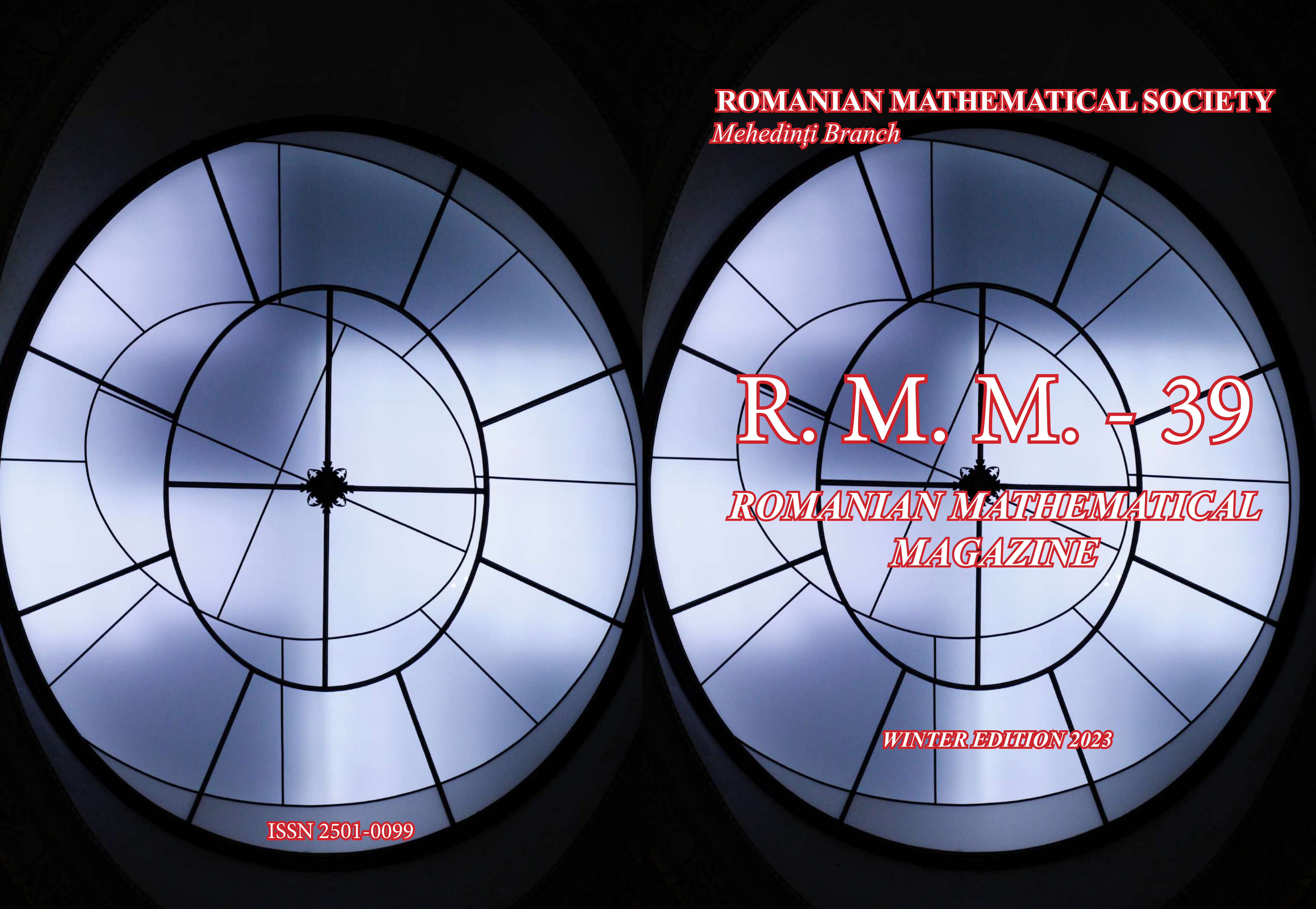

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CONTENT

CRUX MATHEMATICORUM CHALLENGES-(II)

By Daniel Sitaru

4165. Prove that for all real numbers x_1, x_2, x_3 and x_4, we have,

$$|x_1+x_2+x_3+x_4|+2(|x_1|+|x_2|+|x_3|+|x_4|)\geq 6\sqrt[6]{\prod_{1\leq i<j\leq 4}|x_i+x_j|}$$

Proposed by Daniel Sitaru – Romania

Solution with generalization by Michel Bataille-France

We prove the stronger result that for any complex numbers x_1, x_2, x_3 and x_4, we have

$$|x_1+x_2+x_3+x_4|+2(|x_1|+|x_2|+|x_3|+|x_4|)\geq \sum_{1\leq i<j\leq 4}|x_i+x_j| \quad (1)$$

The proposed inequality then follows from (1) by the AM-GM Inequality. To prove (1), we will make use of Hlawka's inequality which states that

$$|a+b+c|+|a|+|b|+|c|\geq |a+b|+|b+c|+|c+a| \quad (2)$$

for all complex numbers a, b, c.

Setting $a=x_1, b=x_2$ and $c=x_3+x_4$, then from (2) we have

$$|x_1+x_2+x_3+x_4|+|x_1|+|x_2|+|x_3+x_4|\geq |x_1+x_2|+|x_2+x_3+x_4|+|x_1+x_3+x_4| \quad (3)$$

Applying (2) again, we obtain:

$$|x_2+x_3+x_4|\geq |x_2+x_3|+|x_3+x_4|+|x_2+x_4|-|x_2|-|x_3|-|x_4| \quad (4)$$

and

$$|x_1+x_3+x_4|\geq |x_1+x_3|+|x_3+x_4|+|x_1+x_4|-|x_1|-|x_3|-|x_4| \quad (5)$$

Adding (4) and (5) and denoting the right side of (3) by R, then we have:

$$R\geq |x_3+x_4|-|x_1|-|x_2|-2|x_3|-2|x_4|+\sum_{1\leq i<j\leq 4}|x_i+x_j| \quad (6)$$

From (3) and (6), we deduce that

$$|x_1+x_2+x_3+x_4|+|x_1|+|x_2|+|x_3+x_4|\geq$$
$$\geq |x_3+x_4|-|x_1|-|x_2|-2|x_3|-2|x_4|+\sum_{1\leq i<j\leq 4}|x_i+x_j|$$

from which (1) follows immediately.

4205. Prove that for $0<a<c<b, a,b,c\in\mathbb{R}$, we have:

$$\frac{1}{c\sqrt{ab}}\int_a^b x\arctan x\,dx>\frac{(c-a)\arctan\sqrt{ac}}{\sqrt{bc}}+\frac{(b-c)\arctan\sqrt{bc}}{\sqrt{ac}}$$

Proposed by Daniel Sitaru – Romania

Solution by Paul Bracken-USA

Let $f(x)=x\arctan x$ for $x>0$. Since $f(0)=f'(0)=0, f'(x)=\arctan x+x(1+x^2)^{-1}$ and $f''(x)=2(1+x^2)^{-2}$, then f is positive, strictly increasing and strictly convex. By the Mean Value Theorem, we have that:

$$f(p)+f'(p)(x-p)<f(x)$$

for distinct positive x and p. Hence

$$(c-a)f(\sqrt{ac})<(c-a)f(\sqrt{ac})+\frac{1}{2}f'(\sqrt{ac})(c-a)(\sqrt{c}-\sqrt{a})^2$$

$$= (c-a)f(\sqrt{ac}) + f'(\sqrt{ac})\int_a^c (x-\sqrt{ac})\,dx < \int_a^c f(x)\,dx,$$

and

$$(b-c)f(\sqrt{bc}) < \int_c^b f(x)\,dx$$

Therefore $(c-a)\sqrt{ac}\arctan\sqrt{ac} + (b-c)\sqrt{bc}\arctan\sqrt{bc} < \int_a^b x\arctan x\,dx$

Dividing by $(\sqrt{ac})(\sqrt{bc})$ yields the desired inequality.

4226. Prove that if $0 < a < b$ then:

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 > (b-a)^2 + \ln^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solutions by – a composite of virtually the same solutions by Arkady Alt; Michel Bataille; M. Bello, M. Benito, O. Ciaurri, E. Fernandez, and L. Roncal (jointly); and Digby Smith.

Note first that

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 > (b-a)^2 + \ln^2\frac{b}{a}$$

$$\Leftrightarrow \left(\int_a^b \frac{\sqrt{1+x^2}}{x}dx\right)^2 - \left(\int_a^b \frac{1}{x}dx\right)^2 > (b-a)^2$$

$$\Leftrightarrow \int_a^b \frac{\sqrt{1+x^2}+1}{x}dx \cdot \int_a^b \frac{\sqrt{1+x^2}-1}{x}dx > (b-a)^2 \quad (1)$$

Let $f(x) = \frac{\sqrt{1+x^2}+1}{x}, x \in [a,b]$. Then $f(x) > 0$ and $\frac{1}{f(x)} = \frac{\sqrt{1+x^2}-1}{x}$. By the integral form of the Cauchy – Schwarz Inequality, we have:

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{1}{f(x)}dx\right) = \left(\int_a^b \left(\sqrt{f(x)}\right)^2 dx\right)\left(\int_a^b \left(\sqrt{\frac{1}{f(x)}}\right)^2 dx\right)$$

$$\geq \left(\int_a^b 1\,dx\right)^2 = (b-a)^2 \quad (2)$$

But equality cannot hold in (2) as f is not a constant on $[a,b]$. Hence, from (1) and (2) the result follows.

4256. Let $a,b,c \in \mathbb{R}$ such that $a+b+c=1$. Prove that:

$$\frac{e^b-e^a}{b-a} + \frac{e^c-e^b}{c-b} + \frac{e^a-e^c}{a-c} > 4$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie-USA

We will prove the slight improvement that

$$\frac{e^b-e^a}{b-a} + \frac{e^c-e^b}{c-b} + \frac{e^a-e^c}{a-c} > 3e^{\frac{1}{3}} > 4$$

for distinct $a, b, c \in \mathbb{R}$, which satisfy the condition $a + b + c = 1$.
Note first that the last inequality follows from the fact that

$$\left(\frac{4}{3}\right)^3 = \frac{64}{27} = 2.\overline{370} < e \quad (1)$$

For the remainder of our solution, we will utilize Hadamard's Inequality which states that if $f(x)$ is continuous and convex on $[p, q]$, then

$$\frac{1}{q-p}\int_p^q f(x)\,dx \geq f\left(\frac{p+q}{2}\right) \quad (2)$$

A proof of this result can be found in R. P. Boas, Jr., *A Primer of Real Functions (3rd ed.)*, Carus Mathematical Monograph No. 13, The Mathematical Association of America, 1981, pg. 174.
Since a and b must be distinct and

$$\frac{e^b - e^a}{b - a} = \frac{e^a - e^b}{a - b},$$

we may assume without loss of generality that $a < b$. Then, since $f(x) = e^x$ is continuous and convex on $\mathbb{R}$, (2) implies that

$$\frac{e^b - e^a}{b-a} = \frac{1}{b-a}\int_a^b e^x\,dx \geq e^{\frac{a+b}{2}} \quad (3)$$

Similar arguments show that

$$\frac{e^c - e^b}{c-b} \geq e^{\frac{b+c}{2}} \quad \text{and} \quad \frac{e^a - e^c}{a-c} \geq e^{\frac{a+c}{2}} \quad (4)$$

Further, because $f(x) = e^x$ is strictly convex on $\mathbb{R}$, Jensen's Theorem and the distinct values of a, b and c imply that

$$e^{\frac{a+b}{2}} + e^{\frac{b+c}{2}} + e^{\frac{a+c}{2}} > 3e^{\frac{1}{3}\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{a+c}{2}\right)} = 3e^{\frac{a+b+c}{3}} = 3e^{\frac{1}{3}} \quad (5)$$

Finally, it follows from (1), (3), (4), and (5) that

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} \geq e^{\frac{a+b}{2}} + e^{\frac{b+c}{2}} + e^{\frac{a+c}{2}} > 3e^{\frac{1}{3}} > 3\left(\frac{4}{3}\right) = 4.$$

Solution 2 by M. Bello, M. Benito, O. Ciaurri, E. Fernandez, and L. Roncal.-Spain

We prove a more general result. Let $a_1, a_2, \dots, a_n \in \mathbb{R}$ such that $a_1 + a_2 + \cdots + a_n = 1$, then

$$\sum_{k=1}^n \frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq ne^{\frac{1}{n}} \quad (1)$$

with $a_{n+1} = a_1$. Moreover, the equality holds if and only if $a_i = \frac{1}{n}$, for $i = 1, \dots, n$ (in this case the left hand side has to be understood as a limit).
The proposed inequality follows taking $n = 3, a_1 = a, a_2 = b$, and $a_3 = c$ and using that $3e^{\frac{1}{3}} = 4.186837 > 4$.
Let us prove (1). From the inequality $\frac{\sinh x}{x} \geq 1$, for $x \in \mathbb{R}$, with equality for $x = 0$ only, taking $x = \frac{a_{k+1} - a_k}{2}$, we deduce that

$$\frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq e^{\frac{a_{k+1} + a_k}{2}},$$

with equality when $a_{k+1} = a_k$. In this way, applying the AM-GM inequality, we have

$$\sum_{k=1}^n \frac{e^{a_{k+1}} - e^{a_k}}{a_{k+1} - a_k} \geq \sum_{k=1}^n e^{\frac{a_{k+1} + a_k}{2}} \geq ne^{\frac{a_1 + \cdots + a_n}{n}} = ne^{\frac{1}{n}}$$

and the equality holds when $a_i = \frac{1}{n}$, for $i = 1, \dots, n$, only.

Solution 3 by Paul Bracken-USA

By Taylor's theorem, we have the expansion with remainder

$$e^b = e^a + e^a(b-a) + \frac{1}{2}e^a(b-a)^2 + \frac{e^{\tau_1}}{6}(b-a)^3,$$

where τ_1 in the remainder is between a and b. This implies that

$$\frac{e^b - e^a}{b-a} = e^a + \frac{1}{2}e^a(b-a) + \frac{e^{\tau_1}}{6}(b-a)^2 \geq e^a + \frac{1}{2}e^a(b-a),$$

since $e^{\tau_1} > 0$ and $(b-a)^2 \geq 0$ always holds. In exactly the same way, we obtain the inequalities

$$\frac{e^c - e^b}{c-b} = e^b + \frac{1}{2}e^b(c-b) + \frac{e^{\tau_2}}{6}(c-b)^2 \geq e^b + \frac{1}{2}e^b(c-b),$$
$$\frac{e^a - e^c}{a-c} = e^c + \frac{1}{2}e^c(a-c) + \frac{e^{\tau_3}}{6}(a-c)^2 \geq e^c + \frac{1}{2}e^c(a-c).$$

Adding these three results, the following lower bound for the function in (1) is obtained,

$$h(a,b,c) = \frac{e^b-e^a}{b-a} + \frac{e^c-e^b}{c-b} + \frac{e^a-e^c}{a-c} \geq e^a + e^b + e^c + \frac{1}{2}\Big(e^a(b-a) + e^b(c-b) + e^c(a-c)\Big) \quad (2)$$

This result holds for all a, b, c and is independent of the constraint which has not been used. Let us minimize the function on the right of (2),

$$f(a,b,c) = e^a + e^b + e^c + \frac{1}{2}\Big(e^b(c-b) + e^b(c-b) + e^c(a-c)\Big),$$

by introducing a Lagrange multiplier λ

$$\mathcal{L} = f(a,b,c) - \lambda(a+b+c-1).$$

Differentiating $\mathcal{L}$ with respect to a, b, c and λ, the following nonlinear system results,

$$e^a + e^c + e^a(b-a) - 2\lambda = 0,$$
$$e^b + e^a + e^b(c-b) - 2\lambda = 0,$$
$$e^c + e^b + e^c(a-c) - 2\lambda = 0,$$
$$a+b+c-1 = 0.$$

This set of equations maps into itself under a cyclic permutation of the variables.
The first three equations of (3) can be put in the form,

$$1 + b - a + e^{c-a} = 2\lambda e^{-a}, 1 + c - b + e^{a-b} = 2\lambda e^{-b}, 1 + a - c + e^{c-b} = 2\lambda e^{-c}$$

For example, adding these three equations, and expression for λ results,

$$\lambda = \frac{e^{a-b} + e^{c-a} + e^{c-b} + 3}{2(e^{-a} + e^{-b} + e^{-c})}.$$

In fact, the solution to the system (3) is given by

$$a = b = c = \frac{1}{3}, \quad \lambda = e^{\frac{1}{3}}.$$

The minimum value of f is found to be

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 3e^{\frac{1}{3}} > 4 \qquad (4)$$

This will correspond to a minimum since a maximum is not expected. Take for example $a = N, b = -N+1$ and $c = 0$, then $e^N \to \infty$ as $N \to \infty$, so h can be made as large as we please. Combining (2) and (4), these imply (1).
Letting $c \to b$ and then $b \to a$ in h and the constraint, or using Taylor's formula, it can be seen that h reduces to $3e^{\frac{1}{3}}$ which matches the minimum (4). Thus the absolute minimum of h under the constraint is $3e^{\frac{1}{3}}$.

