + \sqrt{\left(\sum_{cyc} a\right)^2 + 9\lambda^2} \geq \sum_{cyc} \sqrt{(a+b)^2 + 4\lambda^2}$$

PROOF. In Theorem 1 we take $u = a + i\lambda, v = b + i\lambda, w = c + i\lambda$.

APPLICATION 5. If $x, y, z \in \mathbb{R}$ then

$$3 + \sqrt{3 + 2\sum_{cyc} \cos 2(x-y)} \geq 2\sum_{cyc} |\cos(x-y)|$$

PROOF. In Theorem 1 we take $u = \cos 2x + i \sin 2x, v = \cos 2y + i \sin 2y$,

$$w = \cos 2z + i \sin 2z.$$

APPLICATION 6.

If $a_k > 0, (k = 1, 2, \ldots, n)$ and $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k, G_n = \sqrt[n]{\prod_{k=1}^{n} a_k}$,

$H_n = \frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}}$ then

$$\sqrt{A_n^2 + G_n^2} + \sqrt{G_n^2 + H_n^2} + \sqrt{H_n^2 + A_n^2} + \sqrt{2}(A_n + G_n + H_n) \geq$$

$$\geq \sqrt{(A_n+G_n)^2+(G_n+H_n)^2}+\sqrt{(G_n+H_n)^2+(H_n+A_n)^2}$$
$$+\sqrt{(H_n+A_n)^2+(A_n+G_n)^2}$$

PROOF. In Theorem 1 we take $u=A_n+iG_n, v=G_n+iH_n, w=H_n+iA_n$.

APPLICATION 7. (Daniel Sitaru)

If $u,v,w\in\mathbb{C}$; $|u|=5, |v|=7, |w|=9$ then:

$$|u+v+w|+21>\left|\frac{5}{7}u+\frac{7}{5}v\right|+\left|\frac{7}{9}v+\frac{9}{7}w\right|+\left|\frac{9}{5}w+\frac{5}{9}u\right|$$

PROOF. We prove that:

$$\left|\frac{5}{7}u+\frac{7}{5}v\right|=|u+v|;(1)$$

$$\left|\frac{5}{7}u+\frac{7}{5}v\right|^2=|u+v|^2,\quad \left(\frac{5}{7}u+\frac{7}{5}v\right)\overline{\left(\frac{5}{7}u+\frac{7}{5}v\right)}=(u+v)\overline{(u+v)}$$

$$\left(\frac{5}{7}u+\frac{7}{5}v\right)\left(\frac{5}{7}\bar{u}+\frac{7}{5}\bar{v}\right)=(u+v)(\bar{u}+\bar{v}),\quad u\bar{u}+u\bar{v}+\bar{u}v+v\bar{v}=(u+v)(\bar{u}+\bar{v})$$

$$u(\bar{u}+\bar{v})+v(\bar{u}+\bar{v})=(u+v)(\bar{u}+\bar{v}),\quad (u+v)(\bar{u}+\bar{v})=(u+v)(\bar{u}+\bar{v})$$

Analogous:

$$\left|\frac{7}{9}v+\frac{9}{7}w\right|=|v+w|;(2)$$

$$\left|\frac{9}{5}w+\frac{5}{9}u\right|=|w+u|;(3)$$

By Hlawka's inequality:

$$|u+v+w|+|u|+|v|+|w|\geq|u+v|+|v+w|+|w+u|$$

By (1),(2),(3) it follows that:

$$|u+v+w|+5+7+9>\left|\frac{5}{7}u+\frac{7}{5}v\right|+\left|\frac{7}{9}v+\frac{9}{7}w\right|+\left|\frac{9}{5}w+\frac{5}{9}u\right|$$

$$|u+v+w|+21>\left|\frac{5}{7}u+\frac{7}{5}v\right|+\left|\frac{7}{9}v+\frac{9}{7}w\right|+\left|\frac{9}{5}w+\frac{5}{9}u\right|$$

APPLICATION 8. (Daniel Sitaru)

If $x,y,z\in\mathbb{R}$ we denote:

$$A=|x+y+z|+|x-z|+|z-y|+|y-x|$$

$$B=|x+y-z|+|y+z-x|+|x+z-y|$$

Prove that: $|x|+|y|+|z|\leq\min(A,B)$

Proof. We prove that:

$$|x|+|y|+|z|\leq A;(1)$$

$$|x+y|+|x-y|\ge|x+y+x-y|=2|x|$$

$$|x+y|+|x-y|\ge|x+y-x+y|=2|y|$$

By adding: $2(|x+y|+|x-y|)\ge 2(|x|+|y|)$

$|x+y|+|x-y|\ge|x|+|y|$. Analogously,

$|y+z|+|y-z|\ge|y|+|z|$ and $|z+x|+|z-x|\ge|z|+|x|$.

By adding the last three relationships, we get:

$$|x+y|+|y+z|+|z+x|+|x-y|+|y-z|+|z-x|\ge 2(|x|+|y|+|z|);(2)$$

From (2) and Theorem 1 it follows (1). Now, we prove that $|x|+|y|+|z|\le B;(3)$

We take: $x=\frac{u+v}{2},y=\frac{v+w}{2},z=\frac{w+u}{2}$ then

$$|x|+|y|+|z|=\frac{1}{2}(|u+v|+|v+w|+|w+u|)\le$$

$$\le\frac{1}{2}(|u|+|v|+|v|+|w|+|w|+|u|)\le|u|+|v|+|w|=$$

$$=|x+y-z|+|y+z-x|+|z+x-y|=B$$

From (1),(3) it follows that: $|x|+|y|+|z|\le\min(A,B)$

THEOREM 3. (Daniel Sitaru)

Let $f:[0,\infty)\to\mathbb{R}$ be a differentiable, increasing function such that f' is convexe and $f(0)=0$. For any non-negative numbers x,y,z holds

$$(i)\ f(x)+f(y)+f(z)+f(x+y+z)\ge f\left(2\sqrt{xy}\right)+f\left(2\sqrt{yz}\right)+f\left(2\sqrt{zx}\right)$$

$$(ii)\ f(x)+f(y)+f(z)+f(x+y+z)\ge f\left(\frac{4xy}{x+y}\right)+f\left(\frac{4yz}{y+z}\right)+f\left(\frac{4zx}{z+x}\right)$$

PROOF. Let $g:[0,\infty)\to\mathbb{R}$

$$g(t)=f(t)+f(y)+f(z)+f(t+y+z)-f(t+y)-f(t+z)-f(y+z)$$

$$g'(t)=f'(t)+f'(t+y+z)-f'(t+y)-f'(t+z)$$

$$f(x)+f(y)+f(z)+f(x+y+z)\ge f(x+y)+f(y+z)+f(z+x)$$

f –increasing and $x+y\ge 2\sqrt{xy},y+z\ge 2\sqrt{yz},z+x\ge 2\sqrt{yz}$, then

$$f(x+y)\ge f\left(2\sqrt{xy}\right),f(y+z)\ge f\left(2\sqrt{yz}\right),f(z+x)\ge f\left(2\sqrt{zx}\right)$$

By adding, we get

$$f(x)+f(y)+f(z)+f(x+y+z)\ge f\left(2\sqrt{xy}\right)+f\left(2\sqrt{yz}\right)+f\left(2\sqrt{zx}\right)$$

f –increasing and $x+y\ge\frac{4xy}{x+y}$ then $f(x+y)\ge f\left(\frac{4xy}{x+y}\right)$ and analogously,

$$f(y+z)\geq f\left(\frac{4yz}{y+z}\right) \text{ and } f(z+x)\geq f\left(\frac{4zx}{z+x}\right)$$

By adding, we get

$$f(x)+f(y)+f(z)+f(x+y+z)\geq f\left(\frac{4xy}{x+y}\right)+f\left(\frac{4yz}{y+z}\right)+f\left(\frac{4zx}{z+x}\right)$$

APPLICATION 9. (Marian Dincă)

If $\boldsymbol{x,y,z\geq 0}$ then

$$\sqrt{\boldsymbol{x}}+\sqrt{\boldsymbol{y}}+\sqrt{\boldsymbol{z}}+\sqrt{\boldsymbol{x+y+z}}\geq\sqrt{\boldsymbol{x+y}}+\sqrt{\boldsymbol{y+z}}+\sqrt{\boldsymbol{z+x}}$$

PROOF. Let be $f:[0,\infty)\to\mathbb{R}, f(x)=\sqrt{x}; f(0)=0, g:[0,\infty)\to\mathbb{R}$

$$g(t)=f(t)+f(y)+f(z)+f(t+y+z)-f(t+y)-f(t+z)-f(y+z)$$

then $f'(x)=\frac{1}{2\sqrt{x}}, f''(x)=-\frac{1}{4x\sqrt{x}}, f'''(x)=\frac{4\sqrt{x}+\frac{2x}{\sqrt{x}}}{16x^3}>0.$

$$g'(t)=f'(t)+f'(t+y+z)-f'(t+y)-f'(t+z)$$

$$f(x)+f(y)+f(z)+f(x+y+z)\geq f(x+y)+f(y+z)+f(z+x)$$

Therefore,

$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{x+y+z}\geq\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}$$

APPLICATION 10. (Marian Dincă)

If $\boldsymbol{\alpha}\in\mathbb{R}$ and $\boldsymbol{x,y,z}>0$ then

$$(\boldsymbol{x+y})^{\boldsymbol{\alpha}}+(\boldsymbol{y+z})^{\boldsymbol{\alpha}}+(\boldsymbol{z+x})^{\boldsymbol{\alpha}}\leq\boldsymbol{x}^{\boldsymbol{\alpha}}+\boldsymbol{y}^{\boldsymbol{\alpha}}+\boldsymbol{z}^{\boldsymbol{\alpha}}+(\boldsymbol{x+y+z})^{\boldsymbol{\alpha}}$$

PROOF. If $\alpha\in(0,1)$.

Define $f(x)=(x+y)^\alpha+(y+z)^\alpha+(z+x)^\alpha-x^\alpha-y^\alpha-z^\alpha-(x+y+z)^\alpha$

$$f'(x)=\alpha x^{\alpha-1}+\alpha(x+y+z)^{\alpha-1}-\alpha(x+y)^{\alpha-1}-\alpha(z+x)^{\alpha-1}$$

Now observe that $x\leq x+y\leq x+y+z$ then $\exists\lambda\in[0,1]$ such that

$x+y=\lambda x+(1-\lambda)(x+y+z)$. Similarly, $\exists\mu\in[0,1]$ such that

$x+z=\mu x+(1-\mu)(x+y+z)$. Thus,

$(x+y)+(x+z)=(\lambda+\mu)+(2-\lambda-\mu)(x+y+z)$ or $y+z=(2-\lambda-\mu)(y+z)$

and then $(1-\lambda-\mu)(y+z)=0$.

If $y+z=0$ then $y=z=0$ and inequality becomes $x^\alpha+x^\alpha\leq x^\alpha+x^\alpha$, which is true.

If $\alpha\in(0,1)$ let be the function $g(t)=t^{\alpha-1}$, which is convex on $(0,1)$ and from Jensen's inequality, it follows that:

$$g(x+y)=g\big(\lambda x+(1-\lambda)(x+y+z)\big)\leq\lambda g(x)+(1-\lambda)g(x+y+z)$$

$$g(x+z) = g\big(\mu x + (1-\mu)(x+y+z)\big) \le \mu g(x) + (1-\mu) g(x+y+z)$$

By adding, $g(x+y) + g(x+z) \le (\lambda+\mu) g(x) + (2-\lambda-\mu) g(x+y+z) =$

$= g(x) + g(x+y+z)$, then $f'(x) \ge 0$ and $f(x) \ge f(0) = 0$.

If $\alpha \ge 1$, let $f(t) = t^\alpha$convexe function and from Hardy-Littlewood-Polya inequality for the sequences $\{x+y+z, x, y, z\}$ and $\{x+y, y+z, z+x, 0\}$. Assuming $x \ge y \ge z$, we get

$$f(x+y+z) + f(x) + f(y) + f(z) \ge f(x+y) + f(y+z) + f(z+x).$$

If $\alpha \le 0$, $x^\alpha = (x+y)^\alpha, y^\alpha \ge (y+z)^\alpha, z^\alpha \ge (z+x)^\alpha$. By adding,

$$(x+y)^\alpha + (y+z)^\alpha + (z+x)^\alpha \le x^\alpha + y^\alpha + z^\alpha + (x+y+z)^\alpha.$$

APPLICATION 11. (Daniel Sitaru)

If $x, y, z \ge 0$ then

$$\mathbf{2^x + 2^y + 2^z + 2^{x+y+z} \ge 2^{x+y} + 2^{y+z} + 2^{z+x} + 1}$$

PROOF. Let be $f: [0, \infty) \to \mathbb{R}, f(x) = 2^x - 1; f(0) = 0$ and $g: [0, \infty) \to \mathbb{R}$

$$g(t) = f(t) + f(y) + f(z) + f(t+y+z) - f(t+y) - f(t+z) - f(y+z)$$

$$f'(x) = 2^x \log 2, f''(x) = 2^x \log^2 2, f'''(x) = 2^x \log^3 2. \text{ Then}$$

$$g'(t) = f'(t) + f'(t+y+z) - f'(t+y) - f'(t+z)$$

$$f(x) + f(y) + f(z) + f(x+y+z) \ge f(x+y) + f(y+z) + f(z+x)$$

Therefore, $2^x - 1 + 2^y - 1 + 2^z - 1 + 2^{x+y+z} - 1 \ge 2^{x+y} - 1 + 2^{y+z} - 1 + 2^{z+x} - 1$

$$2^x + 2^y + 2^z + 2^{x+y+z} \ge 2^{x+y} + 2^{y+z} + 2^{z+x} + 1$$

THEOREM 3. (Vasile Cirtoaje-[6])

Let be f be convexe function on a real interval I, and

$x_1, x_2, \dots, x_n \in I$. In these conditions

$$\boldsymbol{f(x_1) + f(x_2) + \dots + f(x_n) + \frac{n}{n-2} f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \ge \frac{2}{n-2} \sum_{i<j} f\left(\frac{x_i + x_j}{2}\right)}$$

THEOREM 4. (Mihaly Bencze[1])

If $z_k \in \mathbb{C}, k = 1, 2, \dots, n$ then

$$\boldsymbol{\left|\sum_{k=1}^n z_k\right| + (n-2)\sum_{k=1}^n |z_k| \ge \sum_{1 \le s < t \le n} |z_s + z_t|}$$

PROOF. We prove by induction. The case $n = 3$ is the Hlawka's inequality. We suppose that the inequality for n holds. Then in the case $n+1$, we have:

$$|z_1 + \dots + z_{n-1} + (z_n + z_{n-1})| + (n-2)(|z_1| + \dots + z_{n-1}| + |z_n + z_{n+1}|) \ge$$

$$\geq \sum_{1\leq s<t\leq n-1}|z_s+z_t|+\sum_{k=1}^{n-1}(|z_k+z_n|+|z_k+z_{n+1}|-|z_k|-|z_{n+1}|)=$$

$$=\sum_{1\leq s<t\leq n+1}|z_s+z_t|+(n-2)|z_n+z_{n+1}|-\sum_{k=1}^{n-1}|z_k|-(n-1)(|z_n|+|z_{n+1}|)$$

where we have used the case $n=3$ for the therm $|z_k+z_n+z_{n+1}|$, So, we get

$$|z_k+\cdots+z_{n-1}+z_n+z_{n+1}|+(n-1)(|z_1|+|z_2|+\cdots+|z_{n-1}|+|z_n|+|z_{n+1}|)\geq$$

$$\geq \sum_{1\leq s<t\leq n+1}|z_s+z_t|$$

APPLICATION 12.

If $a_k\in\mathbb{R}, k=1,2,\dots,n$ then

$$\sqrt{2}\left|\sum_{k=1}^{n}a_k\right|+(n-2)\sum_{cyc}\sqrt{a_1^2+a_2^2}\geq \sum_{1\leq s<t\leq n}\sqrt{(a_s+a_t)^2+(a_{s+1}+a_{t+1})^2}$$

PROOF. In Theorem 4 we take $z_k=a_k+ia_{k+1}, k=1,2,\dots,n$ and $a_{n+1}=a_1$.

Application 13.

If $a_k\in\mathbb{R}, k=1,2,\dots,n$ and $u=a_1+a_2+\cdots+a_n$ then

$$|u|\sqrt{n^2-2n+2}+(n-2)\sum_{k=1}^{n}\sqrt{(u-a_k)^2+a_k^2}\geq$$

$$\geq \sum_{1\leq s<t\leq n}\sqrt{(2u-a_s-a_k)^2+(a_s+a_t)^2}$$

PROOF. In Theorem 4 we take $z=u-a_k+ia_k, k=1,2,\dots,n$.

Theorem 5. (Mihaly Bencze[2])

If $u,v,w\in\mathbb{C}$ then

$$3(|u|+|v|+|w|+|u+v+w|)\geq$$

$$\geq |2u+v|+|2v+w|+|2w+v|+|2v+u|+|2w+u|+|2u+w|$$

Application 8. In ΔABC the following relationship holds:

$$9+\frac{3}{R}\sqrt{s^2+(R+r)^2}\geq 2\sum_{cyc}\sqrt{5+4\cos(A-B)}$$

PROOF. In Theorem 5 we take $u=\sin A+i\cos A, v=\sin B+i\cos B, w=\sin C+i\sin C$

Application 14. (D. Sitaru) In ΔABC the following relationship holds:

$$\mathbf{2\sum_{cyc}\cos\frac{A-B}{2}\le 3+\sqrt{3+2\sum_{cyc}\cos(A-B)}}$$

PROOF. Let $z_1=\cos A+i\sin A\,,z_2=\cos B+i\sin B\,,z_3=\cos C+i\sin C$,

$$|z_1|=|z_2|=|z_3|=1$$

$$|z_1+z_2+z_3|=|\cos A+\cos B+\cos C+i(\sin A+\sin B+\sin C)|=$$

$$=\sqrt{(\cos A+\cos B+\cos C)^2+(\sin A+\sin B+\sin C)^2}=$$

$$=\sqrt{3+2(\cos(A-B)+\cos(B-C)+\cos(C-A))}$$

$$|z_1+z_2|=|\cos A+\cos B+i(\sin A+\sin B)|=$$

$$=\sqrt{(\cos A+\cos B)^2+(\sin A+\sin B)^2}=$$

$$=\sqrt{2+2\cos(A-B)}=\sqrt{2\left(1+2\cos^2\frac{A-B}{2}-1\right)}=2\left|\cos\frac{A-B}{2}\right|$$

Analogously,

$$|z_2+z_3|=2\left|\cos\frac{B-C}{2}\right|\text{ and }|z_3+z_1|=2\left|\cos\frac{C-A}{2}\right|$$

From Theorem 1, it follows that:

$$2\sum_{cyc}\cos\frac{A-B}{2}\le 3+\sqrt{3+2\sum_{cyc}\cos(A-B)}$$

Theorem 6. (M. Bencze, 1979) If $\boldsymbol{u,v,w\in\mathbb{C}}$ then

$$\mathbf{|u|+|v|+|w|+\frac{1}{4}(|2u+v+w|+|2v+w+u|+|2w+u+v|)\ge}$$

$$\mathbf{\ge|u+v|+|v+w|+|w+u|}$$

Application 15. In $\boldsymbol{\Delta ABC}$ the following relationship holds:

$$\mathbf{\sum_{cyc}\sqrt{h_a^2+r_a^2}+\frac{1}{4}\sum_{cyc}\sqrt{(2h_a+h_b+h_c)^2+(2r_a+r_b+r_c)^2}\ge}$$

$$\mathbf{\ge\sum_{cyc}\sqrt{(h_a+h_b)^2+(r_a+r_b)^2}}$$

PROOF. In Theorem 6 we take $u=h_a+ir_a,v=h_b+ir_b,w=h_c+ir_c$.

Theorem 7. (M. Bencze, 1980) If $\boldsymbol{z_k\in\mathbb{C},k=1,2,\dots,n}$ then

$$\mathbf{(n-1)\sum_{k=1}^{n}|z_k|+\left|\sum_{k=1}^{n}z_k\right|\ge\frac{1}{n}\sum_{i,j=1}^{n}|(n-1)z_i+z_j|}$$

Application 16. In all convex polygon $A_1A_2 \dots A_n$ holds

$$(n-1)(n-2)\sqrt{\sum_{k=1}^{n} A_k^2+\sum_{1\le i<j\le n} A_iA_j\cos(A_i-A_j)}\ge$$

$$\ge\frac{1}{n}\sum_{i,j=1}^{n}\sqrt{(n-1)^2A_i^2+A_j^2+2(n-1)A_iA_j\cos(A_i-A_j)}$$

PROOF. In Theorem 7 we take $z_k=A_k(\cos A_k+i\sin A_k), k=1,2\dots,n$.

Theorem 8. (M. Bencze, 1992-The best generalization of Hlawka's inequality)

If $z_k\in\mathbb{C}$ and $\lambda_k>0, k=1,2,\dots,n$ then

$$\sum_{1\le i_1<..<i_p\le n}|\lambda_{i_1}z_{i_1}+\cdots.+\lambda_{i_p}z_{i_p}|\le\binom{n-2}{p-2}\left(\frac{n-p}{p-1}\sum_{k=1}^{n}\lambda_k|z_k|+\left|\sum_{k=1}^{n}\lambda_kz_k\right|\right)$$

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ABOUT FEW INEQUALITIES IN TRIANGLES

By D.M. Bătineţu-Giurgiu,Claudia Nănuți-Romania

Let be $n\in\mathbb{N}, n\ge 2$ and the triangles $T_k=A_kB_kC_k$ with areas F_k and circumradies R_k, $k=\overline{1,n}$. Let us denote a_k, b_k, c_k lengths of sides of the triangles T_k, s_k –semiperimeters and $r_k, k=\overline{1,n}$ inradies.

Theorem 1. If $x,y,z>0$ then:

$$\frac{x+y}{z\cdot\sqrt[n]{a_1a_2\dots a_n}}+\frac{y+z}{x\cdot\sqrt[n]{b_1b_2\dots b_n}}+\frac{z+x}{y\cdot\sqrt[n]{c_1c_2\dots c_n}}\ge\frac{2\sqrt{3}}{\sqrt[n]{R_1R_2\dots R_n}};(1)$$

Proof. We have:

$$\sum_{cyc}\frac{x+y}{z\cdot\sqrt[n]{a_1a_2\dots a_n}}\ge 2\cdot\sum_{cyc}\frac{\sqrt{xy}}{z\cdot\sqrt[n]{a_1a_2\dots a_n}}\ge 2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{xy}}{z\cdot\sqrt[n]{a_1a_2\dots a_n}}}=$$

$$=6\cdot\sqrt[3]{\frac{xyz}{xyz\cdot.\sqrt[n]{(a_1b_1c_1)(a_2b_2c_2)\cdot\ldots\cdot(a_nb_nc_n)}}}=\frac{6}{\sqrt[n]{\sqrt[3]{a_1b_1c_1}\cdot\sqrt[3]{a_2b_2c_2}\cdot\ldots\cdot\sqrt[3]{a_nb_nc_n}}}=$$

$$=\frac{6}{\sqrt[n]{\sqrt[3]{4R_1F_1}\cdot\sqrt[3]{4R_2F_2}\cdot\ldots\cdot\sqrt[3]{4R_nF_n}}}=\frac{6}{\sqrt[n]{\sqrt[3]{4R_1r_1s_1}\cdot\sqrt[3]{4R_2r_2s_2}\cdot\ldots\cdot\sqrt[3]{4R_nr_ns_n}}}\overset{Euler}{\ge}$$

$$\ge\frac{6}{\sqrt[n]{\sqrt[3]{4R_1\frac{R_1}{2}s_1}\cdot\sqrt[3]{4R_2\frac{R_2}{2}s_2}\cdot\ldots\cdot\sqrt[3]{4R_n\frac{R_n}{n}s_n}}}$$

$$=\frac{6}{\sqrt[n]{\sqrt[3]{2R_1^2s_1}\cdot\sqrt[3]{2R_2^2s_2}\cdot\ldots\cdot\sqrt[3]{2R_n^2s_n}}}\overset{Mitrinovic}{\ge}$$

$$\ge\frac{6}{\sqrt[n]{\sqrt[3]{3\sqrt{3}R_1^3}\cdot\sqrt[3]{3\sqrt{3}R_2^3}\cdot\ldots\cdot\sqrt[3]{3\sqrt{3}R_n^3}}}=\frac{6}{\sqrt[n]{\left(\sqrt{3}\right)^n\cdot R_1R_2\cdot\ldots\cdot R_n}}=$$

$$=\frac{6}{\sqrt{3}\cdot\sqrt[n]{R_1R_2\cdot\ldots\cdot R_n}}=\frac{2\sqrt{3}}{\sqrt[n]{R_1R_2\cdot\ldots\cdot R_n}}$$

If $x=y=z$, then the above inequality becomes as:

$$\sum_{cyc}\frac{1}{\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}}\ge\frac{\sqrt{3}}{\sqrt[n]{R_1R_2\cdot\ldots\cdot R_n}};(2)$$

If in (2) $ABC\equiv A_kB_kC_k=T_k, k=\overline{1,n}$, then holds:

$$\sum_{k=1}^{n}\frac{1}{\sqrt[n]{a^n}}\ge\frac{\sqrt{3}}{\sqrt[n]{R^n}}\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{\sqrt{3}}{R}\ (Ionescu-Tiu)$$

Theorem 2. If $m,x,y,z\in[1,\infty), 3m=x+y+z$ then:

$$\frac{x^x+y^x+z^x}{\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}}+\frac{x^y+y^y+z^y}{\sqrt[n]{b_1b_2\cdot\ldots\cdot b_n}}+\frac{x^z+y^z+z^z}{\sqrt[n]{c_1c_2\cdot\ldots\cdot c_n}}\ge\frac{3\sqrt{3}\cdot m^m}{\sqrt[n]{R_1R_2\cdot\ldots\cdot R_n}};(3)$$

Proof. We have:

$$\sum_{cyc}\frac{x^x+y^x+z^x}{\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}}\overset{Radon}{\geq}\sum_{cyc}\frac{(x+y+z)^x}{\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}\cdot 3^{x-1}}\geq$$

$$\geq 3\cdot\sqrt[3]{\prod_{cyc}\frac{(x+y+z)^{x+y+z}}{\sqrt[n]{a_1a_2\ldots a_nb_1b_2\ldots b_nc_1c_2\ldots c_n}\cdot 3^{x+y+z-3}}}=$$

$$=3\cdot\sqrt[3]{\frac{(3m)^{3m}}{3^{3m-3}\cdot\sqrt[n]{(a_1b_1c_1)(a_2b_2c_2)\cdot\ldots\cdot(a_nb_nc_n)}}}$$

$$=3\cdot\frac{(3m)^m}{3^{m-1}\cdot\sqrt[3n]{(4R_1F_1)(4R_2F_2)\ldots(4R_nF_n)}}$$

$$=\frac{9\cdot m^m}{\sqrt[3m]{(4R_1r_1s_1)(4R_2r_2s_2)\ldots(4R_nr_ns_n)}}\overset{Euler}{\geq}\frac{9\cdot m^m}{\sqrt[3n]{2R_1^2s_2\cdot 2R_2^2s_2\cdot\ldots\cdot 2R_n^2s_n}}\overset{Mitrinovic}{\geq}$$

$$\geq\frac{9\cdot m^m}{\sqrt[3n]{3\sqrt{3}R_1^3\cdot 3\sqrt{3}R_2^3\cdot\ldots\cdot 3\sqrt{3}R_n^3}}=\frac{9\cdot m^m}{\sqrt{3}\cdot\sqrt[n]{R_1R_2\ldots R_n}}=\frac{3\sqrt{3}\cdot m^m}{\sqrt[n]{R_1R_2\cdot\ldots\cdot R_n}}$$

If $x=y=z=m$ then (3) becomes:

$$\frac{1}{\sqrt[n]{a_1a_2\ldots a_n}}+\frac{1}{\sqrt[n]{b_1b_2\ldots b_n}}+\frac{1}{\sqrt[n]{c_1c_2\ldots c_n}}\geq\frac{\sqrt{3}}{\sqrt[n]{R_1R_2\ldots R_n}}$$

If $T_k=A_kB_kC_k\equiv ABC, k=\overline{1,n}$, we get:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{\sqrt{3}}{R};(I-T,L)$$

CRUX MATHEMATICORUM CHALLENGES-(I)

By Daniel Sitaru-Romania

4073. Solve the following system:

$$\begin{cases}\sin 2x+\cos 3y=-1\\ \sqrt{\sin^2x+\sin^2y}+\sqrt{\cos^2x+\cos^2y}=1+\sin(x+y)\end{cases}$$

Daniel Sitaru

Solution by Michele Bataille-France

We first show that the second equation is equivalent to $x+y\equiv\frac{\pi}{2}(mod\ 2\pi)$.

If $x+y\equiv\frac{\pi}{2}(mod\ 2\pi)$, then $\sin^2y=\cos^2x$ and $\cos^2y=\sin^2x$. It immediately follows that both sides of the equation equal 2. Conversely, if the equation holds, then squaring gives

$$2\sqrt{(\sin x\cos x-\sin y\cos y)^2+\sin^2(x+y)}=2\sin(x+y)-(1-\sin^2(x+y))$$

and therefore

$$2\sin(x+y) \le 2\sqrt{\sin^2(x+y)} \le 2\sqrt{(\sin x\cos x - \sin y\cos y)^2 + \sin^2(x+y)}$$
$$= 2\sin(x+y) - (1-\sin^2(x+y)) \le 2\sin(x+y)$$

Thus, equality must hold throughout and in particular $\sin(x+y) \ge 0$ and $\sin^2(x+y) = 1$. We deduce that $x+y \equiv \frac{\pi}{2} (mod\ 2\pi)$.

Since $\cos 3\left(\frac{\pi}{2} - x\right) = -\sin 3x$, we are led to seek the solutions to the equation $f(x) = 1$ where $f(x) = \sin 3x - \sin 2x$. Note that $f\left(-\frac{\pi}{2}\right) = 1$ so that the numbers $-\frac{\pi}{2} + 2k\pi (k \in \mathbb{Z})$ are solutions. For other solutions note that f is odd and 2π – periodic; consequently, we may restrict the study of f to the interval $[0,\pi]$ and look for x satisfying either $f(x) = 1$ or $f(x) = -1$ (the latter since then $f(-x) = 1$). Consider first the interval $\left[0, \frac{\pi}{2}\right)$. We have $f(0) = 0$ and if $x \in \left(0, \frac{\pi}{2}\right)$, then $\sin 2x > 0$ and so $f(x) < 1$.

Thus, equality must hold throughout and in particular $\sin(x+y) \ge 0$ and $\sin^2(x+y) = 1$. We deduce that $x+y \equiv \frac{\pi}{2} (mod\ 2\pi)$.

Since $\cos 3\left(\frac{\pi}{2} - x\right) = -\sin 3x$, we are led to seek the solutions to the equation $f(x) = 1$ where $f(x) = \sin 3x - \sin 2x$. Note that $f\left(-\frac{\pi}{2}\right) = 1$ so that the numbers $-\frac{\pi}{2} + 2k\pi (k \in \mathbb{Z})$ are solutions. For other solutions note that f is odd and 2π – periodic; consequently, we may restrict the study of f to the interval $[0,\pi]$ and look for x satisfying either $f(x) = 1$ or $f(x) = -1$ (the latter since then $f(-x) = 1$). Consider first the interval $\left[0, \frac{\pi}{2}\right)$. We have $f(0) = 0$ and if $x \in \left(0, \frac{\pi}{2}\right)$ then $\sin 2x > 0$ and so $f(x) < 1$.

- $x \in \left(0, \frac{\pi}{3}\right]$: $\sin 3x > 0$ for x between 0 and $\frac{\pi}{3}$, hence $f(x) > -1$; since $f\left(\frac{\pi}{3}\right) > -1$, there is no $x \in \left(0, \frac{\pi}{3}\right]$ such that $f(x) = -1$.
- $x \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$: $f''(x) = 4\sin 2x - 9\sin 3x > 0$, hence $f'(x) = 3\cos 3x - 2\cos 2x$ is nondecreasing on the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. For some $x_1 \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, we have $f'(x) \le 0$ for $x \in \left(\frac{\pi}{3}, x_1\right]$ and $f'(x) > 0$ for $x \in \left(x_1, \frac{\pi}{2}\right)$. Since $f\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $f\left(\frac{\pi}{2}\right) = -1$, we have $f(x_1) < -1$ and $f(\alpha) = -1$ for a unique α in $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

In a similar way we treat the interval $\left(\frac{\pi}{2}, \pi\right]$. We have $f(\pi) = 0$ and if $x \in \left(\frac{\pi}{2}, \pi\right)$, then $\sin 2x < 0$, hence $f(x) > -1$.

- $x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$: $f'(x) > 0$ and so f is increasing from -1 to $1 + \frac{\sqrt{2}}{2}$. Thus, $f(\beta) = 1$ for a unique $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$.
- $x \in \left(\frac{5\pi}{6}, \pi\right)$: f is decreasing from $1 + \frac{\sqrt{3}}{2}$ to 0, hence $f(\gamma) = 1$ for a unique γ of $\left(\frac{5\pi}{6}, \pi\right)$.
- $x \in \left[\frac{3\pi}{4}, \frac{5\pi}{6}\right]$: Resorting to $f''(x)$, we see that $f'(x)$ decreases from positive to negative so that $f(x) > 1$.

In conclusion, on the interval $[-\pi,\pi]$ the solutions (x,y) of the system are the pairs $\left(-\frac{\pi}{2}, \pi\right), \left(-\alpha, \frac{\pi}{2} + \alpha\right), \left(\beta, \frac{\pi}{2} - \beta\right)$ and $\left(\gamma, \frac{\pi}{2} - \gamma\right)$.

All other solutions are obtained by adding multiples of 2π to x or y.

4081. Determine all $A,B\in M_2(\mathbb{R})$ such that:

$$\begin{cases} A^2+B^2=\begin{pmatrix}22 & 44\\ 14 & 28\end{pmatrix}\\ AB+BA=\begin{pmatrix}10 & 20\\ 2 & 4\end{pmatrix}\end{cases}$$

Daniel Sitaru

Solution by Joseph DiMuro-Spain

Summing the two equations, we obtain:

$$(A+B)^2=A^2+AB+BA+B^2=\begin{pmatrix}32 & 64\\ 16 & 32\end{pmatrix}$$

We can diagonalize this matrix in order to find its square roots:

$$(A+B)^2=PDP^{-1}=\begin{pmatrix}2 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}64 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & -\frac{1}{2}\end{pmatrix},$$

$$A+B=PD^{\frac{1}{2}}P^{-1}=\begin{pmatrix}2 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}\pm 8 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{4} & \frac{1}{2}\\ \frac{1}{4} & -\frac{1}{2}\end{pmatrix}=\pm\begin{pmatrix}4 & 8\\ 2 & 4\end{pmatrix}.$$

We can also subtract the original two equations to obtain:

$$(A-B)^2=A^2-AB-BA+B^2=\begin{pmatrix}12 & 24\\ 12 & 24\end{pmatrix}$$

As before, we diagonalize this matrix:

$$(A-B)^2=PDP^{-1}=\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}36 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{pmatrix},$$

$$A-B=PD^{\frac{1}{2}}P^{-1}=\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}\begin{pmatrix}\pm 6 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{pmatrix}=\pm\begin{pmatrix}2 & 4\\ 2 & 4\end{pmatrix}.$$

Now we have the two equations

$$A+B=\pm\begin{pmatrix}4 & 8\\ 2 & 4\end{pmatrix}, A-B=\pm\begin{pmatrix}2 & 4\\ 2 & 4\end{pmatrix},$$

which can easily be solved to produce four possible pairs of matrices for A and B.
One solution is

$$A=\begin{pmatrix}3 & 6\\ 2 & 4\end{pmatrix}, B=\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}$$

The other solutions may be obtained by interchanging A and B, and/or replacing A and B with their negatives.

4104. Prove that for $0<a\le b\le c\le d<2$, we have

$$5(ab^4+bc^4+cd^4+16d)<5(b^5+c^5+d^5+16a)+128$$

Daniel Sitaru

Solution 1 by Sefket Arslanagic; and Salem Malikic (independently)-Serbie

By the arithmetic-geometric means inequality, we have
$$a^5+4b^5\ge 5ab^4, b^5+4c^5\ge 5bc^4, c^5+4d^5\ge 5cd^4$$
and
$$d^5+128=d^5+4\cdot 2^5\ge 5\cdot 2^4 d=80d$$
Adding these along with the positive $4a^5$ yields that
$$5(a^5+b^5+c^5+d^5)+128>5(ab^4+bc^4+cd^4+16d)$$
Solution 2 by the proposer.

The function $f(x)=16-x^4$ is nonnegative, decreasing and concave on $[0,2]$. Therefore
$$(b-a)f(b)+(c-b)f(b)+(d-c)f(d)<\int_0^2 f(x)\,dx$$
Hence
$$16(d-a)-(b^5+c^5+d^5)+(ab^4+bc^4+cd^4)<\frac{128}{5}$$
Multiplying by 5 and rearranging the terms gives the result.

4122. Prove that for $\boldsymbol{n}\in\mathbb{N}$, the following holds:
$$\left(\frac{\boldsymbol{e^n-1}}{\boldsymbol{n}}\right)^{\boldsymbol{2n+1}}\le\frac{(\boldsymbol{e}-\boldsymbol{1})(\boldsymbol{e^2-1})(\boldsymbol{e^3-1})\cdot\ldots\cdot(\boldsymbol{e^{2n}-1})}{(\boldsymbol{2n})!}$$

Daniel Sitaru

Solution by Angel Plaza-Spain

Note that the inequality in the statement can be rewritten as
$$\left(\frac{e^n-1}{n}\right)^{2n+1}\le\left(\frac{e-1}{1}\right)\left(\frac{e^2-1}{2}\right)\ldots\left(\frac{e^{2n}-1}{2n}\right)\quad (1)$$
Consider the function
$$f(x)=\ln\left(\frac{e^x-1}{x}\right)$$
defined for $x>0$ and set $f(0)=0$. Then f is continuous for $x\ge 0$ and has second derivative
$$f''(x)=\frac{(e^x-1)^2-x^2e^x}{x^2(e^x-1)^2}$$
To show that $f(x)$ is convex it suffices to prove that $(e^x-1)^2-x^2e^x>0$. This can be reformulated to $e^x\left(e^x+e^{-x}-(2+x^2)\right)>0$. But we have
$$e^x+e^{-x}=\sum_{k=0}^{\infty}\frac{2x^{2k}}{(2k)!}>2+x^2$$
Therefore, the second derivative of f is positive and $f(x)$ is convex for $x>0$.
Rephrasing inequality (1) by taking logarithms we obtain
$$(2n+1)f(n)\le\sum_{k=0}^{2n}f(k)$$
which follows from Jensen's inequality.

4135. Let ABC be a triangle with $BC = a, AC = b, AB = c$. Prove that the following relationship holds:

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \le \sqrt{3\left(\frac{a^2}{b+c-a}+\frac{b^2}{a+c-b}+\frac{c^2}{a+b-c}\right)}$$

Daniel Sitaru

Solution by Dionne Bailey, Elsie Campbell and Charles R. Diminnie-USA

Since $f(x) = \sqrt{x}$ is concave on $(0, \infty)$, Jensen's theorem implies that

$$\sqrt{a}+\sqrt{b}+\sqrt{c} = f(a)+f(b)+f(c) \le 3f\left(\frac{a+b+c}{3}\right) = \sqrt{3(a+b+c)} \quad (1)$$

By the Cauchy-Schwarz inequality, writing $a = \frac{a}{\sqrt{b+c-a}}\sqrt{b+c-a}$ and similarly for b and c, we get

$$a+b+c \le \left(\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c}\right)^{\frac{1}{2}}(a+b+c)^{\frac{1}{2}}$$

which (dividing both sides by $(a+b+c)^{\frac{1}{2}}$ and multiplying by $\sqrt{3}$) yields

$$\sqrt{3(a+b+c)} \le \sqrt{3\left(\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c}\right)} \quad (2)$$

Combining (1) and (2), we get the desired inequality; note that equality holds if and only if $a = b = c$, in other words if and only if ΔABC is equilateral.

4136. Prove that if $a, b, c \in (0, \infty)$ then:

$$b\int_0^a e^{-t^2}\,dt + c\int_0^b e^{-t^2}\,dt + a\int_0^c e^{-t^2} < \frac{\pi}{2}\sqrt{3(a^2+b^2+c^2)}$$

Daniel Sitaru, Mihaly Bencze

Solution by Arkady Alt, Sefket Arslanagic, Paul Bracken and Digby Smith (independently)-USA

We use S to denote the left side of the given inequality. Since

$$\int_0^x e^{-t^2}\,dt \le \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

for $x = a, b$ and c, we have by the Cauchy-Schwarz Inequality that

$$S \le \sqrt{b^2+c^2+a^2}\cdot\sqrt{\left(\int_0^a e^{-t^2}dt\right)^2+\left(\int_0^b e^{-t^2}\,dt\right)^2+\left(\int_0^c e^{-t^2}\,dt\right)^2}$$

$$\le \sqrt{a^2+b^2+c^2}\cdot\sqrt{3\left(\frac{\sqrt{\pi}}{2}\right)^2} = \frac{\sqrt{\pi}}{2}\cdot\sqrt{3(a^2+b^2+c^2)}$$

$$< \frac{\pi}{2}\cdot\sqrt{3(a^2+b^2+c^2)}$$

4142. Prove that if $a, b, c \in (0, \infty)$ then:

$$\left(1+\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}} \le \left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)$$

Daniel Sitaru

Solution by Arkady Alt-USA

Assuming due to the homogeneity of the original inequality, that $a+b+c=1$ and denoting $p = ab+bc+ca, q = abc$,
we obtain $a^2+b^2+c^2 = 1-2p$,

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) = \frac{(a+b)(b+c)(c+a)}{abc} = \frac{p-q}{q},$$

and

$$1+\frac{a^2+b^2+c^2}{ab+bc+ca} = 1+\frac{1-2p}{p} = \frac{1-p}{p}$$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) = \frac{(a+b)(b+c)(c+a)}{abc} = \frac{p-q}{q}$$

and

$$1+\frac{a^2+b^2+c^2}{ab+bc+ca} = 1+\frac{1-2p}{p} = \frac{1-p}{p}$$

The original inequality thus becomes

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}} \le \frac{p}{q}-1$$

Since $0 < q \le \frac{p^2}{3}$, we have $\frac{p}{q} \ge \frac{3}{p}$, and it suffices to prove the inequality

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}} \le \frac{3}{p}-1$$

For $0 < p \le \frac{1}{3}$, this is successively equivalent to

$$\frac{1-p}{p} \le \left(\frac{3-p}{p}\right)^{1-2p}, \quad \left(\frac{3-p}{p}\right)^{2p} \le \frac{3-p}{1-p}, \quad \left(\frac{3}{p}-1\right)^2 \le \left(\frac{\frac{3}{p}-1}{\frac{1}{p}-1}\right)^{\frac{1}{p}}.$$

Denoting $t = \frac{1}{p} \in [3, \infty)$, we obtain the following more convenient equivalent form of the latter inequality.

$$(3t-1)^2 \le \left(\frac{3t-1}{t-1}\right)^t \Leftrightarrow t\ln\left(\frac{3t-1}{t-1}\right) \ge 2\ln(3t-1)$$

Let

$$h(t) = t[\ln(3t-1) - \ln(t-1)] - 2\ln(3t-1)$$

Then

$$h'(t) = \ln(3t-1) - \ln(t-1) + t\left(\frac{3}{3t-1} - \frac{1}{t-1}\right) - \frac{6}{3t-1}$$
$$= \ln(3t-1) - \ln(t-1) - \frac{1}{t-1} - \frac{5}{3t-1}$$

and

$$h''(t)=\frac{3}{3t-1}-\frac{1}{t-1}+\frac{1}{(t-1)^2}+\frac{15}{(3t-1)^2}=\frac{2(9t^2-14t+7)}{(3t-1)^2(t-1)^2}$$

Since $h''(t)>0$ for $t\geq 3$, $h'(t)$ increases on $[3,\infty)$ and, therefore,

$$h'(t)\geq h'(3)=\ln 8-\ln 2-\frac{1}{2}-\frac{5}{8}=2\ln 2-\frac{9}{8}>0.$$

Hence, $h(t)$ increases on $[3,\infty)$ and, therefore,

$$h(t)\geq h(3)=3(\ln 8-\ln 2)-2\ln 8=0.$$

Thus, $t\ln\left(\frac{3t-1}{t-1}\right)\geq 2\ln(3t-1)$, as desired.

4149. Prove that if $[a,b]\subset\left[0,\frac{\pi}{4}\right]$ then:

$$3(a\tan b+b\tan a)\geq ab(6+a\tan a+b\tan b)$$

Daniel Sitaru

Solution by Digby Smith-USA

We first prove that if x is a real number such that $0\leq x\leq 1$, then

$$(3-x^2)\tan x\geq 3x \quad (1)$$

From the Maclaurin series expansion for $\tan x$, we have that

$$\tan x\geq x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7$$

Hence,

$$(3-x^2)\tan x-3x$$
$$\geq(3-x^2)\left(x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7\right)-3x$$
$$=\left(3x+x^3+\frac{6}{15}x^5+\frac{51}{315}x^7\right)-\left(x^3+\frac{1}{3}x^5+\frac{2}{15}x^7+\frac{17}{315}x^9\right)-3x$$
$$=\frac{1}{15}x^5+\frac{9}{315}x^7-\frac{17}{315}x^9=\frac{1}{315}x^5(21+9x^2-17x^4)\geq 0,$$

which establishes (1).

Applying (1) with $x=a$ and b, respectively, we then have

$$(3-a^2)\tan a\geq 3a \text{ and } (3-b^2)\tan b\geq 3b.$$

Therefore: $a(3-b^2)\tan b+b(3-a^2)\tan a\geq 6ab$,

from which the given inequality follows immediately.

4152. Prove that if $a,b,c\in(0,\infty)$ then:

$$\ln(1+a)^{\ln(1+b)^{\ln(1+c)}}\leq\ln^3\left(1+\sqrt[3]{abc}\right)$$

Daniel Sitaru

Solution by Ali Adnan-USA

The left side of the inequality should be bracketed

$$\ln\left[(1+a)^{\ln\left[(1+b)^{\ln(1+c)}\right]}\right]$$

so that it is equal to $\ln(1+a)\ln(1+b)\ln(1+c)$.

Let $f(x)=\ln(\ln(1+e^x))$. Then

$$f''(x) = e^x[(1+e^x)\ln(1+e^x)]^{-2}[\ln(1+e^x) - e^x] < 0,$$

so that $f(x)$ is concave. By Jensen's theorem, for each x, y, z,

$$f(x) + f(y) + f(z) \le 3f\left(\frac{x+y+z}{3}\right)$$

Setting $(x, y, z) = (\ln a, \ln b, \ln c)$ yields that

$$\ln[\ln(1+a)\ln(1+b)\ln(1+c)] \le 3\ln\left[\ln\left(1+e^{\frac{\ln abc}{3}}\right)\right].$$

Exponentiating yields

$$\ln(1+a)\ln(1+b)\ln(1+c) \le \ln^3\left(1+\sqrt[3]{abc}\right)$$

as desired.

FEW OUTSTANDING LIMITS-(IV)

By Florică Anastase-Romania

Abstract: *In this paper it was presented few outstanding limits using special sums.*

Application 1. Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - \sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\right)}$$

Daniel Sitaru

Solution.

$$\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}} = \sum_{k=1}^{n}\left(\sqrt{k}\cdot\frac{1}{\sqrt{k}}\right) + \frac{1}{\sqrt{1}}\left(\sqrt{2}+\dots+\sqrt{n}\right) + \frac{1}{\sqrt{2}}\left(\sqrt{3}+\dots+\sqrt{n}\right) + \cdots$$

$$+\dots+\frac{1}{\sqrt{n-1}}\cdot\sqrt{n} + \sqrt{1}\left(\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}\right) + \sqrt{2}\left(\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}\right) + \cdots$$

$$+\dots+\sqrt{n-1}\cdot\frac{1}{\sqrt{n}} = n + \sum_{1\le i<j\le n}\sqrt{\frac{i}{j}} + \sum_{1\le i<j\le n}\sqrt{\frac{j}{i}} =$$

$$= n + \sum_{1\le i<j\le n}\left(\sqrt{\frac{i}{j}}+\sqrt{\frac{j}{i}}-2+2\right) = n + \sum_{1=i<j\le n}\left(\frac{i+j-2\sqrt{ij}}{\sqrt{ij}}+2\right) =$$

$$= n + \sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}} + 2\cdot\frac{n(n-1)}{2} = n^2 + \sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}$$

Hence,

$$\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}} = n^2 + \sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}$$

And then

$$\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}} - \sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}} = 1$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\right)=1$$

Application 2. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\left(\frac{\sin n}{n^3}\left(\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)\right)}$$

Daniel Sitaru

Solution.