4265. Consider real numbers $a, b, c \in (0, 1)$ such that $a + b + c = 1$. Show that:

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2 - (ab + bc + ca)}$$

Proposed by Daniel Sitaru – Romania

Solution by the team D. Bailey, E. Campbell, and C. Diminnie-USA

Since $\frac{4}{\pi}\arctan x$ is concave for $x \geq 0$ and is equal to x for $x = 0$ and $x = 1$,

$$\frac{4}{\pi}\arctan x \geq x$$

for $0 \leq x \leq 1$. Therefore the left side of the inequality is not less than $a + b + c = 1$

Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2)$$
$$= 1 - (a^2 + b^2 + c^2) \leq 1 - (ab + bc + ca),$$

then $ab + bc + ca \leq \frac{1}{3}$ and $\frac{1}{2-(ab+bc+ca)} \leq \frac{3}{5} < 1$. The result follows.

4276. Let P be a point on the interior of a triangle ABC and let $PA = x, PB = y$ and $PC = z$. Prove that:

$$27(ax + by - cz)(by + cz - ax)(cz + ax - by) \leq (ax + by + cz)^3.$$

Proposed by Daniel Sitaru – Romania

Solution by Digby Smith-USA

Let $p = ax, q = by$, and $r = cz$. Substituting, expanding, then applying Schur's inequality before applying the AM-GM inequality gives

$$(ax + by - cz)(by + cz - ax)(cz + ax - by)$$
$$= (p + q - r)(q + r - p)(r + p - q)$$
$$= pq(p + q) + qr(q + r) + rp(r + p) - p^3 - q^3 - r^3 - 2pqr$$
$$\leq pqr \leq \left(\frac{p + q + r}{3}\right)^3$$

making $27(ax + by - cz)(by + cz - ax)(cz + ax - by) \leq (ax + by + cz)^3$ with equality if and only if $ax = by = cz$.

4298. Compute:

$$L = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n + k}{2 + \sin(n + k) + (n + k)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Missouri State University Problem Solving Group.

Define

$$f(n,k)=\frac{n+k}{2+\sin(n+k)+(n+k)^2} \text{ and } g(n,k)=\frac{1}{n+k}.$$

Since $1\le 2+\sin(n+k)\le 3$, then for $1\le k\le n$, we have

$$|g(n,k)-f(n,k)|=\frac{2+\sin(n+k)}{(n+k)(2+\sin(n+k)+(n+k)^2)}$$
$$\le\frac{3}{(n+k)(1+(n+k)^2)}\le\frac{3}{n^3}.$$

Therefore

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n}g(n,k)-f(n,k)\right|\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{n^3}=\lim_{n\to\infty}\frac{3}{n^2}=0$$

In particular, we now have

$$L=\lim_{n\to\infty}\sum_{k=1}^{n}f(n,k)=\lim_{n\to\infty}\sum_{k=1}^{n}g(n,k)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}.$$

Let $h(x)=\frac{1}{x}$. Since h is continuous on $[1,2]$, it is integrable on $[1,2]$. Therefore

$$L=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n\left(1+\frac{k}{n}\right)}=\lim_{n\to\infty}\sum_{k=1}^{n}h\left(1+\frac{k}{n}\right)\left(\frac{1}{n}\right)=\int_1^2 h(x)\,dx=\ln 2.$$

4309. Let a, b and c be real numbers such that $a+b+c=3$. Prove that:

$$\mathbf{2(a^4+b^4+c^4)\ge ab(ab+1)+bc(bc+1)+ca(ca+1)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sefket Arslanagic-Serbie

Using the inequality

$$3(x^2+y^2+z^2)\ge(x+y+z)^2,$$

we have that

$$a^4+b^4+c^4\ge\frac{1}{3}(a^2+b^2+c^2)(a^2+b^2+c^2)$$
$$\ge\frac{1}{9}(a+b+c)^2(a^2+b^2+c^2)=a^2+b^2+c^2\ge ab+bc+ca$$

Also $a^4+b^4+c^4\ge a^2b^2+b^2c^2+c^2a^2$, so that the desired inequality holds.

Solution 2 by AN-anduud Problem Solving Group.

Using the inequality : $x^2+y^2+z^2\ge xy+yz+zx$,
we have that: $a^4+b^4+c^4\ge(ab)^2+(bc)^2+(ca)^2$.
Since : $x^4-4x+3=(x-1)^2[(x+1)^2+2]\ge 0$, we find that

$$a^4+b^4+c^4\ge 4(a+b+c)-9=3=\frac{1}{3}(a+b+c)^2$$
$$=\frac{1}{6}[(a-b)^2+(b-c)^2+(c-a)^2+6(ab+bc+ca)]\ge ab+bc+ca$$

Adding these two inequalities for $a^4+b^4+c^4$ yields the desired result. Equality holds iff $a=b=c=1$.

Solution 3 by Paolo Perfetti-Italy and Angel Plaza-Spain, independently.

Recall the Muirhead Inequalities for three variables. For $a,b,c>0$ and $p\ge q\ge r$, let

$$[p,q,r]=\frac{1}{6}(a^pb^qc^r+a^pb^rc^q+a^qb^pc^r+a^qb^rc^p+a^rb^pc^q+a^rb^qc^p)$$

Then: $p\ge u, p+q\ge u+v$ and $p+q+r=u+v+w$

together imply that $[p,q,r]\ge[u,v,w]$.

Make the given inequality homogeneous by replacing each 1 by $\frac{1}{9}(a+b+c)^2$. Thus we have to prove that

$$18(a^4+b^4+c^4)\ge 11(a^2b^2+b^2c^2+a^2c^2)+(a^3b+ab^3+b^3c+bc^3+a^3c+ac^3)+$$
$$+5(a^2bc+ab^2c+abc^2)$$

or, equivalently,

$$9[4,0,0]\ge\frac{11}{2}[2,2,0]+[3,1,1]+\frac{5}{2}[2,1,1].$$

This is true since

$$[4,0,0]\ge[2,2,0],\qquad [4,0,0]\ge[3,1,1],\qquad [4,0,0]\ge[2,1,1].$$

B.72 Prove that in triangle ABC, the following relationship holds:

$$\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}}\ge\frac{2s}{r}$$

Proposed by Daniel Sitaru-Romania

Solution by Nguyen Van Canh-Ben Tre-Vietnam

$$\prod\sin\frac{A}{2}=\frac{r}{4R},\prod\cos\frac{A}{2}=\frac{s}{4R}$$

$$\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}}=\sum\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}=$$

$$\frac{2}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\cdot\sum\frac{\left(\sin\frac{A}{2}\right)^2}{\frac{1}{\cos\frac{A}{2}}}\overset{\text{Cauchy-Schwartz}}{\geq}=\frac{2s}{r}\cdot\frac{\left(\sum\sin\frac{A}{2}\right)^2}{\sum\cos\frac{B}{2}\cos\frac{C}{2}}$$

We need to prove that:

$$\frac{\left(\sum\sin\frac{A}{2}\right)^2}{\sum\cos\frac{B}{2}\cos\frac{C}{2}}\ge 1\leftrightarrow\left(\sum\sin\frac{A}{2}\right)^2\ge\sum\cos\frac{B}{2}\cos\frac{C}{2};(1)$$

$$(\exists\Delta A'B'C'\text{ such that: }A=\pi-2A',B=\pi-2B',C=\pi-2C')$$

Now,

$$(1)\leftrightarrow\left(\sum\sin\frac{\pi-2A'}{2}\right)^2\ge\sum\cos\frac{\pi-2B'}{2}\cos\frac{\pi-2C'}{2};$$

$$\leftrightarrow \left(\sum \cos A'\right)^2 \geq \sum \sin B' \sin C' \leftrightarrow \left(1+\frac{r'}{R'}\right)^2 \geq \frac{p'^2+4R'r'+r'^2}{4R'^2};$$

$$\leftrightarrow 4(R'+r')^2 \geq p'^2+4R'r'+r'^2 \leftrightarrow p'^2 \leq 4R'^2+4R'r'+3r'^2$$

(Which is clearly true by Gerretsen's Inequality). So, (1) is true. Proved.

SPECIAL TECHNIQUES FOR DEFINITE INTEGRALS

By Florică Anastase-Romania

Abstract: *In this paper are presented a few general techniques to find definite integrals.*

1. Introduction.

Theorem 1.

If $\boldsymbol{f:[a,b]\to R, g:[a,b]\to R^*, u:[a,b]\to R}$ ***continuous functions and***

$$\boldsymbol{f(x)+f(s-x)=u(x), g(x)=g(s-x), \forall x\in[a,b], s=a+b, then:}$$

$$\boldsymbol{\int_a^b \frac{f(x)}{g(x)}dx=\frac{1}{2}\int_a^b \frac{u(x)}{g(x)}dx}$$

Proof. Using substitution $x=s-t$, we have:

$$\int_a^b \frac{f(x)}{g(x)}dx=\int_a^b \frac{f(s-t)}{g(s-t)}(-dt)=\int_a^b \frac{u(t)-f(t)}{g(t)}dt=\int_a^b \frac{u(t)}{g(t)}dt-\int_a^b \frac{f(t)}{g(t)}dt$$

$$\int_a^b \frac{f(x)}{g(x)}dx=\frac{1}{2}\int_a^b \frac{u(x)}{g(x)}dx$$

Application 1. ***Find:***

$$\boldsymbol{\Omega=\int_0^\pi \frac{(x+1)\sin x}{3+\cos^2 x}dx}$$

Solution.

$$\Omega=\int_0^\pi \frac{(x+1)\sin x}{3+\cos^2 x}dx=\int_0^\pi \frac{x\sin x}{3+\cos^2 x}dx+\int_0^\pi \frac{\sin x}{3+\cos^2 x}dx=I_1+I_2$$

$$I_1=\int_0^{\pi}\frac{x\sin x}{3+\cos^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{3+\cos^2 x}dx=$$

$$=\pi\int_0^{\pi}\frac{\sin x}{3+\cos^2 x}dx-\int_0^{\pi}\frac{x\sin x}{3+\cos^2 x}dx\Rightarrow I_1=\pi I_2-I_1\Rightarrow I_1=\frac{\pi}{2}I_2$$

$$I_2=\int_0^{\pi}\frac{\sin x}{3+\cos^2 x}dx=-\int_0^{\pi}\frac{(\cos x)'}{3+\cos^2 x}dx=-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cos x}{\sqrt{3}}\right)\Bigg|_0^{\pi}=\frac{2\pi}{3\sqrt{3}}$$

Application 2. ***Find:***

$$\boldsymbol{\Omega=\int_{-\sqrt{3}}^{\sqrt{3}}\frac{x\tan^{-1}x}{1+e^{\tan x}}dx}$$

Solution.

$$\Omega=\int_{-\sqrt{3}}^{\sqrt{3}}\frac{x\tan^{-1}x}{1+e^{\tan x}}dx=\int_{-\sqrt{3}}^{\sqrt{3}}\frac{(-x)\tan^{-1}(-x)}{1+e^{\tan(-x)}}dx=\int_{-\sqrt{3}}^{\sqrt{3}}\frac{xe^{\tan x}\cdot\tan^{-1}x}{1+e^{\tan x}}dx$$

$$=\int_{-\sqrt{3}}^{\sqrt{3}}\frac{(1+e^{\tan x}-1)x\cdot\tan^{-1}x}{1+e^{\tan x}}dx=\int_{-\sqrt{3}}^{\sqrt{3}}x\cdot\tan^{-1}x\,dx-I$$

$$2I=\int_{-\sqrt{3}}^{\sqrt{3}}x\cdot\tan^{-1}x\,dx=2\int_0^{\sqrt{3}}x\cdot\tan^{-1}x\,dx$$

$$\Omega=\int_0^{\sqrt{3}}x\cdot\tan^{-1}x\,dx=\left(\frac{x^2}{2}-\tan^{-1}x\right)\Bigg|_0^{\sqrt{3}}-\frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}dx=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$$

Application 3. ***Find:***

$$\boldsymbol{\Omega=\int_{-\sqrt{2}/2}^{\sqrt{2}/2}x\cdot\log\left(1+e^{x\sqrt{1-x^2}}\right)dx}$$

Solution.

$$\Omega=\int_{-\sqrt{2}/2}^{\sqrt{2}/2}x\cdot\log\left(1+e^{x\sqrt{1-x^2}}\right)dx\overset{x=-y}{=}-\int_{-\sqrt{2}/2}^{\sqrt{2}/2}y\cdot\log\left(1+e^{-y\sqrt{1-y^2}}\right)dy=$$

$$= -\int_{-\sqrt{2}/2}^{\sqrt{2}/2} y\cdot\log\left(\frac{1+e^{y\sqrt{1-y^2}}}{e^{y\sqrt{1-y^2}}}\right)dy = -\Omega + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y\cdot\log\left(e^{y\sqrt{1-y^2}}\right)dy =$$

$$= -\Omega + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y^2\sqrt{1-y^2}dy$$

$$2\Omega = 2\int_0^{\sqrt{2}/2} y^2\sqrt{1-y^2}dy \overset{y=\sin t}{=} 2\int_0^{\pi/4} \sin^2 t\sqrt{1-\sin^2 t}\cdot\cos t\,dt$$

$$I = \int_0^{\pi/4} \sin^2 t\cos^2 t\,dt = \frac{1}{4}\int_0^{\pi/4} \sin^2 2t\,dt = \frac{1}{4}\int_0^{\pi/4} \frac{1-\cos 4t}{2}dt = \frac{1}{8}\left(x-\frac{1}{4}\sin 4t\right)_0^{\frac{\pi}{4}} = \frac{\pi}{32}$$

Application 4. ***Find:***

$$\boldsymbol{\Omega = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan x)}{2+\sin 2x+\cos 2x}dx}$$

Solution.

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{\log(1+\mathrm{tg}x)}{2+\sin 2x+\cos 2x}\mathrm{dx} = \int_0^{\frac{\pi}{4}} \frac{\log\left(1+\mathrm{tg}\left(\frac{\pi}{4}-\mathrm{y}\right)\right)}{2+\sin\left(\frac{\pi}{2}-2\mathrm{y}\right)+\cos\left(\frac{\pi}{2}-2\mathrm{y}\right)}\mathrm{dy} =$$

$$= \int_0^{\frac{\pi}{4}} \frac{\log 2-\log(1+tgy)}{2+sin2y+cos2y}dy$$

$$2I = \log 2\int_0^{\frac{\pi}{4}} \frac{1}{2+\frac{2\tan x}{1+\tan^2 x}+\frac{1-\tan^2 x}{1+\tan^2 x}}dx = \log 2\int_0^1 \frac{1}{t^2+2t+3}dt \Rightarrow$$

$$I = \frac{\log 2}{2\sqrt{2}}\left(\tan^{-1}\sqrt{2}-\frac{\pi}{4}\right)$$

Application 5. ***Find:***

$$\boldsymbol{\Omega = \int_0^{2\pi} \frac{x+\tan(\sin x)}{2+\cos x}dx}$$

Solution 1.

$$I=\int_0^{2\pi}\frac{x+\tan(\sin x)}{2+\cos x}dx=\int_0^{2\pi}\frac{2\pi-y-\tan(\sin x)}{2+\cos y}dy=2\pi\int_0^{2\pi}\frac{1}{2+\cos y}dy-\Omega$$

$$\frac{1}{\pi}I=\int_0^{2\pi}\frac{1}{2+\cos x}dx=\int_0^{\pi}\frac{1}{2+\cos x}dx+\int_{\pi}^{2\pi}\frac{1}{2+\cos x}dx=$$

$$\int_0^{\pi}\frac{1}{2+\cos x}dx+\int_0^{\pi}\frac{1}{2-\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{2+\cos x}dx+\int_0^{\frac{\pi}{2}}\frac{1}{2-\cos x}dx+\int_0^{\frac{\pi}{2}}\frac{1}{2+\sin x}dx+$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{2-\sin x}dx=2\int_0^1\frac{1}{t^2+3}dt+\frac{2}{3}\int_0^1\frac{1}{t^2+\frac{1}{3}}dt+\int_0^1\frac{1}{(t+1)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt+$$

$$\int_0^1\frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt=\frac{2\pi\sqrt{3}}{3}\Rightarrow I=\frac{2\pi^2\sqrt{3}}{3}$$

Solution 2.

$$I=\int_0^{2\pi}\frac{x+\tan(\sin x)}{2+\cos x}dx=\int_{-\pi}^{\pi}\frac{\pi+y-\tan(\sin y)}{2-\cos y}dy=\int_{-\pi}^{\pi}\frac{\pi}{2-\cos y}dy+$$

$$+\int_{-\pi}^{\pi}\frac{y-\tan(\sin y)}{2-\cos y}dy=\int_0^{\pi}\frac{2\pi}{2-\cos y}dy=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2+\sin z}dz=$$

$$=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2+\frac{2\tan\left(\frac{z}{2}\right)}{1+\tan^2\left(\frac{z}{2}\right)}}dz=2\pi\int_{-1}^{1}\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt=\frac{2\pi^2\sqrt{3}}{3}$$

2. General result.

Theorem 2. ***If*** $f:\mathbb{R}\to\mathbb{R}$ ***is continuous function with property***

$$\alpha f(x+b)+\beta f(c-x)=g(x+b),\forall x\in\mathbb{R};(1)$$

where $\alpha,\beta\in\mathbb{R}^*,\alpha+\beta\neq 0,b,c\in\mathbb{R}$, ***and*** $g:\mathbb{R}\to\mathbb{R}$ ***continuous function with primitives*** G. ***Then for all*** $m,n\in\mathbb{R}$***, such that*** $m+n=b+c$***,***

$$\boldsymbol{I_{m,n}=\int_m^n f(x)\,dx=\frac{G(n)-G(m)}{\alpha+\beta};(2)}$$

Proof. Replacing x with $x-b$, we get $\alpha f(x)+\beta f(b+c-x)=g(x)$, and hence,

$$f(x)=\frac{1}{\alpha}g(x)-\frac{\beta}{\alpha}f(b+c-x)$$

Integrating on the interval $[m,n]$, we get:

$$I_{m,n}=\int_m^n f(x)\,dx=\frac{1}{\alpha}\int_m^n g(x)\,dx-\frac{\beta}{\alpha}\int_m^n f(b+c-x)\,dx=$$

$$=\frac{1}{\alpha}[G(n)-G(m)]-\frac{\beta}{\alpha}\int_m^n f(b+c-x)\,dx\overset{b+c-x=t}{=}$$

$$=\frac{1}{\alpha}[G(n)-G(m)]+\frac{\beta}{\alpha}\int_n^m f(t)\,dt$$

$$I_{m,n}\left(1+\frac{\beta}{\alpha}\right)=\frac{1}{\alpha}[G(n)-G(m)],\text{then } I_{m,n}=\int_m^n f(x)\,dx=\frac{G(n)-G(m)}{\alpha+\beta};(2)$$

Application 6. If $h:\mathbb{R}\to\mathbb{R}$ is odd function, θ −periodic and with f' continuous, then find:

$$\boldsymbol{\Omega=\int_0^{2\theta}\frac{xh'(x)}{1+h^2(x)}dx}$$

Solution. Let $f(x)=\frac{xh'(x)}{1+h^2(x)}$, then $f(x+2\theta)+f(-x)=2\theta\cdot\frac{f(x)}{1+f^2(x)}$ and for $\alpha=\beta=1$,

$$b=2\theta,c=0,g(x)=2\theta\cdot\frac{h'(x)}{1+h^2(x)}$$

Now, g has the primitives $G:\mathbb{R}\to\mathbb{R},G(x)=2\theta\cdot\tan^{-1}x$. For $m+n=2\theta$ and using **Theorem 2**, we have:

$$I_{m,n}=\int_m^n f(x)\,dx=\theta[\tan^{-1}h(n)-\tan^{-1}h(m)]$$

Application 7. Find:

$$\boldsymbol{\Omega=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx}$$

Solution. Let $(x)=\frac{x\sin x}{1+\cos^2 x}$, then $f(x)+f(\pi-x)=\frac{\pi\sin x}{1+\cos^2 x}$ and using **Theorem 2**, we get:

$$I_{m,n}=\int_m^n f(x)\,dx=\frac{\pi}{2}[\tan^{-1}(\cos m)-\tan^{-1}(\cos n)];m+n=\pi$$

For $m=0,n=\pi\Rightarrow\Omega=\frac{\pi^2}{4}$.

Application 8. Find:

$$\boldsymbol{\Omega = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx}$$

Solution. Let $f(x) = log\ (1+\tan x)$, then

$$f\left(\frac{\pi}{4}-x\right) = \log\left(1+\frac{1-\tan x}{1+\tan x}\right) = \log 2 - \log(1+\tan x), \qquad f(x) + f\left(\frac{\pi}{4}-x\right) = \log 2$$

Using Theorem 2, we get:

$$I_{m,n} = \int_m^n \log(1+\tan x)\, dx = \frac{n-m}{2}\log 2\, ; m+n = \frac{\pi}{4}$$

For $m = 0, n = \frac{\pi}{4} \Rightarrow \Omega = \frac{\pi}{8}\log 2$

Application 9. If $h: [0,1] \to \mathbb{R}$ is continuous function, then prove:

$$\boldsymbol{\int_0^{\pi} x \cdot h(\sin x) dx = \frac{\pi}{2}\int_0^{\pi} h(\sin x)\, dx}$$

Solution. Let $f(x) = x \cdot f(\sin x) - \frac{\pi}{2}h(\sin x)$, then $f(x) + f(\pi - x) = 0$. Using **Theorem 2**, we get:

$$\int_m^n h(x)\, dx = 0; m+n = \pi$$

For $m = 0, n = \pi$, we get the problem.

Application 10. Prove that:

$$\boldsymbol{\int_0^1 \frac{dx}{\sqrt{x^4 - 4x^3 + 6x^2 - 4x + 2}} = \int_0^1 \frac{dx}{\sqrt{1+x^4}}}$$

Solution. Let $f: [0,1] \to \mathbb{R}, f(x) = \frac{1}{\sqrt{x^4-4x^3+6x^2-4x+2}} - \frac{1}{\sqrt{1+x^4}}$, then $f(x) + f(1-x) = 0$. Using **Theorem 2**, we get:

$$\int_m^n f(x)\, dx = 0; m+n = 1$$

For $m = 0, n = 1$, we get the problem.

3. Extension result.

Theorem 3. If $f: [a-\theta, a+\theta] \to \mathbb{R}$ is continuous function with property

$$\boldsymbol{\alpha f(a+x) + \beta f(a-x) = \gamma, \forall x \in [-\theta, \theta]; \alpha, \beta \in \mathbb{R}^*, \gamma \in \mathbb{R}}\textbf{, then:}$$

$$\textbf{(i)}\int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{2\gamma}{\alpha+\beta}\cdot\theta;\alpha+\beta\neq 0$$

$$\textbf{(ii)}\int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{\gamma}{\alpha}\cdot\theta+\frac{\alpha-\beta}{\alpha}\int_{a}^{a+\theta} f(x)\,dx$$

Proof. If $f:[a-\theta,a+\theta]\to\mathbb{R}$ is continuous function and $\varphi,\psi:[-\theta,\theta]\to[a-\theta,a+\theta]$,

$\psi(t)=a+t;\varphi(t)=a-t$, then

$$\text{(i)}\int_{a-\theta}^{a+\theta} f(x)\,dx=\int_{-\varphi(-\theta)}^{\varphi(\theta)} f(x)\,dx=\int_{-\theta}^{\theta} f(\varphi(t))\,\varphi'(t)dt=\int_{-\theta}^{\theta} f(a+t)\,dt=$$

$$=\int_{-\theta}^{\theta}\left(\frac{\gamma}{\alpha}-\frac{\beta}{\alpha}f(a-t)\right)dt=\frac{2\gamma}{\alpha}\cdot\theta-\frac{\beta}{\alpha}\int_{-\theta}^{\theta} f(a-t)\,dt=$$

$$=\frac{2\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{-\theta}^{\theta} f(\psi(t))\,\psi'(t)dt=\frac{2\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{a-\theta}^{a+\theta} f(x)\,dx$$

$$\int_{a-\theta}^{a+\theta} f(x)\,dx+\frac{\beta}{\alpha}\int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{2\gamma}{\beta}\cdot\theta\Rightarrow\int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{2\gamma}{\alpha+\beta}\cdot\theta;\alpha+\beta\neq 0$$

$$\text{(ii}\int_{a-\theta}^{a+\theta} f(x)\,dx=\int_{a-\theta}^{a} f(x)\,dx+\int_{a}^{a+\theta} f(x)\,dx$$

$$\int_{a-\theta}^{a} f(x)\,dx=\int_{\varphi(-\theta)}^{\varphi(0)} f(x)\,dx=\int_{-\theta}^{0} f(\varphi(t))\varphi'(t)\,dt=\int_{-\theta}^{0} f(a+t)\,dt=$$

$$=\int_{-\theta}^{0}\left(\frac{\gamma}{\alpha}-\frac{\beta}{\alpha}f(a-t)\right)dt=\frac{\gamma}{\alpha}\cdot\theta-\frac{\beta}{\alpha}\int_{-\theta}^{0} f(a-t)\,dt=$$

$$=\frac{\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{\psi(-\theta)}^{\psi(0)} f(\psi(t))\psi'(t)\,dt=\frac{\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{a+\theta}^{a} f(x)\,dx$$

So, we have:

$$\int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{a+\theta}^{a} f(x)\,dx+\int_{a}^{a+\theta} f(x)\,dx=\frac{\gamma}{\alpha}\cdot\theta+\frac{\alpha-\beta}{\alpha}\int_{a}^{a+\theta} f(x)\,dx$$

Definition. **Function $f:[a-\theta,a+\theta]\to\mathbb{R}$ is a -*even* function ,**

(a -*odd* function) if $f(a+x)=f(a-x);\forall x\leq|\theta|$,

$$(f(a+x)=-f(a-x);\forall|x|\leq\theta).$$

Application 11. Find:

$$\boldsymbol{\Omega_n = \int_0^1 \frac{4x^3 - 6x^2 + 8x - 3}{(x^2 - x + 1)^n} dx; n \in \mathbb{N}}$$

Solution. $g(x) = x^2 - x + 1$ is $\frac{1}{2} - even$ and $h(x) = 4x^3 - 6x^2 + 8x - 3$ is $\frac{1}{2} - odd$, so

$f(x) = \frac{h(x)}{g^n(x)}$ is $\frac{1}{2} - odd$ and using **Theorem 3**, we get:

$$\Omega_n = \int_0^1 \frac{4x^3 - 6x^2 + 8x - 3}{(x^2 - x + 1)^n} dx = 0.$$

Application 12. Find:

$$\boldsymbol{\Omega_n = \int_0^1 (2x - 1)^{2n+1} e^{x - x^2} dx; n \in \mathbb{N}}$$

Solution. $g(x) = (2x - 1)^{2n+1}$ is $\frac{1}{2} - odd$ function and $h(x) = e^{x - x^2}$ is $\frac{1}{2} - even$ function, then $f(x) = g(x) \cdot h(x)$ is $\frac{1}{2} - odd$ function. Using **Theorem 3**, we get:

$$\Omega_n = \int_0^1 (2x - 1)^{2n+1} e^{x - x^2} dx = 0.$$

Corollary 1. For any function $f: [a - \theta, a + \theta] \to \mathbb{R}$ exist an function $f_1, a - even$ and $f_2, a - odd$ such that $f(x) = f_1(x) + f_2(x); \forall x \in [a - \theta, a + \theta]$.

Corollary 2. **If $f, g: [a - \theta, a + \theta] \to \mathbb{R}$ integrable functions and f is $a - odd$, then**

$$\boldsymbol{\int_{a-\theta}^{a+\theta} f(x) g(x)\, dx = \int_a^{a+\theta} f(x)\big(g(x) + g(2a - x)\big)\, dx}$$

Application 13. Let $f: [-1, 1] \to \mathbb{R}$ continuous with property $f(x) + f(-x) = \pi; \forall x \in [-1, 1]$. Find:

$$\boldsymbol{\Omega_n = \int_0^{(2n+1)\pi} f(\cos x)\, dx; \forall n \in \mathbb{N}}$$

Solution. We have:

$\Omega_n = \Omega_{n-1} + \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x)\, dx$, $g(x) = f(\cos x)$ is $2n\pi - odd$, then:

$$I = \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x)\, dx = 2\int_{2n\pi}^{2n\pi + \pi} f(\cos x)\, dx = 2\int_0^{\pi} f(\cos(t + 2n\pi))\, dt =$$

$$= 2\int_0^{\pi} f(\cos t)\,dt = -2\int_1^{-1} \frac{f(u)}{\sqrt{1-u^2}}du = 2\int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}}du$$

$$g(u) = \frac{1}{\sqrt{1-u^2}} \text{ is } 0 - even \Rightarrow$$

$$I = 2\int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}}du = 2\int_0^1 \frac{f(u)+f(-u)}{\sqrt{1-u^2}}du = 2\pi\int_0^1 \frac{du}{\sqrt{1-u^2}} = \pi^2$$

$$\text{So}, \Omega_n = \Omega_{n-1} + \pi^2 \Rightarrow \Omega_n = (n+1)\pi^2$$

Application 14. Find:

$$\boldsymbol{\Omega_n = \int_0^{\frac{\pi}{4}} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx}$$

Solution. $f(x) = \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}$ is $\frac{\pi}{2} - even$, then using **Theorem 3**, we get:

$$\Omega_n = \int_0^{\frac{\pi}{4}} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx = \int_0^{\pi} xf(x)\,dx = \int_{\frac{\pi}{2}}^{\pi} f(x)(x + \pi - x)\,dx =$$

$$\left(\because g(x) = 1 \text{ is} \frac{3\pi}{4} - even \text{ and } f\left(\frac{3\pi}{2} - x\right) = \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x}\right)$$

$$= \pi\int_{\frac{\pi}{2}}^{\pi} f(x)\,dx = \pi\int_{\frac{3\pi}{4}}^{\pi}\left(f(x) + f\left(\frac{3\pi}{2} - x\right)\right)dx = \pi\int_{\frac{3\pi}{4}}^{\pi} dx = \frac{\pi^2}{4}$$

Application 15. Find:

$$\boldsymbol{\Omega = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan x)}{\sin 2x + \cos 2x}dx}$$

Solution. $f(x) = \frac{1}{\sin 2x + \cos 2x} = \frac{1}{\sqrt{2}\cos\left(2x - \frac{\pi}{4}\right)} \Rightarrow f$ is $\frac{\pi}{8} - even$.