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}=-\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sqrt[3]{i}-\sqrt[3]{j}\right)\left(\frac{1}{\sqrt[3]{i}}-\frac{1}{\sqrt[3]{j}}\right)=$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1+1-\sqrt[3]{\frac{i}{j}}-\sqrt[3]{\frac{j}{i}}\right)=-\sum_{i=1}^{n}\sum_{j=1}^{n}2+\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt[3]{\frac{j}{i}}=$$

$$=-2n^2+\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt[3]{\frac{i}{j}}+\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt[3]{\frac{j}{i}}=-2n^2+2\left(\sum_{k=1}^{n}\sqrt[3]{k}\right)\left(\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)$$

Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}=-2n^2+2n^2\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt[3]{k}\right)\left(\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)$$

$$2\sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}=-2n^2+2n^2\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt[3]{k}\right)\left(\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\right)$$

So, it follows that

$$\frac{1}{n^2}\left(\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)=1;(1)$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(\frac{\sin n}{n^3}\left(\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)\right)=$$

$$=\lim_{n\to\infty}\left(\frac{\sin n}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n}\sqrt[3]{k}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le n}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)\right)\overset{(1)}{=}\lim_{n\to\infty}\frac{\sin n}{n}=0.$$

Application 3. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\frac{\log n}{n^2}\sum_{p=1}^{n}\left(\frac{1}{p^2}\sum_{k=1}^{p}\sqrt[3]{k}\cdot\sum_{k=1}^{p}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le p}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)}$$

Daniel Sitaru

Solution.

Using (1) we have:

$$\frac{1}{p^2}\sum_{k=1}^{p}\sqrt[3]{k}\cdot\sum_{k=1}^{p}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le p}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}=1$$

Hence,

$$\sum_{p=1}^{n}\left(\frac{1}{p^2}\sum_{k=1}^{p}\sqrt[3]{k}\cdot\sum_{k=1}^{p}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le p}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)=n;(2)$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{\log n}{n^2}\sum_{p=1}^{n}\left(\frac{1}{p^2}\sum_{k=1}^{p}\sqrt[3]{k}\cdot\sum_{k=1}^{p}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le p}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)\overset{(2)}{=}$$

$$=\lim_{n\to\infty}\frac{\log n}{n^2}\cdot n=\lim_{n\to\infty}\frac{\log n}{n}\overset{Stolz-C}{=}0.$$

Application 4. Find:

$$\boldsymbol{\Omega=\lim_{m\to\infty}\frac{1}{m^2}\left(\lim_{n\to\infty}\sum_{n=1}^{m}\sum_{i=1}^{n}\sin\frac{(2i-1)j}{n^2}\right)}$$

Florică Anastase

Solution.

For $a>0$. We prove that:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sin\frac{(2i-1)a}{n^2}\right)=a$$

Using the well-known inequality: $x-\frac{x^3}{6}<\sin x<x,\forall x>0\Rightarrow$

$$\frac{a}{n^2}-\frac{1}{6}\cdot\frac{a^3}{n^6}<\sin\frac{a}{n^2}<\frac{a}{n^2}$$

$$\frac{3a}{n^2}-\frac{1}{6}\cdot\frac{3^3a^3}{n^6}<\sin\frac{3a}{n^2}<\frac{3a}{n^2}$$

… … … … … … … … … … … … …

$$\frac{(2n-1)a}{n^2}-\frac{1}{6}\cdot\frac{(2n-1)^3a^3}{n^6}<\sin\frac{(2n-1)a}{n^2}<\frac{(2n-1)a}{n^2}$$

Summing, we get:

$$\frac{a}{n^2}\cdot\sum_{i=1}^{n}(2i-1)-\frac{a^3}{n^6}\cdot\sum_{i=1}^{n}(2i-1)^3<\sum_{i=1}^{n}\sin\frac{(2i-1)a}{n^2}<\frac{a}{n^2}\cdot\sum_{i=1}^{n}(2i-1)$$

Let us denote:

$$a_n=\frac{a}{n^2}\cdot\sum_{i=1}^{n}(2i-1),b_n=x_n-\frac{a^3}{n^6}\cdot\sum_{i=1}^{n}(2i-1)^3$$

$$\Rightarrow a_n=a\Rightarrow b_n=a-\frac{a^3}{n^6}\cdot\sum_{i=1}^{n}(2i-1)^3=a-\frac{a^3}{n^4(2n^2-1)}$$

So, $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=a$, then

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sin\frac{(2i-1)a}{n^2}\right)=a$$

For $a=j\in\{1,2,\dots,m\}$ it follows that:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sin\frac{(2i-1)j}{n^2}\right)=j$$

Therefore,

$$\Omega=\lim_{m\to\infty}\frac{1}{m^2}\left(\lim_{n\to\infty}\sum_{n=1}^{m}\sum_{i=1}^{n}\sin\frac{(2i-1)j}{n^2}\right)=\lim_{m\to\infty}\frac{1}{m^2}\sum_{j=1}^{m}j=\frac{1}{2}.$$

Florică Anastase

Application 5. Find:

$$\boldsymbol{\Omega=\lim_{m\to\infty}\frac{1}{m^2}\left(\lim_{n\to\infty}\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\sin\frac{(2i-1)\sqrt{k}}{n^2}\cdot\sum_{k=1}^{m}\sum_{i=1}^{n}\sin\frac{2i-1}{n^2\sqrt{k}}\right)-\sum_{1\le i<j\le m}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\right)}$$

Florică Anastase

Solution.

Using Application 3, we have:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sin\frac{(2i-1)a}{n^2}\right)=a$$

For $a=\sqrt{k}$ and $a=\frac{1}{\sqrt{k}}$, we get:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{(2i-1)\sqrt{k}}{n^2}=\sqrt{k}\text{ and }\lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{(2i-1)}{n^2\sqrt{k}}=\frac{1}{\sqrt{k}}$$

Hence,

$$\Omega=\lim_{m\to\infty}\frac{1}{m^2}\left(\lim_{n\to\infty}\left(\sum_{k=1}^{m}\sum_{i=1}^{n}\sin\frac{(2i-1)\sqrt{j}}{n^2}\cdot\sum_{k=1}^{m}\sum_{i=1}^{n}\sin\frac{2i-1}{n^2\sqrt{j}}\right)-\sum_{1\le i<j\le m}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\right)=$$

$$=\lim_{m\to\infty}\frac{1}{m^2}\left(\sum_{k=1}^{m}\sqrt{k}\cdot\sum_{k=1}^{m}\frac{1}{\sqrt{k}}-\sum_{1\le i<j\le m}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\right)$$

Using Application 1, it follows that:

$$\Omega=\lim_{m\to\infty}\frac{1}{m^2}\left(\sum_{k=1}^{m}\sqrt{k}\cdot\sum_{k=1}^{m}\frac{1}{\sqrt{k}}-\sum_{1\le i<j\le m}\frac{\left(\sqrt{i}-\sqrt{j}\right)^2}{\sqrt{ij}}\right)=1.$$

Application 6. Find:

$$\boldsymbol{\Omega=\lim_{m\to\infty}\frac{1}{m^3}\left(\lim_{n\to\infty}\left(\sum_{k=1}^{n}\sum_{i=1}^{n}\sin\frac{(2i-1)\sqrt[3]{k}}{n^2}\cdot\sum_{k=1}^{n}\sum_{i=1}^{n}\sin\frac{2i-1}{n\sqrt[3]{k}}\right)-\sum_{1\le i<j\le m}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)}$$

Florică Anastase

Solution.

Using Application 3, we have:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\sin\frac{(2i-1)a}{n^2}\right)=a$$

For $a=\sqrt[3]{k}$ and $a=\frac{1}{\sqrt[3]{k}}$ it follows that:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{(2i-1)\sqrt[3]{k}}{n^2}=\sqrt[3]{k} \text{ and } \lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{(2i-1)}{n^2\sqrt[3]{k}}=\frac{1}{\sqrt[3]{k}}$$

Hence,

$$\Omega=\lim_{m\to\infty}\frac{1}{m^3}\left(\lim_{n\to\infty}\left(\sum_{k=1}^{n}\sum_{i=1}^{n}\sin\frac{(2i-1)\sqrt[3]{k}}{n^2}\cdot\sum_{k=1}^{n}\sum_{i=1}^{n}\sin\frac{2i-1}{n\sqrt[3]{k}}\right)-\sum_{1\le i<j\le m}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)=$$

$$=\lim_{m\to\infty}\frac{1}{m^3}\left(\sum_{k=1}^{m}\sqrt[3]{k}\cdot\sum_{k=1}^{m}\frac{1}{\sqrt[3]{k}}-\sum_{1\le i<j\le m}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)$$

Now, using Application 2, it follows that:

$$\Omega=\lim_{m\to\infty}\frac{1}{m^3}\left(\lim_{n\to\infty}\left(\sum_{k=1}^{n}\sum_{i=1}^{n}\sin\frac{(2i-1)\sqrt[3]{k}}{n^2}\cdot\sum_{k=1}^{n}\sum_{i=1}^{n}\sin\frac{2i-1}{n\sqrt[3]{k}}\right)-\sum_{1\le i<j\le m}\frac{\left(\sqrt[3]{i}-\sqrt[3]{j}\right)^2}{\sqrt[3]{ij}}\right)$$
$$=0$$

Florică Anastase

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ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(IX)

By Bogdan Fuştei-Romania

In ΔABC, F −area, R −circumradius, a,b,c −lengths sides and $M,P\in \mathrm{Int}(\Delta ABC)$.

The following relationship holds:

$$\boldsymbol{a\cdot AP\cdot AM+b\cdot BP\cdot BM+c\cdot CP\cdot CM\ge abc\ (G.Bennett);(1)}$$

Equality holds if and only if P and M are isogonal conjugate.

Let $P=G, G$ −centroid, hence

$$a\cdot AG\cdot AM+b\cdot BG\cdot BM+c\cdot CG\cdot CM\ge abc$$

$$AG=\frac{2}{3}m_a;BG=\frac{2}{3}m_b;CG=\frac{2}{3}m_c$$

$$a\cdot m_a\cdot AM+b\cdot m_b\cdot BM+c\cdot m_c\cdot CM\ge\frac{3}{2}abc=\frac{3}{2}\cdot 4RF=6RF$$

So, we get:

$$\boldsymbol{a\cdot m_a\cdot AM+b\cdot m_b\cdot BM+c\cdot m_c\cdot CM\ge 6RF;(2)}$$

$$2F=a\cdot h_a=b\cdot h_b=c\cdot h_c$$

$$\frac{a\cdot m_a\cdot AM}{a\cdot h_a}+\frac{b\cdot m_b\cdot BM}{b\cdot h_b}+\frac{c\cdot m_c\cdot CM}{c\cdot h_c}\ge\frac{2RF}{2F}$$

$$\frac{\boldsymbol{m_a}}{\boldsymbol{h_a}}\cdot \boldsymbol{AM}+\frac{\boldsymbol{m_b}}{\boldsymbol{h_b}}\cdot \boldsymbol{BM}+\frac{\boldsymbol{m_c}}{\boldsymbol{h_c}}\cdot \boldsymbol{CM}\ge \boldsymbol{3R};(\boldsymbol{3})$$

$$\frac{R}{2r}\ge \frac{m_a}{h_a}\ (Panaitopol)$$

$$\frac{\boldsymbol{R}}{\boldsymbol{2r}}(\boldsymbol{AM}+\boldsymbol{BM}+\boldsymbol{CM})\ge \boldsymbol{3R}\Rightarrow \boldsymbol{AM}+\boldsymbol{BM}+\boldsymbol{CM}\ge \boldsymbol{6r};(\boldsymbol{4})$$

Let K —intersection point of simmedians. If $P=K$,$AK=\frac{2bc}{a^2+b^2+c^2}\cdot m_a$, then

$$\frac{2abc}{a^2+b^2+c^2}(m_a\cdot AM+m_b\cdot BM+m_c\cdot CM)\ge abc$$

$$\boldsymbol{m_a}\cdot \boldsymbol{AM}+\boldsymbol{m_b}\cdot \boldsymbol{BM}+\boldsymbol{m_c}\cdot \boldsymbol{CM}\ge \frac{\boldsymbol{1}}{\boldsymbol{2}}(\boldsymbol{a^2}+\boldsymbol{b^2}+\boldsymbol{c^2});(\boldsymbol{5})$$

If $P=O\Rightarrow AP=BP=CP=R,(a\cdot AM+b\cdot BM+c\cdot CM)\cdot R\ge abc$

$$a\cdot AM+b\cdot BM+c\cdot CM\ge 4F;(abc=4F)\Rightarrow$$

$$\frac{\boldsymbol{AM}}{\boldsymbol{h_a}}+\frac{\boldsymbol{BM}}{\boldsymbol{h_b}}+\frac{\boldsymbol{CM}}{\boldsymbol{h_c}}\ge \boldsymbol{2};(\boldsymbol{6})$$

Let N_a —Nagel's point, $AN_a=\frac{a\cdot n_a}{s}$. If $P=N_a$, then:

$$\frac{a^2\cdot n_a}{s}\cdot AM+\frac{b^2\cdot n_b}{s}\cdot BM+\frac{c^2\cdot n_c}{s}\cdot CM\ge abc$$

$$\boldsymbol{a^2}\cdot \boldsymbol{n_a}\cdot \boldsymbol{AM}+\boldsymbol{b^2}\cdot \boldsymbol{n_b}\cdot \boldsymbol{BM}+\boldsymbol{c^2}\cdot \boldsymbol{n_c}\cdot \boldsymbol{CM}\ge \boldsymbol{s}\cdot \boldsymbol{abc};(\boldsymbol{7})$$

$$\frac{\boldsymbol{a}\cdot \boldsymbol{n_a}}{\boldsymbol{h_a}}\cdot \boldsymbol{AM}+\frac{\boldsymbol{b}\cdot \boldsymbol{n_b}}{\boldsymbol{h_b}}\cdot \boldsymbol{BM}+\frac{\boldsymbol{c}\cdot \boldsymbol{n_c}}{\boldsymbol{h_c}}\cdot \boldsymbol{CM}\ge \boldsymbol{2Rs};(\boldsymbol{8})$$

$$\frac{n_a}{h_a}\cdot\frac{AM}{bc}+\frac{n_b}{h_b}\cdot\frac{BM}{ac}+\frac{n_c}{h_c}\cdot\frac{CM}{ab}\ge \frac{2Rs}{abc}=\frac{1}{2r}$$

But: $bc=2R\cdot h_a, ca=2R\cdot h_b, ab=2R\cdot h_c$, then:

$$\frac{\boldsymbol{n_a}}{\boldsymbol{h_a^2}}\cdot \boldsymbol{AM}+\frac{\boldsymbol{n_b}}{\boldsymbol{h_b^2}}\cdot \boldsymbol{BM}+\frac{\boldsymbol{n_c}}{\boldsymbol{h_c^2}}\cdot \boldsymbol{CM}\ge \frac{\boldsymbol{R}}{\boldsymbol{r}};(\boldsymbol{9})$$

If $P=I$, I —incenter, we have: $AI=\frac{r}{\sin\frac{A}{2}}$,$a=4R\cdot \sin\frac{A}{2}\cos\frac{A}{2}$.

$$\sum_{cyc}4R\cdot \sin\frac{A}{2}\cos\frac{A}{2}\cdot\frac{r}{\sin\frac{A}{2}}\cdot AM\ge 4RF;$$

$$\boldsymbol{AM}\cdot \boldsymbol{\cos}\frac{\boldsymbol{A}}{\boldsymbol{2}}+\boldsymbol{BM}\cdot \boldsymbol{\cos}\frac{\boldsymbol{B}}{\boldsymbol{2}}+\boldsymbol{CM}\cdot \boldsymbol{\cos}\frac{\boldsymbol{C}}{\boldsymbol{2}}\ge \frac{\boldsymbol{F}}{\boldsymbol{r}}=\boldsymbol{s};(\boldsymbol{10})$$

Let Ω —be the first Brocard's point and ω —Brocard's angle.then:

$$A\Omega=2R\cdot\frac{b}{a}\cdot\sin\omega, B\Omega=2R\cdot\frac{c}{b}\cdot\sin\omega, C\Omega=2R\cdot\frac{a}{c}\cdot\sin\omega$$

$$\sin\omega=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$$

$$a\cdot A\Omega\cdot AM+b\cdot B\Omega\cdot BM+c\cdot C\Omega\cdot CM\ge abc$$

$$2R\cdot\sin\omega\,(b\cdot AM+c\cdot BM+a\cdot CM)\ge abc=4RF$$

$$\boldsymbol{b}\cdot \boldsymbol{AM}+\boldsymbol{c}\cdot \boldsymbol{CM}+\boldsymbol{a}\cdot \boldsymbol{CM}\ge \sqrt{\boldsymbol{a^2b^2}+\boldsymbol{b^2c^2}+\boldsymbol{c^2a^2}};(\boldsymbol{11})$$

G_a —Gergonne's point, then $AG_e=\frac{g_a(r_b+r_c)}{4R+r}$

From Bennet's inequality, we have that:

$$\sum_{cyc}\boldsymbol{a}\cdot \boldsymbol{AM}\cdot \boldsymbol{g_a}(\boldsymbol{r_b}+\boldsymbol{r_c})\ge (\boldsymbol{4R}+\boldsymbol{r})\boldsymbol{abc};(\boldsymbol{12})$$

$$bc = 2R \cdot h_a; abc = 4RF \Rightarrow$$
$$\sum_{cyc} \frac{g_a(r_b + r_c)}{h_a} \cdot AM \ge 2R(4R + r); (13)$$

If $M = \Omega, \Omega$ −the first point of Brocard's, it follows that:

$$\sum_{cyc} \boldsymbol{bg_a(r_b + r_c) \ge (4R + r)\sqrt{a^2b^2 + b^2c^2 + c^2a^2}; (14)}$$

If $M \in Int(\Delta ABC)$ then:

$$AM \cdot \cos\frac{A}{2} + BM \cdot \cos\frac{B}{2} + CM \cdot \cos\frac{C}{2} \ge s$$

Let $M = G \Rightarrow AG = \frac{2}{3}m_a; BG = \frac{2}{3}m_b; CG = \frac{2}{3}m_c$, thus,

$$\boldsymbol{m_a \cdot \cos\frac{A}{2} + m_b \cdot \cos\frac{B}{2} + m_c \cdot \cos\frac{C}{2} \ge \frac{3}{2}s; (15)}$$

If $M = \Omega, \Omega$ −the first point of Brocard's, it follows that:

$$\frac{b}{a} \cdot \cos\frac{A}{2} + \frac{c}{b} \cdot \cos\frac{B}{2} + \frac{a}{c} \cdot \cos\frac{C}{2} \ge \frac{s}{2R} \cdot \frac{1}{\sin\omega}$$
$$\frac{1}{\sin\omega} = \frac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{2F}; 2F = 2sr$$
$$\boldsymbol{\frac{b}{a} \cdot \cos\frac{A}{2} + \frac{c}{b} \cdot \cos\frac{B}{2} + \frac{a}{c} \cdot \cos\frac{C}{2} \ge \frac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{4RF}; (16)}$$
$$\cos\frac{A}{2} = \frac{s}{\sqrt{s^2 + r_a^2}}; s^2 = n_a^2 + 2r_ah_a \Rightarrow \cos\frac{A}{2} = \frac{s}{\sqrt{n_a^2 + r_a^2 + 2r_ah_a}}$$
$$\boldsymbol{n_a^2 + r_a^2 \ge 2r_ah_a \Rightarrow \frac{AM}{\sqrt{s^2 + r_a^2}} + \frac{BM}{\sqrt{s^2 + r_b^2}} + \frac{CM}{\sqrt{s^2 + r_c^2}} \ge 1; (17)}$$
$$\boldsymbol{\sum_{cyc} \frac{AM}{\sqrt{r_a(n_a + h_a)}} \ge \sqrt{2}; (18)}$$

Let $M = G$ and from (18), we get: $\boldsymbol{\sum_{cyc} \frac{m_a}{\sqrt{r_a(n_a+h_a)}} \ge \frac{3\sqrt{2}}{2}; (19)}$

If $M = N_a; AN_a = \frac{a \cdot n_a}{s}$ and from (18), we get: $\boldsymbol{\sum_{cyc} \frac{a \cdot n_a}{\sqrt{r_a(n_a+h_a)}} \ge s\sqrt{2}; (20)}$

If $M = N_a; AN_a = \frac{a \cdot n_a}{s}$ and from (10), we get: $\boldsymbol{\sum_{cyc} an_a \cdot \cos\frac{A}{2} \ge s^2; (21)}$

If $M = G_e, G_e$ −Gergonne's point and from (10), we get:

$$\boldsymbol{\sum_{cyc} \frac{g_a(r_b + r_c)}{4R + r} \cdot \cos\frac{A}{2} \ge s \Rightarrow \sum_{cyc} g_a(r_b + r_c)\cos\frac{A}{2} \ge (4R + r)s; (22)}$$

Using $\cos\frac{A}{2} = \frac{s}{\sqrt{s^2+r_a^2}}$ and from (21), (22), we get:

$$\boldsymbol{\sum_{cyc} \frac{an_a}{\sqrt{r_a^2 + s^2}} \ge s; (23) \text{ and } \sum_{cyc} \frac{g_a(r_b + r_c)}{\sqrt{r_a^2 + s^2}} \ge 4R + r; (24)}$$

If $M = G_e; AG_e = \frac{g_a(r_b+r_c)}{4R+r}$ and using (18), it follows that:

$$\boldsymbol{\sum_{cyc} \frac{g_a(r_b + r_c)}{\sqrt{r_a(n_a + h_a)}} \ge (4R + r)\sqrt{2}; (25)}$$

Let a, b, c −be lengths sides of a triangle, then the system

$x+y=c, y+z=a, z+x=b$ has unique solution $x=\frac{b+c-a}{2}, y=\frac{a+c-b}{2}, z=\frac{a+b-c}{2}$

$$x=s-a; y=s-b; z=s-c; 2s=a+b+c=2(x+y+z)$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}; x,y,z>0$$

$$AM\cdot\cos\frac{A}{2}+BM\cdot\cos\frac{B}{2}+CM\cdot\cos\frac{C}{2}\ge s\Rightarrow$$

$$\sqrt{x+y+z}\sum_{cyc}AM\sqrt{\frac{x}{(x+y)(x+z)}}\ge x+y+z$$

Finally, for $M\in Int(\Delta ABC); x,y,z>0$, we have:

$$\boldsymbol{\sum_{cyc}AM\sqrt{\frac{x}{(x+y)(x+z)}}\ge\sqrt{x+y+z}; (26)}$$

$$\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2}$$

In acute $\Delta ABC, \cos A, \cos B, \cos C\ge 0\Rightarrow s^2-(2R+r)^2\ge 0\Rightarrow s\ge 2R+r$. Thus,

$$\sum_{cyc}AM\cdot\cos\frac{A}{2}\ge 2R+r; (27); \{\Delta ABC-acute, M\in Int(\Delta ABC)\}$$

In acute $\Delta ABC, M\in Int(\Delta ABC)$, we have:

$$\boldsymbol{\sum_{cyc}\frac{an_a}{h_a}\cdot AM\ge 2R(2R+r); (28), \sum_{cyc}m_a\cdot\cos\frac{A}{2}\ge\frac{3}{2}(2R+r); (29)}$$

$$\boldsymbol{\sum_{cyc}\frac{an_a}{\sqrt{r_a(n_a+h_a)}}\ge(2R+r)\sqrt{2}; (30), \sum_{cyc}g_a(r_b+r_c)\cos\frac{A}{2}\ge(4R+r)(2R+r); (31)}$$

$$\boldsymbol{\sum_{cyc}\frac{an_a}{\sqrt{r_a^2+s^2}}\ge 2R+r; (32), \sum_{cyc}\frac{an_a}{\sqrt{r_a^2+(2R+r)^2}}\ge s; (33)}$$

$$\boldsymbol{\sum_{cyc}\frac{g_a(r_b+r_c)}{\sqrt{r_a^2+(2R+r)^2}}\ge 4R+r; (34)}$$

If in $\Delta ABC, A\ge B\ge\frac{\pi}{3}\ge C$, then $s\ge(R+r)\sqrt{3}; \tan\frac{A}{2}, \tan\frac{B}{2}\ge\frac{\sqrt{3}}{3}$ and $\tan C\le\frac{\sqrt{3}}{3}$. So,

$$\prod_{cyc}\left(\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\right)\le 0\Leftrightarrow\prod_{cyc}\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}+\frac{1}{3}\sum_{cyc}\tan\frac{A}{2}-\frac{\sqrt{3}}{9}\le 0$$

$$\frac{r}{s}-\frac{\sqrt{3}}{3}+\frac{1}{3}\cdot\frac{4R+r}{s}-\frac{\sqrt{3}}{9}\le 0; \sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}=1; \sum_{cyc}\tan\frac{A}{2}=\frac{4R+r}{s}\Leftrightarrow$$

$$9r+3(4R+r)-4\sqrt{3}s\le 0\Rightarrow s\ge(R+r)\sqrt{3}$$

If in $\Delta ABC, A\ge B\ge\frac{\pi}{3}\ge C$, then

$$\boldsymbol{\sum_{cyc}AM\cdot\cos\frac{A}{2}\ge(R+r)\sqrt{3}; (35), \sum_{cyc}\frac{an_a}{h_a}\cdot AM\ge 2R(R+r)\sqrt{3}; (36)}$$

$$\boldsymbol{\sum_{cyc}m_a\cdot\cos\frac{A}{2}\ge\frac{3\sqrt{3}}{2}(R+r); (37), \sum_{cyc}\frac{an_a}{\sqrt{r_a(n_a+h_a)}}\ge(R+r)\sqrt{6}; (38)}$$

$$\sum_{cyc} \boldsymbol{g_a(r_b+r_c)\cos\frac{A}{2}} \ge \boldsymbol{(4R+r)(R+r)\sqrt{3}; (39)}$$

$$\sum_{cyc} \frac{\boldsymbol{an_a}}{\sqrt{\boldsymbol{r_a^2+s^2}}} \ge \boldsymbol{(R+r)\sqrt{3}; (40)}, \sum_{cyc} \frac{\boldsymbol{g_a(r_b+r_c)}}{\sqrt{\boldsymbol{r_a^2+3(R+r)^2}}} \ge \boldsymbol{4R+r; (41)}$$

$$\sum_{cyc} \frac{\boldsymbol{an_a}}{\sqrt{\boldsymbol{r_a^2+3(R+r)^2}}} \ge \boldsymbol{s; (42)}, \sum_{cyc} \boldsymbol{an_a\cos\frac{A}{2}} \ge \boldsymbol{3(R+r)^2; (43)}$$

In ΔABC the following relationship holds:

$$\frac{(m_a+m_b+m_c)^2}{a^2+b^2+c^2} \le 2+\left(\frac{r}{R}\right)^2; (Sun\ Wen\ Cai)$$

$$(m_a+m_b+m_c)^2 \le \frac{2R^2+r^2}{R^2}(a^2+b^2+c^2)$$

Using (26), we can write:

$$\sum_{cyc} AM\sqrt{\frac{a^2}{(a^2+b62\ )(a^2+c^2)}} \ge \sqrt{a^2+b^2+c^2}$$

$$\sum_{cyc} \frac{a\cdot AM}{\sqrt{(a^2+b^2)(a^2+c^2)}} \ge \sqrt{\frac{R^2}{2R^2+r^2}(m_a+m_b+m_c)^2}$$

$$\sum_{cyc} \frac{a\cdot AM}{\sqrt{(a^2+b^2)(a^2+c^2)}} \ge \frac{R(m_a+m_b+m_c)}{\sqrt{2R^2+r^2}}$$

$$\because a = 2R\cdot\sin A$$

$$\sum_{cyc} \frac{\boldsymbol{\sin A}}{\sqrt{\boldsymbol{(a^2+b^2)(a^2+c^2)}}}\cdot \boldsymbol{AM} \ge \boldsymbol{\frac{1}{2}\cdot\frac{m_a+m_b+m_c}{\sqrt{2R^2+r^2}}; (44)}$$

$$s^2 = n_a^2+2r_ah_a \Rightarrow s^2-n_a^2 = 2r_ah_a \Rightarrow (s+n_a)(s+n_a)+\frac{2r_ah_a}{n_a+s}$$

$$\Rightarrow 3s = n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s} \overset{(10)}{\Longrightarrow}$$

$$\boldsymbol{3\sum_{cyc} AM\cdot\cos\frac{A}{2} \ge n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}; (45)}$$

$$\boldsymbol{2\sum_{cyc} m_a\cdot\cos\frac{A}{2} \ge n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}; (46)}$$

$$\boldsymbol{3\sum_{cyc}\frac{an_a}{\sqrt{r_a(n_a+h_a)}} \ge \sqrt{2}\left(n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}\right); (47)}$$

$$\boldsymbol{3\sum_{cyc} g_a(r_b+r_c)\cos\frac{A}{2} \ge (4R+r)\left(n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}\right); (48)}$$

$$\boldsymbol{3\sum_{cyc}\frac{an_a}{\sqrt{r_a^2+s^2}} \ge n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}; (49)}$$

$$9\sum_{cyc} \boldsymbol{a n_a \cos\frac{A}{2}} \ge \left(\boldsymbol{n_a + n_b + n_c + 2\sum_{cyc}\frac{n_a h_a}{n_a + s}}\right)^2 ;(\mathbf{50})$$

From (5) and Sun Wen Cai's inequality, we get:

$$\boldsymbol{m_a AM + m_b BM + m_c CM \ge \frac{R^2}{2(2R^2 + r^2)}(m_a + m_b + m_c)}; (\mathbf{51})$$

In $\Delta ABC, \Delta A_1B_1C_1, F -$ area of $\Delta ABC, F_1 -$ area of $\Delta A_1B_1C_1, M \in Int(\Delta ABC)$, holds:

$$a_1 \cdot AM + b_1 \cdot BM + c_1 \cdot CM \ge \sqrt{\frac{1}{2}\sum_{cyc} a^2(b_1^2 + c_1^2 - a_1^2) + 8FF_1}; (Bottema)$$

$$a_1 \cdot AM + b_1 \cdot BM + c_1 \cdot CM \ge \sqrt{\frac{1}{2}\sum_{cyc} a_1^2(b^2 + c^2 - a^2) + 8FF_1}; (Bottema)$$

$$b^2 + c^2 = n_a^2 + g_a^2 + 2rr_a, 2rr_a = h_a(r - a - r)$$

$$2F = ah_a = bh_b = ch_c = 2sr, r_a + r_b + r_c = 4R + r$$

$$a^2 = 2R \cdot \frac{h_b h_c}{h_a}$$

$$ah_a = (a + b + c)r \Rightarrow \frac{h_a}{r} = 1 + \frac{b + c}{a}$$

$$b^2 + c^2 = n_a^2 + g_a^2 + 2rr_a \ge 2n_a g_a + 2rr_a$$

$$b^2 + c^2 = 2Rh_a\left(\frac{h_b}{h_c} + \frac{h_c}{h_b}\right) \ge 2n_a g_a + h_a(r_a - r)$$

$$\frac{b}{c} + \frac{c}{b} \ge \frac{2n_a g_a + h_a(r_a - r)}{2Rh_a} \Rightarrow \frac{b}{c} + \frac{c}{b} \ge \frac{1}{R}\left(\frac{n_a g_a}{h_a} + \frac{r_a - r}{2}\right)$$

$$\sum_{cyc}\frac{b + c}{a} \ge \frac{1}{R}\left(\sum_{cyc}\frac{n_a g_a}{h_a} + 2R - r\right)$$

$$\sum_{cyc}\frac{b + c}{a} \ge 2 + \frac{1}{R}\sum_{cyc}\frac{n_a g_a}{h_a} - \frac{r}{R}$$

$$\frac{h_a + h_b + h_c - 3r}{r} \ge 2 + \frac{1}{R}\sum_{cyc}\frac{n_a g_a}{h_a} - \frac{r}{R}$$

$$\frac{h_a + h_b + h_c}{r} \ge \frac{5R - r}{R} + \frac{1}{R}\sum_{cyc}\frac{n_a g_a}{h_a}$$

$$\boldsymbol{\frac{R}{r} \ge \frac{5R - r + \sum\frac{n_a g_a}{h_a}}{h_a + h_b + h_c}}; (\mathbf{51})$$

$$\boldsymbol{5R - r \ge 4R + r \Rightarrow 5R - 4R \ge r + r \Rightarrow R \ge 2r\ (Euler)}$$

$$\boldsymbol{\frac{R}{r} \ge \frac{r_a + r_b + r_c + \sum\frac{n_a g_a}{h_a}}{h_a + h_b + h_c}}; (\mathbf{52})$$

$$\boldsymbol{g_a \ge h_a \Rightarrow \frac{R}{r} \ge \frac{r_a + r_b + r_c + n_a + n_b + n_c}{h_a + h_b + h_c}}; (\mathbf{53})$$

$$\boldsymbol{\frac{R}{r} \ge \frac{5R - r + n_a + n_b + n_c}{h_a + h_b + h_c}}; (\mathbf{54})$$

$$n_a g_a \ge m_a w_a \Rightarrow \frac{R}{r} \ge \frac{5R - r + \sum \frac{m_a w_a}{h_a}}{h_a + h_b + h_c}; (55)$$

$$\frac{R}{r} \ge \frac{r_a + r_b + r_c + \sum \frac{m_a w_a}{h_a}}{h_a + h_b + h_c}; (56)$$

$$m_a w_a \ge s(s-a) = r_b r_c = \frac{h_a}{2}(r_b + r_c); (Panaitopol)$$

$$\frac{R}{r} \ge \frac{5R - r + r_a + r_b + r_c}{h_a + h_b + h_c} = \frac{9R}{h_a + h_b + h_c}; (57)$$

$$\sum_{cyc} \frac{n_a}{h_a^2} AM \ge \frac{5R - r + \sum \frac{n_a g_a}{h_a}}{h_a + h_b + h_c}; (58), \sum_{cyc} \frac{n_a}{h_a^2} AM \ge \frac{r_a + r_b + r_c + \sum \frac{n_a g_a}{h_a}}{h_a + h_b + h_c}; (59)$$

$$\sum_{cyc} \frac{n_a}{h_a^2} AM \ge \frac{r_a + r_b + r_c + n_a + n_b + n_c}{h_a + h_b + h_c}; (60)$$

$$\sum_{cyc} \frac{n_a}{h_a^2} AM \ge \frac{5R - r + n_a + n_b + n_c}{h_a + h_b + h_c}; (61)$$

$$\sum_{cyc} \frac{n_a}{h_a^2} AM \ge \frac{5R - r + \sum \frac{m_a w_a}{h_a}}{h_a + h_b + h_c}; (63), \sum_{cyc} \frac{n_a}{h_a^2} AM \ge \frac{9R}{h_a + h_b + h_c}; (64)$$

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ABOUT AN INEQUALITY BY CONSTANTIN IONESCU-ŢIU

By D.M. Bătineţu-Giurgiu,Daniel Sitaru-Romania

Let $x, y, z > 0; u, v \ge 0, u + v > 0$ and $\Delta ABC, \Delta A_1B_1C_1$ with circumradies R_1, R_2 respectively. If σ is an permutation of the set $\{a_1, b_1, c_1\}$ then:

$$\frac{x+y}{z \cdot a^u(\sigma(a_1))^v} + \frac{y+z}{x \cdot b^u(\sigma(b_1))^v} + \frac{z+x}{y \cdot c^u(\sigma(c_1))^v} \ge \frac{2(\sqrt{3})^{2-u-v}}{R^u \cdot R_1^v}; (1)$$

Proof. Let s, s_1 be the semiperimeters of $\Delta ABC, \Delta A_1B_1C_1$ respectively and F, F_1be the areas of $\Delta ABC, \Delta A_1B_1C_1$ respectively. We have:

$$\sum_{cyc}\frac{x+y}{z\cdot a^u\left(\sigma(a_1)\right)^v}\geq 2\cdot\sum_{cyc}\frac{\sqrt{xy}}{z\cdot a^u\left(\sigma(a_1)\right)^v}\geq 3\cdot 2\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{xy}}{z\cdot a^u\left(\sigma(a_1)\right)^v}}=$$

$$=3\cdot 2\cdot\sqrt[3]{\frac{xyz}{xyz\cdot(abc)^u\cdot(a_1b_1c_1)^v}}=\frac{3\cdot 2}{\sqrt[3]{(4RF)^u(4R_1F_1)}}=$$

$$=\frac{6}{\sqrt[3]{4^{u+v}\cdot R^uR_1^v(rs)^u(r_1s_1)^v}}\overset{Euler}{\geq}\frac{6}{\sqrt[3]{4^{u+v}\cdot R^uR_1^v\cdot\left(\frac{R}{2}\right)^u\left(\frac{R_1}{2}\right)^v\cdot s^us_1^v}}=$$

$$=\frac{6}{\sqrt[3]{2^{u+v}\cdot R^{2u}\cdot R_1^{2v}\cdot s^u\cdot s_1^v}}\overset{Mitrinovic}{\geq}\frac{6}{\sqrt[3]{2^{u+v}\cdot R^{2u}\cdot R_1^{2v}\cdot\left(\frac{3\sqrt{3}R}{2}\right)^u\left(\frac{3\sqrt{3}R_1}{2}\right)^v}}=$$

$$=\frac{3}{\sqrt[3]{\left(3\sqrt{3}\right)^{u+v}\cdot R^{3u}\cdot R_1^{3v}}}=\frac{3}{\left(\sqrt{3}\right)^{u+v}\cdot R^u\cdot R_1^v}=\frac{2\left(\sqrt{3}\right)^{2-u-v}}{R^u\cdot R_1^v}$$

If $x=y=z$ then (1) becomes as:

$$\frac{\mathbf{1}}{a^u\cdot\left(\sigma(a_1)\right)^v}+\frac{\mathbf{1}}{b^u\cdot\left(\sigma(b_1)\right)^v}+\frac{\mathbf{1}}{c^u\cdot\left(\sigma(c_1)\right)^v}\geq\frac{\left(\sqrt{3}\right)^{2-u-v}}{R^u\cdot R_1^v};(\mathbf{2})$$

If in (2) we take $u=v$, we obtain:

$$\frac{\mathbf{1}}{a^u\cdot\left(\sigma(a_1)\right)^u}+\frac{\mathbf{1}}{b^u\cdot\left(\sigma(b_1)\right)^u}+\frac{\mathbf{1}}{c^u\cdot\left(\sigma(c_1)\right)^u}\geq\frac{3^{1-u}}{R^u\cdot R_1^u};(\mathbf{3})$$

If in (3) we take $ABC\equiv A_1B_1C_1$, we get:

$$\frac{\mathbf{1}}{a^u\cdot\left(\sigma(a)\right)^u}+\frac{\mathbf{1}}{b^u\cdot\left(\sigma(b)\right)^u}+\frac{\mathbf{1}}{c^u\cdot\left(\sigma(c)\right)^u}\geq\frac{3^{1-u}}{R^{2u}};(\mathbf{4})$$

If in (4), σ is the identic permutation of the set $\{a,b,c\}$ then:

$$\frac{\mathbf{1}}{a^{2u}}+\frac{\mathbf{1}}{b^{2u}}+\frac{\mathbf{1}}{c^{2u}}\geq\frac{3^{1-u}}{R^{2u}};(\mathbf{5})$$

If in (2) we take $u=1, v=0$ then:

$$\frac{\mathbf{1}}{a}+\frac{\mathbf{1}}{b}+\frac{\mathbf{1}}{c}\geq\frac{\sqrt{3}}{R}\ (\boldsymbol{Ionescu-Tiu})$$

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SSMA-MATH CHALLENGES-(I)

By Daniel Sitaru – Romania

5477. Compute:

$$L=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\right).$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ed Gray, Highland Beach, FL.-USA

We rewrite the expression as:

1. $\lim_{x\to 0}\frac{\left[1-(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}+\cdots+\frac{1}{n}}\right]}{x^2}$.

2. Let $N=\sum_{k=2}^{k=n}\frac{1}{k}$, i.e., the harmonic series -1

3. Now consider $\lim_{x\to 0}\frac{\left[1-(1+x^2)^N\right]}{x^2}$.

We expand $(1+x^2)^N$ by the Binomial Theorem:

4. $(1+x^2)^N=1+Nx^2+\frac{N(N-1)}{2!}x^4+\cdots$

Then

5. $\lim_{x\to 0}\frac{\left[1-\left(1+Nx^2+\frac{N(N-1)}{2}x^4+\cdots\right)\right]}{x^2}$, or

6. $\lim_{x\to 0}\frac{\left[-Nx^2+\frac{-N(N-1)}{2}x^4+\cdots\right]}{x^2}=\frac{-Nx^2}{x^2}=-N$.

The original becomes

7. $\lim_{n\to\infty}(\ln(n)-N)=\lim_{n\to\infty}\left(\ln(n)-\sum_{k=2}^{k=n}\frac{1}{k}\right)=\lim_{n\to\infty}(\ln(n)+1-Harmonic\ series)$.

The Euler-Mascheroni Constant is defined as $\gamma=\lim_{n\to\infty}The\ Harmonic\ series-\ln(n)$.

Therefore our expression in step 7 equals $1-\gamma$.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain.

Since

$$\lim_{x\to 0}\frac{1-(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}}{x^2}=\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\left[\frac{0}{0}=Indet.\right]\underset{L'Hospital}{=}$$

$$\lim_{x\to 0}\frac{0-(H_n-1)(1+x)^{H_n-2}2x}{2x}=(1-H_n)\lim_{x\to 0}(1+x^2)^{H_n-2}=1-H_n,$$

$$L=\lim_{n\to\infty}(\ln n+1-H_n)=1-\lim_{n\to\infty}(H_N-\ln n)=1-\gamma,$$

where H_n is the n-th harmonic number and γ is the Euler Mascheroni constant.

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.

$$\sqrt{1+x^2}\sqrt[3]{1+x^2}\ldots\sqrt[n]{1+x^2}=(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}}=(1+x^2)^{H_n-1}$$

We have

$$\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

Now

$$\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}=-H_n+1$$

thus

$$L=\lim_{n\to\infty}(\ln -\ln n-\gamma+o(1)+1)=-\gamma+1$$

Solution 4 by Julio Cesar Mohnsam and Mateus Gomes Lucas, both from IFSUL, Campus Pelots-RS, Brazil, and Ricardo Capiberibe Nunes of E.E. Amlio de Caravalho Bas, Campo Grande-MS, Brazil

$$L=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)=\lim_{n\to\infty}\left(\ln n+\lim_{x\to 0}(1-H_n)\,(1+x^2)^{H_n-2}\right).$$

because,

$$\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]}{(x^2)}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]'}{(x^2)'}=$$
$$=\lim_{x\to 0}(-H_n+1)(1+x^2)^{H_n-2}=-H_n+1$$

Therefore:

$$L=\lim_{n\to\infty}(\ln n-H_n+1)=\lim_{n\to\infty}(\ln -H_n)+1=1+\lim_{n\to\infty}(\ln n-H_n)=1-\gamma$$

Note: γ is Euler-Mascheroni constant and $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$.

5482. Prove that if n is a natural number then

$$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n+(\tan 3^\circ)^n}+\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}+\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{3}{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Henry Ricardo, Westchester Are Math Circle, NY.-USA

Since, for a fixed natural number n, $(\tan x)^n$ is an increasing positive function for $x\in[0,90^\circ)$, we have

$$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n+(\tan 3^\circ)^n}\geq\frac{(\tan 5^\circ)^n}{(\tan 5^\circ)^n+(\tan 5^\circ)^n}=\frac{1}{2},$$
$$\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}\geq\frac{(\tan 4^\circ)^n}{(\tan 4^\circ)^n+(\tan 4^\circ)^n}=\frac{1}{2},$$
$$\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{(\tan 3^\circ)^n}{(\tan 3^\circ)^n+(\tan 3^\circ)^n}=\frac{1}{2},$$

so that adding these inequalities gives us the desired result. Equality holds if and only if $n=0$ (assuming that 0 is considered a natural number).

Solution 2 by Henry Ricardo, Westchester Are Math Circle, NY.-USA

Since, for a fixed natural number n, $(\tan x)^n$ is an increasing positive function for $x\in[0,90^\circ)$, we have

$$\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{(\tan 3^\circ)^n}{(\tan 4^\circ)^n+(\tan 5^\circ)^n},$$

$$\frac{(\tan 4°)^n}{(\tan 3°)^n + (\tan 2°)^n} \geq \frac{(\tan 4°)^n}{(\tan 3°)^n + (\tan 5°)^n},$$

so that

$$\sum_{cyclic} \frac{(\tan 3°)^n}{(\tan 2°)^n + (\tan 1°)^n} \geq$$

$$\geq \frac{(\tan 5°)^n}{(\tan 4°)^n + (\tan 3°)^n} + \frac{(\tan 4°)^n}{(\tan 3°)^n + (\tan 5°)^n} + \frac{(\tan 3°)^n}{(\tan 4°)^n + (\tan 5°)^n}$$

Setting $a = (\tan 3°)^n, b = (\tan 4°)^n$, and $c = (\tan 5°)^n$, we see that the right-hand side of the last inequality has the form

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b},$$

for $a, b, c > 0$, which is greater than or equal to $\frac{3}{2}$ by Nesbitt's inequality. Equality holds if and only if $n = 0$ (assuming that 0 is considered a natural number).

Solution 3 by Ed Gray, Highland Beach, FL

First we retrieve the required values:

1. $\tan 1° = .017455065$
2. $\tan 2° = .034920769$
3. $\tan 3° = .052407779$
4. $\tan 4° = .069926812$
5. $\tan 5° = .087488664$

We rewrite the problem's equation as:

$$\frac{1}{\frac{\tan 4°}{\tan 5°} + \frac{\tan 3°}{\tan 5°}} + \frac{1}{\frac{\tan 3°}{\tan 4°} + \frac{\tan 2°}{\tan 4°}} + \frac{1}{\frac{\tan 2°}{\tan 3°} + \frac{\tan 1°}{\tan 3°}} \geq \frac{3}{2}$$

Substituting the values from steps 1-5 and performing the indicated divisions we define:

$$f(n) = \frac{1}{(.799267114)^n + (.599023652)^n} + \frac{1}{(.794551256)^n + (.499433116)^n} + $$
$$+ \frac{1}{(.66632797)^n + (.333062483)^n}$$

We note that $f(n)$ is an increasing function of n since the denominators clearly decrease as n increases.

Finally we note that

$$f(1) = .715158838 + 1.248899272 + 1.000609919 = 2.964668029 > \frac{3}{2}$$

Then the equality holds for all n since $f(n)$ is an increasing function.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

Lemma: For fixed positive reals a, b, c with $a < c, b < c$ let $f(x) = \frac{c^x}{b^x + a^x}$ for $x \geq 0$.

Then $f(x) \geq \frac{1}{2}$, for $x \geq 0$, with equality holding only for $x = 0$.

Proof. We calculate the derivative:

$$f'(x) = \frac{(b^x + a^x)c^x \ln c - c^x(a^x \ln a + b^x \ln b)}{(b^x + a^x)^2}$$
$$= c^x \frac{(b^x + a^x)\ln c - (a^x \ln a + b^x \ln b)}{(b^x + a^x)^2}$$

$$= c^x \frac{b^x(\ln c - \ln b) + a^x(\ln c - \ln a)}{(b^x + a^x)^2}.$$

The ln function is increasing, so $\ln c \geq \ln b$ and $\ln c > \ln a$; thus we see that the derivative is positive. Hence the function f is increasing, so $\frac{1}{2} = f(0) \leq f(x)$ for $x \geq 0$. Because the derivative is strictly positive, the function f actually grows: so $f(x) > \frac{1}{2}$ for $x > 0$.

To verify the inequality of the problem, we note that the tangent function is increasing, so in each summand the tangent term in the numerator is larger that each tangent term in the denominator. Hence we can apply the lemma to each of the three summands, forcing the sum $\geq \frac{3}{2}$. Note that equality holds if and only if $n = 0$.

Comment: We can apply the lemma to obtain some ugly inequalities which are clearly true:

$$\frac{3^n}{1^n+2^n} + \frac{4^n}{2^n+3^n} + \frac{5^n}{3^n+4^n} + \cdots + \frac{(n+2)^n}{n^n+(n+1)^n} \geq \frac{n}{2}, \text{ and}$$

$$\frac{[(n+2)!]^n}{[n!]^n + [(n+1)!]^n} \geq \frac{1}{2}.$$

5488. Let a, and b be complex numbers. Solve the following equation:

$$\boldsymbol{x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0.}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith, Angelo State University, San Angelo, TX-USA

To begin, we note that

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3$$

can be re-written as

$$(x^3 - 3ax^2 + 3a^2x - a^3) - 3b^2x + 3ab^2 - 2b^3$$

or

$$(x-a)^3 - 3b^2(x-a) - 2b^3.$$

Hence, if we substitute $y = x - a$, the given equation becomes

$$y^3 - 3b^2y - 2b^3 = 0. \quad (1)$$

Next, the left side of equation (1) can be re-grouped to obtain

$$\begin{aligned} y^3 - 3b^2y - 2b^3 &= (y^3 + b^3) - 3b^2(y+b) \\ &= (y+b)[(y^2 - by + b^2) - 3b^2] \\ &= (y+b)(y^2 - by - 2b^2) \\ &= (y+b)^2(y-2b). \end{aligned}$$

Therefore, the solutions of (1) are $y = 2b$ and $y = -b$ (double solution).

Finally, since $y = x - a$, the solutions of the original equation are $x = a + 2b$ and $x = a - b$ (double solution).

Solution 2 by Michel Bataille, Rouen, France

Let $p(x)$ denote the polynomial on the left-hand side. Then, a short calculation gives

$$p(X+a) = X^3 - 3b^2X - 2b^3 = (X+b)^2(X-2b)$$

which has $2b$ as a simple root and $-b$ as a double one. It immediately follows that the solution of the given equation are $a-b, a-b, a+2b$.