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan x)}{\sin 2x + \cos 2x}dx = \int_0^{\frac{\pi}{4}} f(x)\log(1+\tan x)\,dx =$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} f(x)\left(\log(1+\tan x) + \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right)\right)dx =$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} f(x)\left(\log(1+\tan x) + \log\left(\frac{2}{1+\tan x}\right)\right)dx = \frac{\log 2}{\sqrt{2}}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{dx}{\cos\left(2x - \frac{\pi}{4}\right)}$$

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SSMA-MATH CHALLENGES-(II)

By Daniel Sitaru-Romania

5531. For real numbers x, y, z prove that if $x, y, z > 1$ and $xyz = 2\sqrt{2}$, then

$$x^y + y^z + z^x + y^x + z^y + x^z > 9.$$

Daniel Sitaru

Solution 1 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece.

If $f(t) = t\ln\left(\frac{2\sqrt{2}}{t}\right)$, with $t > 1$, the $f'(t) = \ln\left(\frac{2\sqrt{2}}{t}\right) - 1$, and for $f'(t_k) = 0$, we have $t_k = \frac{2\sqrt{2}}{e} > 1$. So, we have: $f(t) \geq f\left(\frac{2\sqrt{2}}{e}\right) = \frac{2\sqrt{2}}{e}\ln\left(2\sqrt{2}\cdot\frac{e}{2\sqrt{2}}\right) = \frac{2\sqrt{2}}{e}$.

Furthermore, we have:

$$x^y \geq 1 + y\ln x, \qquad y^x \geq 1 + x\ln y, \qquad y^x \geq 1 + z\ln y,$$

$$z^y \geq 1 + y\ln z, \qquad z^x \geq 1 + x\ln z, \qquad x^z \geq 1 + z\ln z,$$

So, we have: $x^y + y^z + z^x + y^x + z^y + x^z \geq 6 + x\ln(yz) + y\ln(xz) + z\ln(xy)$

$$= 6 + x\ln\left(\frac{2\sqrt{2}}{x}\right) + y\ln\left(\frac{2\sqrt{2}}{y}\right) + z\ln\left(\frac{2\sqrt{2}}{z}\right) \geq 6 + 3\cdot\frac{2\sqrt{2}}{e} > 9.$$

Solution 2 by Adrian Naco, Polytechnic University of Tirana, Albania.

Since, $x > 1, y > 1$, and using the Bernoulli inequality, we have that

$$x^y = [1 + (x - 1)]^y > 1 + y(x - 1) \qquad (2)$$

Acting analogously it implies that,

$$x^y+y^z+z^x+y^x+z^y+x^z>6+2(xy+yz+zx)-2(x+y+z)\quad (3)$$

To prove the given inequality (1), it is enough to prove the following equivalent inequalities,

$6+2(xy+yz+zx)-2(x+y+z)>9$ or equivalently

$$(xy+yz+zx)-(x+y+z)>\frac{3}{2}$$

Let $f(x,y,z)=(xy+yz+zx)-(x+y+z)-\frac{3}{2}$. $g(x,y,z)=xyz-2\sqrt{2}$

and using Langrange Multipliers method, we have that,

$$F(x,y,z)=f(x,y,z)-\lambda g(x,y,z)=(xy+yz+zx)-(x+y+z)-\frac{3}{2}-\lambda\left(xyz-2\sqrt{2}\right).$$

$$F_x=y+z-1-\lambda yz=0,\qquad F_y=x+z-1-\lambda xz=0$$

$$F_z=x+y-1-\lambda xy=0,\qquad F_\lambda=-xyz+2\sqrt{2}=0$$

Subtracting side by side, each couple of the last three first equations, we get the following:

$$(z-1)(x-y)=0,\qquad (y-1)(x-z)=0$$

$$(x-1)(z-y)=0,\qquad xyz=2\sqrt{2}$$

So, $x=y=z=\sqrt{2}$, is the only solution (since $x>1,y>1,z>1$). Finally,

$$\min f(x,y,z)=f\left(\sqrt{2};\sqrt{2};\sqrt{2}\right)=2+2+2-3\sqrt{2}-\frac{3}{2}=\frac{3}{2}\left(3-2\sqrt{2}\right)>0.$$

Note. Even if we consider the case when $x=1$, we have that,

$$f(1,y,z)=y+yz+z-1-z-y-\frac{3}{2}=2\sqrt{2}-\frac{5}{2}>f\left(\sqrt{2};\sqrt{2};\sqrt{2}\right)=\frac{3}{2}\left(3-2\sqrt{2}\right)>0.$$

Solution 3 by Moti Levy, Rehovot, Israel.

Let $f(u,v):=u^v+v^u, u,v>1$. By verifying that the Hessian of $f(u,v)$ is positive semi-definite, it becomes evident that $f(u,v)$ is convex function in the domain $u,v>1$.

$$Hess(u^v+v^u)=\begin{bmatrix} u^{v-2}(v-1)v+v^u\ln^2 u & u^{v-1}+v^{u-1}+vu^{v-1}\ln u+uv^{u-1}\ln v \\ u^{v-1}+v^{u-1}+vu^{v-1}\ln u+uv^{u-1}\ln v & (u-1)uv^{u-2}+u^v\ln^2 v\end{bmatrix}$$

Then by Jensen's inequality: $x^y+y^z+z^x+y^x+z^y+x^z$ (1)

$$=f(x,y)+f(y,z)+f(z,x)\geq 3f\left(\frac{x+y+z}{3},\frac{y+z+x}{3}\right)$$

By AM-GM inequality,

$$xyz = 2\sqrt{2} \Rightarrow \frac{x+y+z}{3} \geq \sqrt[3]{2\sqrt{2}} = \sqrt{2} \quad (2)$$

Inequalities (1) and (2) imply the required result,

$$x^y + y^z + z^x + y^x + z^y + x^z \geq 3f(\sqrt{2}, \sqrt{2}) = 6(\sqrt{2})^{\sqrt{2}} > 9.$$

5562. Prove: If $a, b, c \geq 1$, then:

$$\boldsymbol{e^{ab} + e^{bc} + e^{ca} > 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}}.$$

Daniel Sitaru

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY.-USA

The well-known inequality $e^x > 1 + x$ for $x \geq 1$ yields

$$e^{ab} + e^{bc} + e^{ca} > (1 + ab) + (1 + bc) + (1 + ca) = 3 + ab + bc + ca.$$

We also note that since $a, b, c \geq 1$, we have $a \geq \frac{1}{c}, b \geq \frac{1}{a}$, and $c \geq \frac{1}{b}$, so that $ab \geq \frac{b}{c}, bc \geq \frac{c}{a}$, and $ca \geq \frac{a}{b}$. Thus: $e^{ab} + e^{bc} + e^{ca} > 3 + ab + bc + ca \geq 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}$.

Solution 2 by Ed Gray, Highland Beach, FL.

1. $e^{ab} > 1 + ab$

2. $e^{bc} > 1 + bc$

3. $e^{ca} > 1 + ca$

4. $e^{ab} + e^{bc} + e^{ca} > 3 + ab + bc + ca$

5. Claim that $a, b, c \geq 1$ implies that $ab > \frac{a}{b}$ or $ab^2 > a$ because $b \geq 1$; same holds for the others which proves the conjecture.

Solution 3 by Albert Natian, Los Angeles Valley College, Valley Glen, CA.

$$e^{ab} + e^{bc} + e^{ca} = \left[1 + ab + \sum_{k=2}^{\infty} \frac{(ab)^k}{k!}\right] + \left[1 + bc + \sum_{k=2}^{\infty} \frac{(bc)^k}{k!}\right] + \left[1 + ca + \sum_{k=2}^{\infty} \frac{(ca)^k}{k!}\right]$$

$$> [1 + ab] + [1 + bc] + [1 + ca]$$

$$= 3 + \left[ab\left(1 - \frac{1}{b^2}\right) + \frac{a}{b}\right] + \left[bc\left(1 - \frac{1}{c^2}\right) + \frac{b}{c}\right] + \left[ca\left(1 - \frac{1}{c^2}\right) + \frac{c}{a}\right]$$

$$\geq 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}.$$

Solution 4 by Moti Levy, Rehovot, Israel.

Since e^x is convex function, then (Jensen's inequality)

$$\frac{e^{ab}+e^{bc}+e^{ca}}{3} \geq e^{\frac{ab+bc+ca}{3}} \quad (1)$$

Since $e^x \geq 1+x$ for $x \geq 0$, then

$$3e^{\frac{ab+bc+ca}{3}} \geq 3\left(1+\frac{ab+bc+ca}{3}\right) = 3+ab+bc+ca \quad (2)$$

Since $a,b,c \geq 1$: $a^2b^2c+ab^2c^2+a^2bc^2 \geq a^2c+ab^2+bc^2$

or $abc(ab+bc+ca) \geq a^2c+ab^2+bc^2$

Dividing both sides of the inequality by abc results in

$$ab+bc+ca \geq \frac{c}{a}+\frac{b}{c}+\frac{a}{b} \quad (3)$$

The desired inequality follows from (1), (2) and (3).

Solution 5 by Daniel Văcaru, Pitesti, Romania

We know that $e^x \geq x+1, \forall x \geq 0$, or to be accurate $e^x > x+1, \forall x \geq 1$. It follows $e^{ab} > 1+ab \geq 1+\frac{a}{b} \to e^{ab} > 1+\frac{a}{b}$ (1). In the same manner we have $e^{bc} > 1+\frac{b}{c}$ (2) and $e^{ca} > 1+\frac{c}{a}$ (3). Summing, we obtain: $e^{ab}+e^{bc}+e^{ca} > 3+\frac{c}{a}+\frac{b}{c}+\frac{a}{b}$,

5448. Evaluate:

$$\lim_{n\to\infty} \sqrt[n]{\sum_{\substack{0\leq i,j\leq n \\ i+j=n}} \binom{2i}{i}\binom{2j}{j}}.$$

Yubal Barrios, Angel Plaza-Spain

Solution 1 by Brian Bradie, Chrisopher Newport University, Newport News, VA.

The generating function for the central binomial coefficients is $(1-4x)^{-\frac{1}{2}}$; that is,

$$\sum_{i=0}^{\infty} \binom{2i}{i} x^i = \frac{1}{\sqrt{1-4x}}.$$

It follows that

$$\sum_{0\leq i,j\leq n, i+j=n} \binom{2i}{i}\binom{2j}{j} = \sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i}$$

is the coefficient of x^n in the function

$$\frac{1}{\sqrt{1-4x}}\cdot\frac{1}{\sqrt{1-4x}}=\frac{1}{1-4x}=\sum_{n=0}^{\infty}(4x)^n,$$

which is 4^n. Thus,

$$\lim_{n\to\infty}\sqrt[n]{\sum_{0\le i,j\le n,i+j=n}\binom{2i}{i}\binom{2j}{j}}=\lim_{n\to\infty}\sqrt[n]{4^n}=\lim_{n\to\infty}4=4.$$

Solution 2 by Daniel Sitaru – Romania

$$(1+x)^0(1+x)^{2n}+(1+x)^2(1+x)^{2n-2}+(1+x)^4(1+x)^{2n-4}+\cdots$$
$$\ldots+(1+x)^{2n}(1+x)^0=(2n+1)(1+x)^{2n}$$

The coefficient of x^n in LHS and RHS are equal:

$$\binom{2n}{n}+\binom{2}{1}\binom{2n-2}{n-1}+\binom{4}{2}\binom{2n-4}{n-2}+\cdots+\binom{2n}{n}=(2n+1)\binom{2n}{n}$$

$$\sum_{\substack{0\le i,j\le n\\ i+j=n}}\binom{2i}{i}\binom{2j}{j}=(2n+1)\binom{2n}{n}$$

$$\lim_{n\to\infty}\sqrt[n]{\sum_{0\le i,j\le n}\binom{2i}{i}\binom{2j}{j}}=\lim_{n\to\infty}\sqrt[n]{(2n+1)\binom{2n}{n}}$$

$$\overset{CAUCHY-D'ALEMBERT}{\cong}\lim_{n\to\infty}\frac{2n+3}{2n+1}\cdot\frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{((n)!)^2}}=1\cdot\lim_{n\to\infty}\frac{(2n+1)(2n+2)}{(n+1)^2}=4.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China.

It is well known that $|x|\le\frac{1}{4},\sum_{i=0}^{\infty}\binom{2i}{i}x^i=\frac{1}{\sqrt{1-4x}}$, with the usual convention that $0!=1$ and $\binom{0}{0}=1$. Hence,

$$\sum_{i=1}^{\infty}(4x)^i=\frac{1}{1-4x}=\left(\sum_{i=0}^{\infty}\binom{2i}{i}x^i\right)^2=\sum_{n=0}^{\infty}\sum_{\substack{0\le i,j\le n\\ i+j=n}}\binom{2i}{i}\binom{2j}{j}x^n.$$

Thus for nonnegative integers n,

$$\sum_{n=0}^{\infty} \sum_{\substack{0\le i,j\le n \\ i+j=n}} \binom{2i}{i}\binom{2j}{j} = 4^n,$$

so the limit of the problem equals 4.

5568. Given $A \in M_5(\mathcal{R}), \det(A^5 + I_5) \neq 0$, and $A^{20} - I^5 = A^5(A^5 + I_5)$.

Prove that $\sqrt[4]{\det A} \in \mathcal{R}$.

Daniel Sitaru

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We note that: $0 = A^{20} - I_5 - A^5(A^5 - I_5) = (A^5 + I_5)(A^{15} - A^{10} - I^5)$.

By assumption, $\det(A^5 + I_5) \neq 0$. So $A^5 + I_5$ is non-singular and we deduce that

$$A^{15} - A^{10} - I_5 = 0.$$

Let $B = A^5$. So B satisfies the matrix equation: $B^3 - B^2 - I_5 = 0$

which implies that B is non-singular. For if B were singular there would be a non-zero vector x such that $Bx = 0$. But then $0 = (B^3 - B^2 - I_5)x = B\big(B(Bx)\big) - B(Bx) - I_5x = x$ leads to a contradiction.

The equation $x^3 - x^2 - 1 = 0$ has one positive root and two complex conjugate roots. Let us denote by ρ the positive root and by σ and $\overline{\sigma}$ the two complex roots. We find that approximately $p \approx 1.46557, \sigma \approx -0.232786 + 0.792552i, \overline{\sigma} \approx -0.232786 - 0.792552i$. By Vieta's formula,

$$\rho\sigma\overline{\sigma} = 1.$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be the eigenvalues of B. They are the zeros of the characteristic polynomial $p(\lambda) = \det(\lambda I_5 - B)$. Let x_j be an eigenvector associated with the eigenvalue λ_j. So $Bx_j = \lambda_j x_j$. We claim that every eigenvalue of B is a root of $x^3 - x^2 - 1 = 0$. Indeed,

$$0 = (B^3 - B^2 - I_5)x_j = \big(\lambda_j^3 - \lambda_j^2 - 1\big)x_j$$

which implies that $\lambda_j^3 - \lambda_j^2 - 1 = 0$ and so $\lambda_j \in \{\rho, \sigma, \overline{\sigma}\}$ for $1 \le j \le 5$. Finally

$\det(B) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 \cdot \lambda_5$. So either $\det(B) = \rho^5$, or $\det(B) = \rho^3 \cdot \sigma \cdot \overline{\sigma} = \rho^2$ or

$\det(B) = \rho(\sigma \cdot \overline{\sigma})^2$ or $\det(B) = \rho(\overline{\sigma})^2 = \frac{1}{\rho}$, since B is a real matrix and therefore the determinant is real, and

$$\det A = \sqrt[5]{\det B} \in \left\{\rho, \rho^{\frac{2}{5}}, \rho^{-\frac{1}{5}}\right\},$$

So $\sqrt[4]{\det A}$ is positive and can assume at most three (positive) values.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

Since $A^{20} - I_5 - A^5(A^5 + I) = 0_5$, the zero matrix, so $(A^5 + I_5)(A^{15} - A^{10} - I_5) = 0_5$.

Given that $\det(A^5 + I_5) \neq 0$, so the inverse of $A^5 + I_5$ exists and we have

$$A^{15} - A^{10} - I_5 = 0_5 \quad (1)$$

We now show that $\det A > 0$, so that the statement $\sqrt[4]{\det A} \in \mathcal{R}$ holds.

By (1), we have: $A^{10}(A^5 - I_5) = I_5 \quad (2)$

and

$$A^{15} = A^{10} + I_5 \quad (3)$$

By (2), we have $(\det A)^{10} \det(A^5 - I_5) = \det\left(A^{10}(A^5 - I_5)\right) = 1$ ans so $\det A \neq 0$. Let $i = \sqrt{-1}$. By (3), we have: $(\det A)^{15} = \det(A^{10} + I_5) = \det\left((A^5 + iI_5)(A^5 - iI_5)\right)$

$$= \det(A^5 + iI_5)\,(\det(A^5 - iI_5)) = \det(A^5 + iI_5)\,\overline{\det(A^5 + iI_5)}$$

$$= |\det(A^5 + iI_5)|^2 \geq 0.$$

Thus $\det A > 0$ and this completes the solution.

5574. Prove: If $\mathbf{0} < a \leq b \leq c$ then:

$$\frac{1}{1 + e^{1-b+c}} + \frac{1}{1 + e^b} \leq \frac{1}{1 + e^a} + \frac{1}{1 + e^c}$$

Daniel Sitaru

Solution 1 by Titu Zvonaru, Comănești, Romania

Clearing denominators, the given inequality is equivalent to:

$$(2 + e^{a-b+c} + e^b)(1 + e^a)(1 + e^c) \geq (2 + e^a + e^c)(1 + e^{a-b+c})(1 + e^b)$$

$$2 + e^{a-b+c} + e^b + 2e^a + e^{2a-b+c} + e^{a+b} + 2e^c + e^{a-b+2c} + e^{b+c} + 2e^{a+c} + e^{2a-b+2c} + e^{a+b+c}$$

$$\leq 2 + e^a + e^c + 2e^{a-b+c} + e^{2a-b+c} + e^{a-b+2c} + 2e^b + e^{a+b} + e^{b+c} + 2e^{a+c} + e^{2a+c} + e^{a+2c}$$

$$e^a + e^c + e^{2a-b+2c} + e^{a+b+c} \leq e^{a-b+c} + e^b + e^{2a+c} + e^{a+2c}$$

$$(e^b - e^a)(e^{a-b} - 1)(e^{a+c} - 1) \geq 0.$$

The equality holds if and only if $a = b$ or $b = c$.

Solution 2 by Brian Bradie, Chrisopher Newport University, Newport News, VA.-USA

If $a=c$, then $a=b=c=a-b+c$ and

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^{b}}=\frac{1}{1+e^{a}}+\frac{1}{1+e^{c}}=\frac{2}{1+e^{a}}.$$

Now, suppose $0<a<c$, and let

$$f(x)=\frac{1}{1+e^{x}}.$$

Then

$$f''(x)=\frac{e^{2x}-e^{x}}{(1+e^{x})^{3}}>0$$

for $x>0$. Thus, f is convex for $x>0$. Because

$$0\le\frac{c-b}{c-a}\le 1 \text{ and. } a\cdot\frac{c-b}{c-a}+c\left(1-\frac{c-b}{c-a}\right)=b,$$

it follows that

$$f(b)\le\frac{c-b}{c-a}f(a)+\left(1-\frac{c-b}{c-a}\right)f(c) \quad (6)$$

Moreover, because

$$0\le\frac{b-a}{c-a}\le 1 \text{ and } a\cdot\frac{b-a}{c-a}+c\left(1-\frac{b-a}{c-a}\right)=a-b+c,$$

it follows that

$$f(a-b+c)\le\frac{b-a}{c-a}f(a)+\left(1-\frac{b-a}{c-a}\right)f(c) \quad (7)$$

Adding (6) and (7) yields

$$f(a-b+c)+f(b)\le f(a)+f(c),$$

or

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^{b}}\le\frac{1}{1+e^{a}}+\frac{1}{1+e^{c}}.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

Put $x=e^{a}, y=e^{b}, z=e^{c}$. Then $1<x\le y\le z$.

$$\frac{1}{1+e^{a}}+\frac{1}{1+e^{c}}-\frac{1}{1+e^{a-b+c}}-\frac{1}{1+e^{b}}$$

$$=\frac{1}{1+x}+\frac{1}{1+z}-\frac{1}{1+\frac{xz}{y}}-\frac{1}{1+y}=\frac{(y-x)(z-y)(xz-1)}{(1+x)(1+z)(y+xz)}\ge 0.$$

This proof shows that we may even allow $a = 0$.

Solution 4 by Kee-Wai Lau, Hong Kong, China.

Let $x = e^a, y = e^b$ and $z = e^c$, so that $1 < x \le y \le z$ and the inequality of the problem is equivalent to

$$\frac{1}{1+x} + \frac{1}{1+z} - \frac{y}{y+xz} - \frac{1}{1+y} \ge 0 \quad (1)$$

It is easy to check that the left side of (1) equals $\frac{(y-x)(z-y)(xz-1)}{(1+x)(1+y)(1+z)(y+xz)}$ so that (1) and the inequality of the problem indeed hold.

Solution 4 by Daniel Văcaru, Pitesti, Romania.

We prove the following fact: If $f: I \subset \mathbb{R} \to \mathbb{R}$ is convex for all $a, b \in I$ with $0 < a < b < c$, then $f(a - b + c) \le f(a) - f(b) + f(c)$ (*)

We have $\lambda \in (0,1), \lambda = \frac{c-b}{c-a}$ such that $b = \lambda a + (1-\lambda)c$. But f is convex, so

$$f(b) \le \lambda f(a) + (1-\lambda) f(c)$$

We have $a - b + c = (1-\lambda)a + \lambda c$. So again by convexity,

$$f(a - b + c) \le (1-\lambda) f(a) + \lambda f(c) \quad (2)$$

Summing (1) and (2) we obtain $f(a - b + c) + f(b) \le f(a) + f(c)$, proving (*).

Now consider $f: \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{e^x} = e^{-x}$. We have $f''(x) = e^{-x}$, proving the convexity of f. So using (*), we obtain

$$\frac{1}{1 + e^{a-b+c}} + \frac{1}{1 + e^b} \le \frac{1}{1 + e^a} + \frac{1}{1 + e^c},$$

as desired.

5585. In ΔABC the following relationship holds:

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \le \frac{27}{8}$$

Daniel Sitaru

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA.-USA

Note

$$\sin^4\left(x - \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)^4$$

$$=\frac{1}{16}\left(9\sin^4 x-12\sqrt{3}\sin^3 x\cos x+18\sin^2 x\cos^2 x-4\sqrt{3}\sin x\cos^3 x+\cos^4 x\right),$$

$$\sin^4\left(x+\frac{\pi}{6}\right)=\frac{1}{16}\left(9\sin^4 x+12\sqrt{3}\sin^3 x\cos x+18\sin^2 x\cos^2 x+4\sqrt{3}\sin x\cos^3 x\right.$$
$$\left.+\cos^4 x\right),$$

and

$$\sin^4\left(x-\frac{\pi}{6}\right)+\sin^4\left(x+\frac{\pi}{6}\right)=\frac{1}{8}(9\sin^4 x+18\sin^2 x\cos^2 x+\cos^4 x)$$
$$=\frac{1}{8}\left(1+8\sin^2 x\,(\sin^2 x+2\cos^2 x)\right)=\frac{9}{8}-\cos^4 x.$$

Thus,

$$\sin^4\left(x-\frac{\pi}{6}\right)+\sin^4\left(x+\frac{\pi}{6}\right)\le\frac{9}{8}$$

for all x, with equality when $x=\frac{\pi}{2}+n\pi$ for any integer n. Because

$\sin^4 x+\sin^4\left(x+\frac{\pi}{3}\right)$ is just a translation of $\sin^4\left(x-\frac{\pi}{6}\right)+\sin^4\left(x+\frac{\pi}{6}\right)$

by $\frac{\pi}{6}$, it follows that

$$\sin^4 x+\sin^4\left(x+\frac{\pi}{3}\right)\le\frac{9}{8}$$

for all x, with equality when $x=\frac{\pi}{3}+n\pi$ for any integer n. Therefore, for any angles A,B, and C,

$$\sin^4 A+\sin^4 B+\sin^4 C+\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4\left(\frac{\pi}{3}+C\right)\le\frac{27}{8},$$

with equality when $A=\frac{\pi}{3}+n_1\pi, B=\frac{\pi}{3}+n_2\pi$, and $C=\frac{\pi}{3}+n_3\pi$ for any integers n_1,n_2, and n_3. For the special case when A,B, and C are the angles in a triangle, this becomes

$$\sin^4 A+\sin^4 B+\sin^4 C+\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4\left(\frac{\pi}{3}+C\right)\le\frac{27}{8},$$

with equality when $A=B=C=\frac{\pi}{3}$.

Solution 2 by Moti Levy, Rehovot, Israel.

Applying Power Means Inequality $\left(M_{\frac{1}{4}}\le M_1\right)$, we get

$$\sin^4 A+\sin^4 B+\sin^4 C\le\frac{1}{27}(\sin A+\sin B+\sin C)^4 \qquad (1)$$

and

$\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4\left(\frac{\pi}{3}+C\right)\le\frac{1}{27}\Big(\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+$
$\sin\left(\frac{\pi}{3}+C\right)\Big)^4$ (2)

It is well known that: $\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$ (3)

(see, for example, the bible of geometric inequalities, Bottema et al. paragraph 2.2, page 18).

We will prove that $\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+\sin\left(\frac{\pi}{3}+C\right)\le\frac{3\sqrt{3}}{2}$.

$$\sin\left(A+\frac{\pi}{3}\right)+\sin\left(B+\frac{\pi}{3}\right)=2\sin\left(\frac{A+B}{2}+\frac{\pi}{3}\right)\cos\left(\frac{A-B}{2}\right)\le2\sin\left(\frac{A+B}{2}+\frac{\pi}{3}\right)$$

$$=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{\pi}{3}\right)+2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{\pi}{3}\right)$$

$$=2\cos\left(\frac{C}{2}\right)\cos\left(\frac{\pi}{3}\right)+2\sin\left(\frac{C}{2}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+\sin\left(\frac{\pi}{3}+C\right)$$

$$\le2\cos\left(\frac{C}{2}\right)\cos\left(\frac{\pi}{3}\right)+2\sin\left(\frac{C}{2}\right)\sin\left(\frac{\pi}{3}\right)+\sin\left(C+\frac{\pi}{3}\right)$$

$$=\cos\left(\frac{C}{2}\right)+\sqrt{3}\sin\left(\frac{C}{2}\right)+\frac{1}{2}\sin(C)+\frac{\sqrt{3}}{2}\cos(C)$$

$$=\cos\left(\frac{C}{2}\right)+\sqrt{3}\sin\left(\frac{C}{2}\right)+\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)+\frac{\sqrt{3}}{2}\left(2\cos^2\left(\frac{C}{2}\right)-1\right)$$

$$=\cos\left(\frac{C}{2}\right)+\sqrt{3}\sqrt{1-\cos^2\left(\frac{C}{2}\right)}+\sqrt{1-\cos^2\left(\frac{C}{2}\right)}\cos\left(\frac{C}{2}\right)+\frac{\sqrt{3}}{2}\left(2\cos^2\left(\frac{C}{2}\right)-1\right)$$

Now let $t\coloneqq\cos\left(\frac{C}{2}\right)$, and $f(t)\coloneqq t+\sqrt{3}\sqrt{1-t^2}+t\sqrt{1-t^2}+\frac{\sqrt{3}}{2}(2t^2-1)$,

then $\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+\sin\left(\frac{\pi}{3}+C\right)\le f(t)$.

To find the maximum of $f(t)$,

$$\frac{df}{dt}=\frac{1}{\sqrt{1-t^2}}\left(\sqrt{1-t^2}-\sqrt{3}t-2t^2+2\sqrt{3}t\sqrt{1-t^2}+1\right)\quad(4)$$

$$\frac{d^2f}{dt^2}=\frac{2\sqrt{3}(1-t^2)^{\frac{3}{2}}-3t+2t^3-\sqrt{3}}{(1-t^2)^{\frac{3}{2}}}\quad(5)$$

By solving (4), we find that $f(t)$ has critical point at $t=\frac{\sqrt{3}}{2}$ and at $t=-\frac{\sqrt{3}}{2}$. Since

$\frac{d^2f}{dt^2}\left(\frac{\sqrt{3}}{2}\right) = -12\sqrt{3} < 0$, then the critical point $t = \frac{\sqrt{3}}{2}$ is maximum. Since $\frac{d^2f}{dt^2}\left(-\frac{\sqrt{3}}{2}\right) = 0$ then the critical point $t = -\frac{\sqrt{3}}{2}$ is inflection point.

We conclude that: $\sin\left(\frac{\pi}{3}+A\right)+\sin\left(\frac{\pi}{3}+B\right)+\sin\left(\frac{\pi}{3}+C\right) \le f\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$ (6)

By (3) and (6) we get

$$\sin^4 A+\sin^4 B+\sin^4 C+\sin^4\left(\frac{\pi}{3}+A\right)+\sin^4\left(\frac{\pi}{3}+B\right)+\sin^4\left(\frac{\pi}{3}+C\right)$$
$$\le \frac{1}{27}\left(\frac{3\sqrt{3}}{2}\right)^4+\frac{1}{27}\left(\frac{3\sqrt{3}}{2}\right)^4 = \frac{27}{8}$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We prove the stronger inequality: $\sin^4 x+\sin^4\left(x+\frac{\pi}{3}\right) \le \frac{9}{8}$

from which the claimed inequality immediately follows. It is not required that

$A+B+C=\pi, A, B, C$ and be arbitrary.