Solution 3 by Paul M. Harms, North Newton, KS.

The equation can be written as $(x-a)^3-3ab^2(x-a)-2b^3=0$. If $b=0$, the solution is $x=a$. If b is not zero, let $x-a=yb$. Then the equation become $b^3(y^3-3y-2)=0$. We have $y^3-3y-2=(y-2)(y+1)^2=0$ .The y solutions are $2,-1$ and -1. The solutions of the equation in the problem are $x=a+2b$ and $x=a-b$ as a double root.

Solution 4 by G. C. Greubel, Newport News, VA.

$$\begin{aligned}0&=x^3-3ax^2+3(a^2-b^2)x-(a^3-3ab^2+2b^3)\\&=x^3-3ax^2+(a-b)(3a+3b)x-\big((a^2-2ab+b^2)(a+b)\big)\\&=x^3-\big(2(a-b)+(a+2b)\big)x^2+(a-b)\big((a-b)+2(a+2b)\big)x-(a-b)^2(a+2b)\\&=(x^2-2(a-b)x+(a-b)^2)\big(x-(a+2b)\big)\\&=\big(x-(a-b)\big)^2\big(x-(a+2b)\big).\end{aligned}$$

From this factorization the solutions of the cubic equation are

$$x\in\{a-b,a-b,a+2b\}.$$

Editor's comment: David Stone and John Hawkins made an instructive comment in their solution that merits being repeated. They wrote: "We confess – we did not immediately recognize the factorization. We originally used Cardan's Formula to find the solutions. However, there is a line of heuristic reasoning which would lead to the solution. If we consider $a=b$, the equation become $x^3-3ax^2=0$, which has $x=0$ as a double root. Hence, the difference $a-b$ could be significant. Trying $x=a-b$ (via synthetic division) then proves to be productive."

5496. Let a,b,c be real numbers such that $0<a<b<c$. Prove that:

$$\sum_{cyclic}\left(e^{a-b}+e^{b-a}\right)\geq 2a-2c+3+\sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY.-USA

For $x>0$ we apply the known inequality $e^x>x+1$ to $x=a-b,b-c$ and $a-c$ to get

$$e^{a-b}>a-b+1,\qquad e^{b-c}>b-c+1,\qquad e^{a-c}>a-c+1,$$

respectively. Adding these inequalities yields

$$e^{a-b}+e^{b-c}+e^{a-c}>2a-2c+3\quad(1)$$

For $x>y$, we see that

$$e^{x-y}>\left(\frac{x}{y}\right)^{\sqrt{xy}}\Leftrightarrow x-y>\sqrt{xy}\ln\left(\frac{x}{y}\right)\Leftrightarrow\sqrt{xy}<\frac{x-y}{\ln x-\ln y}$$

which is the left-hand member of the *logarithmic mean inequality*. Thus we have, since $0<a<b<c$,

$$e^{b-a}>\left(\tfrac{b}{a}\right)^{\sqrt{ab}},\ e^{c-b}>\left(\tfrac{c}{b}\right)^{\sqrt{bc}},\ e^{c-a}>\left(\tfrac{c}{a}\right)^{\sqrt{ac}}>\left(\tfrac{a}{c}\right)^{\sqrt{ac}}\qquad(2)$$

Adding (1) and (2), we find that

$$\sum_{cyclic}(e^{a-b}+e^{b-a})>2a-2c+3+\sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We will prove the slightly stronger inequality

$$\sum_{cyclic}(e^{a-b}+e^{b-a})\ge a-c+3+\sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

We will use the inequalities

$$e^x\ge 1+x, x \text{ real} \quad (1)$$

$$1\ge\left(\frac{y}{x}\right)^{\sqrt{xy}}, 0\le y\le x \quad (2)$$

$$e^{y-x}\ge\left(\frac{y}{x}\right)^{\sqrt{xy}}, y\ge x \quad (3)$$

(1) and (2) are clear, while (3) is equivalent to each of the following lines:

$$y-x\ge\sqrt{xy}\log\left(\frac{y}{x}\right), \sqrt{\frac{y}{x}}-\sqrt{\frac{x}{y}}\ge\log\left(\frac{y}{x}\right),$$

$$x-\frac{1}{x}-\log x=\int_1^x\left(1+\frac{1}{t^2}-\frac{1}{t}\right)dt\ge 0, x\ge 1 \text{ which holds true.}$$

Thus

$$\sum_{cyclic}(e^{a-b}+e^{b-a})\ge 1+a-b\left(\frac{b}{a}\right)^{\sqrt{ab}}+1+b-c+\left(\frac{b}{c}\right)^{\sqrt{bc}}+1+c-a+a^{a-c}$$

$$=3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+e^{a-c}\ge 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+1+a-c$$

$$\ge 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+\left(\frac{a}{c}\right)^{\sqrt{bc}}+a-c.$$

5502. Prove that if $a,b,c>0$ and $a+b+c=e$ then:

$$e^{ac^e}\cdot e^{ba^e}\cdot e^{cb^e}>e^e\cdot a^{be^2}\cdot b^{ce^2}\cdot c^{ae^2}.$$

Here, $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Proposed by Daniel Sitaru – Romania

Solution by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.

The inequality is equivalent to

$$ac^e+ba^e+cb^e>e+be^2\ln a+ce^2\ln b+ae^2\ln c$$

that is

$$a(c^e-e^2\ln c)+b(a^e-e^2\ln a)+c(b^e-e^2\ln b)>e$$

Let $f(x)=x^e-e^2\ln x$.

$$f''(x)=e(e-1)x^{e-2}+\frac{e^2}{x^2}>0$$

Thus by Jensen's inequality

$$e\sum_{cyc}\frac{a}{e}(c^e-e^2\ln c)\ge e\left[\left(\frac{a+b+c}{e}\right)^e-a^2\ln\frac{a+b+c}{e}\right]=e$$

Solution 2 by Moti Levy, Rehovot, Israel.

The function $\ln x$ is monotone increasing, then by applying log function on both sides of the inequality, we get

$$ac^e + ba^e + cb^e > e + be^2 \ln a + ce^2 \ln b + ae^2 \ln c \quad (1)$$

or

$$\frac{a}{e}c^e + \frac{b}{e}a^e + \frac{c}{e}b^e > 1 + e^2\left(\frac{b}{e}\ln a + \frac{c}{e}\ln b + \frac{a}{e}\ln c\right) \quad (2)$$

The function $\ln x$ is concave, hence

$$\ln\left(\frac{ab+bc+ca}{e}\right) \ge \frac{b}{e}\ln a + \frac{c}{e}\ln b + \frac{a}{e}\ln c \quad (3)$$

Thus we get for the right hand side of inequality (2):

$$1 - e^2 + e^2\ln(ab+bc+ca) \ge 1 + e^2\left(\frac{b}{e}\ln a + \frac{c}{e}\ln b + \frac{a}{e}\ln c\right) \quad (4)$$

The function x^e is convex, hence we get for the left hand side of inequality (2):

$$\frac{a}{e}c^e + \frac{b}{e}a^e + \frac{c}{e}b^e \ge \left(\frac{ab+bc+ca}{e}\right)^e \quad (5)$$

By (4) and (5), to finish the solution, we have to show that

$$\left(\frac{ab+bc+ca}{e}\right)^e > 1 - e^2 + e^2\ln(ab+bc+ca) \quad (6)$$

Let us denote

$$x := (ab+bc+ca)^e \quad (7)$$

Since $ab+bc+ca \le \frac{e^2}{3}$, then

$$0 < x \le \left(\frac{e^2}{3}\right)^e \quad (8)$$

Setting (7) in (6), we need to show that

$$\frac{x}{e^e} > 1 - e^2 + e\ln x, \text{ for } 0 < x \le \left(\frac{e^2}{3}\right)^e,$$

or that

$$f(x) := x - e^{1+e}\ln x + e^e(e^2-1) > 0, \text{ for } 0 < x \le \left(\frac{e^2}{3}\right)^e. \quad (9)$$

One ca easily check that $f'(x) = 1 - \frac{e^{1+e}}{x} < 0$ for $0 < x \le \left(\frac{e^2}{3}\right)^e$. Hence, $f(x)$ is monotone decreasing function for $0 < x \le \left(\frac{e^2}{3}\right)^e$. Moreover, $\lim_{x\to 0} f(x) = +\infty$ and $f\left(\left(\frac{e^2}{3}\right)^e\right) = \left(\frac{e^2}{3}\right)^e - e^{1+e}\left(\ln\left(\frac{e^2}{3}\right)^e\right) + e(e^2-1) \cong 7.4789 > 0$. These and the monotonicity of $f(x)$ imply that $x - e^{1+e}\ln x + e^e(e^2-1) > 0$, for $0 < x \le \left(\frac{e^2}{3}\right)^e$.

Solution 3 by Kee – Wai Lau, Hong Kong, China.

For $0 < x < 1$, let $f(x)$ be the convex function $x^e - e^2\ln x$. By taking logarithms, we see that the inequality of the problem is equivalent to

$$af(c) + bf(a) + cf(b) > e. \quad (1)$$

Let $\gamma_1 = \frac{a}{e}, \gamma_2 = \frac{b}{e}$ and $\gamma_3 = \frac{c}{e}$. By Jensen's inequality, the left side of (1) is greater than or equal to $ef(\gamma_1 c + \gamma_2 a + \gamma_3 b) = ef\left(\frac{ab+bc+ca}{e}\right)$.

Since $f'(x) = \frac{e(x^e - e)}{x} < 0$ and

$ab+bc+ca = \frac{2(a+b+c)^2-(a-b)^2-(b-c)^2-(c-a)^2}{6} \le \frac{e^3}{3}$, so

$$f\left(\frac{ab+bc+ca}{e}\right) \ge f\left(\frac{e}{3}\right) = 1.49\ldots > 1.$$

Thus (1) holds and this completes the solution.

Solution 4 by Michel Bataille, Rouen, France.

Taking logarithms and arranging, we see that the inequality is equivalent to

$$\frac{a}{e}\cdot c^e+\frac{b}{e}\cdot a^e+\frac{c}{e}\cdot b^e>1+e^2\left(\frac{b}{e}\cdot\ln a+\frac{c}{e}\cdot\ln b+\frac{a}{e}\cdot\ln c\right).$$

Since the functions $x\to x^e$ and $x\to\ln x$ are respectively convex and concave on $(0,\infty)$, Jensen's inequality yields

$$\frac{a}{e}\cdot c^e+\frac{b}{e}\cdot a^e+\frac{c}{e}\cdot b^e\ge\left(\frac{ab+bc+ca}{e}\right)^e$$

and

$$\frac{b}{e}\cdot\ln a+\frac{c}{e}\cdot\ln b+\frac{a}{e}\cdot\ln c\le\ln\left(\frac{ab+bc+ca}{e}\right).$$

Therefore, it is sufficient to prove that

$$U^e-e^2\ln U-1>0 \quad (1)$$

where $U=\frac{ab+bc+ca}{e}$.

Since $e^2=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\ge3(ab+bc+ca)$, we have $U\le\frac{e}{3}$, hence $U\in(0,1)$.

Now, let $f(x)=x^e-e^2\ln x-1$. The function f satisfies $f(1)=0$ and $f'(x)=\frac{e(x^e-e)}{x}$.

It follows that f is strictly decreasing on the interval $(0,1]$ and so $f(U)>f(1)$, which is the desired inequality (1).

5506. Find

$$\mathbf{\Omega=det}\left[\begin{pmatrix}\mathbf{1}&\mathbf{5}\\\mathbf{5}&\mathbf{25}\end{pmatrix}^{\mathbf{100}}+\begin{pmatrix}\mathbf{25}&\mathbf{-5}\\\mathbf{-5}&\mathbf{1}\end{pmatrix}^{\mathbf{100}}\right]$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michel Bataille, Rouen, France

Let $A=\begin{pmatrix}1&5\\5&25\end{pmatrix}, B=\begin{pmatrix}25&-5\\-5&1\end{pmatrix}, O_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}, I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

It is readily checked that $AB=BA=O_2$ and $A+B=26I_2$.

Since $AB=BA$, the binomial theorem gives

$$(A+B)^{100}=\sum_{k=0}^{100}\binom{100}{k}A^kB^{100-k} \qquad (1)$$

Now, if $k\in\{1,2,\dots,50\}$, then

$$A^kB^{100-k}=A^kB^kB^{100-2k}=(AB)^kB^{100-2k}=O_2\cdot B^{100-2k}=O_2$$

(note that $A^kB^k=(AB)^k$ since $AB=BA$) and similarly, if $k\in\{51,52,\dots,99\}$, then

$$A^kB^{100-k}=A^{2k-100}(AB)^{100-k}=O_2$$

As a result, (1) gives $(A+B)^{100}=A^{100}+B^{100}$, that is, $26^{100}I_2=A^{100}+B^{100}$. We can conclude:

$$\Omega=\det(26^{100}I_2)=26^{200}.$$

Solution 2 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND-USA

Observe

$$\begin{pmatrix}1 & 5\\5 & 25\end{pmatrix}^{100} = \left(\begin{bmatrix}1\\5\end{bmatrix}\begin{bmatrix}1 & 5\end{bmatrix}\right)^{100} = \begin{bmatrix}1\\5\end{bmatrix}\left(\begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}\right)^{99}\begin{bmatrix}1 & 5\end{bmatrix} = 26^{99}\begin{pmatrix}1 & 5\\5 & 25\end{pmatrix}$$

and

$$\begin{pmatrix}25 & -5\\-5 & 1\end{pmatrix}^{100} = \left(\begin{bmatrix}5\\-1\end{bmatrix}\begin{bmatrix}5 & -1\end{bmatrix}\right)^{100} = \begin{bmatrix}5\\-1\end{bmatrix}\left(\begin{bmatrix}5 & -1\end{bmatrix}\begin{bmatrix}5\\-1\end{bmatrix}\right)^{99}\begin{bmatrix}5 & -1\end{bmatrix} = 2^{99}\begin{pmatrix}25 & -5\\-5 & 1\end{pmatrix}$$

It follows that

$$\Omega = \det\left[26^{99}\begin{pmatrix}1 & 5\\5 & 25\end{pmatrix} + 26^{99}\begin{pmatrix}25 & -5\\-5 & 1\end{pmatrix}\right] = \det\left[\begin{pmatrix}26^{100} & 0\\0 & 26^{100}\end{pmatrix}\right] = 26^{200}.$$

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX-USA

Let $A = \begin{pmatrix}1 & 5\\5 & 25\end{pmatrix}$ and $B = \begin{pmatrix}25 & -5\\-5 & 1\end{pmatrix}$. Matrices A and B are each symmetric, hence orthogonally diagonalizable.

Solving the equation $\det(\lambda I - A) = 0$ yields $\lambda_1 = 0$ and $\lambda_2 = 26$ as the eigenvalues of A.

Solving the equation $(\lambda I - A)\vec{x} = \vec{0}$ successively for $\lambda = 0$ and $\lambda = 26$ yields

$\overrightarrow{x_1} = \begin{pmatrix}-\frac{5}{\sqrt{26}}\\ \frac{1}{\sqrt{26}}\end{pmatrix}$ and $\overrightarrow{x_2} = \begin{pmatrix}\frac{1}{\sqrt{26}}\\ \frac{5}{\sqrt{26}}\end{pmatrix}$ as corresponding unit eigenvectors. So

$$A = \begin{pmatrix}-\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}}\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 26\end{pmatrix}\begin{pmatrix}-\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}}\end{pmatrix}$$

Similarly,

$$B = \begin{pmatrix}\frac{1}{\sqrt{26}} & -\frac{5}{\sqrt{26}}\\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 26\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}}\\ -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\end{pmatrix}.$$

Since for both A and B the matrix of eigenvectors is orthogonal, we have

$$A^{100} = \begin{pmatrix}-\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}}\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 26^{100}\end{pmatrix}\begin{pmatrix}-\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}}\end{pmatrix} = \begin{pmatrix}26^{99} & 5(26^{99})\\5(26^{99}) & 25(26^{99})\end{pmatrix},$$

and

$$B^{100} = \begin{pmatrix}\frac{1}{\sqrt{26}} & -\frac{5}{\sqrt{26}}\\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 26^{100}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}}\\ -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}}\end{pmatrix} = \begin{pmatrix}25(26^{99}) & -5(26^{99})\\-5(26^{99}) & 26^{99}\end{pmatrix}.$$

So $\Omega = \det[A^{100} + B^{100}] = \det\begin{pmatrix}26^{100} & 0\\0 & 26^{100}\end{pmatrix} = 26^{200}$.

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A way to calculate A^n for a 2×2 matrix is to use the Hamilton-Cayley Theorem:

$$A^2 - Tr(A) \cdot A + \det A \cdot I_2 = 0.$$

For example, if we have a 2×2 matrix $A = \begin{pmatrix}1 & a\\a & a^2\end{pmatrix}$ (or $A = \begin{pmatrix}a^2 & -a\\-a & 1\end{pmatrix}$) with

$\det A = 0$ and $Tr\,(A) = a^2 + 1$, then the Hamilton-Cayley theorem becomes:

$$A^2 = Tr(A) = (a^2+1)^2 A.$$
$$A^3 = (a^2+1)A^2 = (a^2+1)^2 A,$$
$$\dots$$
$$A^n = (a^2+1)A^{n-1} = (a^2+1)^{n-1}A.$$

So we have:

$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} = (5^2+1)^{99}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = 26^{99}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix},$$
$$\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100} = (5^2+1)^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = 26^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}^{100} = 26^{99}\left(\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}\right) = 26^{100}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and finally we have:

$$\Omega = \det\left(\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}^{100}\right) = \det\left(26^{100}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 26^{100}.$$

Solution 5 by Paolo Perfetti, Departament of Mathematics, Tor Vergata University, Rome, Italy

Let $c = \sqrt{26}$. We know that

$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = \begin{pmatrix} -\frac{5}{c} & \frac{1}{c} \\ \frac{1}{c} & \frac{5}{c} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} -\frac{5}{c} & \frac{1}{c} \\ \frac{1}{c} & \frac{5}{c} \end{pmatrix} = A\Lambda A^{-1}$$
$$\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & -\frac{5}{c} \\ \frac{5}{c} & \frac{1}{c} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} \frac{1}{c} & \frac{5}{c} \\ -\frac{5}{c} & \frac{1}{c} \end{pmatrix} = B\Lambda B^{-1}$$
$$\Omega = A\ \Lambda^{100}\ A^{-1} + BA\ \Lambda^{100}\ B^{-1}$$
$$A\ \Lambda^{100}\ A^{-1} = \begin{pmatrix} 26^{99} & 5\cdot 26^{99} \\ 5\cdot 26^{99} & 25\cdot 26^{99} \end{pmatrix}$$
$$B\ \Lambda^{100}\ B^{-1} = \begin{pmatrix} 25\cdot 26^{99} & -5\cdot 26^{99} \\ -5\cdot 26^{99} & 26^{99} \end{pmatrix}$$

Thus

$$\Omega = \det\begin{pmatrix} 26^{99}\cdot 26 & 0 \\ 0 & 26^{99}\cdot 26 \end{pmatrix} = 26^{200}.$$

5525. Find real values for x and y such that:

$$\mathbf{4\sin^2(x+y) = 1 + 4\cos^2 x + 4\cos^2 y}.$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Albert Stadler, Herrliberg, Switzerland

Put $u = e^{2ix}, v = e^{2iy}$. Then the given equation reads as

$$0 = \left(e^{2ix+2iy} + e^{-2ix-2iy} - 2\right) + 1 + \left(e^{2ix} + e^{-2ix} + 2\right) + \left(e^{2iy} + e^{-2iy} + 2\right) =$$
$$= u\frac{1}{uv} + u + \frac{1}{u} + v + \frac{1}{v} + 3 = \frac{(uv+u+1)(uv+v+1)}{uv}.$$

So either $v=\frac{1}{u}-1$ or $\frac{1}{v}=-u-1$. If x and y run through the real numbers v and $\frac{1}{v}$ represent circles in the complex plane with radius 1 and center 0, while $-u-1$ and $\frac{-1}{u}-1$ represent circles with radius 1 and center -1. Therefore
$(u,v)\in\left\{\left(e^{\frac{2\pi i}{3}},e^{\frac{2\pi i}{3}}\right),\left(e^{-\frac{2\pi i}{3}},e^{-\frac{2\pi i}{3}}\right)\right\}$ which translates to $x\equiv y\equiv\pm\frac{\pi}{3}(mod\ \pi)$.

Solution 2 by Michael C. Faleski, University Center, MI-USA

Let's rewrite the statement of the problem using several trigonometric indentities. This leads to

$$4(\sin x\cos y+\sin x\cos y)^2=1+4\cos^2x+4\cos^2y$$
$$4(\sin^2x\cos^2y+\sin^2y\cos^2x+2\sin x\sin y\cos x\cos y)=1+4\cos^2x+4\cos^2y$$
$$4\big((1-\cos^2x)\cos^2y+\cos^2x\,(1-\cos^2y)+2\sin x\sin y\cos x\cos y\big)$$
$$=1+4\cos^2x+4\cos^2y$$
$$-8\cos^2x\cos^2y+8\sin x\sin y\cos x\cos y=1$$
$$-8\left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)\left(\frac{1}{2}+\frac{1}{2}\cos(2y)\right)+2\sin 2x\sin 2y=1$$
$$-2(1+\cos 2x+\cos 2y+\cos 2x\cos 2y)+2\sin 2x\sin 2y=1$$
$$-2-2\cos 2x-2\cos 2y-2\cos 2x\cos 2y+2\sin 2x\sin 2y=1$$
$$-2\cos 2x-2\cos 2y-2(\cos 2x\cos 2y-\sin 2x\sin 2y)=3$$
$$\cos 2x+\cos 2y+\cos(2x+2y)=-\frac{3}{2}.$$

And now we use $\cos a=\cos b=2\cos\left(\frac{1}{2}(a+b)\right)\cos\left(\frac{1}{2}(a-b)\right)$
to produce $2\cos(x+y)\cos(x-y)+(2\cos^2(x+y)-1)=-\frac{3}{2}$,
and so we have $2\cos^2(x+y)+2\cos(x-y)\cos(x+y)+\frac{1}{2}=0$, or
$\cos^2(x+y)+\cos(x-y)\cos(x+y)+\frac{1}{4}=0$.
We wil now use the quadratic formula to solve for $\cos(x+y)$.

$$\cos(x+y)=\frac{-\cos(x-y)\pm\sqrt{\cos^2(x-y)-1}}{2}.$$

As we are required to have real solutions, this means that
$\cos^2(x-y)-1\geq 0\rightarrow\cos^2(x-y)\geq 1$. This condition is only true for
$\cos^2(x-y)=1\rightarrow\cos(x-y)=1$.
Letting $y=x-a$, we find $\cos a=1\rightarrow a=2n\pi,\forall\, n\in\mathbb{Z}$.
$\cos(x+y)=-\frac{\cos(x-y)}{2}=-\frac{1}{2}$.
Since $y=\pm 2n\pi$, then for $0\leq x\leq 2\pi, x=y$. Hence, $\cos 2x=-\frac{1}{2}$, which leads to
$2x=\frac{2}{3}\pi,\frac{4}{3}\pi\rightarrow x=\left(\frac{1}{3}\pi,\frac{2}{3}\pi\right)$. So, for $0\leq x,y\leq 2\pi,(x,y)=\left(\frac{1}{3}\pi,\frac{1}{3}\pi\right),\left(\frac{2}{3}\pi,\frac{2}{3}\pi\right)$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain.

$$4\sin^2(x+y)=1+4\cos^2x+4\cos^2y\Leftrightarrow 4(1-\cos^2(x+y))$$
$$=1+2\cos(2x)+2+2\cos(2y)$$
$$\Leftrightarrow 4-4\cos^2(x+y)=5+4\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)$$
$$\Leftrightarrow 0=4-4\cos^2(x+y)+4\cos(x+y)\cos(x-y)+1$$
$$\Leftrightarrow 0=(2\cos(x+y)+\cos(x-y))^2+\sin^2(x-y)$$

$$\Leftrightarrow 2\cos(x+y)+\cos(x-y)=0=\sin(x-y)\Leftrightarrow x-y=k\pi, k\in\mathbb{Z}$$

$$\cos(x+y)+\cos(k\pi)=0\Leftrightarrow x-y=k\pi;\cos(x+y)=\frac{(-1)^{k+1}}{2}, k\in\mathbb{Z}$$

$$\Leftrightarrow x-y=k\pi; x+y=\arccos\frac{(-1)^{k+1}}{2}, \in\mathbb{Z}$$

$$\Leftrightarrow x=\frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2}+k\pi\right), y=\frac{1}{2}\left(\arccos\frac{(-1)^{k+1}}{2}-k\pi\right), k\in\mathbb{Z}$$

Solution 4 by Kee-Wai Lau, Hong Kong, China.

Since $\sin(x+y)=\sin x\cos y+\cos x\sin y$, so the given equation is equivalent to $1-8\sin x\cos x\sin y\cos y+8\cos^2 x\cos^2 y=0$. Clearly $\cos x\neq 0$ and $\cos y\neq 0$. So dividing both sides of the last equation by $\cos^2 x\cos^2 y$, we obtain $\sec^2 x\sec^2 y-8\tan x\tan y+8=0$ or $(1+\tan^2 x)(1+\tan^2 y)-8\tan x\tan y+8=0$, or

$$(\tan x-\tan y)^2+(\tan x\tan y-3)^2=0.$$

Thus $\tan x=\tan y$ and $\tan x\tan y=3$, so that $\tan x=\tan y=\sqrt{3}$ or $\tan x=\tan y=-\sqrt{3}$ It follows that

$$(x,y)=\left(\frac{\pi}{3}+m\pi,\frac{\pi}{3}+n\pi\right),\left(\frac{2\pi}{3}+m\pi,\frac{2\pi}{3}+n\pi\right),$$

where m and n are arbitrary integers.

Solution 5 by Ulrich Abel, Technische Hochshule Mittelhessen, Germany

Using $\cos(2x)=2\cos^2(x)-1=1-2\sin^2(x)$ we see that the equation

$$4\sin^2(x+y)=1+4\cos^2(x)+4\cos^2(y)$$

is equivalent to

$$0=3+2\cos(2x+2y)+2\cos(2x)+2\cos(2y)=:f(x,y).$$

Using $\sin(2a)+\sin(2b)=2\sin(a+b)\cos(a-b)$ we obtain

$$grad\, f(x,y)=-4\cdot(\sin(2x+2y)+\sin(2x),\sin(2x+2y)+\sin(2y))$$
$$=-8\cdot(\sin(2x+y)\cos y,\sin(x+2y)\cos x).$$

Therefore, $grad\, f(x,y)=(0,0)$ happens if

$2x=\pi(mod\ 2\pi)$ and $2y=\pi(mod\ 2\pi)$. The critical points $\left(\frac{2n+1}{2}\pi,\frac{2m+1}{2}\pi\right)$ with integers n,m satisfy

$$f\left(\frac{2n+1}{2}\pi,\frac{2m+1}{2}\pi\right)=3+2\cdot 1+2(-1)^{n+1}+2(-1)^{m+1}>0.$$

$2x=\pi(mod\ 2\pi)$ and $2x+y=0(mod\ \pi)$. The critical points $\left(\frac{2n+1}{2}\pi, m\pi-(2n+1)\pi\right)$ with integers n,m satisfy

$$f\left(\frac{2n+1}{2}\pi,m\pi-(2n+1)\pi\right)=3+2\cdot(-1)+2(-1)^{n+1}+2\cdot 1>0.$$

$2y=\pi(mod\ 2\pi)$ and $x+2y=0(mod\ \pi)$ is symmetrical to the preceding case.

$2x+y=0(mod\ \pi)$ and $x+2y=0(mod\ \pi)$. This implies $3x+3y=(n+m)\pi$ and $x-y=(n-m)\pi$ with integers n,m. We infer that $(x,y)=\frac{\pi}{3}(2n-m,2m-n)$ are the remaining critical points of f.

$$f\left(\frac{2n-m}{3}\pi,\frac{2m-n}{3}\pi\right)$$
$$=3+2\cos\frac{2(n+m)\pi}{3}+2\cos\frac{(4n-2m)\pi}{3}+2\cos\frac{(4m-2n)\pi}{3}$$

$$= 3 + 2\left(2\cos^2\frac{(n+m)\pi}{3} - 1\right) + 4\cos\frac{(n+m)\pi}{3}\cos(n-m)\pi$$

$$= 1 + 4\cos^2\frac{N\pi}{3} + 4(-1)^N\cos\frac{N\pi}{3} = \left(1 + 2(-1)^n\cos\frac{N\pi}{3}\right)^2 \ge 0$$

with $N := n + m$. Consequently, the function value is equal to zero iff N is not a multiple of 3.

In total, we have $f(x,y) \ge 0$ on R^2 and $f(x,y) = 0$ if and only if $(x,y) = (2n-m, 2m-n)\frac{\pi}{3}$, for all integers n, m satisfying $n + m \neq 0(mod\ 3)$. The solutions of the above trigonometric identity are exactly the zeros of f.

ABOUT AN INEQUALITY BY FLORICĂ ANASTASE-IV

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{ab\cdot\sin^2 A} \ge 48r^2$$

Proposed by Florică Anastase-Romania

Solution. Using $\sin A = \frac{a}{2R}$, we get:

$$LHS = \sum_{cyc}\frac{(b^2+c^2-a^2)^2}{ab\cdot\sin A} = \sum_{cyc}\frac{(b^2+c^2-a^2)^2}{ab\cdot\left(\frac{a}{2R}\right)^2} = 4R^2\cdot\sum_{cyc}\frac{\left(\frac{b^2+c^2-a^2}{a}\right)^2}{ab} \overset{CBS}{\ge}$$

$$\overset{CBS}{\ge} 4R^2\frac{\left(\sum\frac{b^2+c^2-a^2}{a}\right)^2}{\sum ab} \overset{(1)}{\ge} 4R^2\cdot\frac{(2s)^2}{\frac{(2s)^2}{3}} = 12R^2 \overset{Euler}{\ge} 48r^2 = RHS$$

where (1) it follows from:

$$\text{(i)} \sum_{cyc}\frac{b^2+c^2-a^2}{a} = \frac{s^4-8s^2-r^2(4R+r)^2}{2Rrs} \overset{(2)}{\ge} 2s, \text{where}$$

$$(2) \Leftrightarrow \frac{s^4-8s^2-r^2(4R+r)^2}{2Rrs} \ge 2s \Leftrightarrow s^4 - 12s^2Rr - r^2(4R+r)^2 \ge 0 \Leftrightarrow$$

$s^2(s^2-12Rr) \ge r^2(4R+r)^2$, which follows from $s^2 \ge 16Rr - 5r^2 \ge \frac{r(4R+r)^2}{R+r}(Gerretsen)$. Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(16Rr - 5r^2 - 12Rr) \ge r^2(4R+r)^2 \Leftrightarrow R \ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

$$\text{(ii)} \sum_{cyc} ab \le \frac{1}{3}\left(\sum_{cyc} a\right)^2 = \frac{1}{3}(2s)^2$$

Equality holds if and only if triangle is equilateral.

Remark. Inequality can be developed.

2) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{ab\cdot\sin^2 A}\ge 12R^2$$

Marin Chirciu

Solution. Using $\sin A=\frac{a}{2R}$, we get:

$$LHS=\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{ab\cdot\sin^2 A}=\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{ab\cdot\left(\frac{a}{2R}\right)^2}=4R^2\sum_{cyc}\frac{\left(\frac{b^2+c^2-a^2}{a}\right)^2}{ab}\overset{CBS}{\ge}$$

$$\overset{CBS}{\ge}4R^2\frac{\left(\sum\frac{b^2+c^2-a^2}{a}\right)^2}{\sum ab}\overset{(1)}{\ge}4R^2\cdot\frac{(2s)^2}{\frac{(2s)^2}{3}}=12R^2,$$

where (1) it follows from:

$$\text{(i)}\ \sum_{cyc}\frac{b^2+c^2-a^2}{a}=\frac{s^4-8s^2-r^2(4R+r)^2}{2Rrs}\overset{(2)}{\ge}2s,\text{where}$$

$$(2)\Leftrightarrow\frac{s^4-8s^2-r^2(4R+r)^2}{2Rrs}\ge 2s\Leftrightarrow s^4-12s^2Rr-r^2(4R+r)^2\ge 0\Leftrightarrow$$

$s^2(s^2-12Rr)\ge r^2(4R+r)^2$, which follows from $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}(Gerretsen)$. Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(16Rr-5r^2-12Rr)\ge r^2(4R+r)^2\Leftrightarrow R\ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

$$\text{(ii)}\sum_{cyc}ab\le\frac{1}{3}\left(\sum_{cyc}a\right)^2=\frac{1}{3}(2s)^2$$

Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$12R^2\le\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc\cdot\sin^2 A}\le\frac{16R^4-208r^4}{r^2}$$

Marin Chirciu

Solution. Lemma. 4) **In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc\cdot\sin^2 A}=\frac{s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3}{s^2r^2}$$

Proof. Using $\sin A=\frac{a}{2R}$, we get:

$$LHS=\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc\cdot\sin^2 A}=\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc\cdot\left(\frac{a}{2R}\right)^2}=4R^2\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{a^2bc}=$$

$$=\frac{4R^2}{abc}\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{a}=\frac{R}{rs}\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{a}=$$

$$=\frac{R}{rs}\cdot\frac{s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3}{Rrs}=$$

$$=\frac{s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3}{s^2r^2},\text{which follows from:}$$

$$\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{a}=\frac{1}{abc}\cdot\sum_{cyc}bc(b^2+c^2-a^2)^2=$$
$$=\frac{4[s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3]}{4Rrs}=$$
$$=\frac{s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3}{Rrs}\text{ and}$$
$$\sum_{cyc}bc(b^2+c^2-a^2)^2=4[s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3]$$

Let's get back to the main problem. For RHS, using Lemma, we have:

$$E=\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc\cdot\sin^2 A}=\frac{s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3}{s^2r^2}=$$
$$=\frac{1}{r^2}\left[s^4-s^2(r^2+12Rr)-r^2(r^2+16Rr)+\frac{r^3(4R+r)^3}{s^2}\right]=$$
$$=\frac{1}{r^2}\left[s^2(s^2-r^2-12Rr)-r^2(r^2+16Rr)+\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\leq}$$
$$\overset{Gerretsen}{\leq}\frac{1}{r^2}\left[(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2-r^2-12Rr)-r^2(r^2+16Rr)\right.$$
$$\left.+\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$
$$=\frac{1}{r^2}[(4R^2+4Rr+3r^2)(4R^2-8Rr+2r^2)-r^2(r^2+16Rr)+r^2(4R+r)(R+r)]=$$
$$=\frac{1}{r^2}[(4R62+4Rr+3r^2)(4R^2-8Rr+2r^2)-r^2(r^2+16Rr)+r^2(4R+r)(R+r)]=$$
$$=\frac{1}{r^2}(16R^4-16R^3r-8R^2r^2-27Rr^3+6r^4)\overset{Euler}{\leq}\frac{16R^4-208r^4}{r^2}=RHS$$

For LHS, using Lemma, we have:

$$E=\sum_{cyc}\frac{(b^2+c^2-a^2)^2}{bc\cdot\sin^2 A}=$$
$$=\frac{s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3}{s^2r^2}\geq 12R^2\Leftrightarrow$$
$$s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr)+r^3(4R+r)^3\geq 12R^2r^2s^2\Leftrightarrow$$
$$s^6-s^4(r^2+12Rr)-s^2r^2(r^2+16Rr+12R^2)+r^3(4R+r)^3\geq 0\Leftrightarrow$$
$$s^2[s^2(s^2-r^2-12Rr)-r^2(r^2+16Rr+12R^2)]+r^3(4R+r)^2\geq 0$$

Distinguish the cases:

Case 1). If $[s^2(s^2-r^2-12Rr)-r^2(r^2+16Rr+12R^2)]\geq 0$, inequality is obviously true.

Case 2). If $[s^2(s^2-r^2-12Rr)-r^2(r^2+16Rr+12R^2)]<0$, we get:

$$r^3(4R+r)^3\geq s^2[r^2(r^2+16Rr+12R^2)-s^2(s^2-r^2-12Rr)],$$

which it follows from Gerretsen Inequality:

$$\frac{r(4R+r)^2}{R+r}\leq 16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

Remains to prove that:

$$r^3(4R+r)^3 \geq \frac{R(4R+r)^2}{2(2R-r)}[r^2(r^2+16Rr+12R^2) - (16Rr-5r^2)(16Rr-5r^2-r^2-12Rr)]$$

$$r(4R+r) \geq \frac{R}{2(2R-r)}[(r^2+16Rr+12R^2)-(16R-5r)(16R-5r-r-12R)]$$

$$2r(2R-r)(4R+r) \geq R[(r^2+16Rr+12R^2)-(16R-5r)(4R-6r)]$$

$$2r(8R^2-2Rr-r^2) \geq R(-52R^2+132Rr-29r^2)$$

$$2r(8R^2-2Rr-r^2) \geq R(-52R^2+132Rr-29r^2)$$

$52R^3-116R^2r+25Rr^2-2r^3 \geq 0 \Leftrightarrow (R-2r)(52R^2-12Rr+r^2) \geq 0$, true from $R \geq 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

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A GENERALIZATION FOR CONSTANTIN IONESCU-ȚIU INEQUALITY

By D.M.Bătineţu-Giurgiu-Romania

Let $\Delta A_1B_1C_1, \Delta A_2B_2C_2$ be two triangles with circumradies R, R_1 respectively, σ an permutation of the set $\{a_1, b_1, c_1\}$ and ζ an permutation of the set $\{a_2, b_2, c_2\}$.
Theorem. If $m, x, y, z \in [1, \infty), 3m = x+y+z$ and $u, v \geq 0, u+v > 0$ then:

$$\frac{x^x+y^x+z^x}{(\sigma(a_1))^u \cdot (\zeta(a_2))^v} + \frac{x^y+y^y+z^y}{(\sigma(a_1))^u \cdot (\zeta(a_2))^v} + \frac{x^z+y^z+z^z}{(\sigma(a_1))^u \cdot (\zeta(a_2))^v} \geq \frac{(\sqrt{3})^{4-u-v} \cdot m^m}{R_1^u \cdot R_2^v}; (*)$$

Proof. Let s, s_1 be the semiperimeters of the triangles $A_1B_1C_1, A_2B_2C_2$ and F_1, F_2 the areas of the triangles $A_1B_1C_1, A_2B_2C_2$ respectively.

$$\sum_{cyc} \frac{x^x+y^x+z^x}{(\sigma(a_1))^u \cdot (\zeta(a_2))^v} \overset{Radon}{\geq} \sum_{cyc} \frac{(x+y+z)^x}{3^{x-1} \cdot (\sigma(a_1))^u \cdot (\zeta(a_2))^v} \geq$$

$$\geq 3 \cdot \sqrt[3]{\prod_{cyc} \frac{(x+y+z)^x}{3^{x-1} \cdot (\sigma(a_1))^u \cdot (\zeta(a_2))^v}} = 3 \cdot \sqrt[3]{\frac{(x+y+z)^{x+y+z}}{3^{x+y+z-3} \cdot (a_1b_1c_1)^u \cdot (a_2b_2c_2)^v}} =$$

$$= 3 \cdot \sqrt[3]{\frac{(3m)^{3m}}{3^{3m-3} \cdot (a_1b_1c_1)^u \cdot (a_2b_2c_2)^v}} = 3 \cdot \frac{(3m)^m}{3^{m-1}} \cdot \frac{1}{\sqrt[3]{(a_1b_1c_1)^u \cdot (a_2b_2c_2)^v}} =$$

$$= \frac{9 \cdot m^m}{\sqrt[3]{(a_1b_1c_1)^u} \cdot \sqrt[3]{(a_2b_2c_2)^v}} = \frac{9 \cdot m^m}{\sqrt[3]{(4R_1F_1)^u} \cdot \sqrt[3]{(4R_2F_2)^v}} =$$

$$= \frac{9 \cdot m^m}{\sqrt[3]{(4R_1s_1r_1)^u} \cdot \sqrt[3]{(4R_2s_2r_2)^v}} = \frac{9 \cdot m^m}{\sqrt[3]{4^{u+v} \cdot R_1^u R_2^v \cdot s_1^2 s_2^2 \cdot r_1^u r_2^v}} \overset{Euler}{\geq}$$

$$\geq \frac{9 \cdot m^m}{\sqrt[3]{4^{u+v} \cdot R_1^2 R_2^v \cdot s_1^u s_2^v \cdot \left(\frac{R_1}{2}\right)^u \left(\frac{R_2}{2}\right)^v}} = \frac{9 \cdot m^m}{\sqrt[3]{2^{u+v} \cdot R_1^u R_2^v \cdot s_1^u s_2^v}} \overset{Mitrinovic}{\geq}$$

$$\geq \frac{9\cdot m^m}{\sqrt[3]{2^{u+v}\cdot R_1^{2u}R_2^{2v}\left(\frac{3\sqrt{3}R_1}{2}\right)^u\left(\frac{3\sqrt{3}R_2}{2}\right)^v}}=\frac{9\cdot m^m}{\sqrt[3]{\left(3\sqrt{3}\right)^{u+v}\cdot R_1^{3u}R_2^{3v}}}=$$

$$=\frac{9\cdot m^m}{\left(\sqrt{3}\right)^{u+v}\cdot R_1^uR_2^v}=\frac{\left(\sqrt{3}\right)^{4-u-v}\cdot m^m}{R_1^u\cdot R_2^v}$$

If $x=y=z=m$ then inequality $(*)$ becomes as:

$$\sum_{cyc}\frac{\mathbf{1}}{\left(\boldsymbol{\sigma}(\boldsymbol{a_1})\right)^{\boldsymbol{u}}\cdot\left(\boldsymbol{\zeta}(\boldsymbol{a_2})\right)^{\boldsymbol{v}}}\geq\frac{\left(\sqrt{\mathbf{3}}\right)^{\mathbf{2}-\boldsymbol{u}-\boldsymbol{v}}}{\boldsymbol{R_1^u}\cdot\boldsymbol{R_2^v}};(\mathbf{1})$$

If $u=v$ then we get:

$$\sum_{cyc}\frac{\mathbf{1}}{\left(\boldsymbol{\sigma}(\boldsymbol{a_1})\cdot\boldsymbol{\zeta}(\boldsymbol{a_2})\right)^{\boldsymbol{u}}}\geq\frac{\mathbf{3}^{\mathbf{1}-\boldsymbol{u}}}{(\boldsymbol{R_1R_2})^{\boldsymbol{u}}};(\mathbf{2})$$

If in (2) we take $A_1B_1C_1\equiv A_2B_2C_2\equiv ABC$, then we get:

$$\sum_{cyc}\frac{\mathbf{1}}{\left(\boldsymbol{\sigma}(\boldsymbol{a})\cdot\boldsymbol{\zeta}(\boldsymbol{a})\right)^{\boldsymbol{u}}}\geq\frac{\mathbf{3}^{\mathbf{1}-\boldsymbol{u}}}{\boldsymbol{R}^{\mathbf{2}\boldsymbol{u}}};(\mathbf{3})$$

If in (3), σ and ζ are identic permutations of the set $\{a,b,c\}$, then

$$\frac{\mathbf{1}}{\boldsymbol{a}^{\mathbf{2}\boldsymbol{n}}}+\frac{\mathbf{1}}{\boldsymbol{b}^{\mathbf{2}\boldsymbol{n}}}+\frac{\mathbf{1}}{\boldsymbol{c}^{\mathbf{2}\boldsymbol{n}}}\geq\frac{\mathbf{3}^{\mathbf{1}-\boldsymbol{u}}}{\boldsymbol{R}^{\mathbf{2}\boldsymbol{n}}};(\mathbf{4})$$

If in (4), we take $u=\frac{1}{2}$ then:

$$\frac{\mathbf{1}}{\boldsymbol{a}}+\frac{\mathbf{1}}{\boldsymbol{b}}+\frac{\mathbf{1}}{\boldsymbol{c}}\geq\frac{\sqrt{\mathbf{3}}}{\boldsymbol{R}}\ \ (\boldsymbol{C.I.T})$$

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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-XVII

By Marin Chirciu-Romania

$\mathbf{1})$ ***In $\boldsymbol{\Delta ABC}$ the following relationship holds*:**

$$\sum\frac{\boldsymbol{w_b}+\boldsymbol{w_c}}{\boldsymbol{h_a^2}}\geq\frac{\mathbf{2}}{\boldsymbol{r}}$$

By Marian Ursărescu – Romania

Solution: Using $w_a\geq h_a$ we have $LHS=\sum\frac{w_b+w_c}{h_a^2}\geq\sum\frac{h_b+h_c}{h_a^2}\overset{(1)}{\geq}RHS$, where (1) it follows from **Lemma 1:**

2) In ΔABC the following relationship holds:

$$\sum\frac{\boldsymbol{h_b}+\boldsymbol{h_c}}{\boldsymbol{h_a^2}}\geq\frac{\mathbf{2}}{\boldsymbol{r}}$$

Proof: We have $\sum\frac{h_b+h_c}{h_a^2}\overset{(2)}{=}\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}\overset{(3)}{\geq}\frac{2}{r}$, where (2) it follows from

Lemma 2:

3) In ΔABC the following relationship holds:

$$\sum \frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4Rr+r)^2}{4Rr^2s^2}$$

Proof: We have $\sum \frac{h_b+h_c}{h_a^2}=\sum \frac{\frac{2S}{b}+\frac{2S}{c}}{\left(\frac{2S}{a}\right)^2}=\frac{1}{2S}\sum \frac{a^2(b+c)}{bc}=\frac{\sum a^3(b+c)}{2sr\cdot abc}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}$, which follows from $\sum a^3(b+c)=2[s^4-4s^2Rr-r^2(4R+r)^2]$

$$\sum \frac{a^2(b+c)}{bc}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{2Rrs}$$

Let's get back to the main problem. Using the above Lemmas it suffices to prove that inequality 3) holds:

$$\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}\ge \frac{2}{r}\Leftrightarrow s^4-12s^2Rr-r^2(4R+r)^2\ge 0\Leftrightarrow$$

$\Leftrightarrow s^2(s^2-12Rr)\ge r^2(4R+r)^2$, which follows from Gerretsen's inequality:

$$s^2\ge 16Rr-5r^2\ge \frac{r(4R+r)^2}{R+r}$$

It remains to prove that: $\frac{r(4R+r)^2}{R+r}(16Rr-5r^2-12Rr)\ge r^2(4R+r)^2\Leftrightarrow R\ge 2r$, (Euler's inequality). Equality holds if and only if the triangle is equilateral.

Remark: The inequality can be strengthened.

4) In ΔABC the following inequality holds:

$$\sum \frac{w_b+w_c}{h_a^2}\ge \frac{1}{4r}\left(11-\frac{6r}{R}\right)\ge \frac{2}{r}$$

Marin Chirciu

Solution: Using $w_a\ge h_a$ we have $LHS=\sum \frac{w_b+w_c}{h_a^2}\ge \sum \frac{h_b+h_c}{h_a^2}\ge \frac{2}{r}=RHS.$

We have $\sum \frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}=\frac{s^2(s^2-4Rr)-r^2(4R+r)^2}{4Rr^2s^2}=$

$$=\frac{1}{4Rr^2}\left[(s^2-4Rr)-\frac{r^2(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\ge}\frac{1}{4Rr^2}\left[(16Rr-5r^2-4Rr)-\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]$$

$$=\frac{1}{4Rr^2}[(12Rr-5r^2)-r(R+r)]=\frac{11Rr-6r^2}{4Rr^2}=\frac{11R-6r}{4Rr}=\frac{1}{4r}\left(11-\frac{6r}{R}\right)\overset{Euler}{\ge}\frac{2}{r}$$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

5) In ΔABC the following relationship holds:

$$\sum \frac{m_b+m_c}{h_a^2}\ge \frac{2}{r}$$

Marin Chirciu

Solution: We use $m_a\ge w_a\ge h_a$ and see above. Equality holds if and only if the triangle is equilateral.