We have

$$\frac{9}{8}-\sin^4 x-\sin^4\left(x+\frac{\pi}{3}\right) = \frac{9}{8}-\sin^4\left(x+\frac{\pi}{6}-\frac{\pi}{6}\right)-\sin^4\left(x+\frac{\pi}{6}+\frac{\pi}{6}\right) =$$
$$= \frac{9}{8}-\left(\frac{\sqrt{3}}{2}\sin\left(x+\frac{\pi}{6}\right)-\frac{1}{2}\cos\left(x+\frac{\pi}{6}\right)\right)^4-\left(\frac{\sqrt{3}}{2}\sin\left(x+\frac{\pi}{6}\right)+\frac{1}{2}\cos\left(x+\frac{\pi}{6}\right)\right)^4 =$$
$$= \frac{9}{8}-\frac{9}{8}\sin^4\left(x+\frac{\pi}{6}\right)-\frac{9}{8}\sin^2\left(x+\frac{\pi}{6}\right)\cos^2\left(x+\frac{\pi}{6}\right)-\frac{1}{8}\cos^4\left(x+\frac{\pi}{6}\right) =$$
$$= \frac{9}{8}-\frac{9}{8}\sin^4\left(x+\frac{\pi}{6}\right)-\frac{9}{8}\sin^2\left(x+\frac{\pi}{6}\right)\left(1-\sin^2\left(x+\frac{\pi}{6}\right)\right)-\frac{1}{8}\left(1-\sin^2\left(x+\frac{\pi}{6}\right)\right)^2 =$$
$$1-\frac{7}{8}\sin^2\left(x+\frac{\pi}{6}\right)-\frac{1}{8}\sin^4\left(x+\frac{\pi}{6}\right) \ge 0$$

Equality holds if and only if $\sin\left(x+\frac{\pi}{6}\right) = \pm 1$, if $x \equiv \frac{\pi}{3} mod\pi$

ABOUT AN INEQUALITY BY JI CHEN

By D.M. Bătineţu-Giurgiu-Romania

If $x, y, z \in (0, \infty)$ then:

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \ge \frac{9}{4}; (J.C.)$$

Theorem. If $\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^*$ and $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}_+^*$ then

$$\left(3m^2+(xy)^{m^2+1}+(yz)^{m^2+1}+(zx)^{m^2+1}\right)\left(n^2+3^{n^2}\left(\frac{1}{(x+y)^{2(n^2+1)}}+\frac{1}{(y+z)^{2(n^2+1)}}+\frac{1}{(z+x)^{2(n^2+1)}}\right)\right)\geq 9mn;(1)$$

Proof. We have:

$$3m^2+\sum_{cyc}(xy)^{m^2+1}=\sum_{cyc}\left(m^2+(xy)^{m^2+1}\right)\overset{AGM}{\geq}$$

$$\geq(m^2+1)\cdot\sum_{cyc}\sqrt[m^2+1]{\underbrace{1\cdot 1\cdot 1\cdot\ldots\cdot 1}_{m^2\ terms}\cdot(xy)^{m^2+1}}=(m^2+1)\cdot\sum_{cyc}xy;(2)$$

$$n^2+3^{n^2}\cdot\sum_{cyc}\frac{1}{(x+y)^{2(n^2+1)}}\overset{Radon}{\geq}n^2+3^{n^2}\cdot\frac{1}{3^{n^2}}\left(\sum_{cyc}\frac{1}{(x+y)^2}\right)^{n^2+1}=$$

$$=n^2+\left(\sum_{cyc}\frac{1}{(x+y)^2}\right)^{n^2+1}\overset{AGM}{\geq}(n^2+1)\cdot\sum_{cyc}\sqrt[n^2+1]{\underbrace{1\cdot 1\cdot\ldots\cdot 1}_{n^2\ terms}\cdot\left(\sum_{cyc}\frac{1}{(x+y)^2}\right)^{n^2+1}}=$$

$$=(n^2+1)\cdot\sum_{cyc}\frac{1}{(x+y)^2};(3)$$

Using (2) and (3) inequality (1) becomes:

$$\left(3m^2+\sum_{cyc}(xy)^{m^2+1}\right)\left(n^2+3^{n^2}\cdot\sum_{cyc}\frac{1}{(x+y)^{2(n^2+1)}}\right)\geq$$

$$\geq(m^2+1)(n^2+1)(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq$$

$$\geq(m^2+1)(n^2+1)\cdot\frac{9}{4}\geq 2m\cdot 2n\cdot\frac{9}{4}=9mn$$

If in (1) we take $m=n=0$, we get Ji Chen’s inequality.

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ABOUT AN INEQUALITY BY G. TSINTSIFAS

By D.M. Bătinețu-Giurgiu and Claudia Nănuți-Romania

Let $ABC, A_1B_1C_1, A_2B_2C_2$ be triangles with areas F, F_1 and F_2, respectively, lengths sides a, b, c, a_1, b_1, c_1 and a_2, b_2, c_2 respectively, s, s_1 and s_2 semiperimeters, r, r_1 and r_2 inradies, R, R_1 and R_2 circumradies. Let's consider the permutation σ of the set $\{a_1, b_1, c_1\}$ and the permutation ζ of the set $\{a_2, b_2, c_2\}$.

Theorem 1. Let $x, y, z > 0$ and $u, v, w \geq 0$ with $u + v + w > 0$, then:

$$\frac{x+y}{z}\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w+\frac{y+z}{x}\cdot b^u\cdot\left(\sigma(b_1)\right)^v\cdot\left(\zeta(b_2)\right)^w+$$

$$+\frac{z+x}{y}\cdot c^u\cdot\left(\sigma(c_1)\right)^v\cdot\left(\zeta(c_2)\right)^w\geq 2^{1+u+v+w}\cdot\left(\sqrt[4]{3}\right)^{4-u-v-w}\cdot\left(\sqrt{F}\right)^u\left(\sqrt{F_1}\right)^v\left(\sqrt{F_2}\right)^w;(1)$$

Proof. We have:

$$\sum_{cyc}\frac{x+y}{z}\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w\geq 2\sum_{cyc}\frac{\sqrt{xy}}{z}\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w\geq$$

$$\geq 2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\left(\frac{\sqrt{xy}}{z}\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w\right)}=$$

$$=6\cdot\sqrt[3]{\frac{xyz}{xyz}\cdot(abc)^u\cdot(a_1b_1c_2)^v\cdot(a_2b_2c_2)^w}=$$

$$=6\cdot\sqrt{\left(\sqrt[3]{(abc)^2}\right)^u\cdot\left(\sqrt[3]{(a_1b_1c_1)^2}\right)^v\cdot\left(\sqrt[3]{(a_2b_2c_2)^2}\right)^w}=$$

$$=6\cdot\sqrt{\frac{1}{3^u}\cdot\left(3\cdot\sqrt[3]{a^2b^2c^2}\right)^u\cdot\frac{1}{3^v}\cdot\left(3\cdot\sqrt[3]{a_1^2b_1^2c_1^2}\right)^v\cdot\frac{1}{3^w}\cdot\left(3\cdot\sqrt[3]{a_2^2b_2^2c_2^2}\right)^w}\overset{Caliz}{\geq}$$

$$\overset{Carliz}{\geq}6\cdot\sqrt{\frac{1}{3^{u+v+w}}\cdot\left(4\sqrt{3}F\right)^u\cdot\left(4\sqrt{3}F_1\right)^v\cdot\left(4\sqrt{3}F_2\right)^w}=$$

$$=6\cdot\sqrt{\left(\frac{4}{3}\right)^{u+v+w}\cdot\left(\sqrt{3}\right)^{u+v+w}\cdot F^u\cdot F_1^v\cdot F_2^w}=$$

$$=\frac{6\cdot 2^{u+v+w}\cdot\left(\sqrt[4]{3}\right)^{u+v+w}}{\left(\sqrt{3}\right)^{u+v+w}}\cdot\left(\sqrt{F}\right)^u\cdot\left(\sqrt{F_1}\right)^v\cdot\left(\sqrt{F_2}\right)^w=$$

$$=\frac{6\cdot 2^{u+v+w}}{\left(\sqrt[4]{3}\right)^{u+v+w}}\cdot\left(\sqrt{F}\right)^{u}\cdot\left(\sqrt{F_1}\right)^{v}\cdot\left(\sqrt{F_2}\right)^{w}=$$

$$=2^{1+u+v+w}\cdot\left(\sqrt[4]{3}\right)^{4-u-v-w}\cdot\left(\sqrt{F}\right)^{u}\cdot\left(\sqrt{F_1}\right)^{v}\cdot\left(\sqrt{F_2}\right)^{w}=$$

Theorem 2: Let $m,x,y,z\in[1,\infty),3m=x+y+z$ and $u,v,w\geq 0$ such that $u+v+w>0$, then:

$$(x^x+y^x+z^x)\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w+(x^y+y^y+z^y)\cdot b^v\cdot\left(\sigma(b_1)\right)^v\cdot\left(\zeta(b_2)\right)^w+$$
$$+(x^z+y^z+z^z)\cdot c^u\cdot\left(\sigma(c_1)\right)^v\cdot\left(\zeta(c_2)\right)^w\geq$$
$$\geq 2^{u+v+w}\cdot\left(\sqrt[4]{3}\right)^{8-u-v-w}\cdot m^m\cdot\left(\sqrt{F}\right)^{u}\cdot\left(\sqrt{F_1}\right)^{v}\cdot\left(\sqrt{F_2}\right)^{w};(2)$$

Proof. We have:

$$\sum_{cyc}(x^x+y^x+z^x)\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w\overset{Radon}{\geq}$$

$$\geq\sum_{cyc}\frac{(x+y+z)^x}{3^{x-1}}\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w\geq$$

$$\geq 3\cdot\sqrt[3]{\prod_{cyc}\left(\frac{(x+y+z)^x}{3^{x-1}}\cdot a^u\cdot\left(\sigma(a_1)\right)^v\cdot\left(\zeta(a_2)\right)^w\right)}=$$

$$=3\cdot\sqrt[3]{\frac{(x+y+z)^{x+y+z}}{3^{x+y+z-3}}\cdot(abc)^u\cdot(a_1b_1c_1)^v\cdot(a_2b_2c_2)^w}=$$

$$=3\cdot\frac{(3m)^m}{3^{m-1}}\cdot\sqrt{\left(\sqrt[3]{a^2b^2c^2}\right)^u\cdot\left(\sqrt[3]{a_1^2b_1^2c_1^2}\right)^v\cdot\left(\sqrt[3]{a_2^2b_2^2c_2^2}\right)^w}=$$

$$=9\cdot m^m\cdot\frac{1}{\left(\sqrt{3}\right)^u}\cdot\left(\sqrt{3\cdot\sqrt[3]{a^2b^2c^2}}\right)^u\cdot\frac{1}{\left(\sqrt{3}\right)^v}\cdot\left(\sqrt{3\cdot\sqrt[3]{a_1^2b_1^2c_1^2}}\right)^v\cdot\frac{1}{\left(\sqrt{3}\right)^w}$$
$$\cdot\left(\sqrt{3\cdot\sqrt[3]{a_2^2b_2^2c_2^2}}\right)^w\overset{Carliz}{\geq}$$

$$\geq\frac{9\cdot m^m}{\left(\sqrt{3}\right)^{u+v+w}}\cdot\left(4\sqrt{3}F\right)^{\frac{u}{2}}\cdot\left(4\sqrt{3}F_1\right)^{\frac{v}{2}}\cdot\left(4\sqrt{3}F_2\right)^{\frac{w}{2}}=$$

$$=\frac{9\cdot m^m\cdot 2^{u+v+w}}{\left(\sqrt{3}\right)^{u+v+w}}\cdot\left(\sqrt[4]{3}\right)^{u+v+w}\cdot\left(\sqrt{F}\right)^{u}\cdot\left(\sqrt{F_1}\right)^{v}\cdot\left(\sqrt{F_2}\right)^{w}=$$

$$= 2^{u+v+w} \cdot \left(\sqrt[4]{3}\right)^{8-u-v-w} \cdot m^m \cdot \left(\sqrt{F}\right)^u \cdot \left(\sqrt{F_1}\right)^v \cdot \left(\sqrt{F_2}\right)^w$$

If $x = y = z, u = 0, v = w = 1$ and σ, ζ are identic permutations, then the inequality (1) becomes:

$$a_1a_2 + b_1b_2 + c_1c_2 \geq 4\sqrt{3} \cdot \sqrt{F_1F_2}; \quad (T)$$

If in (2) we take $x = y = z = m$, we get:

$$3m^m \cdot \sum_{cyc} a^u \cdot \left(\sigma(a_1)\right)^v \cdot \left(\zeta(a_2)\right)^w \geq 2^{u+v+w} \cdot m^m \cdot \left(\sqrt{F}\right)^u \cdot \left(\sqrt{F_1}\right)^v \cdot \left(\sqrt{F_2}\right)^w$$

For $u = 0, v = w = 1$ and σ, ζ are identic permutations, then:

$$3(a_1a_2 + b_1b_2 + c_1c_2) \geq 4 \cdot \left(\sqrt[4]{3}\right)^6 \cdot \sqrt{F_1F_2} \Leftrightarrow$$

$$a_1a_2 + b_1b_2 + c_1c_2 \geq 4\sqrt{3} \cdot \sqrt{F_1F_2}; \quad (T)$$

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ABOUT AN INEQUALITY BY BOGDAN FUȘTEI-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\sum \frac{\mathbf{h_b} + \mathbf{h_c}}{\mathbf{r_b} + \mathbf{r_c}} \leq \mathbf{3}$$

Proposed by Bogdan Fuștei – Romania

Solution We prove the following lemma: **Lemma.**

2) In ΔABC the following relationship holds:

$$\sum \frac{\mathbf{h_b} + \mathbf{h_c}}{\mathbf{r_b} + \mathbf{r_c}} = \mathbf{2} + \frac{\mathbf{2r}}{\mathbf{R}}$$

Proof. Using the formulas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$$\sum\frac{h_b+h_c}{r_b+r_c}=\sum\frac{\frac{2S}{b}+\frac{2S}{c}}{\frac{S}{s-b}+\frac{S}{s-c}}=\frac{2}{abc}\sum(b+c)(s-b)(s-c)=\frac{2}{4Rrs}\cdot 4rs(R+r)=$$

$$=\frac{2(R+r)}{R}=2+\frac{2r}{R}$$

Let’s get back to the main problem.Using the Lemma it follows:

$M_s=\sum\frac{h_b+h_c}{r_b+r_c}=2+\frac{2r}{R}\le 3=M_d$, which follows from Euler’s inequality $R\ge 2r$.

Remark. Let’s find an inequality having an opposite sense:

3) In ΔABC the following inequality holds:

$$\sum\frac{\mathbf{h_b+h_c}}{\mathbf{r_b+r_c}}\ge\frac{\mathbf{6r}}{\mathbf{R}}$$

Proposed by Marin Chirciu – Romania

Solution Using Lemma it follows: $M_s=\sum\frac{h_b+h_c}{r_b+r_c}=2+\frac{2r}{R}\ge\frac{6r}{R}=M_d$, which follows from Euler’s inequality $R\ge 2r$. Equality holds if and only if the triangle is equilateral.

Remark. We can write the double inequality:

4) In ΔABC the following inequality holds:

$$\frac{\mathbf{6r}}{\mathbf{R}}\le\sum\frac{\mathbf{h_b+h_c}}{\mathbf{r_b+r_c}}\le\mathbf{3}$$

Proposed by Marin Chirciu – Romania

Solution See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\mathbf{3}\le\sum\frac{\mathbf{r_b+r_c}}{\mathbf{h_b+h_c}}\le\frac{\mathbf{3R}}{\mathbf{2r}}$$

Proposed by Marin Chirciu – Romania

Solution We prove the following lemma: **Lemma.**

6) In ΔABC the following relationship holds:

$$\sum \frac{r_b + r_c}{h_b + h_c} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$$

Proof. Using the following lemmas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$$\sum \frac{r_b + r_c}{h_b + h_c} = \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\frac{2S}{b} + \frac{2S}{c}} = \frac{abc}{2} \sum \frac{1}{(b+c)(s-b)(s-c)} =$$

$= \frac{4Rrs}{2} \cdot \frac{s^2+5r^2+8Rr}{2r^2 s(s^2+r^2+2Rr)} = \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr}$, which follows from the identities in triangle:

$$\sum \frac{1}{(b+c)(s-b)(s-c)} = \frac{s^2+5r^2+8Rr}{r^2 s(s^2+r^2+2Rr)} \text{ and}$$

$$\sum (a+b)(a+c)(s-a) = s(s^2 + 5r^2 + 8Rr)$$

Let's get back to the main problem.RHS inequality: Using the Lemma it follows:

$M_s = \sum \frac{r_b+r_c}{h_b+h_c} = \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \overset{(1)}{\leq} \frac{R}{r} \cdot \frac{3}{2} = \frac{3R}{2r} = M_a$ where (1) $\Leftrightarrow \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \leq \frac{3}{2} \Leftrightarrow$

$\Leftrightarrow s^2 \geq 10Rr + 7r^2$, which follows Gerretsen's inequality $s^2 \geq 16Rr - 5r^2$.