6) In ΔABC the following inequality holds:

$$\frac{2}{r}\le \sum \frac{r_b+r_c}{h_a^2}\le \frac{R}{r^2}$$

Marin Chirciu

Solution: We prove: **Lemma:**

7) In ΔABC the following relationship holds:

$$\sum \frac{r_b+r_c}{h_a^2}=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}$$

Proof: We have:

$$\sum\frac{r_b+r_c}{h_a^2}=\sum\frac{\frac{S}{s-b}+\frac{S}{s-c}}{\left(\frac{2S}{a}\right)^2}=\frac{1}{4S}\sum\frac{a^3}{(s-b)(s-c)}=$$

$$=\frac{1}{4sr}\cdot\frac{2[s^2(2R+3r)-r(4R+r)^2]}{sr}=$$

$=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}$, which follows from:

$$\sum\frac{a^3}{(s-b)(s-c)}=\frac{2[s^2(2R+3r)-r(4R+r)^2]}{sr}$$

$$\sum\frac{a^3}{(s-b)(s-c)}=\frac{\sum a^3(s-a)}{\prod(s-a)}=\frac{2r[s^2(2R+3r)-r(4R+r)^2]}{sr^2}=$$

$$=\frac{2[s^2(2R+3r)-r(4R+r)^2]}{sr}$$

$$\sum a^3(s-a)=2r[s^2(2R+3r)-r(4R+r)^2]$$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{h_a^2}=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}=\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=\frac{1}{2r^2}\left[(2R+3r)-\frac{2r(2R-r)}{R}\right]=$$

$$=\frac{1}{2r^2}\cdot\frac{R(2R+3r)-2r(2R-r)}{R}=\frac{2R^2-Rr+2r^2}{2Rr^2}\overset{Euler}{\leq}\frac{2R^2}{2Rr^2}=\frac{R}{r^2}$$

Equality holds if and only if the triangle is equilateral.LHS inequality.
Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{h_a^2}=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}=\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\geq\frac{1}{2r^2}\left[(2R+3r)-\frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{1}{2r^2}[(2R+3r)-(R+r)]=\frac{R+2r}{2r^2}\overset{Euler}{\geq}\frac{2}{r}$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

8) In ΔABC the following relationship holds:

$$\frac{2}{r}\leq\sum\frac{h_b+h_c}{h_a^2}\leq\frac{R}{r^2}$$

Marin Chirciu

Proof: We prove: **Lemma:**

9) In ΔABC the following relationship holds:

$$\sum\frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}$$

Proof: We have $\sum\frac{h_b+h_c}{h_a^2}=\sum\frac{\frac{2S}{b}+\frac{2S}{c}}{\left(\frac{2S}{a}\right)^2}=\frac{1}{2S}\sum\frac{a^2(b+c)}{bc}=\frac{\sum a^3(b+c)}{2sr\cdot abc}=\frac{s^4-4s^2Rr-r^2(4Rr+r)^2}{4Rr^2s^2}$

which follows from $\sum a^3\,(b+c)=2[s^4-4s^2Rr-r^2(4R+r)^2]$

$$\sum\frac{a^2(b+c)}{bc}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{2Rrs}$$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$\sum\frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}=\frac{s^2(s^2-4Rr)-r^2(4R+r)^2}{4Rr^2s^2}=$$

$$=\frac{1}{4Rr^2}\left[(s^2-4Rr)-\frac{r^2(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{1}{4Rr^2}\left[(4R^2+4Rr+3r^2-4Rr)-\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{1}{4Rr^2}\left[(4R^2+3r^2)-\frac{2r^2(2R-r)}{R}\right]=\frac{1}{4Rr^2}\cdot\frac{R(4R^2+3r^2)-2r^2(2R-r)}{R}=$$

$$=\frac{1}{4Rr^2}\cdot\frac{R(4R^2+3r^2)-2r^2(2R-r)}{R}=\frac{4R^3-Rr^2+2r^3}{4R^2r^2}\overset{Euler}{\leq}\frac{4R^3}{4R^2r^2}=\frac{R}{r^2}$$

Equality holds if and only if the triangle is equilateral.LHS inequality.Using the Lemma we obtain:

$$\sum\frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}=\frac{s^2(s^2-4Rr)-r^2(4R+r)^2}{4Rr^2s^2}=$$

$$=\frac{1}{4Rr^2}\left[(s^2-4Rr)-\frac{r^2(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\geq}\frac{1}{4Rr^2}\left[(16Rr-5r^2-4Rr)-\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]$$

$$=\frac{1}{4Rr^2}[(12Rr-5r^2)-r(R+r)]=\frac{11Rr-6r^2}{4Rr^2}=\frac{11R-6r}{4Rr}\overset{Euler}{\geq}\frac{2}{r}$$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

10) In ΔABC the following inequality holds:

$$\frac{2}{r}\left(\frac{R}{r}-1\right)\leq\sum\frac{r_b+r_c}{r_a^2}\leq\frac{4}{r}\left(\frac{R}{r}-1\right)^2$$

Marin Chirciu

Solution: We prove: **Lemma:**

11) In ΔABC the following relationship holds:

$$\sum\frac{r_b+r_c}{r_a^2}=\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}$$

Proof: We have:

$$\sum\frac{r_b+r_c}{r_a^2}=\sum\frac{\frac{S}{s-b}+\frac{S}{s-c}}{\left(\frac{S}{s-a}\right)^2}=\frac{1}{S}\sum\frac{a(s-a)^2}{(s-b)(s-c)}=\frac{\sum a(s-a)^3}{sr\cdot\prod(s-a)}=$$

$$=\frac{4Rrs^2-2r^2(4R+r)^2}{sr\cdot sr^2}=$$

$$=\frac{4Rs^2-2r(4R+r)^2}{s\cdot sr^2}=\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}$$

which follows from $\sum a(s-a)^3=4Rrs^2-2r^2(4R+r)^2$

Let's get back to the main problem.RHS inequality.Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{r_a^2}=\frac{2[2Rrs^2-r(4R+r)^2]}{s^2r^2}=\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{2}{r^2}\left[2R-\frac{2r(2R-r)}{R}\right]=\frac{4}{r^2}\cdot\frac{R^2-r(2R-r)}{R}=4\frac{R^2-2Rr+r^2}{Rr^2}=$$

$$=4\frac{(R-r)^2}{Rr^2}=\frac{4}{R}\left(\frac{R}{r}-1\right)^2$$

Equality holds if and only if the triangle is equilateral.LHS inequality.Using the Lemma we obtain:

$$\sum\frac{r_b+r_c}{r_a^2}=\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}=\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\geq\frac{2}{r^2}\left[2R-\frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{2}{r^2}[2R-(R+r)]=\frac{2(R-r)}{r^2}=\frac{2}{r}\left(\frac{R}{r}-1\right)$$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way:

12) In ΔABC the following relationship holds:

$$\frac{2}{r}\leq\sum\frac{h_b+h_c}{r_a^2}\leq\frac{R}{r^2}$$

Marin Chirciu

Solution: We prove: **Lemma:**

13) In ΔABC the following relationship holds:

$$\sum\frac{h_b+h_c}{r_a^2}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}$$

Proof: We have $\sum\frac{h_b+h_c}{r_a^2}=\sum\frac{\frac{2S}{b}+\frac{2S}{c}}{\left(\frac{S}{s-a}\right)^2}=\frac{2}{S}\sum\frac{(b+c)(s-a)^2}{bc}=\frac{2}{sr}\cdot\frac{\sum a(b+c)(s-a)^2}{abc}=$

$$=\frac{2}{sr}\cdot\frac{2r[s^2(4R-r)-r(4R+r)^2]}{4Rrs}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}$$

which follows from:

$$\sum a(b+c)(s-a)^2=2r[s^2(4R-r)-r(4R+r)^2]$$

Let's get back to the main problem.RHS inequality. Using the Lemma we obtain:

$$\sum\frac{h_b+h_c}{r_a^2}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}=\frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=\frac{1}{Rr}\left[(4R-r)-\frac{2r(2R-r)}{R}\right]=$$

$$=\frac{1}{Rr}\cdot\frac{R(4R-r)-2r(2R-r)}{R}=\frac{1}{Rr}\cdot\frac{4R^2-5Rr+2r^2}{R}=$$

$$=\frac{4R^2-5Rr+2r^2}{R^2r}\overset{(1)}{\leq}\frac{R}{r^2},\text{ where (1)}\Leftrightarrow\frac{4R^2-5Rr+2r^2}{R^2r}\leq\frac{R}{r^2}\Leftrightarrow$$

$\Leftrightarrow R^3-4R^2r+5Rr^2-2r^3\geq 0\Leftrightarrow(R-2r)(R-r)^2\geq 0$, obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.LHS inequality .Using Lemma we obtain:

$$\sum\frac{h_b+h_c}{r_a^2}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}=\frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\geq\frac{1}{Rr}\left[(4R-r)-\frac{r(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=\frac{1}{Rr}[(4R-r)-(R+r)]=\frac{1}{Rr}\cdot(3R-2r)=$$

$=\frac{3R-2r}{Rr}\overset{Euler}{\geq}\frac{2}{r}=\frac{4R^2-5Rr+2r^2}{R^2r}\overset{(1)}{\leq}\frac{R}{r^2}$, where (1) $\Leftrightarrow\frac{4R^2-5Rr+2r^2}{R^2r}\leq\frac{R}{r^2}\Leftrightarrow$

$\Leftrightarrow R^3-4R^2r+5Rr^2-2r^3\geq 0\Leftrightarrow(R-2r)(R-r)^2\geq 0$, obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

$$=\frac{2}{r^2}[2R-(R+r)]=\frac{2(R-r)}{r^2}=\frac{2}{r}\left(\frac{R}{r}-1\right)$$

Equality holds if and only if the triangle is equilateral.**Remark:** Between the sums $\sum\frac{h_b+h_c}{h_a^2}$ and $\sum\frac{r_b+r_c}{r_a^2}$ the following relationship exists:

14) In ΔABC:

$$\sum\frac{h_b+h_c}{h_a^2}\leq\sum\frac{r_b+r_c}{r_a^2}$$

Marin Chirciu

Solution: Using the above Lemmas we have the sums:

$\sum\frac{h_b+h_c}{h_a^2}=\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}$ and $\sum\frac{r_b+r_c}{r_a^2}=\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}$

The inequality can be written:

$$\frac{s^4-4s^2Rr-r^2(4R+r)^2}{4Rr^2s^2}\leq\frac{2[2Rs^2-r(4R+r)^2]}{s^2r^2}\Leftrightarrow$$

$\Leftrightarrow s^2(16R^2+4Rr-s^2)\geq r(8R-r)(4R+r)^2$, which follows from Gerretsen's inquality:

$$16Rr-5r^2\leq s^2\leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$$(16Rr-5r^2)(16R^2+4Rr-4R^2-4Rr-3r^2)\geq r(8R-r)(4R+r)^2\Leftrightarrow$$

$\Leftrightarrow(16R-5r)(12R^2-3r^2)\geq(8R-r)(4R+r)^2\Leftrightarrow 16R^3-27R^2r-12Rr^2+4r^3\geq 0$

$\Leftrightarrow(R-2r)(16R^2+5Rr-2r^2)\geq 0$, obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.**Remark:** Between the sums $\sum\frac{h_b+h_c}{r_a^2}$ and $\sum\frac{r_b+r_c}{h_a^2}$ the following relationship exists:

15) In ΔABC:

$$\sum\frac{h_b+h_c}{r_a^2}\leq\sum\frac{r_b+r_c}{r_a^2}$$

Marin Chirciu

Solution: Using the above Lemmas we have the sums:

$\sum\frac{h_b+h_c}{r_a^2}=\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}$ and $\sum\frac{r_b+r_c}{h_a^2}=\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}$

The inequality can be written:

$$\frac{s^2(4R-r)-r(4R+r)^2}{Rrs^2}\leq\frac{s^2(2R+3r)-r(4R+r)^2}{2s^2r^2}\Leftrightarrow$$

$$\Leftrightarrow s^2(2R^2-5Rr+2r^2)\geq(Rr-2r^2)(4R+r)^2\Leftrightarrow$$

$$\Leftrightarrow s^2(R-2r)(2R-r)\geq r(R-2r)(4R+r)^2\Leftrightarrow$$

$$\Leftrightarrow (R-2r)[s^2(2R-r)-r(4R+r)^2] \geq 0$$

Because $(R-2r) \geq 0$, from Euler's inequality $R \geq 2r$ and
$[s^2(2R-r)-r(4R+r)^2] \geq 0 \Leftrightarrow s^2(2R-r) \geq r(4R+r)^2$, which follows from
Gerretsen's inequality: $s^2 \geq 16Rr-5r^2 \geq \frac{r(4R+r)^2}{R+r}$. It remains to prove that:
$\frac{r(4R+r)^2}{R+r}(2R-r) \geq r(4R+r)^2 \Leftrightarrow 2R-r \geq R+r \Leftrightarrow R \geq 2r$, (Euler's inequality).
Equality holds if and only if the triangle is equilateral.

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ABOUT POWER CONVERGENCE OF A SEQUENCE

By Florică Anastase-Romania

Abstract: *In this paper was presented the power convergence of a sequence of real numbers and few criterions to find the limit of sequences.*

Introduction.

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

Then

$$\lim_{n\to\infty} a_n = \log 2\,; \quad (1)$$

Proposition 1. If $\alpha > 1$ and $(a_n)_{n\geq 1}, a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, then

$$\frac{\mathbf{1}}{\mathbf{4n+\alpha}} < \mathbf{\log 2 - a_n + \frac{1}{4n+1}}\,; \quad (\mathbf{2})$$

$$\textbf{and } \mathbf{\Omega_1 = \lim_{n\to\infty} n(\log 2 - a_n) = \frac{1}{4}}\,; \quad (\mathbf{3})$$

Proof. Let us denote $b_n(\alpha) \overset{def}{=} a_n + \frac{1}{4n+\alpha}; \alpha \in \mathbb{R}$, hence, from (2) it follows that:

$$\begin{cases} b_n(\alpha) < \log 2\,; \alpha > 1; & (4) \\ b_n(1) > \log 2\,; & (5) \end{cases} \text{ and } \lim_{n\to\infty} b_n(\alpha) = \log 2\,; \forall \alpha \in \mathbb{R}.$$

It is enough to prove that $\left(b_n(\alpha)\right)_{n\geq 1}$ is increasing sequence and $\left(b_n(1)\right)_{n\geq 1}$ is decreasing sequence. We have:

$$b_n(\alpha) - b_{n+1}(\alpha) = (a_n - a_{n+1}) + \left(\frac{1}{4n+\alpha} - \frac{1}{4n+4+\alpha}\right) =$$

$$= \frac{8(1-\alpha)n - (\alpha^2+4\alpha-8)}{(2n+1)(2n+2)(4n+\alpha)(4n+4+\alpha)} < 0; \forall \alpha > 1$$

So, $b_{n+1}(\alpha) > b_n(\alpha),\ \forall \alpha > 1, n \geq n_0(\alpha) = \left[\frac{\alpha^2+4\alpha-8}{8(1-\alpha)}\right]$. Thus, $\left(b_n(\alpha)\right)_{n\geq 1}$ –increasing.

For $\alpha = 1$, we get: $b_n(1) - b_{n+1}(1) = \frac{3}{(2n+1)(2n+2)(4n+1)(4n+5)} > 0$.

Thus, $\left(b_n(1)\right)_{n\geq 1}$ –is decreasing.

In that conditions, we get the inequality (2).

For $\alpha = 2$, we obtain the well-known double inequality:

$$\frac{1}{4n+2} < \log 2 - a_n < \frac{1}{4n+1}, \forall n \in \mathbb{N}^*.$$

Now, for $\beta > 1$, let's consider $\left(y_n(\beta)\right)_{n\geq 1}$ be a sequence of real numbers such that

$$y_n(\beta) = 1 + \frac{1}{2^\beta} + \cdots + \frac{1}{n^\beta} \text{ and } y(\beta) = \lim_{n\to\infty} y_n(\beta)$$

Let us denote: $c_n(\beta) = y_n(\beta) + \frac{1}{\beta-1}\cdot\frac{1}{(n+1)^{\beta-1}}, d_n(\beta) = y_n(\beta) + \frac{1}{\beta-1}\cdot\frac{1}{n^\beta}, n \in \mathbb{N}^*$

It is easy to prove that $c_n(\beta) < y(\beta) < d_n(\beta)$, because $\left(c_n(\beta)\right)_{n\geq 1}$ −increasing, $(d_n(\beta)_{n\geq 1}$ −decreasing and $\lim_{n\to\infty} c_n(\beta) = \lim_{n\to\infty} d_n(\beta) = y(\beta)$.

So, we have:

$$\frac{1}{\beta-1}\cdot\left(\frac{n}{n+1}\right)^{\beta-1} < n^{\beta-1}[y(\beta) - y_n(\beta)] < \frac{1}{\beta-1}; \forall n \in \mathbb{N}^*, \text{which means}$$

$$\lim_{n\to\infty} n^{\beta-1}[y(\beta) - y_n(\beta)] = \frac{1}{\beta-1} \text{ and } \lim_{n\to\infty}(2n)^{\beta-1}[y(\beta) - y_{2n}(\beta)] = \frac{1}{\beta-1}$$

Therefore,

$$\lim_{n\to\infty} n^{\beta-1}\sum_{k=1}^{n}\frac{1}{(n+k)^\beta} = \lim_{n\to\infty} n^{\beta-1}[y_{2n}(\beta) - y_n(\beta)] =$$

$$= \lim_{n\to\infty}\left[-\frac{1}{2^{\beta-1}}(2n)^{\beta-1}\left(y(\beta) - y_{2n}(\beta)\right) + n^{\beta-1}\left(y(\beta) - y_n(\beta)\right)\right] =$$

$$= -\frac{1}{2^{\beta-1}}\cdot\frac{1}{\beta-1} + \frac{1}{\beta-1} = \frac{1}{\beta-1}\left(1 - \frac{1}{2^{\beta-1}}\right); \quad (3)$$

Case $\beta = 1$ it is well-known.

Remark. Let's extend this result.

Proposition 2. If $f: \mathbb{R}_+ \to \mathbb{R}$ is continuous function such that $\lim_{\substack{x\to 0\\ x>0}} f(x) = l \in \mathbb{R}$. Then

$$\lim_{n\to\infty} n^{\beta-1}\sum_{k=1}^{n} f\left(\frac{1}{n+k}\right) = \begin{cases}\frac{l}{\beta-1}\left(1 - \frac{1}{2^{\beta-1}}\right), & \text{if } \beta > 1\\ l\cdot\log 2, & \text{if } \beta = 1\end{cases}$$

Proof. Let $\varepsilon > 0$, then $\exists\delta > 0$ such that for $x \in (0, \delta)$, we have:

$$\left|\frac{f(x)}{x^\beta} - l\right| < \varepsilon \Leftrightarrow -\varepsilon\cdot x^\beta < f(x) - l\cdot x^\beta < \varepsilon\cdot x^\beta$$

We can find $n_0 \in \mathbb{N}$ for that $\frac{1}{n_0} < \delta$ and then $\frac{1}{n} < \delta, \forall n \in \mathbb{N}^*, n \geq 0$. So,

$$-\varepsilon\frac{1}{n^\beta} < -\varepsilon\frac{1}{(n+k)^\beta} < f\left(\frac{1}{n+k}\right) - l\frac{1}{(n+k)^\beta} < \varepsilon\frac{1}{(n+k)^\beta} < \varepsilon\frac{1}{n^\beta}; \forall n, k \in \mathbb{N}^*, n \geq n_0, k \geq 1$$

By adding for $k \in \{1,2,\ldots,n\}$, it follows that:

$$-\varepsilon\frac{1}{n^\beta}n < \sum_{k=1}^{n} f\left(\frac{1}{n+k}\right) - l\cdot\sum_{k=1}^{n}\frac{1}{(n+k)^\beta} < \varepsilon\frac{1}{n^\beta}n$$

And hence,

$$-\varepsilon < n^{\beta-1}\left(\sum_{k=1}^{n} f\left(\frac{1}{n+k}\right) - l\cdot\sum_{k=1}^{n}\frac{1}{(n+k)^\beta}\right) < \varepsilon, \forall n \geq n_0$$

Therefore,

$$\lim_{n\to\infty} n^{\beta-1}\left(\sum_{k=1}^{n} f\left(\frac{1}{n+k}\right) - l\cdot\sum_{k=1}^{n}\frac{1}{(n+k)^{\beta}}\right) = 0; \quad (4)$$

From (3),(4) the proof is complete.

Remark. Now, let's extend this result.

Proposition 3. If $g:\mathbb{R}\to\mathbb{R}$ is continuous function and second derivable near of the point 0, then:

$$\textbf{(i)}\ \lim_{n\to\infty}\left(\sum_{k=1}^{n} g\left(\frac{1}{n+k}\right) - n\cdot g'(0)\right) = g'(0)\cdot\log 2$$

$$\textbf{(ii)}\ \lim_{n\to\infty} n\left(\sum_{k=1}^{n} g\left(\frac{1}{n+k}\right) - n\cdot g(0) - g'(0)\cdot\log 2\right) = \frac{g''(0)-g'(0)}{4}$$

Proof. (i) Using **Proposition 2** for function $f(x) = g(x) - g(0)$ defined near of the point 0 and $\beta = 1$, because $\lim_{x\to 0}\frac{g(x)-g(0)}{x} = g'(0) = l$.

(ii) Using **Proposition 2** for function $f(x) = g(x) - g(0) - xg'(0)$and $\alpha = 2$, we get:

$$\lim_{x\to 0}\frac{f(x)}{x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{g'(x)-g(0)}{2x} = \frac{g''(0)}{2}$$

Therefore,

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n} g\left(\frac{1}{n+k}\right) - n\cdot g(0) - g'(0)\sum_{k=1}^{n}\frac{1}{n+k}\right) =$$

$$= \lim_{n\to\infty} n\left(\sum_{k=1}^{n} g\left(\frac{1}{n+k}\right) - n\cdot g(0) - g'(0)\log 2\right) =$$

$$= \lim_{n\to\infty}\left[n\left(\sum_{k=1}^{n} g\left(\frac{1}{n+k}\right) - n\cdot g(0) - g'(0)\sum_{k=1}^{n}\frac{1}{n+k}\right) + g'(0)\cdot n\left(\sum_{k=1}^{n}\frac{1}{n+k} - \log 2\right)\right]$$

$$= \frac{g''(0)}{4} - \frac{g'(0)}{4}$$

Applications.

1) For $g(x) = \sin x$, we have $g(0) = 0, g'(0) = 1, g''(0) = 0$ and we get:

$$(i)\ \lim_{n\to\infty}\sum_{k=1}^{n}\sin\left(\frac{1}{n+k}\right) = \log 2$$

$$(ii)\ \lim_{n\to\infty} n\left(\sum_{k=1}^{n}\sin\left(\frac{1}{n+k}\right) - \log 2\right) = -\frac{1}{4}$$

2) For $g(x) = \cos x$, we have $g(0) = 1, g'(0) = 0, g''(0) = -1$ and we get:

$$(i)\ \lim_{n\to\infty}\left(\sum_{k=1}^{n}\cos\left(\frac{1}{n+k}\right) - n\right) = 0$$

$$(ii)\ \lim_{n\to\infty} n\left(\sum_{k=1}^{n}\cos\left(\frac{1}{n+k}\right) - n\right) = -\frac{1}{4}$$

3) For $g(x) = \tan^{-1} x$, we have $g(0) = 0, g'(0) = 1, g''(0) = 0$ and we get:

$$(i)\ \lim_{n\to\infty}\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{n+k}\right) = \log 2$$

$$(ii)\lim_{n\to\infty} n\left(\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{n+k}\right)-\log 2\right)=-\frac{1}{4}$$

4) For $g(x)=a^x, a>0, a\neq 1$ we have $g(0)=1, g'(0)=\log a, g''(a)=\log^2 a$ and we get:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}a^{\frac{1}{n+k}}-n\right)=\log a\cdot\log 2$$

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n}a^{\frac{1}{n+k}}-n-\log a\cdot\log 2\right)=\frac{\log^2 a-\log a}{4}$$

5) For $g(x)=\begin{cases}(1+x)^{\frac{1}{x}}, & \text{if } x\in(-1,\infty)-\{0\}\\ e, & \text{if } x=0\end{cases}$, we have $g(0)=e, g'(0)=-\frac{e}{2}$, $g''(0)=\frac{11}{12}e$, we get:

$$(i)\lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(1+\frac{1}{n+k}\right)^{n+k}-ne\right]=-\frac{e}{2}\log e$$

$$(ii)\lim_{n\to\infty} n\left[\sum_{k=1}^{n}\left(1+\frac{1}{n+k}\right)^{n+k}-ne+\frac{e}{2}\log 2\right]=\frac{17}{48}e$$

Remark. Above applications can be developed.

Proposition 4. (D.M. Bătinețu-Giurgiu)

If $a,b\in\mathbb{R}_+$, let be $f:\mathbb{R}_+^*\to\mathbb{R}$ a bounded function and the sequence $(x_n)_{n\geq 1}$

$$\boldsymbol{x_n=\sum_{k=1}^{n}\frac{(n+k)^a}{(n+k)^{a+1}+b+f(n+k)}},\ \textbf{then}$$

$$\textbf{(i)}\ \lim_{n\to\infty} x_n=\log 2$$

$$\textbf{(ii)}\ \lim_{n\to\infty} n(\log 2-x_n)=\frac{1}{4}$$

Proof. (i) We know that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\log 2,\ \text{hence,}$$

$$0\leq\sum_{k=1}^{n}\frac{1}{n+k}-\sum_{k=1}^{n}\frac{(n+k)^a}{(n+k)^{a+1}+b+f(n+k)}=$$

$$=\sum_{k=1}^{n}\frac{b+f(n+k)}{(n+k)\left((n+k)^{a+1}+b+f(n+k)\right)}\leq$$

$$\leq(b+n_0)\sum_{k=1}^{n}\frac{1}{(n+k)\left((n+k)^{a+1}+b+f(n+k)\right)}\leq$$

$$\leq(b+n_0)\sum_{k=1}^{n}\frac{1}{(n+k)^{a+2}},\ \text{where } n_0>0 \text{ and } f(x)\leq n_0;\forall x\in\mathbb{R}_+^*.$$

Because the sequence $\left(s_n(a+2)\right)_{n\geq 1}, s_n(a+2)=\sum_{k=1}^{n}\frac{1}{k^{a+2}}$ is convergent and $\lim_{n\to\infty} s_n(a+2) = s(a+2) \in \mathbb{R}_+^*$ and from

$$\sum_{k=1}^{n}\frac{1}{(n+k)^{a+2}} = s_{2n}(a+2) - s_n(a+2); \forall n \in \mathbb{N}^*, \text{it follows that:}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(n+k)^{a+2}} = 0, \text{hence,}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(n+k)^a}{(n+k)^{a+1}+b+f(n+k)} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \log 2.$$

$$\text{(ii)} \lim_{n\to\infty} n(\log 2 - x_n) = \lim_{n\to\infty}\frac{\log 2 - x_n}{\frac{1}{n}} \overset{C-S}{=} \lim_{n\to\infty}\frac{-x_{n+1}+x_n}{\frac{1}{n+1}-\frac{1}{n}} = \lim_{n\to\infty} n^2(x_{n+1}-x_n)$$

$$= \lim_{n\to\infty} n^2\left(\sum_{k=1}^{n+1}\frac{(n+1+k)^a}{(n+1+k)^a+f(n+1+k)+b} - \sum_{k=1}^{n}\frac{(n+k)^a}{(n+k)^{a+1}+f(n+k)+b}\right) =$$

$$= \lim_{n\to\infty} n^2\left(\sum_{k=2}^{n+2}\frac{(n+k)^a}{(n+k)^a+f(n+k)+b} - \sum_{k=1}^{n}\frac{(n+k)^a}{(n+k)^{a+1}+f(n+k)+b}\right) =$$

$$= \lim_{n\to\infty} n^2\left(\frac{(2n+1)^a}{(2n+1)^{a+1}+f(2n+1)+b} + \frac{(2n+2)^a}{(2n+2)^{a+1}+f(2n+2)+b} - \frac{(n+1)^a}{(n+1)^{a+1}+f(n+1)+b}\right) =$$

$$= \lim_{n\to\infty} n^2\left(\frac{(2n+1)^a}{(2n+1)^{a+1}+f(2n+1)+b} - \frac{1}{2n+1} + \frac{(2n+2)^a}{(2n+2)^a+f(2n+2)+b} - \frac{1}{2n+2} - \frac{(n+1)^a}{(n+1)^{a+1}+f(n+1)+b} + \frac{1}{n+1} + \frac{1}{2n+2} - \frac{1}{n+1}\right)$$

Now, using that:

$$\lim_{x\to\infty} x^2\left(\frac{x^a}{x^{a+1}+f(x)+b} - \frac{1}{x}\right) =$$

$$= \lim_{x\to\infty}\left(\frac{x}{x^{a+1}+f(x)+b}(x^{a+1}-x^{a+1}-f(x)-b\right) =$$

$$= -\lim_{x\to\infty}\frac{x(f(x)+b)}{x^{a+1}+f(x)+b} = 0, \forall a > 0 (\because f - \text{bounded.})$$

Therefore,

$$\lim_{n\to\infty} n(\log 2 - x_n) = \lim_{n\to\infty} n^2\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = \lim_{n\to\infty}\frac{n^2}{2(n+1)(2n+1)} = \frac{1}{4}.$$

Application 6. If $a \in [1,\infty)$ and $b \in \mathbb{R}_+$ then the sequence $(x_n)_{n\geq 1}$ with

$$x_n = -\frac{1}{a}\cdot\log(n^a+b) + \sum_{k=1}^{n}\frac{k^{a-1}}{k^a+b}$$

is convergent.

D.M. Bătinețu-Giurgiu

Solution.

$$\gamma_n = -\frac{1}{n} + \sum_{k=1}^{n}\frac{1}{k} \text{ and then } \gamma_n - x_n = -\log n + \frac{1}{a}\cdot\log(n^a+b) + \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{k^{a-1}}{k^a+b}\right) =$$

$$=\frac{1}{a}\cdot\log\frac{n^a+b}{n^a}+b\cdot\sum_{k=1}^{n}\frac{1}{k(k^a+b)}=\frac{1}{a}\cdot\log\frac{n^a+b}{n^a}+b\cdot y_n$$

$y_n=\sum_{k=1}^{n}\frac{1}{k(k^a+b)}, n\in\mathbb{N}^*$. We have: $y_{n+1}-y_n=\frac{1}{(n+1)((n+1)^a+b)}>0, \forall n\in\mathbb{N}^*$

Hence, $(y_n)_{n\geq 1}$ —is increasing sequence. On the other hand,

$$y_n=\sum_{k=1}^{n}\frac{1}{k(k^a+b)}\leq\sum_{k=1}^{n}\frac{1}{k^{a+1}}, \forall n\in\mathbb{N}^*; \left(z_n=\sum_{k=1}^{n}\frac{1}{k^{a+1}}-\text{convergent.}\right)$$

Then, $(y_n)_{n\geq 1}$ − is bounded. So, $(y_n)_{n\geq 1}$ − is convergent sequence, let $y=\lim_{n\to\infty}y_n$.

$$\lim_{n\to\infty}(\gamma_n-x_n)=\frac{1}{a}\cdot\log\left(\lim_{n\to\infty}\frac{n^a+b}{n^a}\right)+by=by$$

Therefore,

$$\lim_{n\to\infty}x_n=\gamma-by.$$

Application 7. Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that

$$\boldsymbol{x_n=\frac{1}{\log n}\cdot\sum_{i=1}^{n^m-n^k}\frac{1}{n^k+i}, m,k\in\mathbb{N}, m>k}$$

$$\textbf{Find}: \boldsymbol{\Omega=\lim_{n\to\infty}n\left(a^{\sqrt[n]{x_n}-1}-1\right), a>0}$$

Florică Anastase

Solution.

$$\sum_{i=1}^{n^m-n^k}\frac{1}{n^k+i}=\sum_{i=1}^{n^m}\frac{1}{i}-\sum_{i=1}^{n^k}\frac{1}{i}=\gamma_n\cdot m-\gamma_n\cdot k+\log(n^m)-\log(n^k),$$

$$\text{where } \gamma_n=-\log n+\sum_{k=1}^{n}\frac{1}{i}. \text{ Hence,}$$

$$x_n=\frac{1}{\log n}\cdot\sum_{i=1}^{n^m-n^k}\frac{1}{n^k+i}=\frac{\gamma_{n^m}-\gamma_{n^k}}{\log n}+(m-k)\Rightarrow\lim_{n\to\infty}x_n=m-k.$$

$$\Omega=\lim_{n\to\infty}n\left(a^{\sqrt[n]{x_n}-1}-1\right)=\lim_{n\to\infty}\frac{a^{\sqrt[n]{x_n}-1}-1}{\sqrt[n]{x_n}-1}\cdot n\left(\sqrt[n]{x_n}-1\right)=$$

$$=\log a\cdot\lim_{n\to\infty}n\left(\sqrt[n]{x_n}-1\right); (I)$$

Now, let $y_n=n\left(\sqrt[n]{x_n}-1\right)\Leftrightarrow x_n=\left(1+\frac{y_n}{n}\right)^n=\left[\left(1+\frac{y_n}{n}\right)^{\frac{n}{y_n}}\right]^{y_n}$

$$\Leftrightarrow\log x_n=y_n\cdot\log\left(1+\frac{y_n}{n}\right)^{\frac{n}{y_n}}\Rightarrow\lim_{n\to\infty}y_n=\lim_{n\to\infty}n\left(\sqrt[n]{x_n}-1\right)=m-k.$$

Therefore,

$$\Omega=\lim_{n\to\infty}n\left(a^{\sqrt[n]{x_n}-1}-1\right)=(m-k)\log a.$$

Application 8. If $a\in[1,\infty)$ and $b\in\mathbb{R}_+$, then prove that:

$$\boldsymbol{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(n+k)^{a-1}}{(n+k)^a+b}=\log 2}$$

D.M. Bătinețu-Giurgiu

Solution. Using Application 6, we have that the sequence $(x_n)_{n\ge 1}$ with

$$x_n=-\frac{1}{a}\cdot\log(n^a+b)+\sum_{k=1}^{n}\frac{k^{a-1}}{k^a+b}\text{ is convergent, let }x=\lim_{n\to\infty}x_n.$$

$$\text{Let us denote: }y_n=\sum_{k=1}^{n}\frac{(n+k)^{a-1}}{(n+k)^a+b},n\in\mathbb{N}^*,\text{then:}$$

$$y_n=x_{2n}-x_n+\frac{1}{a}\cdot\log\big((2n^a)+b\big)-\frac{1}{a}\cdot\log(n^a+b)=$$

$$=x_{2n}-x_n+\frac{1}{a}\cdot\log\left(\frac{2^a\cdot n^a+b}{n^a+b}\right),\forall n\in\mathbb{N}^*.$$

Therefore,

$$\lim_{n\to\infty}y_n=x-x+\frac{1}{a}\cdot\log(2^a)=\log 2.$$

Application 9. If $a\in[1,\infty)$ and $b\in\mathbb{R}_+$, then prove that:

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{(n+k)^{a-1}}{(n+k)^a+b}=\log 3$$

D.M. Bătinețu-Giurgiu

Solution. We know that:

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{n+k}=\log 3\text{ and let }x_n=\sum_{k=1}^{2n}\frac{1}{n+k},y_n=\sum_{k=1}^{2n}\frac{(n+k)^{a-1}}{(n+k)^a+b}$$

We get:

$$0\le x_n-y_n=b\cdot\sum_{k=1}^{2n}\frac{1}{(n+k)((n+k)^a+b)}\le b\cdot\sum_{k=1}^{2n}\frac{1}{(n+k)^{a+1}},\forall n\in\mathbb{N}^*;(I)$$

$$\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{(n+k)^{a+1}}=\lim_{n\to\infty}\left(\sum_{k=1}^{2n}\frac{1}{k^{a+1}}-\sum_{k=1}^{n}\frac{1}{k^{a+1}}\right)=0\text{ and from (I), we get:}$$

$$0\le\lim_{n\to\infty}(x_n-y_n)\le 0,\text{therefore,}\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=\log 3.$$

Remark. We can extend this result.

Proposition 5. (Daniel Sitaru)

Let $f:\mathbb{R}\to\mathbb{R}, g:\mathbb{R}\to\mathbb{R}^*$ such that exist $\lim_{x\to 0}\frac{f(x)}{x}=1$. Then find:

$$\Omega=\lim_{n\to\infty}\sum_{i=1}^{n}\left(g\left(\frac{1}{n+i}\right)-g\left(\frac{1}{n+1+i}\right)\right)\left(\sum_{k=1}^{i}\frac{f\left(\frac{1}{n+k}\right)}{g\left(\frac{1}{n+k}\right)}\right)$$

Proof. From $\lim_{x\to 0}\frac{f(x)}{x}=1$ and using Proposition 2, we get:

$$\lim_{n\to\infty}\sum_{k=1}^{n}f\left(\frac{1}{n+k}\right)=\log 2$$

Now, using Abel identity:

$$\sum_{i=1}^{n}a_ib_i=\sum_{i=1}^{n}(a_i-a_{i+1})\left(\sum_{k=1}^{i}b_k\right)$$

For $a_i=g\left(\frac{1}{n+i}\right), b_i=\frac{f\left(\frac{1}{n+i}\right)}{g\left(\frac{1}{n+i}\right)}$, it follows that:

$$\Omega=\lim_{n\to\infty}\sum_{i=1}^{n}\left(g\left(\frac{1}{n+i}\right)-g\left(\frac{1}{n+1+i}\right)\right)\left(\sum_{k=1}^{i}\frac{f\left(\frac{1}{n+k}\right)}{g\left(\frac{1}{n+k}\right)}\right)=$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}g\left(\frac{1}{n+i}\right)\cdot\frac{f\left(\frac{1}{n+i}\right)}{g\left(\frac{1}{n+i}\right)}=\lim_{n\to\infty}\sum_{k=1}^{n}f\left(\frac{1}{n+k}\right)=\log 2$$

Application 10. Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\tan\left(\frac{1}{n+i}\right)-\tan\left(\frac{1}{n+1+i}\right)\right)\left(\sum_{k=1}^{i}\cos\left(\frac{1}{n+i}\right)\right)}$$

Daniel Sitaru

Solution. Using Proposition 2 for $f(x)=\sin x$, we have:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sin\left(\frac{1}{n+k}\right)=\log 2$$

Now, using Proposition 5 for $a_i=\tan\left(\frac{1}{n+i}\right)$ and $b_i=\cos\left(\frac{1}{n+i}\right)$, we get:

$$\Omega=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\tan\left(\frac{1}{n+i}\right)-\tan\left(\frac{1}{n+1+i}\right)\right)\left(\sum_{k=1}^{i}\cos\left(\frac{1}{n+i}\right)\right)=$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\tan\left(\frac{1}{n+i}\right)\cos\left(\frac{1}{n+i}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\sin\left(\frac{1}{n+k}\right)=\log 2.$$

Application 11.Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{2+\sin(n+k)+(n+k)^2}}$$

Daniel Sitaru

Solution. For $a>0$, we prove that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{n+k}{a+(n+k)^2}-\frac{1}{n+k}\right)=0\Leftrightarrow\lim_{n\to\infty}\sum_{k=1}^{n}\frac{-a}{(n+k)(a+(n+k)^2)}=0;(I)$$

From $1\le k\le n$, we have:

$$(n+1)[a+(n+1)^2]\le(n+k)[a+(n+k)^2]\le(n+n)(a+(2n)^2)\Leftrightarrow$$

$$\frac{1}{2(a+(2n)^2)}\le\sum_{k=1}^{n}\frac{1}{(n+k)(a+(n+k)^2)}\le\frac{1}{\left(1+\frac{1}{n}\right)[a+(n+1)^2]}$$

Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(n+k)(a+(n+k)^2)}=0$$

Now, from $-1\le\sin(n+k)\le 1\Rightarrow 1\le 2+\sin(n+k)\le 3$, we get:

$$\frac{n+k}{3+(n+k)^2}\le\frac{n+k}{2+\sin(n+k)+(n+k)^2}\le\frac{n+k}{1+(n+k)^2}\Leftrightarrow$$

$$\sum_{k=1}^{n}\frac{n+k}{3+(n+k)^2}\le\sum_{k=1}^{n}\frac{n+k}{2+\sin(n+k)+(n+k)^2}\le\sum_{k=1}^{n}\frac{n+k}{1+(n+k)^2}$$

From (I), it follows that:

$$\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{n+k}{a+(n+k)^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\log 2$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{2+\sin(n+k)+(n+k)^2}=\log 2.$$

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THE PENTAGON JOURNAL CHALLENGES-(I)

By Daniel Sitaru-Romania

780. Prove that if $a,b,c\in[1,\infty)$, then $ab+bc+ca\ge 3+2\ln\left(a^bb^cc^a\right)$.

Proposed by Daniel Sitaru – Romania

Solution by Richdad Phuc, University of Sciences, Hanoi, Vietnam

We have

$$LHS-RHS=b(a-2\ln a)+c(b-2\ln b)+a(c-2\ln c)-3$$
$$=\frac{b}{a}a(a-2\ln a)+\frac{c}{b}b(b-2\ln b)+\frac{a}{c}c(c-2\ln c)-3.$$

Let $f(x)=x(x-2\ln x)$ for $x\ge 1$. The derivative is $f'(x)=2x-2\ln x-2$ and $f''(x)=2-\frac{2}{x}\ge 0$ for all $x\ge 1$, so $f'(x)\ge f'(1)=0$. This means that $f(x)$ is strictly increasing on $[0,1)$. Then $f(x)\ge f(1)$ for all $x\ge 1$. Hence $a(a-2\ln a)\ge 1$, $b(b-2\ln b)\ge 1$, and $c(c-2\ln c)\ge 1$. Then

$$LHS - RHS \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c} - 3$$

so $LHS - RHS \geq 0$ by the AM-GM inequality. Equality holds if $a = b = c = 1$.

781. Prove that if $a, b, c \in (0, \infty)$, then

$$\sum a\sqrt{\frac{(b^4 + c^4)}{2}} \geq a^2(b + c) + b^2(a + c) + c^2(a + b) - 3abc.$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

We prove that if $x, y \in (0, \infty)$, then

$$x + y - \sqrt{xy} \geq \sqrt{\frac{x^2+y^2}{2}} \qquad (1)$$

We denote $u = \sqrt{\frac{x^2+y^2}{2}}$, which means $2u^2 = x^2 + y^2$, and let $v = \sqrt{xy}$ so $v^2 = xy$. With these notations, we have

$$2u^2 + 2v^2 = x^2 + 2xy + y^2 = (x + y)^2.$$

We can rewrite (1) as $x + y - v \geq u$ or

$$(x + y)^2 \geq (u + v)^2 \Leftrightarrow 2u^2 + 2v^2 - u^2 - v^2 - 2uv \geq 0 \Leftrightarrow (u - v)^2 \geq 0.$$

Now replace x with $\frac{x}{y}$ and y with $\frac{y}{x}$ in (1) to get

$$\frac{x}{y} + \frac{y}{x} \geq \sqrt{\frac{\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2}{2}} + \sqrt{\frac{x}{y} \cdot \frac{y}{x}} \Leftrightarrow \frac{x^2 + y^2}{xy} \geq \frac{1}{xy}\sqrt{\frac{x^4 + y^4}{2}} + 1$$

$$\Leftrightarrow x^2 + y^2 \geq \sqrt{\frac{x^4 + y^4}{2}} + xy.$$

For $x = a$ and $y = b$ and multiplying by c we have

$$a^2c + b^2c \geq c\sqrt{\frac{a^4 + b^4}{2}} + abc.$$

Analogously,

$$b^2a + c^2a \geq a\sqrt{\frac{b^4 + c^4}{2}} + abc$$

and

$$c^2b + a^2b \geq b\sqrt{\frac{c^4 + a^4}{2}} + abc.$$

Adding the last three inequalities gives the desired result.

790. Prove that if $a, b \in \mathbb{R}$ with $a < b$, then

$$\ln\left|\left(\frac{2+\sin 2b}{2+\sin 2a}\right)\right| \le \frac{2\sqrt{3}}{3}(b-a).$$

Proposed by Daniel Sitaru – Romania

Solution by Richdad Phuc, Hanoi, Vietnam.

Let $f(x)=\ln|2+2\sin x|$. Then $f'(x)=\frac{2\cos 2x}{2+\sin 2x}$. By the Mean Value Theorem, there is a $c\in(a,b)$ with $f(b)-f(a)=f'(c)(b-a)$ or

$$\left|\ln\left(\frac{2+\sin 2b}{2+\sin 2a}\right)\right| \le \frac{2\cos 2c}{2+\sin 2c}(b-a).$$

But

$$\frac{2\cos 2c}{2+\sin 2c} \le \frac{2\sqrt{3}}{3} \Leftrightarrow \sqrt{3}\cos 2c-\sin 2c \le 2 \Leftrightarrow \cos\left(2c+\frac{\pi}{6}\right) \le 1,$$

which is clearly true.

799. Prove that if $a,b,c\in(0,2]$ then

$$3\sqrt{2} \le \sum \frac{b\left(\sqrt{a}+\sqrt{2-a}\right)}{c} \le 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Proposed by Daniel Sitaru – Romania

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.-USA

Elementary calculus shows that on the interval $(0,2]$ the function $f(x)=\sqrt{x}+\sqrt{2-x}$ attains a maximum of 2 at $x=1$ and a minimum of $\sqrt{2}$ at $x=2$. Therefore

$$\sum_{cyclic} \frac{b\left(\sqrt{a}-\sqrt{2-a}\right)}{c} \le 2\sum_{cyclic}\frac{b}{c} = 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Similarly, the AM-GM inequality yields

$$\sum_{cyclic} \frac{b\left(\sqrt{a}+\sqrt{2-a}\right)}{c} \ge \sqrt{2}*3\sqrt[3]{\frac{b}{c}*\frac{c}{a}*\frac{a}{b}} = 3\sqrt{2}.$$

Equality holds on the left side of the original inequality if and only if $a=b=c=2$ on the right side if and only if $a=b=c=1$.

800. Prove that if $a\in\mathbb{R}$, then

$$\int_{a+3}^{a+5}\ln(1+e^x)\,dx+\int_{a+6}^{a+8}\ln(1+e^x)\,dx \le \int_{a}^{a+2}\ln(1+e^x)\,dx+\int_{a+9}^{a+11}\ln(1+e^x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain.

Notice that by a change of variables

$$\int_{a+3}^{a+5}\ln(1+e^x)\,dx = \int_{a}^{a+2}\ln(1+e^{x-3})dx,$$

$$\int_a^{a+2} \ln(1+e^{x-3})\,dx + \int_a^{a+2} \ln(1+e^{x-6})\,dx \le$$
$$\le \int_a^{a+2} \ln(1+e^{x})\,dx + \int_a^{a+2} \ln(1+e^{x-9})\,dx.$$

Using the properties of the natural logarithm, we get

$$\int_a^{a+2} \ln\left(\frac{1+e^{x-3}+e^{x-6}+e^{2x-9}}{1+e^{x}+e^{x-9}+e^{2x-9}}\right) dx \le 0.$$

The result will follows by showing that $e^{x-3}+e^{x-6} \le e^{x}+e^{x-9}$ for all x. This is equivalent to $\frac{e^{x-6}-e^{x-9}}{3} \le \frac{e^{x}-e^{x-3}}{3}$ and by Lagrange's Mean Value Theorem the left side is equal to e^{μ} for some μ in $(x-9, x-6)$ and the right side is equal to e^{η} for some η in $(x-3, x)$. Since the function e^{x} is increasing, the inequality is true.

808. Prove that if $a, b, c \in [1, \infty)$ then

$$\frac{e^{a+b+c}}{e^{b/a+c/b+a/c}} \le a^b b^c c^a \le \frac{e^{ab+bc+ca}}{e^{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

Solution by Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC.-USA

Let $a \in (1, \infty)$. Using the fact that $f(x) = \ln x$ is concave on $[1, a]$, we have

$$f'(a) = \frac{1}{a} \le \frac{\ln a - \ln 1}{a-1} \le 1 = f'(1),$$

which yields

$$1 - \frac{1}{a} \le \ln a \le a - 1.$$

Raising e to all three parts of this inequality produces $\frac{e}{e^{\frac{1}{a}}} \le a \le \frac{e^a}{e}$. (Note that this is true for $a = 1$.) Raising this inequality to the power b, we get $\frac{e^b}{e^{\frac{b}{a}}} \le a^b \le \frac{e^{ab}}{e^b}$. Similarly, using the pairs b, c and c, a we have $\frac{e^c}{e^{\frac{c}{b}}} \le b^c \le \frac{e^{bc}}{e^c}$ and $\frac{e^a}{e^{\frac{a}{c}}} \le c^a \le \frac{e^{ca}}{e^a}$. Multiplying all corresponding sides of the three inequalities gives the desired inequality.

809. Prove that if $a, b, c \in (2, \infty)$ then

$$\sqrt{2}\sum\left(\sqrt{a(b-a)} + \sqrt{b(a-2)}\right) \le 3\sqrt{abc}.$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu, National College, "Roman Voda", Roman, Romania.

The inequality is equivalent to

$$\sqrt{2}\sum\left(\sqrt{a}\left(\sqrt{b-2} + \sqrt{c-2}\right)\right) < 3\sqrt{abc}$$
$$\Leftrightarrow \sum\left(\sqrt{a}\left(\sqrt{2(b-2)} + \sqrt{2(c-2)}\right)\right) < 3\sqrt{abc}.$$

Now we use Mahler's inequality:

$$\sqrt{x_1x_2}+\sqrt{y_1y_2}\le\sqrt{(x_1+y_1)(x_2+y_2)}, \forall x_i, y_i>0$$

which says that

$$\sqrt{2(b-2)}+\sqrt{(c-2)\cdot 2}<\sqrt{c\cdot b}\Rightarrow\sqrt{a}\left(\sqrt{2(b-2)}+\sqrt{2(c-2)}\right)<\sqrt{abc}.$$

Similarly,

$$\sqrt{b}\left(\sqrt{2(a-2)}+\sqrt{2(c-2)}\right)<\sqrt{abc}$$

and

$$\sqrt{c}\left(\sqrt{2(a-2)}+\sqrt{2(b-2)}\right)<\sqrt{abc}.$$

Summing the three inequalities gives the desired result.