It remains to prove that $16Rr - 5r^2 \geq 10Rr + 7r^2 \Leftrightarrow R \geq 2r$ (Euler's inequality).

Equality holds if and only if the triangle is equilateral.

LHS inequality: Using the Lemma it follows:

$M_s = \sum \frac{r_b+r_c}{h_b+h_c} = \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \overset{(2)}{\geq} 3 = M_d$, where (1) $\Leftrightarrow \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \geq 3 \Leftrightarrow$

$$\Leftrightarrow s^2(R - 3r) + r(8R^2 - Rr - 3r^2) \geq 0$$

We distinguish the following cases:

Case 1) If $(R - 3r) \geq 0$ the inequality is obvious.

Case 2) If $(R-3r)<0$ the inequality rewrites itself: $r(8R^2-Rr-3r^2)\geq s^2(3r-R)$, which follows from Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(8R^2-Rr-3r^2)\geq(4R^2+4Rr+3r^2)(3r-R)\Leftrightarrow 2R^3-9Rr^2-6r^3\geq 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R^2+4Rr+3r^2)\geq 0$, obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. Between the sums $\sum\frac{r_b+r_c}{h_b+h_c}$ and $\sum\frac{h_b+h_c}{r_b+r_c}$ the following relationship holds:

7) In ΔABC the following relationship holds:

$$\sum\frac{\mathbf{r_b+r_c}}{\mathbf{h_b+h_c}}\geq\sum\frac{\mathbf{h_b+h_c}}{\mathbf{r_b+r_c}}$$

Proposed by Marin Chirciu – Romania

Solution Using the Lemmas $\sum\frac{r_b+r_c}{h_b+h_c}=\frac{R}{r}\cdot\frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr}$ and $\sum\frac{h_b+h_c}{r_b+r_c}=\frac{2(R+r)}{r}$ the inequality can be written:

$$\frac{R(s^2+5r^2+8Rr)}{r(s^2+r^2+2Rr)}\geq\frac{2(R+r)}{R}\Leftrightarrow s^2(R^2-2Rr-2r^2)+r(8R^3+R^2r-6Rr^2-2r^3)\geq 0.$$

We distinguish the following cases:

Case 1) If $(R^2-2Rr-2r^2)\geq 0$ the inequality is obvious.

Case 2) If $(R^2-2Rr-2r^2)<0$ the inequality rewrites itself:

$r(8R^3+R^2r-6Rr^2-2r^3)\geq s^2(2r^2+2Rr-R^2)$, which follows from Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$. It remains to prove that:

$$r(8R^3+R^2r-6Rr^2-2r^3)\geq(4R^2+4Rr+3r^2)(2r^2+2Rr-R^2)\Leftrightarrow$$

$\Leftrightarrow R^4+R^3r-3R^2r^2-5Rr^3-2r^4\geq 0\Leftrightarrow(R-2r)(R+r)^3\geq 0$ obviously from Euler's inequality $R\geq 2r$. Equality holds if and only if the triangle is equilateral.

Remark. We can write the sequence of inequalities:

8) In ΔABC the following relationship holds:

$$\frac{3R}{2r} \geq \sum \frac{r_b + r_c}{h_b + h_c} \geq \sum \frac{h_b + h_c}{r_b + r_c} \geq \frac{6r}{R}$$

Proposed by Marin Chirciu – Romania

Solution See inequalities 4), 5) and 7). Equality holds if and only if the triangle is equilateral.

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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU -XX

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum_{cyc} (b+c) \tan \frac{A}{2} \geq \frac{54R}{2\left(\frac{R}{r}\right)^2 + 1}$$

Proposed by Marian Ursărescu-Romania

Solution. **Lemma. In ΔABC the following relationship holds:**

$$\sum_{cyc} (b+c) \tan \frac{A}{2} = 4(R+r)$$

Proof. Using $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ we get:

$$\sum_{cyc} (b+c) \tan \frac{A}{2} = \sum_{cyc} (b+c) \sqrt{\frac{(s-b)(s-c)}{s(s-c)}} = \sum_{cyc} (b+c) \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)} =$$

$$= \frac{F}{s} \sum_{cyc} \frac{b+c}{s-a} = r \cdot 4\left(1 + \frac{R}{r}\right) = 4(R+r), \text{which follows from:}$$

$$\sum_{cyc} \frac{b+c}{s-a} = 4\left(1 + \frac{R}{r}\right)$$

Let’s get back to the main problem. Using Lemma, inequality becomes as:

$$4(R+r)\geq\frac{54R}{2\left(\frac{R}{r}\right)^2+1}\Leftrightarrow 2(R+r)\left[2\left(\frac{R}{r}\right)^2+1\right]\geq 27R\Leftrightarrow$$

$$2(R+r)\cdot\frac{2R^2+r^2}{r^2}\geq 27R\Leftrightarrow 2(R+r)(2R^2+r^2)\geq 27Rr^2\Leftrightarrow$$

$$2(2R^3+2R^2r+Rr^2+r^3)\geq 27Rr^2\Leftrightarrow 4R^3+4R^2r+2Rr^2+2r^3\geq 27Rr^2\Leftrightarrow$$

$4R^3+4R^2r-25Rr^2+2r^3\geq 0\Leftrightarrow(R-2r)(4R^2+12Rr-r^2)\geq 0$, which is true from $R\geq 2r\ (Euler)$.

2) In ΔABC the following relationship holds:

$$\mathbf{12r\leq\sum_{cyc}(b+c)\tan\frac{A}{2}\leq 6R}$$

Marin Chirciu

Solution. **Lemma. In ΔABC the following relationship holds:**

$$\mathbf{\sum_{cyc}(b+c)\tan\frac{A}{2}=4(R+r)}$$

Proof. Using $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ we get:

$$\sum_{cyc}(b+c)\tan\frac{A}{2}=\sum_{cyc}(b+c)\sqrt{\frac{(s-b)(s-c)}{s(s-c)}}=\sum_{cyc}(b+c)\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)}=$$

$$=\frac{F}{s}\sum_{cyc}\frac{b+c}{s-a}=r\cdot 4\left(1+\frac{R}{r}\right)=4(R+r),\text{which follows from:}$$

$$\sum_{cyc}\frac{b+c}{s-a}=4\left(1+\frac{R}{r}\right)$$

Using Lemma inequality becomes as: $12r\leq 4(R+r)\leq 6R$, which follows from

$R\geq 2r\ (Euler)$. Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\mathbf{36r\leq\sum_{cyc}(b+c)\cot\frac{A}{2}\leq\frac{9R^2}{r}}$$

Marin Chirciu

Solution. Lemma. 4) In ΔABC the following relationship holds:

$$\sum_{cyc}(b+c)\cot\frac{A}{2}=\frac{2(s^2-r^2-4Rr)}{r}$$

Proof. Using identity $\cot\frac{A}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ we get:

$$\sum_{cyc}(b+c)\cot\frac{A}{2}=\sum_{cyc}(b+c)\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\sum_{cyc}(b+c)\frac{\sqrt{s(s-a)(s-b)(s-c)}}{(s-b)(s-c)}=$$

$$=F\cdot\sum_{cyc}\frac{b+c}{(s-b)(s-c)}=sr\cdot\frac{2(s^2-r^2-4Rr)}{sr^2}=\frac{2(s^2-r^2-4Rr)}{r}$$, which follows from

$$\sum_{cyc}\frac{b+c}{(s-b)(s-c)}=\frac{2(s^2-r^2-4Rr)}{sr^2}$$

Let's get back to the main problem. Using **Lemma**, inequality can be written as:

$$36r\le\frac{2(s^2-r^2-4Rr)}{r}\le\frac{9R^2}{r}$$, which is true from

$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen)\ \text{ and }\ R\ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\mathbf{3\sum_{cyc}(b+c)\tan\frac{A}{2}\le\sum_{cyc}(b+c)\cot\frac{A}{2}}$$

Marin Chirciu

Solution. Using these up **Lemmas**, we have that:

$$\sum_{cyc}(b+c)\tan\frac{A}{2}=4(R+r)\text{ and }\sum_{cyc}(b+c)\cot\frac{A}{2}=\frac{2(s^2-r^2-4Rr)}{r}$$

inequality becomes as: $3\cdot 4(R+r)\le\frac{2(s^2-r^2-4Rr)}{r}\Leftrightarrow s^2\ge 10Rr+7r^2$, which follows from $s^2\ge 16Rr-5r^2\ (Gerretsen)$ and $R\ge 2r\ (Euler)$.

Equality holds if and only if triangle is equilateral.

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THE EQUIVALENCE OF THE INEQUALITIES IONESCU – WEITZENBÖCK AND GORDON

By Neculai Stanciu- Romania

- **The inequality *Ionescu* (1897) – *Weitzenböck* (1919)**:

$$a^2+b^2+c^2\geq 4F\sqrt{3};(I-W)$$

- **The inequality of *Gordon* (1966):**

$$ab+bc+ca\geq 4F\sqrt{3};(G)$$

Theorem. If ABC is a triangle with usual notations, then

$$a^2+b^2+c^2\geq 4F\sqrt{3}\Leftrightarrow ab+bc+ca\geq 4F\sqrt{3}$$

Proof. (G)⇒(I-W) it is obvious. Next we will prove tha (I-W) ⇒(G).

$$16F^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$$

$$2ab=\sqrt{(a^2+b^2-c^2)^2+16F^2}$$

$$2bc=\sqrt{(b^2+c^2-a^2)^2+16F^2}$$

$$2ca=\sqrt{(c^2+a^2-b^2)^2+16F^2}$$

Using *Minkovski's* inequality and (I-W) we obtain:

$$2(ab+bc+ca)=\sqrt{(a^2+b^2-c^2)^2+16F^2}+\sqrt{(b^2+c^2-a^2)^2+16F^2}+$$

$$+\sqrt{(c^2+a^2-b^2)^2+16F^2}\overset{(M)}{\geq}$$

$$\geq\sqrt{(a^2+b^2-c^2+b^2+c^2-a^2+c^2+a^2-b^2)^2+(4F+4F+4F)^2}=$$

$$=\sqrt{(a^2+b^2+c^2)^2+144F^2}\overset{I-W}{\geq}$$

$$\geq\sqrt{48F^2+144F^2}=\sqrt{192F^2}=8F\sqrt{3}\text{ i.e. }ab+bc+ca\geq 4F\sqrt{3}$$

Therefore,: $a^2+b^2+c^2\geq 4F\sqrt{3};(I-W)\Rightarrow ab+bc+ca\geq 4F\sqrt{3}$

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A SIMPLE PROOF FOR REDHEFFER-WILLIAMS' INEQUALITY

By Daniel Sitaru-Romania

Abstract: ***In this paper is presented a simple proof for the famous Redheffer-Williams' inequality.***

Lemma: If $x \in [0,1)$ then:

$$(1+x^2)\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right) \ge 1+\frac{x^2}{n}; n \ge 2$$

Proof.

For $n = 2$: $(1+x^2)\left(1-\frac{x^2}{4}\right) \ge 1+\frac{x^2}{2}$

$$1-\frac{x^2}{4}+x^2-\frac{x^4}{4} \ge 1+\frac{x^2}{2}, \qquad \frac{x^2}{2}-\frac{x^2}{4}+x^2-\frac{x^4}{4} \ge 0$$

$2x^2 - x^2 + 4x^2 - x^4 \ge 0$, $x^2(5-x^2) \ge 0$, true for all $x \in [0,1)$.

Equality holds for $x = 0$.

By induction, suppose that its true

$$P(n): (1+x^2)\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right) \ge 1+\frac{x^2}{n}$$

$$P(n+1): (1+x^2)\prod_{k=2}^{n+1}\left(1-\frac{x^2}{k^2}\right) \ge 1+\frac{x^2}{n+1} \text{ to prove.}$$

$$(1+x^2)\prod_{k=2}^{n+1}\left(1-\frac{x^2}{k^2}\right) = (1+x^2)\left(1-\frac{x^2}{n+1}\right)\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right) \overset{P(n)}{\ge}$$

$$\ge \left(1+\frac{x^2}{n}\right)\left(1-\frac{x^2}{(n+1)^2}\right)$$

Remains to prove: $\left(1+\frac{x^2}{n}\right)\left(1-\frac{x^2}{(n+1)^2}\right) \ge 1+\frac{x^2}{n+1}$

$$1-\frac{x^2}{(n+1)^2}+\frac{x^2}{n}-\frac{x^4}{n(n+1)^2} \ge 1+\frac{x^2}{n+1}$$

$$\frac{x^2}{n}-\frac{x^2}{(n+1)^2}-\frac{x^2}{n+1}-\frac{x^4}{n(n+1)^2}\geq 0$$

$$x^2\left(\frac{1}{n}-\frac{1}{(n+1)^2}-\frac{1}{n+1}-\frac{x^2}{n(n+1)^2}\right)\geq 0$$

$$\frac{x^2}{n(n+1)^2}\left((n+1)^2-n-n(n+1)-x^2\right)\geq 0$$

$$\frac{x^2}{n(n+1)^2}\left(n^2+2n+1-n-n^2-n-x^2\right)\geq 0$$

$$\frac{x^2}{n(n+1)^2}(1-x^2)\geq 0 \text{ true } \forall x\in[0,1)$$

Equality holds for $x=0$.

Theorem (REDHEFFER-WILLIAMS' INEQUALITY)

If $x>0$ then:

$$\boldsymbol{\frac{\sin x}{x}\geq\frac{\pi^2-x^2}{\pi^2+x^2}}$$

Proof.

If $x\in(0,1)$ by Lemma:

$$(1+x^2)\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right)\geq 1+\frac{x^2}{n}\geq 1\geq 1-x^2; n\geq 2$$

$$\prod_{k=2}^{n}\left(1-\frac{x^2}{k^2}\right)\geq\frac{1-x^2}{1+x^2},\qquad \prod_{k=2}^{\infty}\left(1-\frac{x^2}{k^2}\right)\geq\frac{1-x^2}{1+x^2}$$

$$\frac{\sin(\pi x)}{\pi x}\geq\frac{1-x^2}{1+x^2}\Leftrightarrow\frac{\sin x}{x}\geq\frac{1-\left(\frac{x}{\pi}\right)^2}{1+\left(\frac{x}{\pi}\right)^2}$$

$$\frac{\sin x}{x}\geq\frac{\pi^2-x^2}{\pi^2+x^2}$$

If $x>1$,

$$\frac{1-x^2}{1+x^2}-\frac{\sin(\pi x)}{\pi x}=\frac{1-x^2}{1+x^2}+\frac{\sin(\pi x-\pi)}{\pi x}=\frac{1-x^2}{1+x^2}+\frac{\pi(x-1)}{\pi x}\cdot\frac{\sin\pi(x-1)}{\pi(x-1)}\leq$$

$$\le \frac{1-x^2}{1+x^2}+\frac{x-1}{x}=\frac{x-x^3+x-1+x^3-x^2}{x(1+x^2)}=$$

$$=\frac{-x^2+2x-1}{x(1+x^2)}=\frac{-(x-1)^2}{x(1+x^2)}<0$$

$$\frac{1-x^2}{1+x^2}-\frac{\sin(\pi x)}{\pi x}\le 0 \Leftrightarrow \frac{\sin(\pi x)}{\pi x}\ge \frac{1-x^2}{1+x^2}\Leftrightarrow$$

$$\frac{\sin x}{x}\ge \frac{1-\left(\frac{x}{\pi}\right)^2}{1+\left(\frac{x}{\pi}\right)^2}\Leftrightarrow \frac{\sin x}{x}\ge \frac{\pi^2-x^2}{\pi^2+x^2}$$

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ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro.