810. Compute

$$\boldsymbol{L=\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx.}$$

Proposed by Daniel Sitaru – Romania

Solution by Andrea Fanchini, Cantu, Italy.

$$I=\int_1^n\frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx=\int_1^n\frac{x(x+1)(x^2+3x+9)}{15x(x+3)(x^2+3x+9)}dx$$
$$=\frac{1}{15}\int_1^n\frac{x+1}{x+3}dx$$
$$=\frac{1}{15}\int_1^n 1-\frac{2}{x+3}dx=\frac{1}{15}(n-2\log(n+3)-1+\log 16).$$

Finally,

$$\frac{1}{15}\lim_{n\to\infty}\frac{n-2\log(n+3)-1+\log 16}{n}=\frac{1}{15}.$$

821. Prove that if $\boldsymbol{a,b,c\in\mathbb{R}}$ then

$$\boldsymbol{4\sum_{cyclic}a\left|b(1-b^2)\right|\le\sum_{cyclic}a(1+b^2)^2.}$$

Proposed by Daniel Sitaru – Romania

Solution by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania.

We have

$$4a|b(1-b^2)|\le a(1+b^2)^2\Leftrightarrow 4a\sqrt{\left(b(1-b^2)\right)^2}\le a(1+b^2)^2$$
$$\Leftrightarrow 16a^2b^2(1-b^2)^2\le a^2(1+b^2)^4\Leftrightarrow a^2(b^8-12b^6+38b^4-12b^2+1)\ge 0$$
$$\Leftrightarrow a^2(b^2+2b-1)^2(b^2-2b-1)^2\ge 0.$$

The last inequality is true so the first is true and then summing, we get the desired result.

822.Prove that in any acute-angled $\boldsymbol{\Delta ABC}$ you have

$$\mathbf{2\sum_{cyclic}\tan^3 A \geq \sum_{cyclic}\sqrt{\frac{\tan^6 A+\tan^6 B}{2}}+3(\tan A+\tan B+\tan C).}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer.

Lemma. If $a,b\in(0,1)$ then $a+b\geq\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}$.

Proof. Denote $\begin{cases} x=\sqrt{\frac{a^2+b^2}{2}} \\ y=\sqrt{ab}\end{cases} \Rightarrow \begin{cases} a^2+b^2=2x^2 \\ ab=y^2\end{cases}$. Then

$$a+b\geq x+y \Leftrightarrow (a+b)^2\geq(x+y)^2\Leftrightarrow 2x^2+2y^2\geq(x+y)^2$$
$$\Leftrightarrow 2x^2+2y^2\geq x^2+2xy+y^2\Leftrightarrow x^2-2xy+y^2\geq 0\Leftrightarrow(x-y)^2\geq 0,$$

which is true. Now replace a and b in the Lemma with $a=\tan^3 A$; $b=\tan^3 B$ and get

$$\tan^3 A+\tan^3 B\geq\sqrt{\frac{\tan^6 A+\tan^6 B}{2}}+\sqrt{\tan^3 A\tan^3 B}$$

so that

$$\sum\tan^3 A+\tan^3 B\, g\geq\sum\sqrt{\frac{\tan^6 A+\tan^6 B}{2}}+\sum\tan A\tan B\sqrt{\tan A\tan B}$$

and by the AM-GM

$$2\sum\tan^3 A\geq\sum\sqrt{\frac{\tan^6 A+\tan^6 B}{2}}+3\sqrt[3]{\tan^3 A\tan^3 B\tan^3 C}$$
$$=\sum\sqrt{\frac{\tan^6 A+\tan^6 B}{2}}+3(\tan A+\tan B+\tan C)$$

829.Let $\Omega_n=\binom{n}{7}+2\binom{n-1}{7}+3\binom{n-2}{7}+\cdots(n-6)\binom{7}{7}$ for all $n\geq 7$. Find

$$\mathbf{\Omega=\lim_{n\to\infty}\sqrt[n]{\Omega_n}.}$$

Proposed by Daniel Sitaru – Romania

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.-USA

Let $n\geq 7$ and

$$\Omega_n=\binom{n}{7}+2\binom{n-1}{7}+3\binom{n-2}{7}+\cdots(n-6)\binom{7}{7}=\sum_{j=7}^{n}\sum_{i=7}^{j}\binom{i}{7}.$$

By the Hockey Stick Identity,

$$\sum_{i=7}^{j}\binom{i}{7}=\binom{j+1}{8}\ \ and\ \sum_{j=7}^{n}\binom{j+1}{8}=\sum_{j=8}^{n+1}\binom{j}{8}=\binom{n+2}{9}.$$

Thus

$$\Omega_n=\frac{(n+2)(n+1)n(n-1)\dots(n-6)}{9!}=\frac{1}{9!}n^9\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)\dots\left(1-\frac{6}{n}\right),$$

and

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\Omega_n}=\lim_{n\to\infty}\left(\sqrt[n]{n}\right)^9\sqrt[n]{\frac{1}{9!}\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)\dots\left(1-\frac{6}{n}\right)}=1.$$

830. If $x\in\left(0,\frac{\pi}{2}\right)$, prove that $2(\sin x)^{1-\sin x}\cdot(1-\sin x)^{\sin x}\le 1$.

Proposed by Daniel Sitaru – Romania

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.-USA

Noting that $0<\sin x<1$ for $x\in\left(0,\frac{\pi}{2}\right)$, we apply the weighted AGM inequality twice to see that

$$2(\sin x)^{1-\sin x}\cdot(1-\sin x)^{\sin x}\le 2[(1-\sin x)\sin x+\sin x(1-\sin x)]$$
$$=4\sin x(1-\sin x)$$
$$\le 4\left(\frac{\sin x+(1-\sin x)}{2}\right)^2=1,$$

with equality when $x=\frac{\pi}{6}$.

831.If $\Delta ABC\sim\Delta A'B'C'$, prove that

$$\sum\frac{(a'+b')(a'+c')}{b'c'}+3\ge\frac{15(b+c)(c'+a')(a'+b')}{8ab'c'}.$$

Proposed by Daniel Sitaru – Romania

Solution by Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania.

With $\Delta ABC\sim\Delta A'B'C'$, we have $a=ka',b=kb',c=kc'$ and obtain

$$\frac{(a'+b')(a'+c')}{b'c'}=\frac{(a+b)(a+c)}{bc}$$

and

$$\frac{15(b+c)(c'+a')(a'+b')}{8ab'c'}=\frac{15(b+c)(c+a)(a+b)}{8abc}.$$

That is

$$\sum\frac{(a+b)(a+c)}{bc}+3\ge\frac{15(b+c)(c+a)(a+b)}{8abc}.$$

Multiplying by $\frac{abc}{(b+c)(c+a)(a+b)}$, we obtain

$$\sum\frac{a}{b+c}+\frac{3abc}{(a+b)(b+c)(c+a)}\ge\frac{15}{8}$$

and we write this as

$$\sum\frac{a}{b+c}-\frac{3}{2}\ge\frac{3}{8}-\frac{3abc}{(a+b)(b+c)(c+a)}.$$

By a calculation, we obtain the LHS is equal to $\frac{\sum[(a+b)(a-b)^2]}{2(a+b)(b+c)(c+a)}$ and the RHS is equal to $\frac{\sum a(b-c)^2}{8(a+b)(b+c)(c+a)}$. But

$$4(a+b) \geq c \Rightarrow 4(a+b)(a-b)^2 \geq c(a-b)^2$$

which implies the required inequality.

843.Prove that in ΔABC you have

$$\sqrt{(2^{h_a}+2^{h_b}+2^{h_c})(2^{m_a}+2^{m_b}+2^{m_c})} < 2^a+3^b+4^c.$$

Proposed by Daniel Sitaru – Romania

Solution by Ioannis Sfikas, Athens, Greece.

It is well-known in every triangle: $h_s \leq m_a < \frac{b+c}{2}$. If we assume that function $f(x)=2^x$, then $f'(x)=2^x \ln 2 > 0$ and $f''(x)=2^x(\ln 2)^2 > 0$. So, the function $f(x)$ is increasing and convex function. Also, we have:

$$2^{h_a} \leq 2^{m_a} < 2^{\frac{b+c}{2}} \leq \frac{2^b+2^c}{2},$$

and $2^{m_a}+2^{m_b}+2^{m_c} < 2^a+2^b+2^c$ and

$$\sqrt{(2^{h_a}+2^{h_b}+2^{h_c})(2^{m_a}+2^{m_b}+2^{m_c})} \leq 2^{m_a}+2^{m_b}+2^{m_c}$$
$$< 2^a+2^b+2^c < 2^a+3^b+4^c.$$

844. Prove that if $0<a<b<c<1$, then

$$2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a\ln a & b\ln b & c\ln c \end{vmatrix} >$$
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ (a-1)\ln(a^2+1) & (b-1)\ln(b^2+1) & (c-1)\ln(c^2+1) \end{vmatrix}.$$

Proposed by Daniel Sitaru – Romania

Solution by Michel Bataille, Rouen, France.

Let $f:(0,1)\to\mathbb{R}$. Then

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-b \\ f(a) & f(b)-f(c) & f(c)-f(b) \end{vmatrix}$$
$$=(b-a)(c-b)\left(\frac{f(c)-f(b)}{c-b}-\frac{f(b)-f(a)}{b-a}\right).$$

Applying this result first with $f(x)=2x\ln x$ and then with $f(x)=(x-1)\ln(x^2+1)$ and observing that $(b-a)(c-b)>0$, we obtain that the proposed inequality is equivalent to

$$\frac{g(c)-g(b)}{c-b} > \frac{g(b)-g(a)}{b-a} \quad (1)$$

where g denotes the function defined by $g(x)=2x\ln x-(x-1)\ln(x^2+1)$.
Now we calculate the first two derivatives of g:

$$g'(x) = 2 + 2\ln x - \ln(x^2+1) - \frac{2x^2-2x}{x^2+1}$$
$$g''(x) = \frac{2(1-x^2)(1+x)}{x(x^2+1)^2}.$$

We deduce that $g''(x)$ is positive when x in in $(0,1)$. Thus g is convex on the interval $(0,1)$ and (1) follows since $a < b < c$.

845. If $a, b, c \in [0,1)$, then

$$8\int_0^a \left(\int_0^b \left(\int_0^c \frac{\sin^{-1} x \cdot \sin^{-1} y \cdot \sin^{-1} z}{(1+\sin^{-1} x)(1+\sin^{-1} y)(1+\sin^{-1} z)} dz\right) dy\right) dx$$

Proposed by Daniel Sitaru – Romania

Solution by the Missouri State University Problem Solving Group, Springfield, MO.-USA

Let $t \in [0,1)$. Consider the function $f(t) = t(1+\arcsin t) - \arcsin t$.

Since we have that

$$\frac{t-1}{\sqrt{1-t^2}} = -\frac{1-t}{\sqrt{(1-t)(1+t)}} = -\sqrt{\frac{1-t}{1+t}} \geq -1,$$

we see

$$f'(t) = 1 + \arcsin t + \frac{t}{\sqrt{1-t^2}} - \frac{1}{\sqrt{1-t^2}} \geq 0.$$

Since $f(0) = 0$, it follows that $t(1+\arcsin t) \geq \arcsin t$ and so $\frac{\arcsin t}{1+\arcsin t} \leq t$. We therefore have

$$8\int_0^a \int_0^b \int_0^c \frac{\arcsin x \arcsin y \arcsin z}{(1+\arcsin x)(1+\arcsin y)(1+\arcsin z)} dzdydx$$
$$= \int_0^a 2\frac{\arcsin x}{1+\arcsin x} dx \int_0^b 2\frac{\arcsin y}{1+\arcsin y} dy \int_0^c 2\frac{\arcsin z}{1+\arcsin z} dz$$
$$\leq \int_0^a 2t\,dt \int_0^b 2t\,dt \int_0^c 2t\,dt$$
$$= a^2b^2c^2.$$

PROBLEMS FOR JUNIORS

J.1263 Solve in real numbers the following equation:

$$x^2-2x+29=2\sqrt{x^2-x+1}+4\sqrt{x+3}+6\sqrt{11-2x}$$

Proposed by Nguyen Viet Hung – Vietnam

J.1264 In $\Delta ABC, AB=\sqrt{2(\tan 20^\circ+\tan 33^\circ)}, b=\sqrt{2(\tan 33^\circ+\tan 37^\circ)}$,

$c=\sqrt{2(\tan 37^\circ+\tan 20^\circ)}$. Find $\Omega=[ABC]$

Proposed by Adil Abdullayev-Azerbaijan

J.1265 In $\Delta ABC, M$ point in plane, the following relationship holds:

$$\frac{AM}{h_a}+\frac{BM}{h_b}+\frac{CM}{H_c}\geq 2$$

Proposed by Bogdan Fuștei – Romania

J.1266 Find all functions $f:\mathbb{N}\to\mathbb{R}$ such that

$$\forall x\geq y\geq 0, f(x+y)+f(x-y)=f(5x)+2f(3x)$$

Proposed by Rajeev Rastogi-India

J.1267 If in $\Delta ABC, H$ −orthocenter, then

$$\frac{AH^2}{b^2+c^2}+\frac{BH^2}{c^2+a^2}+\frac{CH^2}{a^2+b^2}\leq\frac{R}{r}+\frac{2r}{R}-\frac{5}{2}$$

Proposed by Eldeniz Hesenov-Georgia

J.1268 In ΔABC prove that:

$$\frac{3R}{\sqrt{r}}\geq\frac{a}{\sqrt{h_a}}+\frac{b}{\sqrt{h_b}}+\frac{c}{\sqrt{h_c}}\geq 3\sqrt{r}$$

Proposed by Mokhtar Kassani-Algerie

J.1269 Solve for integers:

$$x^2+y^2=2020(x-y)$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1270 Solve the equation: $a^b=ab, a,b\in\mathbb{N}$.

Proposed by Ilir Demiri-Macedonia

J.1271 If $\alpha>0$ then in ΔABC holds:

$$\sqrt[3]{abc}\leq\frac{e^\alpha}{3^{\alpha+1}}(a+2^\alpha\cdot b+3^\alpha\cdot c)$$

Proposed by Khaled Abd Imouti-Syria

J.1272 In acute $\Delta ABC, AD, BE, CF$ −altitudes, r_1,r_2,r_3,r_4,r_5,r_6 −inradii of

$\Delta ABD,\Delta ACD,\Delta BCE,\Delta BAE,\Delta ACF$ −respectively $\Delta BCF, O$ −circumcentre, I −incentre,

H −orthocentre. Prove that:

$$[OIH]=|(r_1-r_2)(r_3-r_4)(r_5-r_6)|\cdot\frac{4R}{F}$$

Proposed by Mehmet Șahin-Turkiye

J.1273 If $x, y, z > 0$, then in any ABC triangle the following inequality holds:

$$\frac{xa}{\sqrt{yz}h_a}+\frac{yb}{\sqrt{zx}h_b}+\frac{zc}{\sqrt{xy}h_c}\geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1274 If $x, y, z > 0$, then in any ABC triangle the following inequality holds:

$$\frac{y+z}{x}m_a+\frac{z+x}{y}m_b+\frac{x+y}{z}m_c\geq\frac{4\sqrt{3}F}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1275 In any ABC triangle having the semiperimeter s the following inequality holds:

$$\frac{a^2b^2}{(s-c)^2}+\frac{b^2c^2}{(s-a)^2}+\frac{c^2a^2}{(s-b)^2}\geq 144r^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1276 If $x, y, z \in \mathbb{R}_+^* = (0,\infty)$ and a, b, c are the lengths sides of ABC triangle with the area F, then:

$$(x^2\cdot a^4+1)(y^2b^4+1)(z^2c^4+1)\geq 12(xy+yz+zx)F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1277 In any ABC triangle with the area F the following inequality holds:

$$a^2\tan\frac{B}{2}+b^2\tan\frac{C}{2}+c^2\tan\frac{A}{2}\geq 4F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1278 In any ABC triangle the following inequality holds:

$$\frac{a^2b^2}{h_ah_b}+\frac{b^2c^2}{h_bh_c}+\frac{c^2a^2}{h_ch_a}\geq\frac{16\sqrt{3}}{3}F$$

where F is the triangle's area.

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

J.1279 In any ABC triangle with the area F the following inequality holds:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

J.1280 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{\sqrt{R}}{2r}\cdot\sum_{cyc}\sqrt{m_a+h_a}\geq\sum_{cyc}\left(\frac{n_a}{h_a}+\frac{2r_a}{s+n_a}\right)$$

Proposed by Bogdan Fuștei – Romania

J.1281 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\cos\left(\frac{A-B}{2}\right)\geq 2\cdot\sqrt{\frac{2r}{R}}\cdot\sum_{cyc}\frac{m_a}{n_a+g_a}$$

Proposed by Bogdan Fuștei – Romania

J.1282 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sqrt{\frac{R}{2r}}\geq\frac{1}{3}\sum_{cyc}\frac{r_b+r_c}{n_a+g_a}$$

Proposed by Bogdan Fuștei – Romania

J.1283 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$3\sum_{cyc}ar_a\geq\sqrt{2}(2R-r)\sum_{cyc}\left(n_a+2\sqrt{r_ah_a}\right)$$

Proposed by Bogdan Fuștei – Romania

J.1284 In ΔABC the following relationship holds:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq\sum_{cyc}\frac{m_b+m_c-m_a}{\sqrt{s_bs_c}}$$

Proposed by Bogdan Fuștei – Romania

J.1285 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$m_a^2\geq\frac{1}{2}(n_ag_a+r_br_c)\geq r_br_c$$

Proposed by Bogdan Fuștei – Romania

J.1286 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_bh_c}{a^2}\left(\sum_{cyc}\frac{r_a-r}{w_a}\sqrt{\frac{h_a}{r_a}}\right)^2\leq\frac{s_a+s_b+s_c}{r}$$

Proposed by Bogdan Fuștei – Romania

J.1287 In ΔABC the following relationship holds:

$$\cos A+\cos B+\cos C\geq\frac{h_a}{r_b+r_c}+\frac{h_b}{r_c+r_a}+\frac{h_c}{r_a+r_b}$$

Proposed by Bogdan Fuștei-Romania

J.1288 In ΔABC the following relationship holds:

$$\sum \sqrt{\frac{n_a g_a(n_a+g_a)}{2w_a} - w_a^2} \geq \frac{1}{2}\sum |b-c|$$

Proposed by Bogdan Fuștei-Romania

J.1289 In ΔABC the following relationship holds:

$$\sum \frac{m_a w_a}{(r_b+r_c)(n_a+g_a)} \leq \frac{3}{4}$$

Proposed by Bogdan Fuștei-Romania

J.1290 In ΔABC the following relationship holds:

$$\sum \sqrt{n_a g_a h_a} \geq s\sum \sqrt{h_a - 2r}$$

Proposed by Bogdan Fuștei-Romania

J.1291 In ΔABC the following relationship holds:

$$\sum \frac{n_a g_a}{a} \geq \sum \frac{F}{r_a - r}$$

Proposed by Bogdan Fuștei-Romania

J.1292 In ΔABC the following relationship holds:

$$\sum \left(a + \sqrt{h_a(h_b - 2r)}\right)\left(n_a + \sqrt{2r_a h_a}\right) \leq 3\sqrt{2}s^2$$

Proposed by Bogdan Fuștei-Romania

J.1293 If $x, y, z \in \mathbb{R}, \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1} = 3\sqrt{2}$ then

$$x^2 + y^2 + z^2 + 6 \geq 3(x+y+z)$$

Proposed by Daniel Sitaru-Romania

J.1294 Solve for real numbers:

$$32(2x^{12} + x^8 + x^6 + x^4) + 19 = 64x^2(x^8+1)$$

Proposed by Daniel Sitaru-Romania

J.1295 If in $\Delta ABC, \Delta A'B'C', 2s = 2s' = 3$ then:

$$\prod_{cyc}\left(1 + \frac{1}{\sqrt[3]{a}}\right)\left(1 + \frac{1}{\sqrt[3]{a'}}\right) \geq \frac{128\sqrt{3}}{3(R+R')}$$

Proposed by Daniel Sitaru-Romania

J.1296 If $x, y, z \in \mathbb{R}$ then:

$$\frac{25x}{21} + \frac{13y}{15} + \frac{33z}{35} \leq \sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{z^2+x^2}$$

Proposed by Daniel Sitaru-Romania

J.1297 In ΔABC the following relationship holds:

$$\sum \frac{1}{bc}\cot^2\frac{A}{2} \geq \left(\frac{6r}{R}\right)^2 \sum \frac{1}{bc}\tan^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.1298 If $a, b, c > 0$ and $\lambda \geq 0$ then:

$$(\lambda+2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\lambda \geq 2\sqrt{\lambda+1}\left(\sqrt{\frac{a+\lambda b}{c}}+\sqrt{\frac{b+\lambda c}{a}}+\sqrt{\frac{c+\lambda a}{b}}\right)$$

Proposed by Marin Chirciu – Romania

J.1299 In ΔABC the following relationship holds:

$$\frac{1}{R} \leq \frac{1}{h_a+h_b}+\frac{1}{h_b+h_c}+\frac{1}{h_c+h_a} \leq \frac{1}{2r}$$

Proposed by Marin Chirciu – Romania

J.1300 In ΔABC the following relationship holds:

$$\left(\frac{1}{2}+\csc\frac{A}{2}\right)^2+\left(\frac{1}{2}+\csc\frac{B}{2}\right)^2+\left(\frac{1}{2}+\csc\frac{C}{2}\right)^2 \geq \frac{75}{4}$$

Proposed by Marin Chirciu – Romania

J.1301 In ΔABC the following relationship holds:

$$\sum h_a \sum \frac{1}{h_b+h_c} \geq \frac{2r}{R}\sum r_a \sum \frac{1}{r_b+r_c}$$

Proposed by Marin Chirciu – Romania

J.1302 In ΔABC the following relationship holds:

$$\left(\frac{1}{2}+\cos\frac{A}{2}\right)^2+\left(\frac{1}{2}+\cos\frac{B}{2}\right)^2+\left(\frac{1}{2}+\cos\frac{C}{2}\right)^2 \geq \frac{11+3\sqrt{3}}{4}+\frac{p+r}{2R}$$

Proposed by Marin Chirciu – Romania

J.1303 If $x, y > 0, x^2+2y^2=3$ then:

$$\left(\frac{x+y}{\sqrt{y^2+3}}+\frac{y}{\sqrt{x^2+3}}\right)^2 \leq \frac{3(x+2y)}{4y\sqrt{x}}$$

Proposed by Daniel Sitaru – Romania

J.1304 In ΔABC the following relationship holds:

$$(s-a)^5+(s-b)^5+(s-c)^5+10Rrs(a^2+b^2+c^2)=s^5$$

Proposed by Daniel Sitaru – Romania

J.1305 Find $x, y > 0$ such that:

$$\begin{cases} 2^x + \log_2 y = 5 \\ \dfrac{x^2}{x^2+2y^2} + \dfrac{y^2}{y^2+2x^2} + \dfrac{2x}{x+2y} + \dfrac{2y}{y+2x} = 3 \end{cases}$$

Proposed by Daniel Sitaru – Romania

J.1306 In ΔABC holds:

$$\frac{a^3}{bc+a^2} + \frac{b^3}{ca+b^2} + \frac{c^3}{ab+c^2} = s \Leftrightarrow 2s = 3\sqrt{3}R$$

Proposed by Daniel Sitaru – Romania

J.1307 If $a,b,c > 0, abc = 1$ then:

$$\frac{a^2+b^2}{a^7+b^7} + \frac{b^2+c^2}{b^7+c^7} + \frac{c^2+a^2}{c^7+a^7} \le a^5 + b^5 + c^5$$

Proposed by Daniel Sitaru – Romania

J.1308 If $a,b,x,y > 0$ then:

$$32ab(ax+by)^4 \le (a+b)^4(8a^2x^4 + ab(x+y)^4 + 8b^2y^4)$$

Proposed by Daniel Sitaru – Romania

J.1309 In ΔABC the following relationship holds:

$$\sum_{cyc} (b+c)\csc\frac{A}{2} = 24\sqrt{3}r \Leftrightarrow a = b = c$$

Proposed by Daniel Sitaru – Romania

J.1310 If $m,n \in \mathbb{R}_+^* = (0,\infty)$ and $x,y,z \in (0,1)$, then:

$$\frac{1}{(my+nz)(1-x^2)} + \frac{1}{(mz+nx)(1-y^2)} + \frac{1}{(mx+ny)(1-z^2)} \ge \frac{9\sqrt{3}}{2(m+n)}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1311 In any ABC triangle with the semiperimeter s the following inequality holds:

$$\frac{y+z}{x}\cdot\frac{bc}{(s-a)(s-b)} + \frac{z+x}{y}\cdot\frac{ca}{(s-b)(s-c)} + \frac{x+y}{z}\cdot\frac{ab}{(s-c)(s-a)} \ge 24, \forall x,y,z > 0$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1312 If $x,y,z \in (0,1)$ and ABC is a triangle with the area F, then:

$$\frac{a^2b^2}{x^2(1-x)} + \frac{b^2c^2}{y^2(1-y)} + \frac{c^2a^2}{z^2(1-z)} \ge 108F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1313 Let be $m \in \mathbb{R}_+ = [0,\infty)$ and ABC a triangle with the area F, then the following inequality holds:

$$\sum_{cyc}\left(\sqrt{\frac{a^6+b^6}{2}}+\frac{2\cdot a^{2m+2}\cdot b^{2m+2}}{a^3+b^3}\right)\geq 2^{2m+3}\left(\sqrt{3}\right)^{1-m}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1314 If $m,n\in\mathbb{R}_+=[0,\infty)$ and ABC is a triangle with the area F, then:

$$\sum_{cyc}\left(\sqrt{\frac{a^{2n}+b^{2n}}{2}}+\frac{2\cdot a^{2m+2}\cdot b^{2m+2}}{a^n+b^n}\right)\geq 2^{2m+3}\left(\sqrt{3}\right)^{1-m}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1315 If $m,n,x,y\geq 0, m+n, x+y>0$ then in any ABC triangle with the area F the following inequality holds:

$$\sum_{cyc}\left(\frac{a^2-ab+b^2}{b^2+ab+a^2}\right)^m(xa^2+yb^2)^{n+1}\geq 4^{n+1}(x+y)^{n+1}\left(\sqrt{3}\right)^{1-2m-n}F^{n+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1316 If $x,y,z>0$ and ABC is a triangle with the area F, then:

$$(x^2a+y^2b+z^2c)\left(\frac{a^2}{(y+z)^2h_a}+\frac{b^2}{(z+x)^2h_b}+\frac{c^2}{(x+y)h_c}\right)\geq 6F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1317 If $x,y,z>0$ such that $xy+yz+zx=3xyz$ and $\lambda\leq 2$ then:

$$x^2y+y^2z+z^2x+3(\lambda-1)\geq\lambda(x+y+z)$$

Proposed by Marin Chirciu – Romania

J.1318 In ΔABC the following relationship holds:

$$\sum\frac{1}{bc}\cot\frac{B}{2}\cot\frac{C}{2}\geq 9\sum\frac{1}{bc}\tan\frac{B}{2}\tan\frac{C}{2}$$

Proposed by Marin Chirciu – Romania

J.1319 In ΔABC the following relationship holds

$$\frac{1}{\lambda+\left(\frac{r}{r_a}\right)^2}+\frac{1}{\lambda+\left(\frac{r}{r_b}\right)^2}+\frac{1}{\lambda+\left(\frac{r}{r_c}\right)^2}\leq\frac{27}{9\lambda+1},\lambda\geq\frac{7}{9}$$

Proposed by Marin Chirciu – Romania

J.1320 If $a_1,a_2,\dots,a_n>0$ such that $a_1+a_2+\cdots+a_n\leq n$ and $k\in\mathbb{N}^*$ then:

$$\frac{1}{a_1^k}+\frac{1}{a_2^k}+\cdots+\frac{1}{a_n^k}\geq n$$

Proposed by Marin Chirciu – Romania

J.1321 If $a, b, c \geq 0$ such that $ab + bc + ca = 1$ then find the minimum value of expression

$$P = \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} + \frac{20}{a+b+c}$$

Proposed by Marin Chirciu – Romania

J.1322 In ΔABC the following relationship holds

$$3\sum a^3 \tan\frac{A}{2} \geq \sum a^3 \cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.1323 Let be M an interior point in ABC triangle and x, y, z the distances from M to the apices A, B, C and u, v, w the distances from M to the sides BC, CA, AB. If F is the triangle's area, then:

$$(x+y+z)\left(\frac{a^4}{v+w} + \frac{b^4}{w+u} + \frac{c^4}{u+v}\right) \geq 48F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

J.1324 Let be ABC a triangle with the area F and M an interior point in triangle. If x, y, z are the distances from M to the apices A, B, C and u, v, w the distances from M to the sides BC, CA respectively AB, then:

$$\frac{y+z}{w}a^4 + \frac{z+x}{u}b^4 + \frac{x+y}{v}c^4 \geq 64F^2$$

Proposed by D.M. Bătinețu-Giurgiu– Romania

J.1325 Let be $t \geq 0$ and ABC a triangle with the area F and M an interior point in triangle. If x, y, z are the distances from M to the apices A, B, C respectively, and u, v, w the distances from M to the sides BC, CA, AB respectively, then:

$$\frac{x^2 a^{2t+2}}{v \cdot w} + \frac{y^2 b^{2t+2}}{w \cdot u} + \frac{z^2 \cdot c^{2t+2}}{u \cdot v} \geq t^{t+1}\left(\sqrt{3}\right)^{1-t} F^{t+1}$$

Proposed by D.M. Bătinețu-Giurgiu– Romania

J.1326 If $m \in \mathbb{R}_+ = [0, \infty); x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{y+z}{x}\left(a+b-\sqrt{ab}\right)^{m+1} + \frac{z+x}{y}\left(b+c-\sqrt{bc}\right)^{m+1} + \frac{x+y}{z}\left(c+a-\sqrt{ca}\right)^{m+1} \geq$$

$$\geq 2^{m+2}\left(\sqrt[4]{3}\right)^{3-m}\left(\sqrt{F}\right)^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1327 If $m>0, M$ is an interior point in ABC triangle with the area F and x,y,z are the distances from M to the apices A,B,C and u,v,w the distances from M to the sides BC,CA,AB, then:

$$(x+y+z)\left(\frac{a^m}{v+w}+\frac{b^m}{w+u}+\frac{c^m}{u+v}\right)\geq 2^m\left(\sqrt[4]{3}\right)^{8-m}F^{\frac{m}{2}}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1328 Let be $a,b,c\in\mathbb{R}_+^*=(0,\infty)$ and $f:\mathbb{R}_+^*\to\mathbb{R}_+^*, f(x)=(a^2+x^2)(b^2+x^2)(c^2+x^2)$, then

$$f(x)+f(y)+f(z)\geq\frac{9}{4}(x^4+y^4+z^4)(ab+bc+ca)$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.1329 Let be $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{y+z}{x}m_a^2+\frac{z+x}{y}m_b^2+\frac{x+y}{z}m_c^2\geq\frac{8F^2}{R^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1330 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then:

$$\frac{(x+2y)(y+2z)}{(z+2x)(x+3y+2z)}+\frac{(y+2z)(z+2x)}{(x+2y)(y+3z+2x)}+\frac{(z+2x)(x+2y)}{(y+2z)(z+3x+2y)}\geq\frac{3}{2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

J.1331 If $m,n>0, p\geq 0$ then in ABC triangle the following inequality holds:

$$\frac{a}{(mh_a+nr)^p}+\frac{b}{(mh_b+nr)^p}+\frac{c}{(mh_c+nr)^p}\geq\frac{6\sqrt{3}}{(3m+n)^p r^{p-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.1332 Let x,y be positive real numbers. Prove that:

$$\frac{2xy}{x+y}\leq\sqrt{(1+x)(1+y)}$$

Proposed by Jalil Hajimir-Canada

J.1333 Let α,β,γ be the roots of the equation $2x^3-3x^2-12x-1=0$. Find the value of $\left[\sqrt{|\alpha|}\right]+\left[\sqrt{|\beta|}\right]+\left[\sqrt{|\gamma|}\right]$, where $[*]$ – is the greatest integer part of $*$.

Proposed by Jalil Hajimir-Canada

J.1334 Find the sum of the roots of:

$1-\frac{x[x]}{2}=\cos x$, where $[*]$ –is the greatest integer part of $*$.

Proposed by Jalil Hajimir-Canada

J.1335 In $\Delta ABC, \alpha \le 1$ prove that:

$$\left(\sum_{cyc} r_a h_a\right)^2 + \alpha r^3(R-2r) \le \left(\sum_{cyc} r_a^2\right)\left(\sum_{cyc} h_a^2\right)$$

Proposed by Nguyen Van Canh-Vietnam

J.1336 In $\Delta ABC, p_a$ –Spieker's cevian, $\lambda \le 4$ prove that:

$$\sum_{cyc} m_a^2 + \lambda r(R-2r) \le \sum_{cyc} p_a^2$$

Proposed by Nguyen Van Canh-Vietnam

J.1337 Find all functions

$$f:\mathbb{R}\to\mathbb{R}, f(x-y^2) = f(y-x^2) + (y-x)(y+x+1), \forall x,y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

J.1338 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}} \le \sqrt{6}\sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}}$$

Proposed by Nguyen Van Canh-Vietnam

J.1339 In $\Delta ABC, V$ –Bevan's point, I –incenter. Prove that:

$$[AIV]+[BIV]+[CIV] < 3R(R-r)$$

Proposed by Nguyen Van Canh-Vietnam

J.1340 Find all functions

$$f:(0,\infty)\to\mathbb{R}, f(x) = \max\left\{f\left(\frac{y}{2021}\right) - x^\beta y^\alpha \middle| \forall y \ge x\right\}, \alpha,\beta > 0$$

Proposed by Nguyen Van Canh-Vietnam

J.1341 If $a,b,c,d>0, a+b+c+d=4$ then

$$\left(\sum a^3\right)\left(\sum \frac{1}{a(bc+d)}\right) \ge 8$$

Proposed by George Apostolopoulos-Greece

J.1342 If $a,b,c>0, abc=1$ then:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{3(a+b+c)}{(ab+bc+ca)^2} \ge 2$$

Proposed by George Apostolopoulos-Greece

J.1343 If $a, b, c > 0, a^2 + b^2 + c^2 = 3$ then:

$$2(ab + bc + ca) - \frac{3abc(a + b + c)}{ab + bc + ca} \leq 3$$

Proposed by George Apostolopoulos-Greece

J.1344 In ΔABC the following relationship holds:

$$\sum \frac{a}{2b + c} - \frac{1}{9}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leq \frac{R}{2r}$$

Proposed by George Apostolopoulos-Greece

J.1345 In ΔABC the following relationship holds:

$$\frac{4Rr}{r^2} \geq \prod_{cyc}\left(\frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C\right)$$

Proposed by Bogdan Fuștei-Romania

J.1346 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$\sum_{cyc} \frac{m_a}{h_a} \cdot x \geq \sqrt{xy + yz + zx} \cdot \sqrt{\left(\sum_{cyc} \sqrt{\frac{m_a}{h_a}}\right) \prod_{cyc}\left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} - \sqrt{\frac{m_c}{h_c}}\right)}$$

Proposed by Bogdan Fuștei-Romania

J.1347 For $x, y, z, u, v, w > 0$ prove that:

$$x(v + w) + y(w + u) + z(u + v) \geq 2\sqrt{(xy + yz + zx)(uv + vw + wu)}$$

Proposed by Bogdan Fuștei-Romania

J.1348 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\sum_{cyc} \frac{y + z}{\sqrt{a}} \geq ax + by + cz + 2\sqrt{r(4R + r)(xy + yz + zx)}$$

Proposed by Bogdan Fuștei-Romania

J.1349 For $x, y, z > 0$ prove that:

$$1 + 2 \cdot \frac{x^2 + y^2 + z^2}{xy + yz + zx} \geq \sqrt{3 \sum_{cyc} \frac{x^2 + xy + y^2}{y^2 + yz + z^2}}$$

Proposed by Bogdan Fuștei-Romania

J.1350 In ΔABC the following relationship holds:

$$\frac{s}{r}-\frac{n_a+n_b+n_c+h_a+h_b+h_c}{s}\geq\sum_{cyc}\cot A$$

Proposed by Bogdan Fuștei-Romania

J.1351 In ΔABC the following relationship holds:

$$n_a\geq\sqrt{r\left[(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-2(2r-a+h_a)\right]}$$

Proposed by Bogdan Fuștei-Romania

J.1352 Solve for real numbers:

$$|\cos x|+|\cos 2x|+|\cos 4x|+|\cos 4x|=\frac{12019}{10000}$$

Proposed by Rovsen Pirguliyev-Azerbaijan

J.1353 If $x\in\left(0,\frac{\pi}{2}\right), n\geq 2$ then:

$$(\sin^{2n}x+\tan^{2n}x)(\cos^{2n}x+\tan^{2n}x)+(\sin^{2n}x+\cot^{2n}x)(\cos^{2n}x+\cot^{2n}x)>2^{3-n}$$

Proposed by Rovsen Pirguliyev-Azerbaijan

J.1354 If in acute $\Delta ABC, AD, BE, CF$ –altitudes, H –orthocenter, $k\geq 1$ then:

$$HD^k+HE^k+HF^k\geq 3\left(\frac{2r^2}{R}\right)^k$$

Proposed by Rovsen Pirguliyev-Azerbaijan

J.1355 Solve for real numbers:

$$\left\{\frac{\tan x}{202}\right\}+\left\{\frac{\cot x}{202}\right\}+\left\{\frac{\tan x+1010}{2020}\right\}+\left\{\frac{\cot x+1010}{2020}\right\}=1, \{*\}=*-[*]-GIF.$$

Proposed by Rovsen Pirguliyev-Azerbaijan

J.1356 Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that $\forall n\in\mathbb{Z}, k\in\mathbb{R}, k$ –fixed:

$$kf^2(n)+kf(n)-f\big(f(n)\big)=kn^2+(k-1)n.$$

Proposed by Rovsen Pirguliyev-Azerbaijan

J.1357 In acute ΔABC the following relationship holds:

$$4\sum\frac{a}{b^2+c^2}\geq\sum\frac{b+c}{a^2+bc}+\frac{\sum b+c-a}{bc}$$

Proposed by Alex Szoros-Romania

J.1358 In acute ΔABC holds:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq\frac{a^2+b^2+c^2}{2abc}+\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}$$

Proposed by Alex Szoros-Romania

J.1359 Let $z_1, z_2, z_3 \in \mathbb{C}$ such that $z_1 + z_2 + z_3 = 0, |z_1^2 + z_2^2 + z_3^2| = 2$ and $\left|z_1^5 + z_2^5 + z_3^5\right| = 5$.

Find: $|z_1^7 + z_2^7 + z_3^7|$.

Proposed by Alex Szoros-Romania

J.1360 In ΔABC the following relationship holds:

$$\frac{r_a r_b}{r_c^2} + \frac{r_b r_c}{r_a^2} + \frac{r_c r_a}{r_b^2} + 3 \geq 3\left(\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c}\right)$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1361 In ΔABC the following relationship holds:

$$w_a r_a + w_b r_b + w_c r_c \leq s^2$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1362 In ΔABC the following relationship holds:

$$\frac{1}{r_a^n} + \frac{1}{r_b^n} + \frac{1}{r_c^n} \geq \frac{1}{h_a^n} + \frac{1}{h_b^n} + \frac{1}{h_c^n}, n \geq 1$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1363 If $x_1 = x_2 = 2, x_3 = 8, x_{n+3} + x_{n+1} = 2x_{n+2} + 2x_n, n \in \mathbb{N} - \{0\}$.

Find: $\Omega = x_{2020}$.

Proposed by Rahim Shahbazov-Azerbaijan

J.1364 In ΔABC the following relationship holds:

$$\frac{m_a^2}{a^2} + \frac{m_b^2}{b^2} + \frac{m_c^2}{c^2} \geq \frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab}$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1365 Solve for real numbers:

$$\frac{(x+2)^2 - 20}{2(x+1)} = \sqrt{(x+1)(x-3)}$$

Proposed by George Florin Şerban-Romania

J.1366 If $\sqrt{(x+y)(4-z)} + \sqrt{(y+z)(2-t)} + \sqrt{(z+t)(4-x)} + \sqrt{(t+x)(2-y)} = \sqrt{\frac{5s^2 - 24s + 144}{2}}, s = x + y + z + t$. Then find: $\Omega = \overline{xyzt}$.

Proposed by George Florin Şerban-Romania

J.1367 Find $n \in \mathbb{N}$ such that

$$\sqrt[3]{\frac{n+27}{(n+1)(n+8)}}\in\mathbb{Q}$$

Proposed by George Florin Şerban-Romania

J.1368 Solve for natural numbers:

$$(3x+xy+2)(3y+xy+2)=4(x+2)(y+2)\sqrt{xy}$$

Proposed by George Florin Şerban-Romania

J.1369 Solve for real numbers:

$$x+\frac{x}{\sqrt{x^2-1}}=2\sqrt{2+\sqrt{2}}$$

Proposed by George Florin Şerban-Romania

J.1370 In ΔABC holds:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{19s^2-3r^2-12Rr}{8s^3}$$

Proposed by George Florin Şerban-Romania

J.1371 Solve for real numbers:

$$(x^2+x+1)^2+1=\frac{(8x^2+8x+17)^3}{2160}$$

Proposed by George Florin Şerban-Romania

J.1372 In $\Delta ABC, K$ —Lemoine's point, G —centroid,Γ —Gergonne's point. Prove that:

(K,G,Γ) —collinears if and only if $(a-b)(b-c)(c-a)=0$ or $a=b=c$.

Proposed by Gheorghe Alexe, George Florin Şerban-Romania

J.1373 In ΔABC holds:

$$\frac{1}{r_a^2r_b}+\frac{1}{r_b^2r_c}+\frac{1}{r_c^2r_a}\geq\frac{4r}{3R^2}\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)$$

Proposed by George Florin Şerban-Romania

J.1374 Find $x\in\mathbb{Z}$ such that $\sqrt{(x-3)(5x-1)(5x^2-12x+3)}\in\mathbb{N}$.

Proposed by George Florin Şerban-Romania

J.1375 In ΔABC the following relationship holds:

$$m_a^2\geq\left(\frac{b^2+c^2}{4R}\right)^2+\frac{4(b-a)^2(b-c)^2}{ac}$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.1376 In ΔABC the following relationship holds

$$(ab+bc+ca)^2+2(a^2+b^2+c^2)\geq 16\sqrt{3}\cdot s^3r$$

Proposed by D.M.Bătinețu-Giurgiu –Romania

J.1377 In ΔABC the following relationship holds

$$\sum\frac{a^6}{(bx+cy)(a-b)^2(a-c)^2}>\frac{6\sqrt{3}}{x+y}r$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți –Romania

J.1378 If $x,y\in\mathbb{R}_+^*$ then in ΔABC the following relationship holds:

$$\frac{\cos^{2m+2}\frac{A}{2}}{\left(x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2}\right)^m}+\frac{\cos^{2m+2}\frac{B}{2}}{\left(x\sin^2\frac{C}{2}+y\sin^2\frac{A}{2}\right)^m}+\frac{\cos^{2m+2}\frac{C}{2}}{\left(x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}\right)^m}$$
$$\geq\frac{(4R+r)^{m+1}}{2R(x+y)^m(2R-r)^m}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți –Romania

J.1379 If $x,y\in(0,\infty)$, then in acute ΔABC the following relationship holds:

$$2(3m-2n)R^2+4Rrn+(m+n)r^2+(n-m)s^2\geq\frac{9}{2}\sqrt[3]{mn^2R^2(s^2-(2R+r)^2)^2}.$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1380 For $a,b,c>1$ prove that

$$\log_{ab^2c^2}a+\log_{a^2bc^2}b+\log_{a^2b^2c}c\geq\frac{3}{5}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1381 If $m\in[0,\infty),x,y,z,t\in(0,\infty)$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(xa^2+ym_b^2)^{m+1}}{(zw_c^2+tw_a^2)^m}\geq\frac{(4x+3y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}F$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1382 If $A=\left\{x\middle|x\in\mathbb{R},\frac{2x^2+2x-4}{x^2-x+2}\in\mathbb{N}\right\}$ then find

$$\Omega=card(A)+\sum_{x\in A}x^2$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1383 If $x,y\in\mathbb{R}_+$ then in ΔABC holds:

$$(4R+r)^3x+s^2(y-12xR)\geq 4s\cdot\sqrt[4]{3xs^2y^3r^3}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1384 If $x\in\mathbb{R}_+$ then in ΔABC holds:

$$\frac{m_a^{x+2}}{(Rm_b+rm_c)^x}+\frac{m_b^{x+2}}{(Rm_c+rm_a)^x}+\frac{m_c^{x+2}}{(Rm_a+rm_b)^x}\geq\frac{3\sqrt{3}}{(R+r)^x}F$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1385 If $a,b,c,m,n,p,x,y,z\in(0,\infty)$ such that $x+y+z=s(a+b+c)$, then prove that:

$$\frac{x^2}{a(mb+nc)}+\frac{y^2}{b(mc+na)}+\frac{z^2}{c(ma+nb)}\geq\frac{3s^2}{m+n}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1386 If $x,y,z>0$ then prove:

$$\frac{x^2}{z^3(zx+y^2)}+\frac{y^2}{x^3(xy+z^2)}+\frac{z^2}{y^3(yz+x^2)}\geq\frac{3}{2xyz}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1387 Prove that in any triangle:

$$\sqrt{m_a\cos^2\frac{A}{2}}+\sqrt{m_b\cos^2\frac{B}{2}}+\sqrt{m_c\cos^2\frac{C}{2}}\leq\frac{3}{2}\sqrt{4R+r}$$

Proposed by Rajeev Rastogi – India

J.1388 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x+y)\cdot f(x-y)+x(x+y)f(y)=\big(f(x)+f(y)\big)^2,\forall x,y\in\mathbb{R}$$

Proposed by Rajeev Rastogi – India

J.1389 Find a closed form without using Beta Function:

$$\Omega=\left(\int_0^1\sqrt[2019]{1-x^{2021}}\,dx\right)\left(\int_0^1\sqrt[2021]{1-x^{2019}}\,dx\right)^{-1}$$

Proposed by Rajeev Rastogi – India

J.1390 $x,y,z>0,(x+y+z)^3=32xyz$. Find:

$$\Omega=\min\left(\frac{(x^2+y^2+z^2)^3}{(x+y+z)^6}\right)$$

Proposed by Rajeev Rastogi – India

J.1391 Solve for real numbers:

$$\begin{cases}a+b+c=2\\ab+2c=bc+2a=ca+2b\end{cases}$$

Proposed by Rajeev Rastogi – India

J.1392 S_p – Spieker point in $\Delta ABC, d_a=d(S_p,BC),d_b=d(S_p,CA),d_c=d(S_p,AB)$

Prove that:

$$d_a + d_b + d_c = \frac{s^2 + r^2 - 2Rr}{4R}$$

Proposed by Mehmet Șahin – Turkey

J.1393

$$P \in Int\,(\Delta ABC), E, F \in [AB], M, D \in [BC], K, L \in [CA]$$

$AF = AK, BE = BD, CM = CL, (F, P, K), (M, P, L), (E, P, D)$ – collinears

$PX \perp MD, PY \perp KL, PZ \perp EF, PX = h_1, PY = h_2, PZ = h_3$. Prove that:

$$h_1 + h_2 + h_3 \leq \frac{3}{4}\sqrt{ML^2 + FK^2 + DE^2}$$

Proposed by Mehmet Șahin - Turkey

J.1394 S_p – Spieker point in ΔABC. If $D \in (BC), E \in (CA), F \in (AB), DS_p = d_a, ES_p = d_b$, $FS_p = d_c$ then:

$$\frac{1}{h_a d_a} + \frac{1}{h_b d_b} + \frac{1}{h_c d_c} \leq \frac{1}{r^2}$$

Proposed by Mehmet Șahin - Turkey

J.1395 In ΔABC the following relationship holds:

$$\frac{a^3}{m_a} + \frac{b^3}{m_b} + \frac{c^3}{m_c} \geq 12\sqrt{3}Rr$$

Proposed by Mehmet Șahin - Turkey

J.1396 If $x, y, z \geq 1$ then:

$$\frac{xy\sqrt{z}}{x^3 + y^3} + \frac{yz\sqrt{x}}{y^3 + z^3} + \frac{zx\sqrt{y}}{z^3 + x^3} \leq \frac{3}{2}$$

Proposed by Mehmet Șahin – Turkey

J.1397 In acute $\Delta ABC, I$ – incenter, O – circumcenter

$m(\sphericalangle IAO) = x, m(\sphericalangle IBO) = y, m(\sphericalangle ICO) = z$. Prove that:

$$1 + \frac{4r}{R} \leq \cos^2 x + \cos^2 y + \cos^2 z \leq 2 + \frac{3r}{2R} + \left(\frac{r}{R}\right)^2$$

Proposed by Eldeniz Hesenov – Georgia

J.1398 I_a, I_b, I_c – excenters in ΔABC. Prove that:

$$\frac{1}{[ABI_c]} + \frac{1}{[BCI_a]} + \frac{1}{[CAI_b]} \geq \frac{3}{F}$$

Proposed by Eldeniz Hesenov - Georgia

J.1399 In ΔABC the following relationship holds:

$$2R\sum_{cyc}\frac{1}{w_a}\cos\frac{B-C}{2}\geq 3+\sum_{cyc}\left(\frac{a}{b+c}\right)^2$$

Proposed by Eldeniz Hesenov - Georgia

J.1400 O – circumcenter, Ω – first Brocard's point in $\Delta ABC, O_a, O_b, O_c$ – circumcenters of $\Delta B\Omega C, \Delta C\Omega A, \Delta A\Omega B$. Find $f(r), g(R)$ such that:

$$27\cdot\max f(r)\leq OO_a\cdot h_c+OO_b\cdot h_a+OO_c\cdot h_b\leq\frac{9}{2}\cdot\min g(r)$$

Proposed by Eldeniz Hesenov - Georgia

J.1401 In ΔABC the following relationship holds:

$$\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{h_b}{r_b}}+\sqrt{\frac{h_c}{r_c}}\leq\sqrt{\frac{R}{2r}\left(\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\right)}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1402 In ΔABC the following relationship holds:

$$\frac{m_am_bm_c(m_a+m_b+m_c)}{9F^2}\geq\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1403 In ΔABC the following relationship holds:

$$\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\geq 1+\frac{4r}{R}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1404 $\Delta DEF, \Delta MNK$ –circumcevian triangle of orthocenter and orthic triangle of acute ΔABC.Prove that: $[MNK]\leq[DEF]$

Proposed by Adil Abdullayev-Azerbaijan

J.1405 Prove that in any triangle ABC the following inequality holds:

$$\sum_{cyc}\frac{\sin\frac{A}{2}}{1+\cos A}\geq 1$$

Proposed by Neculai Stanciu – Romania

J.1406 If $a,b,c>0$ and $n,k>0$ such that $4k\geq n^2$ then:

$$\sum\sqrt{a^2+nab+kb^2}\geq\sqrt{n+k+1}(a+b+c)$$

Proposed by Marin Chirciu – Romania

J.1407 In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\le\frac{s\sqrt{3}-AI-BI-CI}{r}$$

Proposed by Bogdan Fuștei – Romania

J.1408 Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ and $\alpha\beta\gamma\delta\neq 0$. Exists a function
$f(x)=ax^3+\beta x^2+\gamma x+\delta$ such that $|f(x)|\le 1,\forall x\in[-1,1]$?

Proposed by Nguyen Van Canh-Vietnam

J.1409 If $x,y,z>0$ then:

$$\sum\sqrt[3]{x^5(y+z)}\ge\frac{5}{4}(xy+yz+zx)$$

Proposed by Marin Chirciu – Romania

J.1410 If $x,y,z,t>0$ then:

$$xyzt(x+y+z+t)^2\le 2(xy+zt)(xz+yt)(xt+yz)$$

Proposed by Daniel Sitaru-Romania

J.1411 Solve for real numbers:

$$\begin{cases}xy+yz+zx=-7\\ \sum_{cyc}x^2(y-z)(y^2+z^2-x^2)=0\\ xyz=6\end{cases}$$

Proposed by Daniel Sitaru,Claudia Nănuți-Romania

J.1412 Solve for real numbers:

$$\begin{cases}x+y+z=xyz\\ \frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=0\end{cases}$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

J.1413 If $x,y,a,b>0$ then:

$$\frac{2x^2}{\sqrt{ab}}+\frac{4y^2}{a+b}\ge\frac{4(x+y)^2}{\left(\sqrt{a}+\sqrt{b}\right)^2}+\sqrt{ab}\left(\frac{x}{\sqrt{ab}}-\frac{2y}{a+b}\right)^2$$

Proposed by Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.1113 In ΔABC the following relationship holds:

$$\left(2-\frac{r}{R}\right)^2 \leq \frac{r_b r_c}{a^2}+\frac{r_c r_a}{b^2}+\frac{r_a r_b}{c^2} \leq \left(\frac{R}{r}\right)^2-\frac{R}{r}+\left(\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

S.1114 Find the value of α, such that:

$$\int_0^{\frac{\pi}{2}} \sin\left(\frac{x}{4}\right)\tanh^{-1}(\sin(x))\,dx = \log(\alpha)$$

Proposed by K. Srinivasa Raghava – India

S.1115 Let be the function $f:[0,\infty)\to[0,\infty)$ continuous such that:

$\left(f(f(x))\right) = (2^{x+1}+x-1)f(x), \forall x \geq 0$. Prove that: f invertible and find

$$\Omega = \lim_{x\to 0}\frac{f^{-1}(x)}{x}$$

Proposed by Florică Anastase – Romania

S.1116 Find F if in ΔABC holds:

$$\begin{cases} c^2 = 8+\sqrt{2a^2-13a+21}+\sqrt{8b^2-14b+5} \\ 64c\cdot\cos\frac{\pi}{18}\cos\frac{5\pi}{18}\cos\frac{7\pi}{18} = 6b+3a+24 \end{cases}$$

Proposed by Samir Cabiyev-Azerbaijan

S.1117 If $2\cdot E\left(\frac{x}{x-1}\right)-5\cdot E\left(\frac{5x-4}{4x+1}\right)=\frac{-10x^2-23x+8}{5x^2-4x}, (\forall)x\in R\setminus\left\{0,\frac{4}{5},1,-\frac{1}{4},\frac{1}{5},-4,\frac{5}{4}\right\}$

Then find $E(2021)$

Proposed by Neculai Stanciu, George Florin Șerban – Romania

S.1118 If $x,y,z,t>0$ such that $\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}+\frac{t}{1+t}=3$ then:

$$\sqrt{xy}+\sqrt{xz}+\sqrt{xt}+\sqrt{yz}+\sqrt{yt}+\sqrt{zt}\leq 2\sqrt{xyzt}$$

Proposed by Marin Chirciu – Romania

S.1119 If $f:\mathbb{N}^*\to\mathbb{Q}_+^*, f(n)=\frac{\sum_{i=1}^n f(i)}{n^2}, (\forall)n\in\mathbb{N}^*$ and $f(1)=\frac{1}{2}$, then find

$$\sum_{i=1}^{2021} f(i)$$

Proposed by Neculai Stanciu – Romania

S.1120 In acute ΔABC holds:

$$\sum_{cyc} \tan^2 A \cdot \cot^5 B \geq \frac{1}{\sqrt{3}}\left(\frac{8}{3}\left(2-\frac{r}{R}\right)^3 - 2\left(\frac{R}{r}\right)^2\right)$$

Proposed by Marian Ursărescu – Romania

S.1121 In ΔABC the following relationship holds:

$$2\left(\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\right) \geq 2+\left(\frac{a}{m_a}\right)^2+\left(\frac{b}{m_b}\right)^2+\left(\frac{c}{m_c}\right)^2$$

Proposed by Eldeniz Hesenov-Georgia

S.1122 In scalene ΔABC. True or false:

$$\frac{1}{2r}\sqrt{\sum_{cyc}\frac{1}{(r_a-r_b)^2}} = \sum_{cyc}\frac{1}{(a-b)(a+b-c)}$$

Proposed by Ertan Yildirim-Turkiye

S.1223 In $\Delta ABC, S_p$ – Spieker point, the following relationship holds:

$$a \cdot AS_p + b \cdot BS_p + c \cdot CS_p \leq 3R\sqrt{5R^2-4Rr}$$

Proposed by Mehmet Șahin – Turkyie

S.1224 If $a+b=1, a, b>0$ and $n \in \mathbb{N}$ then prove:

$$\sum_{k=1}^{n}\frac{1}{a^k+b^k} \leq 2^n-1$$

Proposed by Amrit Awasthi-India

S.1225 $\Omega_1(n)=\int_{\sqrt[n]{2}}^{\sqrt[n]{8}} x^{n+1}\sqrt{x^n\sqrt{x^{n-1}\ldots\sqrt{x}}}\,dx, \Omega_2(n)=\int_1^2 x\sqrt{x^2\sqrt{x^3\ldots\sqrt{x^{n+1}}}}\,dx, n \in \mathbb{N}^*, n \geq 2$ then find: $\Omega = \lim_{n\to\infty} 2021^{\frac{\Omega_1(n)}{\Omega_2(n)}}$.

Proposed by Costel Florea-Romania

S.1226 In ΔABC holds the following relationship holds:

$$\sum \frac{1}{r_a^{2n} r_b^n} \geq \frac{4^n r^n}{3^{2n-1}R^{2n}}\left(\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)^n, n \in \mathbb{N}^*$$

Proposed by George Florin Șerban-Romania

S.1227 Solve for real numbers:

$$y^6+6y^5+18y^4+32y^3+36y^2+24y=117, y=\log(\log x)$$

Proposed by Haxverdiyev Tarverdi-Azerbaijan

S.1228 If $a,b,c>0, ab+bc+ca=3$ then

$$\frac{a^a}{11+2b^2+3c^2}+\frac{b^b}{11+2c^2+2a^2}+\frac{c^c}{11+2a^2+3b^2}\geq\frac{3}{16}$$

Proposed by Pavlos Trifon-Greece

S.1229 In ΔABC the following relationship holds:

$$\sqrt{\frac{a}{c}}\cos A+\sqrt{\frac{b}{a}}\cos B+\sqrt{\frac{c}{a}}\cos C\leq\frac{3R}{4r}$$

Proposed by Ionuț Florin Voinea-Romania

S.1230 If $a_1=1$ and $a_{N+1}(1+a_n)=\sqrt{n}$ for all $n\geq 1$, then: $\sum_{k=1}^n a_k^4\leq\frac{n^2-n+2}{2}\leq\sum_{k=1}^n a_{k^4}$

and

$$\lim_{n\to\infty}\sqrt[4]{n}\left(\frac{a_n^4}{n}-1\right)\in(-4,0]$$

Proposed by Mihály Bencze-Romania

S.1231 In ΔABC the following relationship holds:

$$\sum\frac{m_a r_a}{r_b+r_c}\geq 9r\cdot\sqrt[3]{\frac{r}{4R}}$$

Proposed by Celal Ersoy-Turkiye

S.1232 Solve for real numbers: $\sqrt{2019-135x-225x^2-125x^3}=$

$$\sqrt{179+945x+1575x^2+875x^3}+\sqrt{113+540x+900x^2+500x^3}$$

Proposed by Mokhtar Kassani-Algerie

S.1233 In acute $\Delta ABC, H$ –orthocenter, the following relationship holds:

$$\frac{HA}{HA_1}+\frac{HB}{HB_1}+\frac{HC}{HC_1}\geq 4\left(\frac{HA_1}{HA}+\frac{HB_1}{HB}+\frac{HC_1}{HC}\right).$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1234 In ΔABC the following relationship holds:

$$\left(\frac{a}{m_a}+\frac{b}{m_b}+\frac{c}{m_c}\right)(\cot A+\cot B+\cot C)\geq 6$$

Proposed by Eldeniz Hesenov-Georgia

S.1235 Solve for real numbers: $\begin{cases} x+y+z=\pi, x,y,z>0 \\ \cos x+\cos(y-z)=\sqrt{2(2+\sqrt{3})}\cdot\sin x\cdot\tan x \\ \cos 2x-\cos 2y-\cos 2z+1=(3+\sqrt{3})\cdot\sin 2x\cdot\tan x \end{cases}$

Proposed by Samir Cabiyev-Azerbaijan

S.1236 Prove that:

$$\frac{1}{\sin^{14}\left(\frac{\pi}{7}\right)}+\frac{1}{\sin^{14}\left(\frac{2\pi}{7}\right)}+\frac{1}{\sin^{14}\left(\frac{3\pi}{7}\right)}+\cdots+\frac{1}{\sin^{14}\left(\frac{13\pi}{7}\right)}+1=\frac{12019}{4096}$$

Proposed by Sergio Esteban-Argentina

S.1237 If $a,b,c,n>0, a+b+c=1$ then

$$e^n(a+1)^{nb}(c+1)^{nc}(c+a)^{na}<e^4(na)^{na}(nb)^{nb}(nc)^{nc}.$$

Proposed by Nikos Ntorvas-Greece

S.1238 If $a,b,c>0, \frac{a}{3}<b<\frac{a}{2}, c=\frac{2b-a}{a-3b}, 0\le x\le 1$ then:

$$(a+b)\sqrt{c(c+1)(x-x^2)}+\sqrt{ab(x+x^2)}\le\frac{(a+b)(c+1)+b}{2}$$

Proposed by Carlos Paiva-Brazil

S.1239 In ΔABC the following relationship holds:

$$\frac{\sqrt[4]{a^2b^2(w_a^4+w_b^4)}+\sqrt[4]{b^2c^2(w_b^4+w_c^4)}+\sqrt[4]{c^2a^2(w_c^4+w_a^4)}}{\sin 2A+\sin 2B+\sin 2C}\ge 3\sqrt[4]{2}R^2$$

Proposed by Haxverdiyev Tarverdi-Azerbaijan

S.1240 Let $a,b\in\mathbb{R}$. Prove that:

$$\frac{a+b-1}{\sqrt{a+b}}\ge\log(a+b)$$

Proposed by Jalil Hajimir-Canada

S.1241 Let be $m,n,x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{mx+ny}{mn+my+(m+n)z}a^2+\frac{my+nz}{ny+mz+(m+n)x}b^2+\frac{mz+nx}{nz+mx+(m+n)y}c^2\ge 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1242 If $a,b,c,x>0$, then:

$$(ax^2+bx+c)(a+bx+cx^2)(bx^2+ax+c)(b+ax+cx^2)\ge(a+b+c)^4x^4$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1243 If $x, y, z \in (0,1)$, then:

$$\frac{1}{(y+z)(1-x^2)}+\frac{1}{(z+x)(1-y^2)}+\frac{1}{(x+y)(1-z^2)} \geq \frac{9\sqrt{3}}{4}$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

S.1244 If $x, y, z \in (0,1)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{a^4}{(1-x^2)(y+z)}+\frac{b^4}{(1-y^2)(z+x)}+\frac{c^4}{(1-z)(x+y)} \geq 12\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1245 If M is an interior point in ABC triangle with the area F and x, y, z are the distances from M to the apices A, B, C and u, v, w the distances from M to the sides BC, CA, AB then:

$$(x+y+z)\left(\frac{a^2}{v+w}+\frac{b^2}{w+u}+\frac{c^2}{u+v}\right) \geq 12\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.1246 If $x, y, z > 0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{x^2a^3}{h_a}+\frac{y^2b^3}{h_b}+\frac{z^2c^3}{h_c} \geq \frac{8}{3}(xy+yz+zx)F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.1247 If $x, y, z, d \in \mathbb{R}_+^*$ and $x \cdot y \cdot z \geq d^3$ then in any ABC triangle with the area F the following inequality holds:

$$x \cdot ab + y \cdot bc + z \cdot ca \geq 4\sqrt{3}dF$$

Proposed by D.M. Bătinețu-Giurgiu– Romania

S.1248 Let be $t \geq 0$ and ABC any triangle, then if $x, y, z > 0$ the following inequlality holds:

$$\frac{x \cdot m_a^t}{(y+z)h_a^{t+2}}+\frac{y \cdot m_b^t}{(z+x)h_b^{t+2}}+\frac{z \cdot m_c^t}{(x+y)h_c^{t+2}} \geq \frac{2\sqrt{3}}{2F}$$

where F is the triangle's area.

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.1249 Let be $m \in \mathbb{R}_+ = [0,\infty), n \in \mathbb{N}$ and $a, b, c, x, y \in \mathbb{R}_+^* = (0,\infty)$, then:

$$3n+\frac{a^{n+1}}{(bx+cy)^{(m+1)(n+1)}}+\frac{b^{n+1}}{(cx+ay)^{(m+1)(n+1)}}+\frac{c^{n+1}}{(ax+by)^{(m+1)(n+1)}} \geq$$

$$\geq \frac{3^{m+1}(n+1)}{(x+y)^{m+1}(a+b+c)^m}$$

Proposed by D.M. Bătinețu-Giurgiu– Romania

S.1250 If $m, a, b, c \in \mathbb{R}_+^* = (0, \infty)$, then:

$$(a^2 + 2m^2)(b^2 + 2m^2)(c^2 + 2m^2) \geq 3m^2(a + b + c)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1251 In any ABC triangle the following inequality holds:

$$\frac{a}{h_b} + \frac{b}{h_c} + \frac{c}{h_a} \geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1252 In any ABC triangle, the following inequality holds:

$$\frac{(b+c)^4}{w_b w_c} + \frac{(c+a)^4}{w_c w_a} + \frac{(a+b)^4}{w_a w_b} \geq 768r^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1253 Let be M an interior point of ABC triangle with the area F and x_a, x_b, x_c the distances from M to the apices A, B, C and d_a, d_b, d_c the distances from M to the sides BC, CA, AB, then:

$$\frac{a^3 \cdot x_a^2}{d_a} + \frac{b^3 \cdot x_b^2}{d_b} + \frac{c^3 \cdot x_c^2}{d_c} \geq \frac{32}{3}(d_a d_b + d_b d_c + d_c d_a)F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1254 In ΔABC the following relationship holds:

$$\frac{m_a w_a + m_b w_b + m_c w_c}{h_a h_b + h_b h_c + h_c h_a} \geq \left(\frac{r_a + r_b + r_c + s\sqrt{3}}{m_a + m_b + m_c + h_a + h_b + h_c}\right)^2$$

Proposed by Bogdan Fuștei – Romania

S.1255 In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\sum_{cyc}\left(\frac{m_a}{w_a} + \sqrt{\frac{m_a}{r_a}} + \sqrt{\frac{h_a}{h_b}} + \sqrt{\frac{h_b}{h_c}}\right) \leq \frac{2(AI + BI + CI)}{r}$$

Proposed by Bogdan Fuștei – Romania

S.1256 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\frac{n_a - g_a}{r_b + r_c - 2h_a} \leq \sum_{cyc}\frac{r_a}{n_a}$$

Proposed by Bogdan Fuștei – Romania

S.1257 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\frac{1}{2}\left(\frac{r}{R}+\sum_{cyc}\frac{h_b+h_c}{h_a}\right)\geq 1+\sum_{cyc}\frac{n_a g_a}{bc}$$

Proposed by Bogdan Fuștei – Romania

S.1258 In $\Delta ABC, \omega$ – Brocard's angle, n_a – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$4\cos\omega\sqrt{\sum_{cyc}a^2b^2}\geq 2r(r+4R)+\sum_{cyc}(n_a n_b+g_a g_b)$$

Proposed by Bogdan Fuștei – Romania

S.1259 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\cot\frac{A}{4}=\frac{n_a+n_b+n_c}{3r}+\frac{2}{3}\cdot\sum_{cyc}\frac{2r_a+h_a}{n_a+s}+\sum_{cyc}\frac{w_b+w_c}{h_a}$$

Proposed by Bogdan Fuștei – Romania

S.1260 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$n_a n_b+n_b n_c+n_c n_a\geq\sqrt[3]{n_a^2 n_b^2 n_c^2}+2\sqrt[3]{h_a h_b h_c r_a r_b r_c}$$

Proposed by Bogdan Fuștei – Romania

S.1261 In ΔABC the following relationship holds:

$$\frac{m_a^2}{h_a^2}\geq 1+\frac{3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}$$

Proposed by Bogdan Fuștei – Romania

S.1262 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{1+\frac{a(m_a-h_a)}{2r^2}}\geq\sum_{cyc}\frac{n_a}{h_a}$$

Proposed by Bogdan Fuștei – Romania

S.1263 In $\Delta ABC, n_a$ – Nagel's cevian, I – incenter, the following relationship holds:

$$\frac{a}{2r}\geq\frac{1}{\sqrt{2}}\left(\frac{m_a}{h_a}+\frac{|b-c|}{2r}\sqrt{1-\frac{AI^2}{w_a^2}}\right)+\frac{2r_a}{n_a+s}$$

Proposed by Bogdan Fuștei – Romania

S.1264 In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\frac{m_a m_b m_c}{r_a r_b r_c} \le \frac{n_a n_b + n_b n_c + n_c n_a}{h_a h_b + h_b h_c + h_c h_a}$$

Proposed by Bogdan Fuștei – Romania

S.1265 In $\Delta ABC, n_a$ – Nagel's cevian, I – incenter, the following relationship holds:

$$\frac{s^2}{2r^2} - \frac{3}{2} \le \frac{R}{r}\left(3 + \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c}\right)$$

Proposed by Bogdan Fuștei – Romania

S.1266 In ΔABC the following relationship holds:

$$\frac{1}{2}\sum \frac{n_a + g_a}{w_a} \ge \sqrt{\frac{2r}{R}} \cdot \sum \frac{m_a}{h_a}$$

Proposed by Bogdan Fuștei-Romania

S.1267 In ΔABC the following relationship holds:

$$\sum \frac{1}{1 - \cos A} \ge 2\left(\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1268 In ΔABC the following relationship holds:

$$\prod \left(\frac{2m_a}{h_a} - \sin^2 \frac{A}{2}\right) \ge \prod \frac{n_a^2 + g_a^2}{a^2}$$

Proposed by Bogdan Fuștei-Romania

S.1269 In ΔABC the following relationship holds:

$$\frac{\sqrt{R}}{2r} \cdot \sum \sqrt{m_a + h_a} \ge \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1270 In ΔABC the following relationship holds:

$$\frac{n_a n_b + n_b n_c + n_c n_a}{h_a h_b + h_b h_c + h_c h_a} \ge \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c}\right)^2$$

Proposed by Bogdan Fuștei-Romania

S.1271 In $\Delta ABC, I$ –incenter, the following relationship holds:

$$\prod \left(\frac{4(n_a + g_a - m_a)}{AI}\right) \ge \prod \left(2 + \frac{a}{b} + \frac{a}{c}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1272 In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{n_a g_a(n-a+g_a)}{2w_a}-r_a r_b}\geq\frac{1}{2}\sum|b-c|$$

Proposed by Bogdan Fuștei-Romania

S.1273 In ΔABC the following relationship holds:

$$\sum n_a g_a(r_a-r)\geq 2r_a r_b r_c$$

Proposed by Bogdan Fuștei-Romania

S.1274 In ΔABC the following relationship holds:

$$\sum w_a\geq s+\left(6-3\sqrt{3}+\sum\frac{w_a}{h_a}\right)r$$

Proposed by Bogdan Fuștei-Romania

S.1275 In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{R}{r_a-r}}\geq\sqrt{\frac{m_a}{r_a}}+\sqrt{\frac{m_b}{r_b}}+\sqrt{\frac{m_c}{r_c}}$$

Proposed by Bogdan Fuștei-Romania

S.1276 In ΔABC the following relationship holds:

$$\sum\frac{n_a g_a(r_a-r)}{h_b h_c}\geq 3R$$

Proposed by Bogdan Fuștei-Romania

S.1277 In ΔABC the following relationship holds:

$$\sum\frac{g_a h_a}{w_a}\geq\sqrt{\frac{2r}{R}}\cdot\sum\frac{r_b r_c}{n_a}$$

Proposed by Bogdan Fuștei-Romania

S.1278 In ΔABC the following relationship holds:

$$\sum\frac{r_a+r}{r_a-r}=\sum\frac{h_a}{w_a}\sqrt{\frac{h_a}{r_a}}\cdot\sum\frac{r_a-r}{w_a}\cdot\sqrt{\frac{h_a}{r_a}}$$

Proposed by Bogdan Fuștei-Romania

S.1279 In ΔABC the following relationship holds:

$$\sum\frac{r_b+r_c}{a}\geq\frac{1}{2}\sum\frac{b+c}{\sqrt{(r_b-r)(r_c-r)}}$$

Proposed by Bogdan Fuștei-Romania

S.1280 In $\Delta ABC, I$ —incenter, the following relationship holds:

$$\sum \frac{AI}{h_a - r} \geq \sum \frac{s_a}{h_a}$$

Proposed by Bogdan Fuștei-Romania

S.1281 Solve for real numbers:

$$\sqrt[9]{2x^{27} - 2(x^3 - x^2 + 3x - 2)^9} = 2(x^2 - 3x + 2)$$

Proposed by Daniel Sitaru-Romania

S.1282 Find without any software:

$$\Omega = \int \frac{3xe^x + 12}{(2\log x + 3e^x - 1)(2\log x^x + 3xe^x + x)} dx$$

Proposed by Daniel Sitaru-Romania

S.1283 If $a, b, c > 1$ then find:

$$\Omega = \lim_{x\to 0} (\log_a(\log_b(\log_c(c^b \cdot e^{-x})))) (\log_c(\log_b(\log_a(a^b \cdot e^{-x})))^{-1}$$

Proposed by Daniel Sitaru-Romania

S.1284 In $\Delta ABC, K$ —Lemoine's point, the following relationship holds:

$$\frac{AK}{\sqrt{BK}} + \frac{BK}{\sqrt{CK}} + \frac{CK}{\sqrt{AK}} \geq 3\sqrt{2}r$$

Proposed by Daniel Sitaru-Romania

S.1285 If $a, b, c > 0$ then:

$$abc(a+b)^2(b+c)^2(c+a)^2 \leq 64\left(\frac{a+b+c}{3}\right)^9$$

Proposed by Daniel Sitaru-Romania

S.1286 If $x, y, z \geq 1$ and $n \in \mathbb{N}, n \geq 2$ then:

$$\sqrt{x^n - 1} + \sqrt{y^n - 1} + \sqrt{z^n - 1} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$$

Proposed by Marin Chirciu – Romania

S.1287 In ΔABC the following relationship holds:

$$\left(\frac{1}{2} + \sec\frac{A}{2}\right)^2 + \left(\frac{1}{2} + \sec\frac{B}{2}\right)^2 + \left(\frac{1}{2} + \sec\frac{C}{2}\right)^2 \geq \frac{19}{4} + 2\sqrt{3}$$

Proposed by Marin Chirciu – Romania

S.1288 $a > b + 2 > c > d > 0$. Solve for real numbers:

$$(a+6)^x + b^x + c^x + d^x = a^x + (b+2)^x + (c+2)^x + (d+2)^x$$

Proposed by Marin Chirciu – Romania

S.1289 If $a,b,c,d>0$ such that $a\sqrt[3]{bcd}+b\sqrt[3]{cda}+c\sqrt[3]{dab}+d\sqrt[3]{abc}\geq a+b+c+d$ then find the minimum of the expression:

$$E=a+b+c+d$$

Proposed by Marin Chirciu – Romania

S.1290 In ΔABC the following relationship holds:

$$\frac{9R}{2}\leq\sum\frac{m_b^2+m_c^2}{r_b+r_c}\leq\frac{9R^2}{4r}$$

Proposed by Marin Chirciu – Romania

S.1291 If $x,y,z>0$ such that $x\sqrt[3]{yz^2}+y\sqrt[3]{zx^2}+z\sqrt[3]{xy^2}\geq x+y+z$ then find the minimum of the expression:

$$E=x+y+z$$

Proposed by Marin Chirciu – Romania

S.1292 If $a,b,c>0$ and $\lambda\geq 0$ then:

$$\frac{1+a^2}{\lambda+b+\lambda c^2}+\frac{1+b^2}{\lambda+c+\lambda a^2}+\frac{1+c^2}{\lambda+a+\lambda b^2}\geq\frac{6}{2\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.1293 In ΔABC the following relationship holds:

$$\sum\frac{m_b^2+m_c^2}{h_b+h_c}\geq\frac{2r}{R}\sum\frac{m_b^2+m_c^2}{r_b+r_c}$$

Proposed by Marin Chirciu – Romania

S.1294 In ΔABC the following relationship holds:

$$9r\leq\sum\sqrt{\frac{h_b^2+h_c^2}{2}}\leq 4R+r$$

Proposed by Marin Chirciu – Romania

S.1295 In $\Delta ABC, o_a$ – circumcevian. Prove that:

$$\frac{2r^2}{R^2}+\frac{3r}{R}+4\leq\sum\frac{h_a+r_a}{o_a}\leq\frac{2R}{r}+\frac{13r}{R}-\frac{2r^2}{R^2}-4$$

Proposed by Marin Chirciu – Romania

S.1296 In $\Delta ABC, o_a$ – circumcevian. Prove that:

$$11-\frac{2r}{R}-\frac{2R}{r}\leq\sum\frac{r_b+r_c}{o_a}\leq 5+\frac{2r}{R}$$

Proposed by Marin Chirciu – Romania

S.1297 $\Omega(n) = (1+2^2)(1+2^4)(1+2^8)\cdot\ldots\cdot\left(1+2^{2^{n-1}}\right), n \geq 1$

Find:

$$\Omega = \lim_{n\to\infty}\left(1+\frac{1}{3\Omega(n)}\right)^{2^{2^n}}$$

Proposed by Daniel Sitaru – Romania

S.1298 If $a,b,c,d > 0, a+b+c+d = 4$ then:

$$3^{12}\cdot a^a\cdot b^b\cdot c^c\cdot d^d \geq (4-a)^{4-a}\cdot(4-b)^{4-b}\cdot(4-c)^{4-c}\cdot(4-d)^{4-d}$$

Proposed by Daniel Sitaru – Romania

S.1299 If $X,Y,Z,U,V,W \in M_5(\mathbb{C})$ then:

$$rank\,(XYZUVW) - rank\,(WVUZYX) \leq 4$$

Proposed by Daniel Sitaru – Romania

S.1300 If $X \in M_{3,4}(\mathbb{C}), Y \in M_{4,2}(\mathbb{C}), Z \in M_{2,3}(\mathbb{C}), \det(XYZ) \neq 0, rank\,Y = 1$ then:

$$rank\,(XY) + rank\,(YZ) \leq 4$$

Proposed by Daniel Sitaru – Romania

S.1301 If $x \leq y \leq e \leq z \leq t$ then:

$$e^x + e^y + e^z + e^t \geq e^{x+y-e} + 3\sqrt[3]{e^{z+t+e}}$$

Proposed by Daniel Sitaru – Romania

S.1302 If $e < a \leq b$ then:

$$4^{a+3b}\cdot(a+3b)^{3a+b} \leq 4^{3a+b}\cdot(3a+b)^{a+3b}$$

Proposed by Daniel Sitaru – Romania

S.1303 If $a,b,c,d > 0, a+b+c+d = 1$ then:

$$a^4+b^4+c^4+d^4+252abcd \leq 1$$

Proposed by Daniel Sitaru – Romania

S.1304 Solve for real numbers:

$$\frac{256^{x^2}}{256^y}+\frac{256^{y^2}}{256^z}+\frac{256^{z^2}}{256^t}+\frac{256^{t^2}}{256^x} = 1$$

Proposed by Daniel Sitaru – Romania

S.1305 Find $x,y,z > 0$ such that:

$$\frac{6(x+y)}{\sqrt{xy}}+\frac{6(x+y+z)}{\sqrt[3]{xyz}}+\frac{6\sqrt{xy}}{x+y}+\frac{6\sqrt[3]{xyz}}{x+y+z} = 35$$

Proposed by Daniel Sitaru – Romania

S.1306 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \frac{x^2}{4y^2}+\frac{y^2}{6z^2}+\frac{19}{12}=\sqrt{\frac{x}{y}}+\sqrt[3]{\frac{y}{z}} \\ 2x^4+18z+54=y^3+39y^2 \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.1307 Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ x^4+560x^2+x=56y^4+10z^3+56 \\ x+y+z=2\sqrt[3]{xyz}+\frac{3xyz}{xy+yz+zx} \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.1308 If $a,b,c>0, a+b+c=3$ then:

$$\sum_{cyc}\sqrt{(a+3b)^2+(a+3b)(3a+b)+(3a+b)^2}\ge 12\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

S.1309 Find:

$$\Omega=\lim_{n\to\infty}\frac{(-1)^{n-1}\cdot(n-1)!}{2^n}\sum_{k=0}^{n}\frac{(-1)^{k-1}\cdot 4^k}{k!}$$

Proposed by Daniel Sitaru – Romania

S.1310 Solve for real numbers:

$$\tan 4x+3\tan x+3\cot 2x=\cot x+\tan 2x+3\cot 4x$$

Proposed by Daniel Sitaru – Romania

S.1311 Solve for real numbers:

$$\begin{vmatrix} 1 & 3 & x & x \\ 5 & 9 & x & x \\ x & x & 1 & 3 \\ x & x & 5 & 9 \end{vmatrix}=0$$

Proposed by Daniel Sitaru – Romania

S.1312 If $0<a\le b$ then:

$$\int_a^b\left(\frac{1}{x}\cdot\tan^{-1}x\right)dx\ge\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Proposed by Daniel Sitaru – Romania

S.1313 Solve for real numbers:

$$\begin{cases} x^4 = \sqrt{y^4+8} - \sqrt{y^4+3} \\ y^4 = \sqrt{z^4+8} - \sqrt{z^4+3} \\ z^4 = \sqrt{t^4+8} - \sqrt{t^4+3} \\ t^4 = \sqrt{x^4+8} - \sqrt{x^4+3} \end{cases}$$

Proposed by Daniel Sitaru – Romania

S.1314

$$a,b,c > 0, a+b+c = 3, \Omega(a) = \prod_{k=1}^{n}\left(1+\frac{k}{an^2}\right), n \in \mathbb{N}, n \geq 1$$

Prove that:

$$a\Omega(a) + b\Omega(b) + c\Omega(c) \leq e^{\frac{1}{2a}} + e^{\frac{1}{2b}} + e^{\frac{1}{2c}}$$

Proposed by Daniel Sitaru – Romania

S.1315 If $m,n > 0$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{1}{\sqrt{3}}\sqrt{(ma+nb)^2+(mb+nc)^2+(mc+na)^2} + \frac{3a^2b^2c^2}{ab+bc+ca} \geq 4\sqrt{3}\sqrt[3]{mn}F$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.1316 If $x,y,z \in (0,1)$, then in ABC triangle with the area F the following inequality holds:

$$\frac{a^4}{(y+z)(1-x^2)} + \frac{b^4}{(z+x)(1-y^2)} + \frac{c^4}{(x+y)(1-z^2)} \geq 12\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1317 If $x,y > 0, n \in \mathbb{N}^*$, then in any ABC triangle the following inequality holds:

$$x^{n+1}\sin^n A + y^n\sqrt[n]{y}\sin B\sin C \geq (n+1)xy\sqrt[n+1]{\left(\frac{sr}{2nR^2}\right)^n}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.1318 If $x,y,z \in \mathbb{R}_+^* = (0,\infty)$ and ABC is a triangle with the area F, then:

$$\frac{(2x+3y)(2y+5z+3x)}{(2y+3z)(2z+3x)}a^2 + \frac{(2y+3z)(2z+5x+3y)}{(2z+3x)(2x+3y)}b^2 +$$

$$+\frac{(2z+3x)(2x+5y+3z)}{(2x+3y)(2y+3z)}c^2 \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1319 If $x,y > 0$ then in any ABC triangle the following inequality holds:

$$\frac{x^2ya}{(xy^2+x+y)h_a}+\frac{xy^2b}{(x^2y+x+y)h_b}+\frac{c}{xyh_c}\geq\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.1320 If $x,y>0$ then in any triangle with the area F the following inequality holds:

$$\frac{x^2ya^2}{xy^2+x+y}+\frac{xy^2b^2}{x^2y+x+y}+\frac{c^2}{xy}\geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Mihály Bencze – Romania

S.1321 If $x\in\left(0,\frac{\pi}{2}\right)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{a^2}{h_a^2}\sin^2x+\frac{b^2}{h_b^2}\cos^2x+\frac{c^2}{h_c^2}\sin^2 2x\geq\frac{2}{3}\sin 2x\left(1+(\sin x+\cos x)\right)$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

S.1322 If $m\in\mathbb{N}, M$ is an interior point in ABC triangle and $x_A=MA, x_B=MB, x_C=MC$, then:

$$3m+\left(\frac{x_A}{h_a}\right)^{m+1}+\left(\frac{x_B}{h_b}\right)^{m+1}+\left(\frac{x_C}{h_c}\right)^{m+1}\geq 2(m+1)$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1323 Let be M be an interior point in ABC triangle with the area F, x_A,x_B,x_C respectively the distances from M to the apices $A,B,C, X=x_A+x_B+x_C$ and y_a,y_b,y_c respectively the distances from M to the sides BC,CA,AB, then:

$$X\frac{a^4}{y_a}+X\frac{b^4}{y_b}+X\frac{c^4}{y_c}\geq 96F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1324 Let be $x,y,z>0$ and ABC is a triangle with the area F, then:

$$\frac{x\cdot m_a+ym_b}{z}c^3+\frac{ym_b+xm_c}{x}a^3+\frac{zm_c+xm_a}{y}b^3\geq 16\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

S.1325 Let be $m,n\in(0,\infty)$ and $x,y,z\in(0,1)$, then:

$$\frac{x}{(my+nz)^2(1-x^2)}+\frac{y}{(mz+nx)^2(1-y^2)}+\frac{z}{(mx+ny)^2(1-z^2)}\geq\frac{9\sqrt{3}}{2(m+n)^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1326 If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then in any ABC triangle with the area F the following inequality holds:

$$\frac{a^4}{(\sin y+\sin z)\cdot\cos^2 x}+\frac{b^4}{(\sin z+\sin x)\cos^2 y}+\frac{c^4}{(\sin x+\sin y)\cos^2 z}\ge 12\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1327 If $x,y,z\in(0,1)$ and ABC is a triangle with the area F, then:

$$\frac{a^4}{(xy+xz)(1-x)}+\frac{b^4}{(yz+yx)(1-y)}+\frac{c^4}{(zx+zy)(1-z)}\ge 54F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1328 In ΔABC the following relationship holds

$$3\sum a^5\tan\frac{A}{2}\ge\sum a^5\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.1329 In acute ΔABC the following relationship holds

$$\sum\sec^3 A\sec B\ge 48$$

Proposed by Marin Chirciu – Romania

S.1330 In ΔABC the following relationship holds

$$\sum\frac{1}{h_b+h_c}\ge\sum\frac{1}{r_b+r_c}$$

Proposed by Marin Chirciu – Romania

S.1331 In ΔABC the following relationship holds

$$\sum\cos\frac{A}{2}\ge\frac{1}{2}\sum\frac{h_b+h_c}{a}$$

Proposed by Marin Chirciu – Romania

S.1332 In ΔABC the following relationship holds

$$\frac{4R}{r}\le\frac{(h_a+r_a)(h_b+r_b)(h_c+r_c)}{h_ah_bh_c}\le\frac{2R^2}{r^2}$$

Proposed by Marin Chirciu – Romania

S.1333 In ΔABC the following relationship holds

$$2\le\frac{m_a+m_b+m_c+r_a+r_b+r_c}{h_a+h_b+h_c}\le\frac{R}{r}$$

Proposed by Marin Chirciu – Romania

S.1334 In ΔABC the following relationship holds

$$\frac{9r}{4R^2}\le\sum\frac{h_a}{bc}\cos^2\frac{A}{2}\le\frac{9}{8R}$$

Proposed by Marin Chirciu – Romania

S.1335 If $a,b,c>0$ such that $a+b+c=3$ and $\lambda\geq 0$ then:

$$\sum\frac{a^3}{(\lambda+a)(\lambda+b)}\geq\frac{a^2+b^2+c^2}{(\lambda+1)^2}$$

Proposed by Marin Chirciu – Romania

S.1336 In ΔABC the following relationship holds

$$\frac{1}{2R}\left(\frac{8R}{r}-7\right)\leq\sum\frac{h_a}{bc}\cot^2\frac{A}{2}\leq\frac{1}{2R}\left(\frac{2R}{r}-1\right)^2$$

Proposed by Marin Chirciu – Romania

S.1337 If $A=\frac{\tan^3 x}{\sqrt{1+\cot^2 x}}+\frac{\cot^3 x}{\sqrt{1+\tan^2 x}}, x\in\left(0,\frac{\pi}{2}\right)$ then find $\min(A)$.

Proposed by Marin Chirciu – Romania

S.1338 In ΔABC the following relationship holds

$$6\left(\frac{2r}{R}\right)^2\leq\sum m_am_b\left(\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)\leq\frac{R}{r}\left(\frac{2R^2}{r^2}-\frac{3R}{r}+1\right)$$

Proposed by Marin Chirciu – Romania

S.1339 If $a,b,c,d>0$ then:

$$\frac{a+b}{c^2}+\frac{b+c}{d^2}+\frac{c+d}{a^2}+\frac{d+a}{b^2}\geq\frac{16}{a+b+c+d}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

Proposed by Marin Chirciu – Romania

S.1340 In ΔABC the following relationship holds

$$12\left(\frac{2r}{R}\right)^2\leq\sum m_am_b\left(\frac{1}{h_b}+\frac{1}{h_c}\right)^2\leq\frac{R^2}{r^2}+\frac{7R}{2r}+1$$

Proposed by Marin Chirciu – Romania

S.1341 If ABC is a triangle with the area F and semiperimeter s then for any $m\geq 0$ the following inequality holds:

$$\frac{a^{m+2}}{(2s-a)^m}+\frac{b^{m+2}}{(2s-b)^m}+\frac{c^{m+2}}{(2s-c)^m}\geq 2^{2-m}\sqrt{3}F+\frac{1}{2^m}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1342 Prove that in any triangle with the area F the following inequality holds:

$$a^2+b^2+c^2\geq\frac{12}{5}\sqrt{3}F+\frac{1}{10}(3a-b)^2+\frac{1}{10}(3b-c)^2+\frac{1}{10}(3c-a)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1343 If $x,y,z\in(0,1)$ then in ABC triangle with the area F the following inequality holds:

$$\frac{a^2}{(y+z)(1-x^2)}+\frac{b^2}{(z+x)(1-y^2)}+\frac{c^2}{(x+y)(1-z^2)}\geq 9F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.1344 If $x,y,z>0$ then in any ABC triangle with the semiperimeter s the following inequality holds:

$$\frac{y+z}{x}\cdot\frac{bc}{s-a}+\frac{z+x}{y}\cdot\frac{ca}{s-b}+\frac{x+y}{z}\cdot\frac{ab}{s-c}\geq 24\sqrt{3}r$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

S.1345 If $m,n,x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle with the area F the following inequality holds:

$$\left(\frac{1}{3}\left(\frac{y+z}{x}a^m+\frac{z+x}{y}b^m+\frac{x+y}{z}c^m\right)\right)^{\frac{n}{m}}\geq 2^{\frac{(m+1)n}{m}}\left(\sqrt[4]{3}\right)^{-n}\left(\sqrt{F}\right)^n$$

Proposed by D.M. Bătinețu-Giurgiu, Flaviu – Cristian Verde – Romania

S.1346 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{x}{\sqrt{yz}}\cdot\frac{a^2b}{h_b}+\frac{y}{\sqrt{zx}}\cdot\frac{b^2c}{h_c}+\frac{z}{\sqrt{xy}}\cdot\frac{c^2a}{h_a}\geq 8F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1347 Let be $m\geq 0$ and ABC a triangle with the semiperimeter s, and $x,y>0$, then:

$$\frac{r_a\cdot r_b^{m+1}}{(xr_b+yr_c)^m}+\frac{r_b\cdot r_c^{m+1}}{(xr_c+yr_a)^m}+\frac{r_c\cdot r_a^{m+1}}{(xr_a+yr_b)^m}\geq\frac{s^2}{(x+y)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1348 Let be $A_1B_1C_1, A_2B_2C_2$ two triangles with the area F_1, respectively F_2 and the sides having the lengths a_1,b_1,c_1 respectively a_2,b_2,c_2 and σ a permutation of the set $\{a_2,b_2,c_2\}$, then:

$$\left(a_1+\sigma(a_2)\right)^2+\left(b_1+\sigma(b_2)\right)^2+\left(c_1+\sigma(c_2)\right)^2\geq 16\sqrt{3}\sqrt{F_1F_2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1349 In any non-right triangle the following inequality holds:

$$\frac{1}{(\sin A+\sin B)\cos^2 C}+\frac{1}{(\sin B+\sin C)\cos^2 A}+\frac{1}{(\sin C+\sin A)\cos^2 B}\geq\frac{9\sqrt{3}}{4}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1350 If $x, y, z \in (0,1)$, then:

$$\left(\frac{x}{1-y^2}+\frac{y}{1-z^2}+\frac{z}{1-x^2}\right)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{27}{8}\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

S.1351 Solve:

$$x[\sin x] + [x]\sin x = \frac{\pi}{4}$$

$[*]$ –is the greatest integer part of $*$.

Proposed by Jalil Hajimir-Canada

S.1352 Minimize $\sum_{i=1}^{3}\sqrt{\frac{x_i}{x_i+3}}$. Subject to $\prod_{i=1}^{3} x_i = 1, x_i \geq 0, i = 1,2,3$.

Proposed by Jalil Hajimir-Canada

S.1353 If $x, y, z \in \mathbb{R}$ then:

$$\frac{|x-y|}{\sqrt{1+x^2+y^2+x^2y^2}} \leq \frac{|x-z|}{\sqrt{1+x^2+z^2+x^2z^2}} + \frac{|y-z|}{\sqrt{1+y^2+z^2+y^2z^2}}$$

Proposed by Jalil Hajimir-Canada

S.1354 Maximize $\prod_{i=1}^{n}(x_i^3 + x_i + 1)$ subject to $\sum_{i=1}^{n} x_i^2 = n, x_i \geq 0, i \in \overline{1,n}$.

Proposed by Jalil Hajimir-Canada

S.1355 In $\Delta ABC, v_a$ –Bevan's cevian, p_a –Spieker's cevian, prove that:

$$\sum_{cyc} p_a^2 \leq \sum_{cyc} v_a^2 \leq \sum_{cyc} p_a^2 + 9(R^2 - 4r^2)$$

Proposed by Nguyen Van Canh-Vietnam

S.1356 In ΔABC the following relationship holds:

$$\sum_{cyc} h_a^2 + 3r(R-2r) \leq \max\left\{\sum_{cyc} r_a^2, \sum_{cyc} m_a^2\right\}$$

Proposed by Nguyen Van Canh-Vietnam

S.1357 In ΔABC the following relationship holds:

$$\sum_{cyc} w_a^2 + 3r(R-2r) \leq \max\left\{\sum_{cyc} r_a^2, \sum_{cyc} m_a^2\right\}$$

Proposed by Nguyen Van Canh-Vietnam

S.1358 In $\Delta ABC, g_a$ –Gergonne's cevian, p_a –Spieker's cevian, n_a –Nagel's cevian, v_a –Bevan's cevian. Prove that:

$$h_a \leq g_a \leq w_a \leq m_a \leq p_a \leq n_a \leq v_a$$

Proposed by Nguyen Van Canh-Vietnam

S.1359 Let $m, n \in \mathbb{N}, m, n \geq 1$. In ΔABC the following relationship holds:

$$\frac{9}{(m+n)(8R^2+2r^2)} \leq \frac{1}{ma^2+nb^2} + \frac{1}{mb^2+nc^2} + \frac{1}{mc^2+na^2} \leq \frac{1}{4(m+n)r^2}$$

Proposed by Nguyen Van Canh-Vietnam

S.1360 Let $m, n \in \mathbb{N}, m, n \geq 1$. In ΔABC the following relationship holds:

$$\frac{2}{(m+n)R} \leq \frac{1}{mh_a+nh_b} + \frac{1}{mh_b+nh_c} + \frac{1}{mh_a+nh_c} \leq \frac{1}{(m+n)r}$$

Proposed by Nguyen Van Canh-Vietnam

S.1361 In $\Delta ABC, p_a$ –Spieker's cevian, n_a –Nagel's cevian, the following relationship holds:

$$\sum_{cyc} p_a \leq \sum_{cyc} n_a \leq \sum_{cyc} p_a + s\sum_{cyc} \frac{|b-c|}{2s+a}$$

Proposed by Nguyen Van Canh-Vietnam

S.1362 If $a, b, c > 0, ab+bc+ca \geq 3$ then:

$$\left(a^3+\frac{2}{3}\right)\left(b^3+\frac{2}{3}\right)\left(c^3+\frac{2}{3}\right) \geq \frac{125}{27}$$

Proposed by George Apostolopoulos-Greece

S.1363 If $a, b, c > 0, a^2+b^2+c^2 = 12$ then:

$$\frac{a^4}{\sqrt{a^3+1}} + \frac{b^4}{\sqrt{b^3+1}} + \frac{c^4}{\sqrt{c^3+1}} \geq 16$$

Proposed by George Apostolopoulos-Greece

S.1364 In ΔABC the following relationship holds:

$$\frac{4\sqrt{3}}{3R} \leq \frac{\csc A}{m_a} + \frac{\csc B}{m_b} + \frac{\csc C}{m_c} \leq \frac{\sqrt{3}\cdot R}{3r^2}$$

Proposed by George Apostolopoulos-Greece

S.1365 In ΔABC the following relationship holds:

$$\frac{3+\sum\left(\frac{\sin A+\sin B}{\sin C}\right)^3}{\frac{1}{\sin^4 A}+\frac{1}{\sin^4 B}+\frac{1}{\sin^4 C}} \leq \frac{81}{16}$$

Proposed by George Apostolopoulos-Greece

S.1366 If $a,b,c>0, a+b+c=3$ then:

$$\frac{5a-17}{a^2-3a-1}+\frac{5b-17}{b^2-3b-1}+\frac{5c-17}{c^2-3c-1}\le 3\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by George Apostolopoulos-Greece

S.1367 If α,β and γ be the three roots of the Equation then find Ω:

$$x^3-\phi x+\sqrt{\phi}=0\Rightarrow\Omega=\alpha^9+\beta^9+\gamma^9$$

ϕ – Golden Ratio.

Proposed by Asmat Qatea – Afghanistan

S.1368 Solve in term of hypergeometric function:

$$\int\frac{dx}{\sqrt{3+3x+3x^2+x^3}}$$

Proposed by Asmat Qatea – Afghanistan

S.1369 Solve for real numbers:

$$\frac{3}{2}(x^2+y^2+z^2)=xy+xz+zy$$

Proposed by Asmat Qatea - Afghanistan

S.1370 Prove that:

$$\Omega=\sqrt[3]{4+24\cos(20^\circ)}+\sqrt[3]{4-24\cos(40^\circ)}+\sqrt[3]{4-24\cos(80^\circ)}$$

Proposed by Asmat Qatea - Afghanistan

S.1371 If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\sum_{cyc}(y+z)\frac{h_a}{m_a}\ge\sum_{cyc}x\frac{m_a^2}{h_a^2}+\sqrt{xy+yz+zx}\sqrt{2\sum_{cyc}\frac{m_a^2m_b^2}{h_a^2h_b^2}-\sum_{cyc}\left(\frac{m_a}{h_a}\right)^4}$$

Proposed by Bogdan Fuștei-Romania

S.1372 In ΔABC the following relationship holds:

$$s\sqrt{3}+w_a-m_a\ge\sqrt{\left(\sum_{cyc}w_aw_b\right)\left(\sum_{cyc}\frac{w_a+w_b}{w_b+w_c}\right)}$$

Proposed by Bogdan Fuștei-Romania

S.1373 If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{y+z}{m_a}\ge\sum_{cyc}xm_a^2+3F\sqrt{xy+yz+zx}$$

Proposed by Bogdan Fuștei-Romania

S.1374 In ΔABC the following relationship holds:

$$\frac{w_a+w_b+w_c}{s^2}\le\frac{R}{2}\sum_{cyc}\frac{1}{m_am_b}$$

Proposed by Bogdan Fuștei-Romania

S.1375 In ΔABC the following relationship holds:

$$\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\ge\frac{1}{2}\left(3+\sqrt{10-\frac{2r}{R}}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1376 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{n_a}{h_a}\ge\frac{3\sqrt{2}}{2}+\frac{1}{2\sqrt{2}r}\sum_{cyc}|a-b|$$

Proposed by Bogdan Fuștei-Romania

S.1377 In ΔABC the following relationship holds:

$$\frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}}-\sqrt{\frac{R}{r}-2}\right)\le\sum_{cyc}\sqrt{\frac{b+c-a}{a}}\le\frac{3\sqrt{2}}{2}\left(\sqrt{\frac{R}{r}}+\sqrt{\frac{R}{r}-2}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1378 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a}{s-n_a}\ge 3\sqrt{3}+\sum_{cyc}\frac{n_a}{h_a}$$

Proposed by Bogdan Fuștei-Romania

S.1379 In ΔABC the following relationship holds:

$$\frac{r_a-r}{\sqrt{r_aw_a}}+\frac{r_b-r}{\sqrt{r_bw_b}}+\frac{r_c-r}{\sqrt{r_cw_c}}\ge\sqrt{\frac{2R}{r}}$$

Proposed by Bogdan Fuștei-Romania

S.1380 In ΔABC the following relationship holds:

$$4\sum_{cyc}\frac{r_br_c}{w_a^2}=3+(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1381 In ΔABC the following relationship holds:

$$s^2 g_a g_b g_c \geq r^2 \prod_{cyc} \left(n_a + \frac{2r_a h_a}{n_a}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1382 If $x > 0$ then:

$$2 \cdot \tan^{-1} x \cdot \cos^{-1} x + 3\sqrt[3]{2} < \frac{\pi^2}{2 \cdot \tan^{-1} x \cdot \cot^{-1} x} + \frac{\pi}{4}$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1383 Solve for real numbers:

$\tan^9 x - 9\tan^7 x - 27\tan^6 x + 27\tan^5 x + 27\tan^4 x - 26\tan^3 x - 12\tan^2 x - 3\tan x + 2 = 0.$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1384$\sum_{k=1}^{1010} \alpha^{2k+1} = 1010.$ Solve for real numbers:

$$\frac{\sin 2020x + \cos 2020x - \sqrt{2}}{\alpha - 1} \leq 0$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1385 Find $x > 0$ such that:

$$\begin{cases} \sin^4 x \cdot \cos^3 x = \dfrac{1}{512} \\ 4\sin x + 3\cos x = \dfrac{9\sin 2x - 4}{2\sin 2x} \end{cases}$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1386 Find:

$$\Omega(n) = \int_0^1 \frac{\sin^{2n} x + \sin^2 x}{1 + \sin^{2n} x + \cos^{2n} x} dx, n \in \mathbb{N}$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1387 Prove without softs:

$$e + \pi/\cos 1 > \pi + e \cdot \cos\left(\frac{\pi}{e}\right)$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1388 If $x \in \left(0, \frac{\pi}{2}\right)$ then:

$$(\sin x)^{\sin x} + (\cos x)^{\cos x} + \frac{1}{\sqrt{2}} < 3$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1389 Find $x \in \left(0, \frac{\pi}{2}\right)$ such that:

$$\sqrt[4]{(1+\sin x)(1+\cos x)\left(1+\frac{\sqrt{2}}{2}\right)^2} = 1 + \sqrt[4]{\sin x \cdot \cos x}$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1390 If $x \in \left(0, \frac{\pi}{2}\right), k \geq 2, k \in \mathbb{N}, k$ −fixed then:

$$\left(k^{\sin x} + k^{-\sin x}\right)^{\frac{1}{\sin x}} + \left(k^{\cos x} + k^{-\cos x}\right)^{\frac{1}{\cos x}} > \left(k^{\tan x} + k^{-\tan x}\right)^{\cot x} + \left(k^{\cot x} + k^{-\cot x}\right)^{\tan x}.$$

Proposed by Rovsen Pirguliyev-Azerbaijan

S.1391 In ΔABC the following relationship holds:

$$\sum \frac{1}{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^2} \geq \frac{9}{4}$$

Proposed by Alex Szoros-Romania

S.1392 In ΔABC the following relationship holds:

$$\frac{3}{8}\left(\frac{1}{r} + \frac{3}{4R+r}\right) \geq \sum \frac{\tan\frac{A}{2}}{a} \geq \frac{1}{R}$$

Proposed by Alex Szoros-Romania

S.1393 Let $f: [0, \infty) \to \mathbb{R}, f(x) = \int_0^x \frac{\log(1+t)}{1+t^2} dt$. Find:

$$\Omega = \int_0^1 x f(x)\, dx$$

Proposed by Alex Szoros-Romania

S.1394 In ΔABC the following relationship holds:

$$\frac{R}{r} \geq \frac{cs}{2}\left(\frac{1}{h_a w_b} + \frac{1}{h_b w_a}\right) \geq 2$$

Proposed by Alex Szoros-Romania

S.1395 If in $\Delta ABC, abc = \frac{1}{8}$, then

$$\left(\sqrt{\frac{r_a}{r_b}} + \sqrt{\frac{r_b}{r_a}}\right)\left(\sqrt{\frac{r_b}{r_c}} + \sqrt{\frac{r_c}{r_a}}\right)\left(\sqrt{\frac{r_c}{r_a}} + \sqrt{\frac{r_c}{r_a}}\right) \geq \sqrt{\left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{c} + \frac{1}{a}\right)}$$

Proposed by Alex Szoros-Romania

S.1396 If $x, y, z > 0$ then:

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x} \geq \sqrt{9(x^2+y^2+z^2)-6(xy+yz+zx)}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1397 If $x, y, z > 0$ then:

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x} \geq \sqrt{2x^2-3xy+2y^2}+\sqrt{2y^2-3xy+2z^2}+\sqrt{2z^2-3zx+2x^2}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1398 In ΔABC the following relationship holds:

$$5+\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a} \leq \frac{4R}{r}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1399 If $x, y, z, t > 0, \frac{1}{3x+2}+\frac{1}{3y+2}+\frac{1}{3z+2}+\frac{1}{3t+2}=\frac{1}{2}$ then:

$$\frac{1}{48+xyz^2}+\frac{1}{48+yzt^2}+\frac{1}{48+ztx^2}+\frac{1}{48+txy^2} \leq \frac{1}{16}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1400 If $a, b, c > 0$ then:

$$\frac{a^2+bc}{a^2+2bc}+\frac{b^2+ca}{b^2+2ca}+\frac{c^2+ab}{c^2+2ab} \geq 2$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1401 If $x, y > 0, xy = 1$ then:

$$\frac{1}{x+3}+\frac{1}{y+3} \geq \frac{1}{x^2+y^2}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1402 Find:

$$\lim_{n\to\infty}\sum_{k=3}^{n}\frac{3k^2+10k+6}{2^k(k^4+4)}$$

Proposed by George Florin Şerban-Romania

S.1403 $A, B \in M_n(\mathbb{R}), \det A \neq 0, \det B \neq 0, AB = -BA, n \in \mathbb{N}, n \geq 2$. Prove that:

$$\det(A^{4n}+2A^{2n}+I_n+A^6B^6) \geq 0.$$

Proposed by Gheorghe Alexe, George Florin Şerban-Romania

S.1404 In $\Delta ABC, I$ —incentre, K —Lemoine's point. Prove that:

$$\overrightarrow{IK} = \vec{0} \Leftrightarrow AB = BC = CA$$

Proposed by Gheorghe Alexe, George Florin Şerban-Romania

S.1405 If $a \in \left(0, \frac{\pi}{2}\right), b \in [1, \infty)$ and $m, n \in \mathbb{N}^*$, then find

$$\Omega = \int_{-a}^{a} \frac{\sin x + \tan x}{(b - \cos x)^m + \sin^{2n} x} dx$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți –Romania

S.1406 If $a \in (1, \infty), b \in (0, \infty)$ then find

$$\Omega = \lim_{n \to \infty} n \left(2 - e^{\sum_{k=1}^{n} \frac{(n+k)^{a-1}}{(n+k)^a + b}}\right).$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți –Romania

S.1407 If $n \in \mathbb{N}, n \geq 3, a, b, c, x_k \in \mathbb{R}_+^*, k = \overline{1, n}, x_{n+1} = x_1$ such that $\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \prod_{k=1}^{n} x_k =$ a, then

$$\sum_{k=1}^{n} (b x_k^{n-1} + c) \frac{x_k^3 + x_{k+1}^3}{x_k^2 + x_k x_{k+1} + x_{k+1}^2} \geq \frac{2n}{3a} (ab + cn) \prod_{k=1}^{n} x_k.$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

S.1408 Find: $\Omega = \int_0^{\frac{\pi}{2}} \frac{3 \sin x + 4 \cos x}{3 \cos x + 4 \sin x} dx$ and then prove that

$$\pi < 14 \log\left(\frac{4}{3}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1409 If $(A, +, \cdot)$ is a ring with $0 \neq 1$ and $1 + 1$ is invertible, then prove that if $a, b \in A$ and $(a + b)^2 = a^2 + b^2, (a + b)^4 = a^4 + b^4$, then $(ab)^2 = 0$.

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1410 If $a, b \in \mathbb{R}, a + b = 1, e_n = \left(1 + \frac{1}{n}\right)^n, c_n = -\log n + \sum_{k=1}^{n} \frac{1}{k}$ then find:

$$\Omega = \lim_{n \to \infty} \left((n+1)^a \sqrt[n+1]{\left((n+1)!\, c_n\right)^b} - n^a \sqrt[n]{(n!\, e_n)^b}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1411 If $(u_n)_{n \geq 1}, (v_n)_{n \geq 1}$ are real positive sequences with $\lim_{n \to \infty} \frac{u_{n+1}}{n^2 \cdot u_n} = u \in \mathbb{R}_+^*$, $\lim_{n \to \infty} \frac{v_{n+1}}{n^b \cdot v_n} = v \in \mathbb{R}_+^*$, where $a, b \in \mathbb{R}_+^*, ax + by = 1$ then find

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{u_{n+1}^x v_{n+1}^x}-\sqrt[n]{u_n^x v_n^x}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1412 In ΔABC the following relationship holds:

$$\frac{1}{6}\sum\frac{a^2+b^2}{\sqrt{2(b^2+c^2)-a^2}}\le R$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1413 If $x,y\in\mathbb{R}_+^*, m\in\mathbb{R}_+$ then in ΔABC holds:

$$\frac{a^{m+2}}{(xa+yb)^m}+\frac{b^{m+2}}{(xb+yc)^m}+\frac{c^{m+2}}{(xc+ya)^m}\ge\frac{2(s^2-r^2-4Rr)}{(x+y)^m}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1414 If $a,b>0$ then:

$$\frac{b}{a}\left(1-\sqrt{\frac{b}{a+b}}\right)+1>\frac{\sqrt{3}}{2}\left(1+\frac{a}{b}\right)^{\frac{b}{2a}}$$

Proposed by Mohammed Bouras-Morocco

S.1415 Find the number of 2 – element subsets $\{x,y\}$ of set $S=\{1920,1921,\dots,2020\}$ such that:

$$(x+y)(x+y+1)-xy\equiv 6(mod\ 7)$$

Proposed by Rajeev Rastogi – India

S.1416 If $\frac{1^5+2^5+\dots+11^5}{(1^5+2^5+\dots+11^5)+(1^7+2^7+\dots+11^7)}=\frac{m}{n}$ where $\gcd(m,n)=1$, then find $(100m-n)$

Proposed by Rajeev Rastogi – India

S.1417 Let ABC be an equilateral triangle of side length $\sqrt{3}$, let P be a point insides ΔABC and x,y,z be the feet of perpendicular of p consider BC,CA,AB respectively, then prove that:

$$\frac{1+(px)^2}{py+pz}+\frac{1+(py)^2}{pz+px}+\frac{1+(pz)^2}{px+py}\ge\frac{15}{4}$$

Proposed by Rajeev Rastogi – India

S.1418 Given $a,b,c\in\mathbb{R}^+$ such that $a^2+b^2+c^2=18$. Prove that:

$$\sqrt[3]{\frac{1+a^3}{(a^2+3)^{\frac{3}{2}}}}+\sqrt[3]{\frac{1+b^3}{(b^2+3)^{\frac{3}{2}}}}+\sqrt[3]{\frac{1+c^3}{(c^2+3)^{\frac{3}{2}}}}\le 3$$

Proposed by Rajeev Rastogi – India

S.1419 Prove without softs:

$$\frac{\pi}{4}+\log\sqrt{\sqrt{2}-1}<\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sqrt{\sin^2 x-\sqrt{\sin^2 x-\sqrt{\sin^2 x-\cdots\ldots}}}}{\sin x}<\frac{\pi}{4}$$

Proposed by Rajeev Rastogi – India

S.1420 Find:

$$\Omega=\lim_{n\to\infty}\left[\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n^{2020}}}{\log n}\right],[*]-GIF$$

Proposed by Rajeev Rastogi – India

S.1421 In $\Delta ABC, V$ – Bevan's point, I_a, I_b, I_c – excenters, R_a, R_b, R_c – circumradii in $\Delta VI_aI_b, \Delta VI_bI_c,$

ΔVI_cI_a. Prove that:

$$rR_aR_bR_c=4R^4$$

Proposed by Mehmet Șahin - Turkey

S.1422 V – Bevan's point, I_a, I_b, I_c – excenters in $\Delta ABC, R_a, R_b, R_c$ – circumradii of $\Delta VI_bI_c, \Delta VI_cI_a,$

ΔVI_aI_b. Prove that:

$$\frac{a}{R_a^2}+\frac{b}{R_b^2}+\frac{c}{R_c^2}=\frac{[I_aI_bI_c]-2F}{2R^3}$$

Proposed by Mehmet Șahin - Turkey

S.1423 Let γ – be area of pedal triangle of Spielker's point in ΔABC. Find the value of γ in terms of s, r, R .

Proposed by Mehmet Șahin - Turkey

S.1424 V – Bevan's point in $\Delta ABC, I_a, I_b, I_c$ – excenters, $VK \perp I_bI_c, K \in (I_bI_c)$, $VL \perp I_cI_a, L \in (I_cI_a), VM \perp I_aI_b, L \in (I_aI_b)$. Prove that:

$$\frac{h_bh_c}{VK^2}+\frac{h_ch_a}{VL^2}+\frac{h_ah_b}{VM^2}=\frac{s^2}{R^2}$$

Proposed by Mehmet Șahin - Turkey

S.1425 In acute $\Delta ABC, r_A$ – radii of circle tangent simultaneous to BC in the middle of BC and circumcircle of ΔABC (internal tangent). If r_B, r_C – are similarly defined then:

$$4\left(\frac{2}{r}-\frac{1}{R}\right) \le \frac{1}{r_A}+\frac{1}{r_B}+\frac{1}{r_C} \le \frac{1}{r^2}(5R-4r)$$

Proposed by Mehmet Șahin - Turkey

S.1426 Let $\Delta K_aK_bK_c, \Delta DEF$ – be pedal triangles of Lemoine's point and circumcenter in acute ΔABC. Prove that:

$$\frac{KK_a}{OD}+\frac{KK_b}{OE}+\frac{KK_c}{OF}=\left(\frac{a^2+b^2+c^2}{4F}\right)^2$$

Proposed by Mehmet Șahin – Turkey

S.1427 S_p – Spieker point, O – circumcenter in ΔABC. If $d(O,BC)=d_a, d(O,CA)=d_b$ $d(O,AB)=d_c, d(S_p,BC)=d_1, d(S_p,CA)=d_2, d(S_p,AB)=d_3$ then:

$$d_1d_a+d_2d_b+d_3d_c=\frac{s^2-6Rr-3r^2}{4}$$

Proposed by Mehmet Șahin – Turkey

S.1428 V – Bevan's point in $\Delta ABC, I_a, I_b, I_c$ – excenters, $VK \perp (I_bI_c), K \in (I_bI_c), VL \perp (I_cI_a)$, $L \in (I_cI_a), VM \perp (I_aI_b), M \in (I_aI_b)$. Prove that:

$$w_a^2 \cdot VK^2 + w_b^2 \cdot VL^2 + w_c^2 \cdot VM^2 = 48R^2F^2$$

Proposed by Mehmet Șahin - Turkey

S.1429 V – Bevan's point, I_a, I_b, I_c – excenters in $\Delta ABC, R_a, R_b, R_c$ – circumradii of $\Delta VI_bI_c, \Delta VI_cI_a, \Delta VI_aI_b, \varphi_a, \varphi_b, \varphi_c$ – circumradii of $\Delta BCI_a, \Delta CAI_b, \Delta ABI_c$. Prove that:

$$\varphi_aR_a=\varphi_bR_b=\varphi_cR_c=2R^2$$

Proposed by Mehmet Șahin - Turkey

S.1430 O – circumcenter, Ψ – area of orthic triangle of acute $\Delta ABC, O_a, O_b, O_c$ – circumcenters of $\Delta BOC, \Delta COA, \Delta AOB$. Prove that:

$$F=2\sqrt{\Psi \cdot [O_aO_bO_c]}$$

Proposed by Mehmet Șahin - Turkey

S.1431 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\sum_{cyc}\frac{AI}{R_a} \ge \frac{2r}{R}\sum_{cyc}\frac{m_a}{r_a}$$

Proposed by Eldeniz Hesenov - Georgia

S.1432 In ΔABC the following relationship holds:

$$\frac{m_bm_c}{m_a}+\frac{m_cm_a}{m_b}+\frac{m_am_b}{m_c} \le \frac{9R^2}{4r}$$

Proposed by Eldeniz Hesenov - Georgia

S.1433 Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ – orthocenter. Prove that:

$$\frac{AH\cdot r_a}{EF}+\frac{BH\cdot r_b}{FD}+\frac{CH\cdot r_c}{DE}\le\frac{4(r_a^2+r_b^2+r_c^2)}{a+b+c}$$

Proposed by Eldeniz Hesenov - Georgia

S.1434 In ΔABC the following relationship holds:

$$\frac{(b+c)^4}{r_b+r_c}+\frac{(c+a)^4}{r_c+r_a}+\frac{(a+b)^4}{r_a+r_b}\le 144R^3$$

Proposed by Eldeniz Hesenov - Georgia

S.1435 In $\Delta ABC, R_a, R_b, R_c$ – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ – incenter holds:

$$\frac{(b+c)^2}{R_a^2}+\frac{(c+a)^2}{R_b^2}+\frac{(a+b)^2}{R_c^2}\le\left(\frac{4R}{r}-2\right)^2$$

Proposed by Eldeniz Hesenov - Georgia

S.1436 In ΔABC the following relationship holds:

$$am_a^2+bm_b^2+cm_c^2\le\frac{3\sqrt{3}R^2}{2}(r_a+r_b+r_c)$$

Proposed by Eldeniz Hesenov - Georgia

S.1437 In ΔABC the following relationship holds:

$$\frac{12\sqrt{3}r^3}{R^2}\le\sum_{cyc}\frac{a^3}{b^2+c^2}\le\frac{3\sqrt{3}R(R-r)}{2r}$$

Proposed by Eldeniz Hesenov - Georgia

S.1438 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\sqrt[4]{\frac{a^3b+ab^3}{2}}\right)^3\left(\sum_{cyc}\cos^3\frac{A}{2}\right)^{-1}\le 8R^3$$

Proposed by Eldeniz Hesenov - Georgia

S.1439 In acute ΔABC the following relationship holds:

$$\frac{AH\cdot\cos B}{ab}+\frac{BH\cdot\cos C}{bc}+\frac{CH\cdot\cos A}{ca}\le\frac{1}{4r}$$

Proposed by Eldeniz Hesenov - Georgia

S.1440 If $a>1, p\ge 2, p\in\mathbb{N}$ then:

$$e^{a-\sqrt[p]{a}}<a^{\left(1-\frac{1}{p}\right)a}$$

Proposed by Seyran Ibrahimov – Azerbaijan

S.1441 In ΔABC the following relationship holds:

$$R \geq \left(\sum_{cyc} \frac{1}{m_a + m_b}\right)^{-1} \geq 2r, R \geq \left(\sum_{cyc} \frac{1}{w_a + w_b}\right)^{-1} \geq 2r$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1442 In ΔABC the following relationship holds:

$$\frac{1}{\sin A}\sqrt{\frac{2 \cdot \sum_{cyc} \sin A \cdot \prod_{cyc}(\sin A + \sin B - \sin C)}{3 + 2\cos 2A - 2\cos 2B - 2\cos 2C}} \leq 1$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1443 If $0 < a \leq t_1 \leq (1-\lambda)a + \lambda b \leq t_2 \leq b, t \in [0,1]$ then:

$$e^{\frac{b-a}{\lambda t_1 + (1-\lambda)t_2}} \leq \frac{b}{a}$$

Proposed by Seyran Ibrahimov – Azerbaijan

S.1444 If $0 < m \leq a_i \leq M, i \in \overline{1,n}, n \in \mathbb{N}, n \geq 2$ then:

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{\sum_{\substack{i=1 \\ j \neq 1}}^{n} a_j^2}} \leq \frac{n}{2\sqrt{n-1}}\left(\frac{m}{M} + \frac{M}{m}\right)$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1445 In ΔABC the following relationship holds:

$$\frac{R}{2r} \geq \frac{a^3 + b^3 + c^3}{ab(a+b) + bc(b+c) + ca(c+a) - 3abc} \geq 1$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1446

$$(n!)^{-\frac{1}{n}} \geq \frac{2}{n + 1 - 2\sqrt[n]{n!}\ln\left(\frac{n+1}{2\sqrt[n]{n!}}\right)}, n \in \mathbb{N}, n \geq 1$$

Proposed by Seyran Ibrahimov – Azerbaijan

S.1447 If $a_i, b_i > 0, i \in \overline{1,n}, p \geq 2$ then:

$$n^{p-2} \cdot \sum_{i=1}^{n} \frac{a_i^p}{b_i} \cdot \sum_{i=1}^{n} b_i \geq \left(\sum_{i=1}^{n} a_i\right)^p$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1448 In ΔABC the following relationship holds:

$$\frac{R}{2r} \geq \frac{4(a^3 + b^3 + c^3)}{(a+b)(a^2+b^2) + (b+c)(b^2+c^2) + (c+a)(c^2+a^2)} \geq 1$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1449 If $a,b,c,d>0$ then:

$$3^6(abcd)^2\left(\sum_{cyc}\frac{a}{\sqrt{(b+c+d)^3}}\right)^4\le 4\left(\sum_{cyc}a^2\right)^3$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1450 If $a_i>0, i\in\overline{1,n}, n\in\mathbb{N}, n\ge 2$ then:

$$(n-1)^{2(n-1)}\left(\prod_{i=1}^{n}a_i\right)^2\left(\sum_{i=1}^{n}\frac{a_i}{\left(\sum_{\substack{j=1\\ j\ne 1}}^{n}a_j\right)^{\frac{n-1}{2}}}\right)^4\le n\left(\sum_{i=1}^{n}a_i^2\right)^3$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1451 If $x_i>0, i\in\overline{1,n}$ then:

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)+\max_{1\le k\le n}\left(\frac{1}{k}\sum_{i=1}^{k}\ln x_i-\ln\left(\frac{1}{k}\sum_{i=1}^{k}x_i\right)\right)\ge\frac{1}{n}\sum_{i=1}^{n}\ln x_i$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1452 If $a,b,c,d>0$ then:

$$\frac{a+b+c+d}{4}\le\frac{1}{24}\sum_{sym}\left(\sqrt{a}+\sqrt{b}\right)^2\ge\sqrt[4]{abcd}$$

Proposed by Seyran Ibrahimov - Azerbaijan

S.1453 If $a_i>0, i\in\overline{1,n}, n\in\mathbb{N}, n\ge 2$ then:

$$\sum_{i=1}^{n}\frac{a_i}{\left(\sum_{\substack{j=1\\ j\ne 1}}^{n}a_j\right)^n}\ge\left(\frac{n}{n-1}\right)^n\cdot\left(\sum_{i=1}^{n}a_i\right)^{1-n}$$

Proposed by Seyran Ibrahimov – Azerbaijan

S.1454 If $a_i>0$ prove that the inequality:

$$\sum_{i=1}^{2n-1}a_i^{2n}\ge 2(n-1)\prod_{i=1}^{2n-1}a_i\left(\sum_{i=1}^{2n-1}\frac{a_i^n}{\left(\sum_{j=1,i\ne j}^{2n-1}a_j\right)^{n-1}}\right)$$

Proposed by Seyran Ibrahimov – Azerbaijan

S.1455 In ΔABC the following relationship holds:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{(m_a+m_b+m_c)^4}{(m_a+m_b+m_c)^4+2(m_am_bm_c(m_a+m_b+m_c)-9F^2)}\ge 2$$

Proposed by Adil Abdullayev-Azerbaijan

S.1456 In ΔABC the following relationship holds:

$$5+\frac{(m_a+m_b+m_c)^4}{9F^2}\ge$$
$$\ge\frac{32m_am_bm_c(m_a+m_b+m_c)}{9F^2}+\frac{2(m_am_bm_c(m_a+m_b+m_c)-9F^2)}{2m_am_bm_c(m_a+m_b+m_c)-9F^2}$$

Proposed by Adil Abdullayev-Azerbaijan

S.1457 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{1}{m_a^2}-\frac{1}{m_bm_c}\right)\ge 6\left(\frac{m_b-m_c}{m_a^2+m_b^2+m_c^2}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

S.1458 In ΔABC the following relationship holds:

$$m_a^3+m_b^3+m_c^3\ge 3F\sqrt{m_a^2+m_b^2+m_c^2}$$

Proposed by Adil Abdullayev-Azerbaijan

S.1459 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^2+\frac{9F^2}{4m_am_bm_c(m_a+m_b+m_c)}\ge 1$$

Proposed by Adil Abdullayev-Azerbaijan

S.1460 In ΔABC the following relationship holds:

$$1+\left(1+\frac{4m_am_bm_c(m_a+m_b+m_c)}{9F^2}\right)^2\ge$$
$$\ge\frac{(m_a+m_b+m_c)^4}{9F^4}+\frac{2(m_am_bm_c(m_a+m_b+m_c)-9F^2)}{2m_am_bm_c(m_a+m_b+m_c)-9F^2}$$

Proposed by Adil Abdullayev-Azerbaijan

S.1461 In ΔABC the following relationship holds:

$$2\left(\frac{r_a}{r_b+r_c}+\frac{r_b}{r_c+r_a}+\frac{r_c}{r_a+r_b}\right)\le\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}$$

Proposed by Adil Abdullayev-Azerbaijan

S.1462 Let p_1,p_2,p_3,p_4 prime numbers successive

$$\begin{cases} p_2+p_3+p_4=p_5=prime\ number\\ (p_1+1)^{(p_1+1)}-p_1^{(p_1+1)}=p_2\cdot p_3\cdot p_4\cdot p_5\\ p_4+p_1=p_3+p_2\\ 4\cdot(p_2-p_1)^2=p_3+p_1\\ p_1\cdot p_2\cdot p_3=p_4\cdot p_5-(p_2+p_3)\end{cases}$$

Find all prime numbers (p_1,p_2,p_3,p_4,p_5)

Proposed by Mohamed Bouras-Morocco

S.1463 In ΔABC the following relationship holds:

$$\max\{\mu^2(A),\mu^2(B),\mu^2(C)\}+\sum_{cyc}\left(\sin^2\frac{A}{2}+\frac{1}{2}\mu(A)\tan\frac{A}{2}\right)\geq\frac{5\pi^2}{18}$$

Proposed by Radu Diaconu – Romania

S.1464 Prove that:

$$\frac{1}{e-1}\left|e^{(\cos x)^{2n}}-e^{(\sin x)^{2n}}\right|\leq|\cos(2x)|, n\in\mathbb{N}, x\in\mathbb{R}$$

Proposed by Mohamed Bouras-Morocco

S.1465 If $a_1,a_2,\dots,a_n>0$ then:

$$\prod_{i=1}^{n}\left(1+a_i^{1+a_i^{1+a_i}}\right)\geq 2^n\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}\sum_{i=1}^{n}a_i}$$

Proposed by Florică Anastase – Romania

S.1466 Let $\lambda\geq 0$. Find:

$$\int\frac{x\cos x-\sin x+\lambda}{(x+\lambda\cos x)^2}dx, x\in\left(0,\frac{\pi}{2}\right)$$

Proposed by Marin Chirciu – Romania

S.1467 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i(i+2)(i+4)\right]^{-1}$$

Proposed by Vasile Mircea Popa – Romania

S.1468 Let $a>0$. Find:

$$\int\frac{1}{x^4\sqrt{x^2-a^2}}dx, x\in(0,a)$$

Proposed by Marin Chirciu – Romania

S.1469 If $a,b>0$ then:

$$\ln^2\left(\frac{a}{b}\right)+\frac{4ab}{a^2+b^2}\geq 2$$

Proposed by Asmat Qatea-Afghanistan

S.1470 If $0<a\leq b\leq c$ then:

$$e^{a^2+a+b^2}\cdot\left(\frac{b}{a}\right)^{a^2+a}\cdot\left(\frac{c}{b}\right)^{b^2+b}\leq e^{ab+bc+ca}$$

Proposed by Lazaros Zachariadis – Greece

S.1471 Solve for real numbers:

$$\begin{cases} x+y+z=xyz \\ \dfrac{x(3-x^2)}{1-3x^2}+\dfrac{y(3-y^2)}{1-3y^2}+\dfrac{z(3-z^2)}{1-3z^2}=0 \end{cases}$$

Proposed by Daniel Sitaru-Romania

S.1472 In acute ΔABC holds:

$$\frac{a^4b^4}{c^5}+\frac{b^4c^4}{b^5}+\frac{c^4a^4}{b^5}\geq 9\sqrt{3}R^3$$

Proposed by Daniel Sitaru-Romania

S.1473 If $a,b,c>0$ then:

$$e^{\left(\frac{a}{a+b+c}\right)^2}+e^{\left(\frac{b}{a+b+c}\right)^2}+e^{\left(\frac{c}{a+b+c}\right)^2}\geq 3\sqrt[9]{e}$$

Proposed by Daniel Sitaru-Romania

S.1474 If $a,b,c\geq\frac{\sqrt{2}}{2}$ then:

$$\sum_{cyc}\exp\left(-\frac{(a^2+bc)^2}{(a+b+c)^4}\right)\geq\frac{3}{\sqrt[3]{9}}$$

Proposed by Daniel Sitaru-Romania

S.1475 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{\log^3 n}\left(\sum_{i=1}^{n^3-n^2}\frac{1}{i+n^2}\right)\left(\sum_{i=1}^{n^4-n^3}\frac{1}{i+n^3}\right)\left(\sum_{i=1}^{n^5-n^4}\frac{1}{i+n^4}\right)$$

Proposed by Daniel Sitaru-Romania

S.1476

If $\lim\limits_{n\to\infty}x_n=3,\lim\limits_{n\to\infty}y_n=5,\lim\limits_{n\to\infty}z_n=7.$ Find:

$$\Omega=\lim_{n\to\infty}n^3\left(\frac{4^{\sqrt[n]{x_n}}}{4}-1\right)\left(\frac{6^{\sqrt[n]{y_n}}}{6}-1\right)\left(\frac{8^{\sqrt[n]{z_n}}}{8}-1\right)$$

Proposed by Daniel Sitaru-Romania

S.1477 Find:

$$\Omega=\lim_{n\to\infty}n\left(\prod_{k=2}^{n}\sqrt[k]{k!}\right)^{\frac{1-n}{n^2}}$$

Proposed by Daniel Sitaru-Romania

S.1478 Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(1-\frac{1}{n}\right)^{\frac{H_k}{k}} - n\right)$$

Proposed by Daniel Sitaru-Romania

S.1479 If $0 \le a \le b$ then:

$$ab(4+(a+b)^2)\tan^{-1}(\sqrt{ab}) \le (1+ab)(a+b)^2\tan^{-1}\left(\frac{a+b}{2}\right)$$

Proposed by Daniel Sitaru, Claudia Nănuți -Romania

S.1480 If $0 < a \le b < \frac{\pi}{2}$ then:

$$3\int_a^b \sin x \cdot \sinh x\, dx \le b^3 - a^3$$

Proposed by Daniel Sitaru ,Claudia Nănuți -Romania

S.1481 If $n \in \mathbb{N}$ then in ΔABC holds:

$$\sqrt{n(n+1)}\cos\left(A-\frac{\pi}{7}\right)+\sqrt{n(n+2)}\cos\left(B-\frac{\pi}{7}\right)+\sqrt{(n+1)(n+2)}\cos\left(C-\frac{\pi}{7}\right) <$$

$$< 3(n+1)\cos\frac{\pi}{21}$$

Proposed by Daniel Sitaru ,Claudia Nănuți -Romania

S.1482 If $1 < a \le b$ then:

$$\left(\left(\frac{1}{\sqrt{2}}\right)^{\sqrt{ab}}+\left(1-\frac{1}{\sqrt{2}}\right)^{\sqrt{ab}}\right)^{\frac{1}{\sqrt{ab}-1}} \le \left(\left(\frac{1}{\sqrt{2}}\right)^{\frac{a+b}{2}}+\left(1-\frac{1}{\sqrt{2}}\right)^{\frac{a+b}{2}}\right)^{\frac{2}{a+b-2}}$$

Proposed by Daniel Sitaru-Romania

S.1483 If $a, b, c \ge 0$ then :

$$a(e^{2b}+e^{-2c}) + b(e^{2c}+e^{-2a}) + c(e^{2a}+e^{-2b}) \ge 2(a+b+c)$$

Proposed by Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.589 Prove that:

$$\int_0^\infty \frac{dt}{\sqrt{4+(4-z)t^2}(1+t^2)} = \frac{1}{\sqrt{z}}\tan^{-1}\left(\sqrt{\frac{z}{4-z}}\right), z<4$$

Proposed by Abdulhafeez Ayinde-Nigeria

U.590 Prove that:

$$\int_0^\infty \frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}} = \frac{2}{\pi}\log\left(3+2\sqrt{2}\right)$$

Proposed by K. Srinivasa Raghava – India

U.591 Prove that:

$$\int_0^\infty e^{-2\pi mx}\sin(\pi x^2)\,dx = (-1)^{m+1}\left(\frac{C\left(m\sqrt{2}\right)}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\right), \forall m \in \mathbb{N}$$

Where

$$C(z) = \int_0^z \cos\left(\frac{1}{2}\pi x^2\right)dx$$

Proposed by Surjeet Singhania – India

U.592 Prove that for $n \geq 1$

$$\int_0^\infty \frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}} = \frac{4}{\pi}\frac{\sinh^{-1}\left(\sqrt{n}\right)}{\sqrt{n}}$$

Proposed by K. Srinivasa Raghava – India

U.593 Prove:

$$\lim_{\substack{x\to 0\\ x>0}} \sum_{n=2}^{\infty}\left(\frac{1}{n!}\prod_{k=0}^{n-1}(1-kx)\right) = e-2$$

Proposed by Ghazaly Abiodun-Nigeria

U.594 For $\alpha,\beta,\varphi>0$ and $\alpha\neq\beta$

$$\int_{\alpha}^{\beta}\frac{\ln(\varphi^n x^m)\,dx}{x^2+(\alpha+\beta)x+\alpha\beta}=\frac{\ln(\varphi^{2n}(\alpha\beta)^m)}{2(\beta-\alpha)}\ln\left(\frac{(\alpha+\beta)^2}{\alpha\beta}\right)$$

Proposed by Kaushik Mahanta – India

U.595 If $f(x)=17x^6+72x^5-19x^4+29x^3+71x^2-21x-1$

Find the remainder when $f(1729)$ is divided by 5

Proposed by Amrit Awasthi-India

U.596

$$f(n)=\int_0^1\int_0^1\left(\frac{x^n-y^n}{\ln x-\ln y}\right)dxdy$$

Find the closed form for

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}$$

Proposed by Ghazaly Abiodun-Nigeria

U.597 Prove the following result:

$$Li_2(\sqrt{2}-1)+Li_2(\sqrt{5}-2)-\frac{1}{4}\left(Li_2(3-2\sqrt{2})+Li_2(9-4\sqrt{5})\right)=$$

$$=\frac{5\pi^2}{48}-\frac{1}{4}\ln^2(\sqrt{2}-1)-\frac{3}{4}\ln^2\left(\frac{1+\sqrt{5}}{2}\right)$$

where $Li_2(x)$ is dilogarithm function

Proposed by Narendra Bhandari – Nepal

U.598 Prove:

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\cos\left(\frac{\pi n}{3}\right)=\frac{\pi}{\sqrt{12}}$$

Proposed by Asmat Qatea-Afghanistan

U.599 Prove or disprove that:

$$\int_0^1\int_0^1\left(\frac{\ln\left(\frac{1}{x}\right)-\ln\left(\frac{1}{y}\right)}{\ln\ln\left(\frac{1}{y}\right)-\ln\ln\left(\frac{1}{y}\right)}\right)^n dx\,dy=$$

$$=\frac{1}{\pi^{n-1}}\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\prod_{m=0}^{n}\left(m-\pi\sum_{k=1}^{n}t_k\right)\frac{dt_1dt_2\dots dt_n}{\sin\left(\sum_{k=1}^{n}t_k\right)}$$

Proposed by Ghazaly Abiodun - Nigeria

U.560 Find the closed form for:

$$\Omega=\int_0^1\int_0^1 \frac{\ln\left(\frac{1+x}{1+y}\right)}{\ln x-\ln y}dxdy$$

Proposed by Ghazaly A Abiodun - Nigeria

U.561 Prove that:

$$\sum_{n=0}^{\infty}\frac{1}{n+3k}\cos\left(\frac{\pi n}{3}\right)$$

$$=\sum_{p}^{\frac{k}{2}-1}\left(\frac{3(2p+1)}{36p^2+6p+8}-\frac{3(2p+1)}{36p^2+6p+5}\right)+\frac{1}{3}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\cdots-\frac{1}{k-1}\right)$$

Proposed by Amrit Awasthi-India

U.562 Let

$$I_n(x)=\int_0^{\infty} t\cdot e^{-tx-tn}\cdot\ln(t)\,dt, x\in\mathbb{R}^+$$

$$x_n=e^{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}-e^{1+\frac{1}{2}+\cdots+\frac{1}{n}}$$

Find:

$$\Omega=\lim_{n\to\infty}\left(1-\frac{I_n(0)}{\ln\left(\sqrt[n]{n}\right)}\right)^{\sqrt[n]{n}\cdot x_n}$$

Proposed by Mikael Bernardo-Mozambique

U.563 Suppose $X_n=\underbrace{\overline{111\ldots1}}_{n\ times}$ where $n\in\mathbb{N}$ find all prime number $p\in U(41)$ such that

i) $X_p\equiv 1(mod\ 41)$

ii) $X_p\equiv 0(mod\ 41)$

Here $U(n)=\{x\in\mathbb{N}|1\le x\le n-1,\gcd(n,x)=1\}$

For example $X_1=1, X_2=11, X_4=1111$

Proposed by Surjeet Singhania – India

U.564 If $a,b>2, n>0$ then:

$$\frac{\log(an+bn+1-2n)-\log 2n}{\log(an+bn+1-4n)-\log 2n}\le\frac{\log\left(2n\sqrt{ab}+1-2n\right)-\log 2n}{\log\left(2n\sqrt{ab}+1-4n\right)-\log 2n}$$

Proposed by Sudhir Jha-India

U.565 Prove that:

$$\int_{|z|=1}\frac{1}{\sqrt{10z^2-7z+1}}=\frac{\sqrt{2}\pi i}{\sqrt{5}}$$

Proposed by Surjeet Singhania-India

U.566

$$\begin{cases} a+b+c=345 \\ ab+bc+ca=27992 \\ abc=27648 \end{cases}, ab,c,x,y,z\in\mathbb{N}, \begin{cases} a=x!^{y!+z!} \\ b=y!^{z!+x!} \\ c=z!^{x!+y!} \end{cases}$$

Find:

$$\Omega=\sqrt{x^3+y^3+z^3+3x^2y^2z^2}$$

Proposed by Haxverdiyev Tarverdi-Azerbaijan

U.567 Find without softwares:

$$\Omega=\int_0^1\left\{\frac{x-1}{x+1}\right\}\cdot\frac{xdx}{1-x}$$

Proposed by Timson Folorunsho-Nigeria

U.568

$$a,b,c\geq 1\Rightarrow\frac{a^a}{(b+c)\left(\frac{1}{b}+\frac{1}{c}+\ln(bc)\right)}+\frac{b^b}{(c+a)\left(\frac{1}{c}+\frac{1}{a}+\ln(ca)\right)}$$
$$+\frac{c^c}{(a+b)\left(\frac{1}{a}+\frac{1}{b}+\ln(ab)\right)}\geq\frac{3}{4}$$

Proposed by Pavlos Trifon – Greece

U.569

$$I(\psi)=\int_0^\infty\frac{e^{-\psi x}}{x^2+1}dx$$

$$Ci(\psi)\sin(\psi)-si(\psi)\cos(\psi)+\frac{\pi\cos(\psi)}{2}$$

Proopsed by Qusay Yousef-Algerie

U.570 Prove:

$$\int_0^\infty\frac{x^2\sinh x}{3+4\sinh^2 x}dx=\frac{7}{9}\zeta(3),$$

$$\int_0^\infty\frac{x^2\cosh x}{4\cosh^2 x-3}dx=\frac{5\pi^3}{144},$$

$$\int_0^\infty \frac{3x\cosh x}{8\cosh^2 x-6}=G$$

where $G,\zeta(.)$ is Catalan's constant and Riemann zeta function. It is required to prove the result without the use of any standard definitions and integral formulae.

Proposed by Narendra Bhandari – Nepal

U.571 Let $x^{2020}+a_{2019}x^{2019}+a_{2018}x^{2018}+\cdots+a_0\in Z_{[x]}$ and all roots of this polynomials are positive real numbers.

Find the smallest possible value of coefficient a_{1010}.

Proposed by Gantumur Choijilsuren-Mongolia

U.572

$$\sum_{m,n\ge 0}\frac{m!\,n!\,(m+n+4)}{(m+n+3)(m+n+2)!}<\sum_{m,n,p\ge 0}\frac{m!\,n!\,p!}{(m+n+p+2)!}<Ei(1)-\gamma+\zeta(2)$$

Where $Ei(\,.\,)$ is Exponential integral

Proposed by Narendra Bhandari – Nepal

U.573 If all the derivatives of $f(x)$ are defined at $x=1$ then prove that $f(e^{-x})=\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}[B_k(D)f(x)]|_{x=1}$, where $D:=\frac{d}{dx}$ and $B_k(x)$ is the Bell polynomial.

Proposed by Angad Singh-India

U.574 Prove that $\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sqrt{1+\sin^2 x}}=\frac{\pi}{2}\left(\frac{\Gamma^4\left(\frac{1}{4}\right)-8\pi^2}{(2\pi)^{\frac{3}{2}}\Gamma^2\left(\frac{1}{4}\right)}\right)=\frac{\pi}{2}\left(G-\frac{1}{\pi G}\right)$, where $G=\frac{\Gamma^2\left(\frac{1}{4}\right)}{(2\pi)^{\frac{3}{2}}}$ is Gauss Constant.

Proposed by Narendra Bahandari-Nepal

U.575 Given that $\Phi(n)=\int_0^\infty|Ei(-x)x^{n-1}|\,dx$, where $Ei(x)$ is the exponential integral $Ei(x)=-\int_{-x}^{\infty}\frac{e^{-t}}{t}dt$, then find: $\sum_{n=1}^{\infty}\frac{1}{\Phi(n)}$.

Proposed by Ty Halpen-USA

U.576 If $f(x)=\frac{\sin x}{x}$ then prove that $\sum_{k=1}^{4n}(x+k)\cdot f^{k-1}(x)=x\big(f(x)-f^{4n}(x)\big)$, where $f^k(x)$ denotes the k^{th} order derivative w.r.t. x and $n\in\mathbb{N}$.

Proposed by Amrit Awasthi-India

U.577 Find:

$$\int_0^1 x^2\cdot\sin^{-1}x\cdot\sin^{-1}(4x^3-3x)\,dx$$

Proposed by Ty Halpen-USA

U.578 If $A_n=\sqrt{2+\sqrt{2+\sqrt{2}+\cdots+\sqrt{6}}}$; $(n-roots)\phi$ − golden ratio, the prove that:

$$\Omega=\prod_{k=1}^{\infty}\frac{1+(\phi-1)^{5^k}+(2-\phi)^{5^k}+(2\phi-3)^{5^k}+(5-3\phi)^{5^k}}{\frac{1}{2}A_k}=\frac{(4+5\phi)\log(2+\sqrt{3})}{11\sqrt{3}}$$

Proposed by Asmat Qatea-Afghanistan

U.579 Find:

$$\int\frac{(1+\sin x)e^x}{1+\cos x}\tan^{-1}\left(e^x\tan\frac{x}{2}\right)dx$$

Proposed by Mihály Bencze-Romania

U.580 $[*]-GIF,\{x\}=x-[x]$. Find: $\Omega_1=\lim_{n\to\infty}\left(\left\{\left(2+\sqrt{5}\right)^{2n}\right\}\cos\left(\pi\left[\left(2+\sqrt{5}\right)^{2n}\right]\right)\right)$ and

$$\Omega_2=\lim_{n\to\infty}\left(\left\{\left(2+\sqrt{5}\right)^{2n+1}\right\}\cos\left(\pi\left[\left(2+\sqrt{5}\right)^{2n+1}\right]\right)\right)$$

Proposed by Rajeev Rastogi-India

U.581 Find closed forms $a,b\geq 1, b+1\neq a$,

$$\Omega_1(a,b)=\int_{-\infty}^{\infty}\frac{\sin\left(ax-\frac{b}{x}\right)}{x+\frac{1}{x}}dx,\ \Omega_2(a,b)=\int_{-\infty}^{\infty}\frac{\cos\left(ax-\frac{b}{x}\right)}{\left(x+\frac{1}{x}\right)^2}dx$$

Proposed by Zaharia Burghelea-Romania

U.582 Find:

$$\int_0^{\infty}\frac{1}{\left(x+\sqrt{1+x^2}\right)^2}dx$$

Proposed by Pranesh Pyara Shrestha-Nepal

U.583 Find:

$$\int_0^{\frac{\pi}{2}}(\log(\tan x)+\log^2(\tan x)+\log^3(\tan x))\sin^3(2x)\,dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.584 In ΔABC holds:

$$\frac{1}{a}\int_{h_a}^{m_a}\frac{\cos x}{x}dx+\frac{1}{b}\int_{h_b}^{m_b}\frac{\cos x}{x}dx+\frac{1}{c}\int_{h_c}^{m_c}\frac{\cos x}{x}dx\leq 9F$$

Proposed by Mokhtar Kassani-Algerie

U.585 Prove that:

$$\prod_{n=1}^{\infty}\frac{n+\cos\left(\frac{n\pi}{3}\right)}{n+\cot\left(\frac{\pi}{3}\right)\sin\left(\frac{n\pi}{3}\right)}=\frac{\pi\sqrt{2}\cdot\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{12}\right)}{\Gamma^2\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{7}{6}\right)\Gamma\left(\frac{11}{12}\right)}$$

Proposed by Asmat Qatea-Afghanistan

U.586 Prove that:

$$\int_0^{\frac{\pi}{2}} \log(x^2+\log^2(\cos x))\,dx = \pi\log(\log 2)$$

Proposed by Simon Peter-Madagascar

U.587 For a,b,c,ω, find:

$$\Phi(a,b,c,\omega)=\int_0^{\infty}\frac{xe^{-\omega x^2}}{ax^4+bx^2+c}dx$$

Proposed by Simon Peter-Madagascar

U.588 Prove that:

$$\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{3}\right)\psi\left(n+\frac{1}{3}\right)}{3^n n!}=-\sqrt[3]{\frac{3}{2}}\Gamma\left(\frac{1}{3}\right)\left\{\log\left(\frac{108}{9}\right)+\gamma+\frac{\pi}{2\sqrt{3}}\right\}$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

U.589 For $\mathrm{erf}(*)$ –error function, find:

$$\varphi(t)=\int_0^t x^{-1}e^{-\frac{a^2}{x}}\,\mathrm{erf}\left(\frac{b}{\sqrt{x}}\right)dx$$

Proposed by Simon Peter-Madagascar

U.590 Prove that:

$$\int_0^1\sqrt[6]{x^2\left(1-\sqrt[3]{x^2}\right)^3}\log x\log\left(1-\sqrt[3]{x^2}\right)dx=\frac{1532}{375}-\frac{32}{25}\log 2-\frac{3\pi^2}{10}$$

Proposed by Abdul Mukhtar-Nigeria

U.591 Prove that:

$$\frac{1}{\sum_{n\ge1}\frac{H_n^{(4)}}{n^2}}+\frac{1}{\sum_{n\ge1}\frac{H_n^{(2)}}{n^4}}\ge\frac{15120}{11\pi^6}\ge\frac{2}{\sqrt{\sum_{n\ge1}\frac{H_n^{(4)}}{n^2}}+\sqrt{\sum_{n\ge1}\frac{H_n^{(2)}}{n^4}}}$$

Proposed by Samir HajAli-Syria

U.592 Find:

$$\Omega=\int\frac{dx}{\sin x+\cos x+\tan x+\cot x+\sec x+\csc x}$$

Proposed by Nawar Alasadi-Iraq

U.593 For any Δabc prove that:

$$\sin^{-1}\left(\frac{a}{c}\sqrt{\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}\right)+\sin^{-1}\left(\frac{b}{c}\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}\right)+$$

$$+\sin^{-1}\left(\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}\right)\geq\pi-\Psi,\Psi\in[0,\pi].$$

Proposed by Qusay Yousef-Algerie

U.594 Find:

$$\Omega=\int_0^n\sum_{i=0}^n\left[x+\frac{i}{n}\right]dx,[x]-\text{GIF}$$

Proposed by Rajesh Darbi-India

U.595 Prove that:

$$\lim_{x\to1}\left(\sin\frac{(1-x)\pi}{2}\cdot\Gamma(1-x)\right)=\frac{\pi}{2}$$

Proposed by Ghazaly Abiodun-Nigeria

U.596 Prove that:

$$\int_0^\infty\tan^{-1}(\pi t)\,(1-\tanh(\gamma t))\,dt=\pi\log\left(\frac{2^{\frac{\gamma}{2}}\pi^{-\frac{2}{\pi}}\gamma^{\frac{1}{\pi}}2^{\frac{2}{\pi^2}-\frac{1}{\gamma}}\left(\frac{\gamma}{\pi^2}-1\right)!^{\frac{1}{\gamma}}}{\left(\frac{2\gamma}{\pi^2}-1\right)!^{\frac{1}{\gamma}}e^{\frac{1}{\pi^2}}}\right)$$

Proposed by Abdulhafeez Ayinde-Nigeria

U.597 Find:

$$\Omega=\lim_{n\to\infty}n\left(\sqrt[n]{\frac{2\left(e^{\frac{1}{n}}+e^{\frac{2}{n}}+\cdots+e^{\frac{2019}{n}}-2019\right)}{\left(1+\frac{1}{n}\right)^{2019}-1}}-\left(\cos\frac{1}{n}\right)^{\cot\frac{1}{n}}\right).$$

Proposed by Hikmat Mammadov-Azerbaijan

U.598 Find the closed form:

$$\Omega=\sum_{k=0}^n\frac{(n-k)^kx^k\log x^{(n-k)}\log x^k}{(n-k)!\,k!}$$

Proposed by Simon Peter-Madagascar

U.599 Evaluate:

$$\Omega=\int_0^1\sin^{-1}\left(\frac{1+x^2}{\sqrt{1+x^4}}\right)dx$$

Proposed by Abdul Mukhtar-Nigeria

U.600 Prove that:

$$1+\frac{1}{2}\left(\frac{1}{2}\right)^2+\frac{1}{4}\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2+\frac{1}{8}\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2+\cdots=\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{\pi^3}}$$

Proposed by Ngulmun George Baite-India

U.601 Find:

$$\Omega = \int \frac{2x^3 + 10x^2 + 8x}{2x^5 + 2x^4 - 5x^3 - 5x^2 + 3x + 3} dx$$

Proposed by Pranesh Pyara Shrestha-Nepal

U.602 Find:

$$\Omega = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\zeta_n(3)}{n}$$

Proposed by Probal Chakraborty-India

U.603 Prove that:

$$\Phi = \int_0^1 \int_0^{\pi} \int_0^{2\pi} \dots \int_0^{71\pi} 72\pi d72\pi \dots d\pi + \sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n+2} + \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)^{(-1)^{n-1}} =$$

$$= \frac{\Gamma(2)}{\Gamma(74)} + \frac{11}{6} - 2\log A - \frac{\gamma}{3} - \frac{\log(2\pi)}{2} + \frac{\pi}{2}\tanh\left(\frac{\pi}{2}\right)$$

Proposed by Ajenikoko Gbolahan-Nigeria

U.604 Find:

$$\Omega = \sum_{n=1}^{\infty} \left(\frac{\log \phi}{\phi^{n+3}} + \frac{1}{\phi^{n+3}}\left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}\right)\right), \phi - \text{golden ratio.}$$

Proposed by Abdul Mukhtar-Nigeria

U.605 Prove that: $\int_0^{\infty} \frac{\tanh^3 x}{e^{2x}} \frac{dx}{x} = 2 + \frac{7}{\pi^2}\zeta(3) + \log\left(\frac{8\pi^3}{A^{24}\sqrt[3]{1024}}\right)$, where $\zeta(\cdot)$ –is Riemann zeta function and A is Glashier-kinkelin constant.

Proposed by Narendra Bhandari-Nepal

U.606 Define the set $M = \{2p | p \in \mathbb{N}\}$ and $\Omega(n) = \sum_{k=0}^{N} \frac{(-1)^k}{2k+M}\binom{N}{k}$, $Ei(x)$ –exponential integral, then show that

$$\sum_{N=1, j\in M}^{\infty} \sum_{i=1}^{\frac{j}{2}} \frac{2\Omega(j)}{N\left(\frac{j}{2} - 1\right)!} \frac{1}{\left(N - \frac{j}{2} + 1\right)^j} \left(\lim_{\frac{j}{2}\to\infty} \left(\sum_{j\in M}^{\frac{j}{2}} \frac{j\Omega(j)}{\left(\frac{j}{2} - 1\right)!}\right)\right) = (e-1)(Ei(1) - \gamma)\zeta(j)$$

Proposed by Narendra Bhandari-Nepal

U.607 Find:

$$\Omega = \int_0^1 \frac{\log(1 + x^{\alpha})}{x^{\beta}} dx, \alpha, \beta \in \mathbb{N}, \beta < \alpha + 1$$

Proposed by Precious Itsuokor-Nigeria

U.608 Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \frac{(-1)^{1^k+2^k+\cdots+n^k}}{1+2+3+\cdots+n}, k \in \mathbb{N}$$

Proposed by Ghazaly Abiodun-Nigeria

U.609 If $A = (x^2+1)(x^2+2)(x^2+3), B = (x^2+2)(x^2+3)(x^2+4)$,

$C = (x^2+3)(x^2+4)(x^2+5)$, find: $\Omega = \int_0^{\infty} x^2 \left(\frac{1}{A} + \frac{2}{B} + \frac{3}{C}\right) dx$.

Proposed by Hussain Reza Zadah-Afhanistan

U.610 Find a closed form:

$$\Omega = \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{(n^3+6n^2+11n+5)\cdot n!}{1+(n^3+6n^2+11n+6)\cdot(n!)^2}\right)$$

Proposed by Daniel Sitaru-Romania

U.611 $S(p) = \{(x,y)|x^3+y^3 \le p^3, x \ge 0, y \ge 0, p > 0\}$. Find:

$$\Omega = \lim_{n\to\infty} \frac{Area\big(S(p)\big)}{p^2}$$

Proposed by Daniel Sitaru-Romania

U.612 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n!} \int_0^{\infty} \frac{x^n \sin\left(x+\frac{\pi}{4}\right)}{e^x} dx$$

Proposed by Daniel Sitaru-Romania

U.613 Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \frac{n(n+1)^2(2n+1)}{4^n \cdot n!}$$

Proposed by Daniel Sitaru – Romania

U.614 If $\log_5(7^a - 2) = \log_7(5^a + 2)$. Find:

$$\Omega = \int_a^e \frac{1+\log x}{x^x + x^{-x}} dx$$

Proposed by Daniel Sitaru – Romania

U.615 If $0 < a \le b < \frac{\pi}{2}$ then:

$$\int_a^b \int_a^b \frac{1-\cos(x+y)}{1+\cos(x+y)} dxdy \le \log^2\left(\frac{\cos b}{\cos a}\right)$$

Proposed by Daniel Sitaru – Romania

U.616 Let $F(x)$ is a monotonically decreasing function and continuous function in the closed-interval $[0,1]$, the prove the inequality:

$$\left(\int_0^1 xF(x)\,dx\right)^2+\int_0^1 F(x)\,dx+1\geq 0$$

Proposed by Srinivasa Raghava-India

U.617 If define $f(m,n)=$

$$=\sum_{k=0}^{n-1}\frac{\frac{\pi(1-2k)\sin\left(\frac{\pi(2k+1)(m+1)}{n}\right)}{2n}-\cos\left(\frac{\pi(2k+1)(m+1)}{n}\right)\log\left(2\sin\left(\frac{\pi(2k+1)}{2n}\right)\right)}{n}$$

Find the value of $f(6,14)$.

Proposed by Srinivasa Raghava-India

U.618 If define $f(m,n)=$

$$=\sum_{k=0}^{n-1}\frac{\frac{\pi(1-2k)\sin\left(\frac{\pi(2k+1)(m+1)}{n}\right)}{2n}-\cos\left(\frac{\pi(2k+1)(m+1)}{n}\right)\log\left(2\sin\left(\frac{\pi(2k+1)}{2n}\right)\right)}{n}$$

Find the value of $f(6,14)$.

Proposed by Srinivasa Raghava-India

U.619 Prove the integral relation

$$\int_0^\infty\left(\frac{\sin x}{x^3}-\frac{\sinh x}{x^3}-\frac{\cos x}{x^2}+\frac{\cosh x}{x^2}\right)\frac{e^{-x}}{\sqrt{x}}\,dx=\frac{4}{15}\left(\sqrt{2\pi}+\sqrt{\left(-17+13\sqrt{2}\right)\pi}\right)$$

Proposed by Srinivasa Raghava-India

U.620 Let $f(t)=(-1)^{[t]}$, where $[t]$ —is greatest integer function and if

$A(m)=|f(\pi)-f(2\pi)|+|f(2\pi-f(3\pi)|+\cdots+|f(\pi m)-f\big(\pi(m+1)\big)|$ then find

$$\Omega=\lim_{m\to\infty}\frac{A(m)}{2m}$$

Proposed by Srinivasa Raghava-India

U.621 For any complex numbers a,b with $Re(a),Re(b)>0$. If

$$f_m(a,b)=\int_0^\infty\left(e^{-ax}-e^{-bx}\right)\frac{\sin(mx)}{x}\,dx$$

then:

$$\frac{\partial}{\partial a}\left(\int_0^1 f_m(a,b)\,dm\right)+\frac{\partial}{\partial b}\left(\int_0^1 f_m(a,b)\,dm\right)=\frac{1}{2}\log\left(\frac{a^2(1+b^2)}{b^2(1+a^2)}\right)$$

Proposed by Srinivasa Raghava-India

U.622 Prove that:

$$\int_0^{\infty}\left(\sin\left(\frac{x}{2}\right)+\sin\left(\frac{x}{3}\right)+\sin x\right)e^{-x}\frac{dx}{\sqrt{x}}$$
$$=\frac{\sqrt{\pi}}{10}\left(5\sqrt{\sqrt{2}-1}+2\sqrt{5(\sqrt{5}-2)}+\sqrt{15(\sqrt{10}-3)}\right)$$

Proposed by Srinivasa Raghava-India

U.623 Let the matrices $p_0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, p_1=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, p_2=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Then find the Determinant and Trace of the following:

$$\frac{(p_0+p_1p_2)+(p_1+p_0p_2)+(p_2+p_0p_1)}{p_0+p_1+p_2}$$

Proposed by Srinivasa Raghava-India

U.624 If, for $n\geq 1$ $\Psi(n)=\frac{1}{n}-\int_{-\infty}^{\infty}\frac{e^{-n\pi x}+xe^{\pi x}}{\cosh^2(n\pi x)}dx$ then prove the following

$$\int_1^{\infty}\psi(n)\,dn=\frac{1}{\pi}-\frac{2}{\pi^2};\ \int_1^{\infty}\frac{\psi(n)}{n}dn=\frac{1}{\pi}-\frac{8G}{\pi^3}$$

Proposed by Srinivasa Raghava-India

U.625 Prove that:

$$\int_0^1\int_0^1\int_0^1\sqrt[3]{\frac{x^2y^2z^2}{(1+x)^2y^2z^2+(1+y)^2z^2x^2+(1+z)^2x^2y^2}}dxdydz<\frac{1}{30}$$

Proposed by Jalil Hajimir-Canada

U.626 Find:

$$\Omega=\lim_{x\to 1}\left(\frac{x^{\Gamma(x)}-1}{\Gamma(x)^{\Gamma(x)}-1}\cdot\frac{x^{x\Gamma(x)}-1}{\Gamma(x)^{x\Gamma(x)}-1}\right)$$

Proposed by Jalil Hajimir-Canada

U.627 Evaluate:

$$\Omega=\int_0^{\infty}\frac{x^{2p}}{x^{2q}+1}dx$$

, where p and q are two integers such that $0\leq p<q$.

Proposed by Jalil Hajimir-Canada

U.628 Prove:

$$\int_0^1\int_0^1\frac{2^x+2^y}{x+y+2}dxdy<1$$

Proposed by Jalil Hajimir-Canada

U.629 Prove that: $\frac{1}{e} < \left|\int_0^\infty \log(1-e^{-x})J_0(2\sqrt{x})dx\right| < \frac{\pi^2}{6}$, where $J_n(x)$ –is the Bessel function of order n.

Proposed by Angad Singh-India

U.630 For all $n > 0$, let a_n be the largest natural number such that, $e^{na_n} > a_n!$ then prove that $a_n \cong 5e^{n-1}$.

Proposed by Angad Singh-India

U.631 Prove that:

$$\int_0^\infty \left(\frac{\sin(\pi x^2) - x^3\cos(\pi x^2)}{x^6+1} - \frac{\cos(\pi x^2)}{x^3+1}\right)dx = \frac{\pi e^{-\frac{\pi\sqrt{3}}{2}}}{3}$$

Proposed by Angad Singh-India

U.632 If $n \in \mathbb{N}$, then:

$$\int_0^\infty (1-\tanh(x))^n\, dx = 2^{n-1}\left(\log 2 - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right)$$

Proposed by Angad Singh-India

U.633 If $\phi(m) \coloneqq \phi(x,m) = \sum_{k=1}^\infty \frac{e^{-kx^2}}{m-\log(k^2)}$ then:

$$\int_0^\infty x^5 \log x \cdot \phi(3-2\gamma)dx = \frac{\zeta(3)}{4}$$

Proposed by Angad Singh-India

U.634 If H_n is the n^{th} harmonic number then:

$$\int_0^\infty \frac{e^{-(n+1)x}\sin(nx) + \sin x - e^{-nx}\sin((n+1)x)}{\cosh(x) - \cos x}dx = H_n$$

Proposed by Angad Singh-India

U.635 If $m > 0$, then prove:

$$\lim_{x\to 0}\frac{1}{x}\left(\psi\left(\frac{m+x}{2x}\right) - \psi\left(\frac{m}{2x}\right)\right) = \frac{1}{m}$$

Proposed by Angad Singh-India

U.636 Prove that:

$$\int_0^\infty \cos(\pi x^2)\frac{\sinh(\sqrt{2}\pi x) - \sin(\sqrt{2}\pi x)}{\sinh^2\left(\frac{\pi x}{\sqrt{2}}\right) + \sin^2\left(\frac{\pi x}{\sqrt{2}}\right)}dx = \frac{1}{\sqrt{2}}\left(\frac{\pi^{\frac{1}{4}}}{\Gamma\left(\frac{3}{4}\right)} - 1\right)$$

Proposed by Angad Singh-India

U.637 If $\psi(x)=\sum_{k=1}^{n}\frac{x^k}{d(k)}$ then: $\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\psi\left(\frac{e^{\frac{2\pi ik}{n}}}{2}\right)=1$, where $|x|<1$ and $d(n)$ is the number of divisors of n.

Proposed by Angad Singh-India

U.638 If $n\in\mathbb{N}$, then $\int_0^{\infty}(1-\tanh^{2n}x)dx=\sum_{k=0}^{n-1}\binom{2n}{2k+1}\phi(k,n)$, where (x_k) is the falling factorial,

$$\phi(k,n)=\frac{(2k)!}{(2n-1)_{2k}(2n-2k-1)}\left(1-\frac{1}{2^{2n-2k-1}}\right)-\sum_{m=1}^{2k}\frac{(2k)_{m-1}}{(2n-1)_m 2^{2n-m}}$$

Proposed by Angad Singh-India

U.639 If $\phi(x)=\sum_{k=1}^{n}\frac{x^k}{\sigma(k)}$ then $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\psi\left(\frac{e^{\frac{2\pi ik}{n}}}{2}\right)=1,\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{xe^{\frac{2\pi ik}{n}}}{1-xe^{\frac{2\pi ik}{n}}}=\sum_{k=1}^{\infty}\frac{kx^k}{1-x^k}$

and

$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{x^m}{\sigma^2(m)}e^{\frac{2\pi imk}{n}}=\psi(x)$, where $|x|<1$ and $\sigma(n)$ is the sum of divisors of n.

Proposed by Angad Singh-India

U.640 Prove that: $\lim\limits_{x\to 0}\frac{1}{x^2}\left(\psi(x)+\omega\psi(\omega x)+\omega^2\psi(\omega^2 x)+\frac{3}{x}\right)=-3\zeta(3)$, where $\omega^2+\omega+1=0$ and $\psi(x)$ is the digamma function.

Proposed by Angad Singh-India

U.641 If $n>0$, then: $\int_0^{\infty}\frac{\log(\cosh(x))}{\cosh^n(x)}dx=\int_0^{\infty}\frac{\psi(n)-\psi\left(\frac{n}{2}\right)-\log 2}{\cosh^n(x)}dx$, where $\psi(n)$ is the digamma function.

Proposed by Angad Singh-India

U.642 Prove that:

$$\int_0^{\infty}\frac{\cosh x}{\cosh(5x)}dx=\frac{\pi\phi}{5\sqrt{4\phi^2-1}},\int_0^{\infty}\frac{\cosh^2 x}{\cosh(5x)}dx=\frac{\pi(\pi+2)}{20\phi}$$

Proposed by Angad Singh-India

U.643 Prove that:

$$\int_0^{\infty}\frac{x-x^3}{(4x-4x^3)^2+(2-6x^2+x^4)^2}dx=\frac{1}{16\sqrt{2}}\left(\pi-\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

Proposed by Angad Singh-India

U.644 If n is a perfect number, then prove that $\oint_C \frac{\log(q)_\infty}{q^{n+1}} dq = 4\pi i$, where C is the unit circle centered at the origin.

Proposed by Angad Singh-India

U.645 If $\psi(m,a) = \int_0^\infty \frac{e^{-x^2} x^{m-1}}{(1+x^2)^a} dx$ and $a \in \mathbb{N}$, then: $\psi(2,a) = \frac{1}{2}\sum_{k=1}^{a-1} \frac{(-1)^{k-1}}{(a-1)_k} - \frac{(-1)^a e E_1(1)}{2\Gamma(a)}$, where $E_1(x)$ is the generalised exponential integral.

Proposed by Angad Singh-India

U.646 Prove that:

$$\sum_{k=1}^{\infty} \frac{1}{k\binom{3k}{k}5202^k} = \frac{1}{53}\left(\frac{6}{5}\tan^{-1}\left(\frac{5}{99}\right) - \log\left(\frac{18}{17}\right)\right)$$

Proposed by Angad Singh-India

U.647 If $0 < n < \frac{1}{2}$ and $K(n) = \frac{\Gamma(n)}{\int_0^\infty x^{n-1} e^x erfc(\sqrt{x}) dx}$ then:

$$K^2(n) + K^2\left(\frac{1}{2} - n\right) = 1$$

Proposed by Angad Singh-India

U.648 If $a \in \mathbb{R}$, then:

$$\int_0^1 \frac{e^{ax}}{1+x} dx = e^{-a}\left(\log 2 + \sum_{k=1}^{\infty} \frac{2^k - 1}{k \cdot k!} a^k\right)$$

Proposed by Angad Singh-India

U.649 If $|a| < 1, x^{(0)} = 1, x^{(k)} = x(x+1)(x+2) \dots (x+k-1)$ and $R(a) = \frac{1}{\Gamma^2\left(\frac{a}{2}\right)\Gamma(1-a)} \sum_{k=0}^{\infty} \frac{a^{(k)}}{k!\left(k+\frac{a}{2}\right)^2}$ then $R^2(a) + R^2(1-a) = 1$.

Proposed by Angad Singh-India

U.650 Find:

$$\Omega = \int_1^2 \int_1^2 \int_1^2 \frac{x^2 - yz}{x^3 + y^3 + z^3 - 3xyz} dxdydz$$

Proposed by Asmat Qatea – Afghanistan

U.651 Prove that:

$$\frac{1}{1^5} + \frac{1}{1^5 + 2^5} + \frac{1}{1^5 + 2^5 + 3^5} + \dots = 60 - 4\pi^2 + 8\pi\sqrt{3}\tan\left(\frac{\pi\sqrt{3}}{2}\right)$$

Proposed by Asmat Qatea - Afghanistan

U.652 Find:

$$\Omega = \int_0^1 \frac{x + \ln(1-x)}{x \cdot \ln(1-x)} dx$$

Proposed by Asmat Qatea – Afghanistan

U.653 If:

$$I_n = \int_0^1 \sin^n(\pi x) \ln\big(\Gamma(x)\big)\, dx, P_n = \int_0^1 \frac{x^n}{1+x} dx$$

then prove that:

$$2\sqrt{\pi} \cdot I_n \cdot \Gamma\left(\frac{n+2}{2}\right) = \Gamma\left(\frac{n+1}{2}\right)(\ln(\pi) + P_n)$$

Proposed by Asmat Qatea - Afghanistan

U.654 Evaluate the integral:

$$\Omega = \int_0^1 \frac{\sqrt[3]{x} - 1}{\ln(x)\big(1+\sqrt{x}\big)\big(1+\sqrt[4]{x}\big)\big(1+\sqrt[8]{x}\big)\big(1+\sqrt[16]{x}\big)} dx$$

Proposed by Asmat Qatea - Afghanistan

U.655 Prove:

$$\sum_{k=0}^{\infty} \frac{\cos^5\left(\frac{nk}{3}\right)}{(2k)!} = \frac{11}{16}\cosh\left(\frac{\sqrt{3}}{2}\right)\cos\left(\frac{1}{2}\right) + \frac{5}{16}\cos(1)$$

Proposed by Asmat Qatea – Afghanistan

U.656 Find:

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\tan x - 1}{\ln(\tan x)} dx$$

Proposed by Asmat Qatea - Afghanistan

U.657 Find a closed form:

$$\Omega = \int_0^{\infty} x^{-2} \cdot e^{-4x} \cdot \sin^2(2x)\, dx$$

Proposed by Abdul Mukhtar-Nigeria

U.658 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{16^n}{\left(\binom{2}{1}\binom{4}{2}\binom{6}{3} \cdot \ldots \cdot \binom{2n}{n}\right)^2} - n\right)$$

Proposed by Abdul Mukhtar-Nigeria

U.659 If $\frac{1}{2}\omega_n =$

$$= \frac{\pi^{\frac{n}{2}} r^n \int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{n}{2}-\frac{1}{2}\right)} x \cos^{2\left(r^{r=0}\right)-1} x\, dx}{\int_0^\infty x^{\frac{n}{2}-\sin^2 t - \cos^2 t} e^{-x} dx \int_0^\infty \left(-e^{-t \log e} t \sin^2 t - e^{-t \log e} t \cos^2 t\right)(-1)^3 dt \int_0^\infty \frac{1}{2x+1+x^2} dx}$$

Evaluate ω_n and hence show that $\omega_{n=0} + 2019 = 2020$.

Proposed by Abdul Mukhtar-Nigeria

U.660 Find without any software:

$$\Omega(x) = \int_0^x \frac{\log(\cos y)}{\sin(2y)} dy$$

Proposed by Abdul Mukhtar-Nigeria

U.661 Find:

$$\Omega = \int \frac{1}{x}\left(\frac{1-3x+x^2}{2-e^x}\right) dx$$

Proposed by Abdul Mukhtar-Nigeria

U.662 Find a closed form:

$$\Omega = \int_0^1 \log^2\big(\Gamma(x)\Gamma(1-x)\big)\, dx$$

Proposed by Abdul Mukhtar-Nigeria

U.663 Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \left\{\frac{n!}{e}\right\} \cdot \frac{1}{n!}, \{*\} = x - [x], [*] - GIF.$$

Proposed by Abdul Mukhtar-Nigeria

U.664 Find:

$$\Omega = \int_0^1 \frac{\log^2 x \sqrt{\log\left(\frac{1}{x}\right)}}{1+x} dx$$

Proposed by Abdul Mukhtar-Nigeria

U.665 Prove that:

$$\sum_{k=1}^{\infty} \frac{\{(2n-1)!\sinh(1)\}}{(2n-1)!} = \frac{1}{2e}$$

,where $\{\cdot\}$ –fractional part.

Proposed by Abdul Mukhtar-Nigeria

U.666 Find:

$$\Omega=\int_0^1 \log^2\left(\frac{1-x}{1+x}\right)\log^3 x\,\frac{dx}{1-x^2}$$

Proposed by Abdul Mukhtar-Nigeria

U.667 Prove that:

$$\int_0^{\frac{\pi}{2}} Li_2(-x^2\tan^2 y)dy=2\pi Li_2(-x), y\geq 0$$

, where $Li_2(x)$ −dilogarithmic function.

Proposed by Abdul Mukhtar-Nigeria

U.668 Find a closed form:

$$\Omega(x)=\sum_{n=1}^{\infty}\frac{(n!\cdot e-[n!\cdot e])x^n}{n!}, x\in\mathbb{R}-\{1\}$$

, where $[*]$ − great integer function.

Proposed by Abdul Mukhtar-Nigeria

U.669 Evaluate:

$$\Omega=\lim_{x\to 0}\left\{e^{\pi}\sqrt[12]{(1+\sin(2x))^{\frac{\pi^3}{x}}\sqrt[80]{(1+\sin(4x))^{\frac{\pi^5}{x}}\sqrt[252]{(1+\sin(6x))^{\frac{x^7}{x}}\dots}}}\right\}$$

Proposed by Mikael Bernardo – Mozambique

U.670 If $f(x)=\frac{2}{1+\sqrt{x}}\cdot\frac{2}{1+\sqrt{\sqrt{x}}}\cdot\frac{2}{1+\sqrt{\sqrt{\sqrt{x}}}}\dots$ and $S=-1+\frac{1}{16}-\frac{1}{81}+\frac{1}{256}-\cdots$ find

$$L=\log\left(\lim_{x\to 1}\left(1+f(x)(x-1)\right)^{\frac{S}{\log(x)}}\right)$$

Proposed by Mikael Bernardo - Mozambique

U.671 Let $H_n=\psi(n+1)+\gamma, n\in\mathbb{R}^+, Li_2(n)=\sum_{k=1}^{\infty}\frac{n^k}{k^2}; |n|<1$

Find:

$$\lim_{n\to\frac{1}{2}}\left(\frac{\log\left(1+\sqrt{H_n}-\sqrt{H_{\frac{1}{2}}}\right)}{\sqrt[3]{Li_2(n)}-\sqrt[3]{Li_2\left(\frac{1}{2}\right)}}\right)$$

Proposed by Mikael Bernardo - Mozambique

U.672

$$\{x_n\}_{n\geq 1}$$
$$x_n=\frac{\pi}{2}+\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{3}{5}\right)+\cdots+\sin^{-1}\left(\frac{2n-2}{n^2-2n+2}\right)$$

Find:

$$\Omega=\lim_{n\to\infty}\left((n^2-2n+2)\cdot x_n\right)$$

Proposed by Mikael Bernardo - Mozambique

U.673 Evaluate:

$$\Omega=\int_0^1\sqrt{x^{2n-1}}\log(1-x)\,dx, n\in\mathbb{N}$$

Proposed by Mikael Bernardo - Mozambique

U.674 Find:

$$\Omega=\int_{1018}^{5060}\frac{\log((6078-x)^2)}{\log(36942084x-12156x^2+x^3)+\log(x)}dx$$

Proposed by Mikael Bernardo - Mozambique

U.675 Evaluate:

$$\int_0^\infty\frac{\ln(1+px)}{\sqrt[n]{x^{n+1}}}dx, n,p\in\mathbb{N}$$

Proposed by Mikael Bernardo - Mozambique

U.676 Prove that: $\Omega=\int_0^1\frac{x}{1+x^2}\Big[\log(1+x+x^2+x^3)^2+\frac{1}{3}\log\Big((1-x)^{2\sqrt{2}+\sqrt{5}}\cdot$ $(1+x)^{\sqrt{2}+2\sqrt{5}}\Big)\cdot\log\Big((1-x)^{2\sqrt{2}-\sqrt{5}}\cdot(1+x)^{\sqrt{2}-2\sqrt{5}}\Big)\Big]dx$, $\Omega=\frac{\pi}{2}C+\frac{9}{32}\zeta(3)$.

Proposed by Mohammed Bouras-Morocco

U.677 If $a_i\geq 0, i=\overline{1,n}, a_1+a_2+\cdots+a_n=n$, $\begin{cases}\phi_n=\frac{a_1}{1+a_1}+\frac{a_2}{1+a_2^2}+\cdots+\frac{a_n}{1+a_n^n}\\ \varphi=\frac{a_1}{1+a_1}+\frac{a_2}{1+\sqrt{a_2}}+\cdots+\frac{a_n}{1+\sqrt[n]{a_n}}\end{cases}$. Prove that:

$\max(\varphi_n)=\max(\phi_n)$ and $\min(\varphi_n)\geq\min(\phi_n)$.

Proposed by Mohammed Bouras-Morocco

U.678 Let $A(n)=\lim\limits_{n\to\infty}\frac{1}{2^n}\prod_{k=1}^{n}\left(\sqrt[2^n]{x}+\frac{1}{\sqrt[2^n]{x}}\right)$. Prove that:

$$\int_0^1 \frac{1}{A(x)}\left(1+\frac{1}{3x}\right)dx=\frac{\pi^2}{12}$$

Proposed by Mohammed Bouras-Morocco

U.679 Let $P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$. If $P^{(k)}(0)=P^{(k)}(1), k\in\{1,2,3\}$ and $\int_0^1 P(x)\log(\Gamma(x))\,dx=\frac{\zeta(5)}{40a_0\pi^2}$. Then find: $(a_0,a_1,\dots,a_4)$.

Proposed by Mohammed Bouras-Morocco

U.680 Find:

$$\Omega=\lim_{n\to\infty} n^{m+2}\cdot\int_0^1\left(\frac{1}{\left(n-\sqrt[m]{x}\right)^{m+1}}-\frac{1}{\left(n+\sqrt[m]{x}\right)^{m+1}}\right)dx, m\in\mathbb{N}^*$$

Proposed by Mohammed Bouras-Morocco

U.681 Prove that:

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)=1$$

Proposed by Mohammed Bouras-Morocco

U.682 Prove that:

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}\left(\frac{2}{n+1}+\frac{3}{n+2}+\frac{4}{n+3}\right)=1$$

Proposed by Mohammed Bouras-Morocco

U.683 $x_1=1, x_2=\frac{1}{4}, x_{n+1}=\frac{(n-1)x_n}{n-x_n}, n\in\mathbb{N}, n\geq 2$

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{n^2\tan^{-1}(x_nx_{n+1})+\sin^{-1}\left(\frac{x_n}{x_{n+1}}\right)}{e^{\frac{x_n}{x_{n+1}}}}\right)$$

Proposed by Rajeev Rastogi – India

U.684 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sin^{-1}\frac{1}{\sqrt{n^2+2k-1}}+\sin\frac{k\pi}{n}\sin\frac{k\pi}{n^2}\right)$$

Proposed by Rajeev Rastogi – India

U.685 Prove without softwares:

$$\sum_{k=2}^{99}\int_0^1 \frac{k^2+k+2}{x^2-(2k+2)x+k^2+2k}dx > \frac{4949}{50}$$

Proposed by Rajeev Rastogi – India

U.686 $a_0 = 2, (4+a_n)(2-a_{n+1}) = 8, b_n = \frac{1}{a_n}, n \in \mathbb{N}, \emptyset(n)$ – Euler's totient function. Find $k \in \mathbb{R}$ such that:

$$\prod_{m=0}^{n}(1+2b_n)^{\emptyset(2^{2m+1})} = 2^k$$

Proposed by Rajeev Rastogi – India

U.687 Study the convergence:

$$\Omega = \int_0^\infty \frac{dx}{1-\cos x + x^2}$$

Proposed by Seyran Ibrahimov - Azerbaijan

U.688 $f: [0,\infty) \to [0,\infty)$ Lebesque measurable then:

$$\int_0^\infty f(x)\,dx \le \sqrt{\frac{\pi}{2}}\left(\int_0^\infty f^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^\infty f^2(x)(1+x^2)^2dx\right)^{\frac{1}{4}}$$

Proposed by Seyran Ibrahimov - Azerbaijan

U.689 Evaluate the limit:

$$\lim_{n\to\infty}\left(\frac{4}{3}\sum_{m=0}^{n-1}\frac{n^{\frac{3}{2}}(2m)!\,2^{2n-4m}(n-m-1)!^2}{(2m+1)(2n-2m)!}\right)$$

Proposed by K. Srinivasa Raghava – India

U.690 Prove the relation:

$$\mathcal{R}\left(\int_{-i}^{i} e^{i\pi n x}\, Li_2\left(e^{\frac{i\pi x}{n}}\right)dx\right) = \left(\frac{e^{n\pi}-1}{n^3} - \frac{\pi}{n^2}\right)e^{-\pi n}$$

$Li_2(x)$ is Poly-Log function and $\mathcal{R}$ is Real part.

Proposed by K. Srinivasa Raghava – India

U.691 If we consider the lines $(d): bcx = cay = abz$ $(e): cax = aby = bcz$ then prove that

(i) : $\sin\angle(d,e) = \sin\angle(e,f) = \sin\angle(f,d)$ where we denote by $\angle(d,e)$ the angle from the lines (d) and (e), and similarly;

(ii) : $\sin\angle\big((d,e),(e,f)\big) = \sin\angle\big((e,f),(f,d)\big) = \sin\angle\big((f,d),(d,e)\big)$, where we denote by

$\angle((d,e),(e,f))$ the angle from the planes (d,e) and (e,f) and similarly.

Proposed by Neculai Stanciu – Romania

U.692 Show that:

$$\int_0^1 \log\left(\frac{(\sqrt{1-x}-1)(\sqrt{1-x^2}+1)}{(\sqrt{1-x}+1)(\sqrt{1-x^2}-1)}\right)\sin^{-1}(x)^2\,dx$$
$$=\frac{\pi^3}{12}-\pi\left(6\sqrt{2}+2+\log\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)\right)-4(2C-7)$$

C is Catalan's constant

Proposed by K. Srinivasa Raghava – India

U.693 If $0<a\le b$ then:

$$12\int_a^b\int_a^b\int_a^b \frac{dxdydz}{x^4y+y^4z+z^4x}\le(b-a)^3(a^3+a^2b+ab^2+b^3)$$

Proposed by Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-AUTUMN 2023

PROBLEMS FOR JUNIORS

JP.436 In ΔABC the following relationship holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{18R}{ab+bc+ca}\cdot\sqrt[3]{w_a w_b w_c}\geq 6$$

Proposed by Alex Szoros-Romania

JP.437 If $a,b>0$ then:

$$\left(\frac{ab}{a+b}+\frac{\sqrt{ab}}{2}+\frac{a+b}{4}+\sqrt{\frac{a^2+b^2}{8}}\right)^2\geq 2ab+\sqrt{2ab(a^2+b^2)}$$

Proposed by Daniel Sitaru-Romania

JP.438 Find $\lambda>0$ so that the following relationship holds in any ΔABC:

$$\frac{3R}{r}\geq\lambda\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{9\sqrt[3]{abc}}{\lambda(a+b+c)}\geq 6$$

Proposed by Alex Szoros-Romania

JP.439 If x,y,z are natural numbers such that $2x^x+y^y=3z^z$, then find

$$\frac{2021x+2022y+2023z}{x+y+z}$$

Proposed by Neculai Stanciu-Romania

JP.440 Solve in positive integers the equation

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2$$

Proposed by Neculai Stanciu-Romania

JP.441 Prove that if $a, b, c > 0$ then holds:

$$\frac{a^2+2b^2}{b\sqrt{a^2+b^2}}+\frac{b^2+2c^2}{c\sqrt{b^2+c^2}}+\frac{c^2+2a^2}{a\sqrt{c^2+a^2}}\geq 6$$

Proposed by Florică Anastase-Romania

JP.442 For $x, y, z > 0$ prove that:

$$\sqrt{\frac{x^2z}{xy^2+yz^2}}+\sqrt{\frac{xy^2}{yz^2+x^2z}}+\sqrt{\frac{yz^2}{x^2z+xy^2}}>2$$

Proposed by Florică Anastase-Romania

JP.443 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{w_a^4}{w_b^2+w_c(w_a+w_b)}}+\sqrt[3]{\frac{w_b^4}{w_c^2+w_a(w_b+w_c)}}+\sqrt[3]{\frac{w_c^4}{w_a^2+w_b(w_c+w_a)}}\geq 3\sqrt{3r^2}$$

Proposed by Marin Chirciu-Romania

JP.444 If $a, b, c > 0$ such that $a+b+c=1$ and $\lambda \geq 1$ then:

$$\frac{a}{\sqrt{b+\lambda c}}+\frac{b}{\sqrt{c+\lambda a}}+\frac{c}{\sqrt{a+\lambda b}}\geq\sqrt{\frac{3}{\lambda+1}}$$

Proposed by Marin Chirciu-Romania

JP.445 If $a, b, c > 0$ such that $a+b+c=\lambda$ and $\lambda > 0$ then:

$$\sum_{cyc}\frac{\lambda+a}{\lambda-a}\leq 2\sum_{cyc}\frac{a}{b}$$

Proposed by Marin Chirciu-Romania

JP.446 Let $a > 0$ fixed. Solve for real numbers:

$$x\left(1+\frac{1}{\sqrt{a^2x^2-1}}\right)=\frac{35}{12a}$$

Proposed by Laura and Gheorghe Molea-Romania

JP.447 If $4R+r=1$, then prove that:

$$\sum_{cyc}\frac{\sqrt{r_ar_b}}{r_a+r_b}\leq\frac{1}{2}\left(1+\frac{R}{r}\right)$$

Proposed by Laura and Gheorghe Molea-Romania

JP.448 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^8\frac{A}{2}}{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}\ge\frac{3}{16}\left(\frac{s}{R}\right)^2$$

Proposed by Marian Ursărescu-Romania

JP.449 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cot^4\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\ge 3\sqrt{3}\left(\frac{4R}{r}-5\right)$$

Proposed by Marian Ursărescu-Romania

JP.450 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt[3]{\frac{r_a^2+r_br_c}{r_b^2+r_c^2}}\ge\frac{27r}{4R+r}$$

Proposed by Marian Ursărescu-Romania

PROBLEMS FOR SENIORS

SP.436 In ΔABC the following relationship holds:

$$\frac{1}{\left(b+\sqrt{bc}+\sqrt{ca}\right)h_a^3}+\frac{1}{\left(c+\sqrt{ca}+\sqrt{ab}\right)h_b^3}+\frac{1}{\left(a+\sqrt{ab}+\sqrt{bc}\right)h_c^3}\ge\frac{\sqrt{3}}{6F}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

SP.437 Prove that in acute ΔABC the following relationship holds:

$$\max\{h_a^2,h_b^2,h_c^2\}-\min\{h_a^2,h_b^2,h_c^2\}\ge\frac{1}{2}(s^2-2R^2-8Rr-3r^2)$$

Proposed by Cristian Miu-Romania

SP.438 In ΔABC the following relationship holds:

$$\frac{a^3}{b+\sqrt{bc}+\sqrt{ca}}+\frac{b^3}{c+\sqrt{ca}+\sqrt{ab}}+\frac{c^3}{a+\sqrt{ab}+\sqrt{bc}}\ge\frac{4\sqrt{3}}{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

SP.439 If $x, y, z > 0$, then:

$$\frac{1+x+x^2}{1+y+z+z^2}+\frac{1+y+y^2}{1+z+x+x^2}+\frac{1+z+z^2}{1+x+y+y^2}\geq\frac{9}{4}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

SP.440 If $a, b, c > 0$, then prove that:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\frac{\sqrt{3(a^2+b^2+c^2)}}{a+b+c}$$

Proposed by Neculai Stanciu-Romania

SP.441 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_a}{r_a+2r_b}\leq\frac{3(4R-5r)}{4R+r}$$

Proposed by Marian Ursărescu-Romania

SP.442 In ΔABC the following relationship holds:

$$\sqrt{\frac{a+c-b}{a}}+\sqrt{\frac{b+a-c}{b}}+\sqrt{\frac{c+b-a}{c}}\leq\sqrt{8+\frac{2r}{R}}$$

Proposed by Marian Ursărescu-Romania

SP.443 Determine all functions $f: \mathbb{R} \to (0, +\infty)$ continuous such that

$$f(4x)\cdot f(3x)=2^x;\ \forall x\in\mathbb{R}$$

Proposed by Marian Ursărescu-Romania

SP.444 If $a, b, c.\ 0$ and $abc = 1$ then:

$$\sum_{cyc}\frac{a^8+28a^6+70a^4+28a^2+1}{b^6+7b^4+7b^2+1}\geq 24$$

Proposed by Marin Chirciu-Romania

SP.445 If $x, y, z > 0$ and $\lambda \geq 3$ then:

$$\sum_{cyc}\frac{x}{\sqrt{4y^2+(\lambda^2-8)yz+4z^2}}\geq\frac{3}{\lambda}$$

Proposed by Marin Chirciu-Romania

SP.446. If $x,y\geq 0, x+y>0$ then in any convex quadrilateral with sides a,b,c,d and r −inradii, holds:

$$(a^2x+b^2y)^6+(b^2x+c^2y)^6+(c^2x+d^2y)^6+(d^2x+a^2y)^6\geq 4^7(x+y)^6r^{12}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

SP.447 If $x\geq 0$ then:

$$e^x\cdot 2^{e^{-x}-1}+e^{-x}\cdot 2^{e^x-1}\geq \cosh x\cdot 2^{\operatorname{sech} x}$$

Proposed by Daniel Sitaru-Romania

SP.448 If $x,y,z>0$ and $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=\frac{3}{2}$ then:

$$\frac{\sqrt{x^2+1}}{x^2-x+1}+\frac{\sqrt{y^2+1}}{y^2-y+1}+\frac{\sqrt{z^2+1}}{z^2-z+1}\leq 3\sqrt{2}$$

Proposed by Daniel Sitaru-Romania

SP.449 If $a,b\geq 0$ then:

$$\sqrt{2^{a+b}}-2^{\sqrt{ab}}+3^{\sqrt{a+b}}-3^{\sqrt{ab}}+\frac{1}{\sqrt{6^{a+b}}}-\frac{1}{6^{\sqrt{ab}}}\geq 0$$

Proposed by Daniel Sitaru-Romania

SP.450 In ΔABC the following relationship holds:

$$4s^2+2\left(s-\frac{3c}{2}\right)^2+\frac{3}{2}(a-b)^2\geq 108r^2$$

Proposed by Daniel Sitaru-Romania

UNDERGRADUATE PROBLEMS

UP.436 Let $f,g:(0,\infty)\to(0,\infty)$ be continuous functions such that

$$g(x)\int_0^x f(t)\,dt=f(x)\int_0^x g(t)\,dt=1;\forall x\in(0,\infty)$$

Prove that exists $a,b\in(0,\infty)$ such that

$$\frac{1}{3}\cdot\sum_{cyc}\big(f(\alpha)+g(\alpha)\big)\ge\frac{a+1}{\sqrt{2a+b}},\text{where } \alpha+\beta+\gamma=3$$

Proposed by Florică Anastase-Romania

UP.437 Let $ABCD$ be a bicentric quadrilateral , R and r its exradius and inradius , a , b , c , and d its sides (in this order) and e , f its diagonals.

Prove that :

$$\frac{R^2-Rr+r^2\sqrt{2}}{r^2}\ge\frac{e^2}{f^2}+\frac{f^2}{e^2}$$

Proposed by Vasile Jiglău-Romania

UP.438 If $t\in\mathbb{R}_+^*$ and $(a_n)_{n\ge1},(b_n)_{n\ge1}$ be positive real sequences such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in\mathbb{R}_+^*,\lim_{n\to\infty}\frac{b_{n+1}}{n^tb_n}=b\in\mathbb{R}_+^*.\text{ Find}:$$

$$\Omega=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^t}-\frac{\sqrt[n]{a_nb_n}}{n^t}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP.439 If $f(x)=\frac{1-\sqrt{1-2x}}{2}$, $f_n^{-1}=\underbrace{f^{-1}\circ f^{-1}\circ\ldots\circ f^{-1}}_{n-times}$ then find:

$$\Omega=\lim_{n\to\infty}\int_0^{\frac{1}{2}}f_n^{-1}(x)\,dx$$

Proposed by Neculai Stanciu-Romania

UP.440 Find a closed form:

$$\Omega=\left(\int_0^1\frac{x^{29}-x^9}{x^{40}+1}dx\right)\left(\int_0^1\frac{x^{29}-2x^9}{x^{40}+4}dx\right)$$

Proposed by Daniel Sitaru-Romania

UP.441 If $f:\mathbb{R}\to\left[-\frac{5}{2},\frac{5}{2}\right]$, f –continuous, then:

$$\int_{-\frac{5}{2}}^{\frac{5}{2}}\sqrt{50-8f^2(x)}\,dx+\int_{-\frac{5}{2}}^{\frac{5}{2}}f(x)\,dx\le\frac{75}{2}$$

Proposed by Daniel Sitaru-Romania

UP.442 If $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n} \frac{1}{k}$, then compute

$$\Omega = \lim_{n\to\infty} e^{-2H_n} \sum_{k=2}^{n} \sqrt[k]{k!}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP.443 Find all derivable functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\frac{f(x^3) - f(0)}{f(x) - f(0)} = x^2, \forall x \neq 0$$

Proposed by Neculai Stanciu-Romania

UP.444 Find:

$$\Omega = \lim_{n\to\infty} \left(e^{2H_{n+1}} - e^{2H_n}\right) \cdot \frac{1}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.445 Solve for real numbers: $e^2\Omega^2(a) - 6e\Omega(a) + 8 = 0$, where

$$\Omega(a) = \lim_{n\to\infty} \left(\sqrt[n]{a} - 1\right) \cdot \sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.446 Find:

$$\Omega = \lim_{n\to\infty} n^3 \cdot \sqrt[n]{n!} \cdot \sqrt[n]{(2n-1)!!} \cdot \sin\frac{1}{n^2} \cdot \sin\frac{1}{n^3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.447 Find:

$$\Omega(a) = \lim_{n\to\infty} e^{5H_n} \cdot \left(\sqrt[n^2]{a} - 1\right) \cdot \sin^3\frac{1}{n}; a > 0$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.448 If $m \in [1,\infty), a_k \in \mathbb{R}_+^*, k = \overline{1,n}, n \in \mathbb{N}^* - \{1,2\}$ and

$\sum_{k=1}^{n} a_k = s \in \mathbb{R}_+^*$, then:

$$\sum_{k=1}^{n} a_k^m \geq s\left(\prod_{k=1}^{n} a_k\right)^{\frac{m-1}{n}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

UP.449 Prove that:

$$\sum_{k=1}^{n+1}\frac{\sqrt[k]{n+1}+\sqrt[k]{n-1}}{2\sqrt[k]{n}}\le\sum_{k=1}^{n}\sqrt[k]{k+1}\text{, where } n,k\in\mathbb{N}^*$$

Proposed by Florică Anastase-Romania

UP.450 If $A_1A_2\dots A_n(n\ge 3)$ is a convex polygon with the inradius r and the lengths sides $A_kA_{k+1}=a_k, k=\overline{1,n}, A_{n+1}=A_1$. If $h_k\in\mathbb{R}_+^*$ such that $a_kh_k=(n-1)F, k=\overline{1,n}$, then:

$$\sum_{k=1}^{n}\frac{h_k-(n-1)r}{h_k+(n-1)r}\ge\frac{n(n-2)}{n+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com