MADHAVA-LEIBNIZ FORMULA

By Amrit Awasthi-India

The Leibniz formula for π, named after Gottfried Leibniz, states that

$$\mathbf{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4};\ (1)}$$

an alternating series. It is also called the Leibniz-Madhava series as it is a special case of a more general series expansion for the inverse tangent function, first discovered by the Indian mathematician Madhava of Sangamagrama in the 14th century, the specific case first published by Leibniz around 1676.

Proof. We start with the integral

$$I_{4n}=\int_0^{\frac{\pi}{4}}\tan^{4n}x\,dx$$

Now in order to evaluate we establish the reduction formula as follows:

$$I_{4n}=\int_0^{\frac{\pi}{4}}\tan^{4n-2}x\tan^2x\,dx=\int_0^{\frac{\pi}{4}}\tan^{4n-2}x\,(\sec^2x-1)\,dx=$$

$$=\int_0^{\frac{\pi}{4}}\tan^{4n-2}x\sec^2x\,dx-\int_0^{\frac{\pi}{4}}\tan^{4n-2}x\,dx$$

$$I_{4n}=\int_0^1 t^{4n-2}\,dt-I_{4n-2}$$

Where for first integral on right hand side we made the substitution $\tan x=t$, where as the second integral is nothing but I_{4n-2}. Hence, we get:

$$I_{4n}=\frac{1}{4n-1}-I_{4n-2};\text{(i)}$$

Similarly, we have:

$$I_{4n-2}=\frac{1}{4n-3}-I_{4n-4}$$

Putting back in (i) yields

$$I_{4n}=\frac{1}{4n-1}-\frac{1}{4n-3}+I_{4n-4}$$

Proceeding in a similar way, gives

$$I_{4n}=\frac{1}{4n-1}-\frac{1}{4n-3}+\frac{1}{4n-5}-\cdots+\cdots+\frac{1}{3}-\int_0^{\frac{\pi}{4}}\tan^2 x\,dx=$$

$$=\frac{1}{4n-1}-\frac{1}{4n-3}+\frac{1}{4n-5}-\cdots+\cdots+\frac{1}{3}-(\tan x-x)\Big|_0^{\frac{\pi}{4}}$$

$$I_{4n}=\frac{1}{4n-1}-\frac{1}{4n-3}+\frac{1}{4n-5}-\cdots+\cdots+\frac{1}{3}-1+\frac{\pi}{4};\text{(ii)}$$

Now, it's easy to follow that $\forall 0\le x\le\frac{\pi}{4}, 0\le\tan x<1$. Henceforth, we must have:

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{4}}\tan^{4n}x\,dx=0$$

Using (ii) gives $\lim_{n\to\infty}\left(\frac{1}{4n-1}-\frac{1}{4n-3}+\frac{1}{4n-5}-\cdots+\cdots+\frac{1}{3}-1+\frac{\pi}{4}\right)=0$

Transposing the terms gives:

$$\lim_{n\to\infty}\left(\frac{1}{4n-1}-\frac{1}{4n-3}+\frac{1}{4n-5}-\cdots+\cdots+\frac{1}{3}-1\right)=\frac{\pi}{4}$$

Therefore, we get:

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4};(2)$$

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-WWW.SSMRMH.RO

ABOUT PROBLEM 5638-SSMA-April 2021

By Marin Chirciu and Daniel Văcaru-Romania

1) If $a,b,c\geq -1$ such that $a+b+c=3$, then:

$$\left(\frac{a+1}{a+3}\right)^2+\left(\frac{b+1}{b+3}\right)^2+\left(\frac{c+1}{c+3}\right)^2\leq\frac{3}{4}$$

Proposed by Daniel Sitaru-Romania

Solution. Because $a,b,c\geq -1$, denoting $a+1=x, b+1=y, c+1=z$, the proposed problem can be written as

2) If $x,y,z\geq 0$ such that $x+y+z=6$, then:

$$\left(\frac{x}{x+2}\right)^2+\left(\frac{y}{y+2}\right)^2+\left(\frac{z}{z+2}\right)^2\leq\frac{3}{4}$$

Proof. We have

$$LHS=\sum_{cyc}\left(\frac{x}{x+2}\right)^2\overset{(1)}{\leq}\sum_{cyc}\frac{x}{8}=\frac{1}{8}\sum_{cyc}x=\frac{1}{8}\cdot 6=\frac{3}{4}=RHS$$

$(1)\Leftrightarrow\left(\frac{x}{x+2}\right)^2\leq\frac{x}{8}\Leftrightarrow x(x-2)^2\geq 0$ with quality for $x\in\{0,1\}$.

Equality holds for $x=y=z=2$.

Finally, we deduce that the problem is true. Equality holds for $a=b=c=1$.

Remark. The problem can be developed.

3) If $a,b,c\geq -1$ such that $a+b+c=3$ and $0\leq\lambda\leq 3$, then:

$$\left(\frac{a+1}{a+\lambda}\right)^2+\left(\frac{b+1}{b+\lambda}\right)^2+\left(\frac{c+1}{c+\lambda}\right)^2\leq\frac{12}{(\lambda+1)^2}$$

Marin Chirciu and Daniel Văcaru-Romania

Solution. Because $a,b,c\geq -1$, denoting $a+1=x, b+1=y, c+1=z$, the proposed problem can be written as

4) If $x,y,z\geq 0$ such that $x+y+z=6$, then:

$$\left(\frac{x}{x+\lambda-1}\right)^2+\left(\frac{y}{y+\lambda-1}\right)^2+\left(\frac{z}{z+\lambda-1}\right)^2\leq\frac{12}{(\lambda+1)^2}$$

Proof. We have:

$$LHS=\sum_{cyc}\left(\frac{x}{x+\lambda-1}\right)^2\overset{(1)}{\leq}\sum_{cyc}\frac{2x}{(\lambda+1)^2}=\frac{2}{(\lambda+1)^2}\sum_{cyc}x=\frac{2}{(\lambda+1)^2}\cdot 6=\frac{12}{(\lambda+1)^2},$$

$$(1)\Leftrightarrow\left(\frac{x}{x+\lambda-1}\right)^2\leq\frac{2x}{(\lambda+1)^2}\Leftrightarrow 2x^2-(\lambda^2-2\lambda+5)x+2(\lambda-1)^2\geq 0\Leftrightarrow$$

$$2\left[x-\frac{(\lambda-1)^2}{2}\right](x-2)\geq 0,\text{which follows from}$$

$$x-\frac{(\lambda-1)^2}{2}\geq x-2\Leftrightarrow-\frac{(\lambda-1)^2}{2}\geq -2\Leftrightarrow$$

$$2\geq\frac{(\lambda-1)^2}{2}\Leftrightarrow(\lambda-1)^2\leq 4\Leftrightarrow\lambda\leq 3.$$

For $\lambda\leq 3$ from $x-\frac{(\lambda-1)^2}{2}\geq x-2$ we have $\left[x-\frac{(\lambda-1)^2}{2}\right](x-2)\geq(x-2)^2\geq 0$. Equality for $x=2$. Equality holds if and only if $x=y=z=2$.

Finally, we deduce that the proposed inequality is true, with equality for $a=b=c=1$.

Note: For $\lambda=2$ we get the Proposed Problem 5638-SSMA-April-2021 by Daniel Sitaru-Romania.

5) If $a,b,c\geq -1$ such that $a+b+c=3$, then:

$$\left(\frac{a+1}{a+3}\right)^2+\left(\frac{b+1}{b+3}\right)^2+\left(\frac{c+1}{c+3}\right)^2\leq\frac{3}{4}$$

Proposed by Daniel Sitaru-Romania

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GERGONNE'S POINT AND OUTSTANDING DISTANCES

By Jose Ferreira Queiroz-Olinda-Brazil

Abstract: *In this article is proved a metric relationship for the distance between Gergonne's point in a fixed triangle and any point in the plane of triangle.*

Keywords: Gergonne's Point, Stewart's Theorem, Gergonne's cevians.

1. Introduction.

We call Gergonne point of the triangle to the meeting point of the lines containing the vertex of a triangle and the point of tangency with the inscribed circle. The Gergonne's Point was discovered by Joseph Diaz Gergonne (1771-1859) French mathematician. The identity described here, gives us the distance between the Gergonne's point of a triangle and any point on the plane that contains the triangle.

2. Notations.

Let ABC be an acute triangle. We denote its side-lengths by $BC = a, AC = b, AB = c$, it's semi perimeter by $s = \frac{1}{2}(a + b + c)$, it's area by F, it's circumradius by R and inradius by r. Its classical centres are the Centroid G, the Incenter I, the Circumcentre O, the Orthocentre H and Nagel's point N_a.

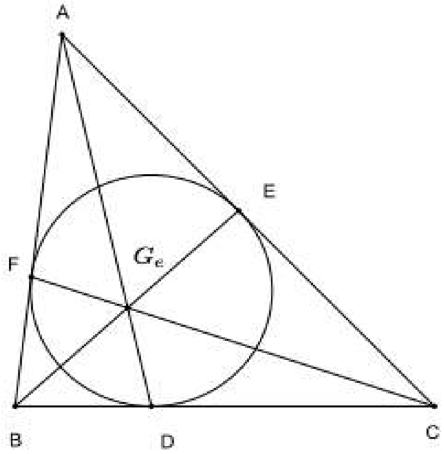

Figure 1.

3. Propositions.

Proposition 3.1

If AD, BE, CF are the Gergonne's cevians, then $AE = AF = s - a, BF = BD = s - b, CE = CD = s - c$.

Proof. Denoting $AF = AE = x, BF = BD = y$ and $CE = CD = z$. We have $y + z = a$,

$z + x = b$ and $x + y = c$. Solving the system we find $x = s - a, y = s - b$ and $z = s - c$.

Proposition 3.2

If AD, BE and CF are the Gergonne's cevians of the triangle ABC then they are concurrent and the point of concurrence is the Gergonne's point G_e of the triangle ABC.

Proof. By Proposition 3.1, we have: $AF = AE = s - a, BF = BD = s - b$ and $CE = CD = s - c$. Using Ceva's theorem and replacing AF with AE, BD with F and C with CE, we soon find that

$$\frac{AE}{CE}\cdot\frac{CD}{BD}\cdot\frac{BF}{AF}=1$$

Then

$$\frac{AE}{CE}\cdot\frac{CE}{BF}\cdot\frac{BF}{AE}=1$$

Hence the three cevians that connect the vertices to the point of tangency of the circumference intersect at a point called the point of Gergonne G_e.

Proposition 3.3

If AD, BE and CF are the Gergonne's cevians of the triangle ABC then the length of each cevian is given by $AD=\sqrt{(s-a)^2+\frac{4F^2}{as}}$, $BE=\sqrt{(s-b)^2+\frac{4F^2}{bs}}$ and $CF=\sqrt{(s-c)^2+\frac{4F^2}{cs}}$

Proof. We apply Stewart's theorem to triangle ABC in which AD is a cevian. See Figure 4. We get:

$$BC\cdot AD^2=BD\cdot AC^2+CD\cdot AB^2=BC\cdot BD\cdot CD$$

$$a\cdot AD^2=c^2(s-c)+b^2(s-b)-a(s-b)(s-c)$$

$$AD^2=\frac{c^2(s-c)}{a}+\frac{b^2(s-b)}{a}-(s-b)(s-c);(1)$$

After simplifying a few steps, we obtain $AD=\sqrt{(s-a)^2+\frac{4F^2}{as}}$. Similarly, we can prove that:

$BE=\sqrt{(s-b)^2+\frac{4F^2}{bs}}$ and $CF=\sqrt{(s-c)^2+\frac{4F^2}{cs}}$.

Proposition 3.4

The Gergonne's Point G_e of the triangle ABC divides each cevian in the ratio given by

$$\frac{AG_e}{G_eD}=\frac{a(s-a)}{(s-b)(s-c)},\frac{BG_e}{G_eE}=\frac{b(s-b)}{(s-a)(s-c)}\ and\ \frac{CG_e}{G_eF}=\frac{c(s-c)}{(s-a)(s-b)}.$$

Proof. We have by Proposition 3.1, $BD=s-b$ and $CD=s-c$.

Now in the triangle ABD, the line CF as transversal. Applying Menelaus' Theorem we have

$$\frac{AF}{FB}\cdot\frac{BC}{CD}\cdot\frac{DG_e}{G_eA}=1$$

$$\frac{s-a}{s-b}\cdot\frac{a}{s-c}\cdot\frac{G_eD}{AG_e}=1$$

$$\frac{AG_e}{G_eD}=\frac{a(s-a)}{(s-b)(s-c)};(2)$$

Similarly we can prove that $\frac{BG_e}{G_eE}=\frac{b(s-b)}{(s-a)(s-c)}$ and $\frac{CG_e}{G_eF}=\frac{c(s-c)}{(s-a)(s-b)}$.

From expression (2) and the fact that $AD=AG_e+G_eD$, we will have

$$\frac{AG_e}{AD}=\frac{a(s-a)}{a(s-a)+(s-b)(s-c)}=\frac{a(s-a)}{bc-(s-a)^2};(3)$$

$$\frac{G_eD}{AD}=\frac{(s-b)(s-c)}{a(s-a)+(s-b)(s-c)}=\frac{(s-b)(s-c)}{bc-(s-a)^2};(4)$$

Proposition 3.5

Let a, b and c the sides of an triangle ABC, and s, r, R and F are, respectively, its semi-perimeter, inradius, circumradius and are of that triangle, then

$$\mathbf{1.}\ \boldsymbol{ab+bc+ca=s^2+r^2+4Rr}$$

$$\mathbf{2.}\ \boldsymbol{a^2+b^2+c^2=2s^2-2r^2-8Rr}$$

$$\mathbf{3.}\ \boldsymbol{a^3+b^3+c^3=2s^3-6r^2s-12Rrs}$$

Proof. Using Heron's formula for the area of the triangle and the fact that $abc=4RF=4Rrs$, we have $F^2=s(s-a)(s-b)(s-c)$.

$$s^2r^2=s^2(-s^2+ab+bc+ca-4Rr)$$

Hence, $a^2+b^2+c^2=2s^2-2r^2-8Rr$.

Now, we know that

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc$$

$$a^3+b^3+c^3=2s[(2s^2-2r^2-8Rr)-(s^2+r^2+4Rr)]+12Rrs$$

So, $a^3+b^3+c^3=2s^3-6r^2s-12Rrs$.

Theorem 3.6

Let M be any point in the plane of an acute triangle ABC which Gergonne's point G_e. Then

$$\boldsymbol{MG_e^2=\frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot MA^2+(s-a)(s-c)\cdot MB^2+(s-a)(s-b)\cdot MC^2]-}$$

$$\boldsymbol{-\frac{4rs^2(R+r)}{(4R+r)^2}}$$

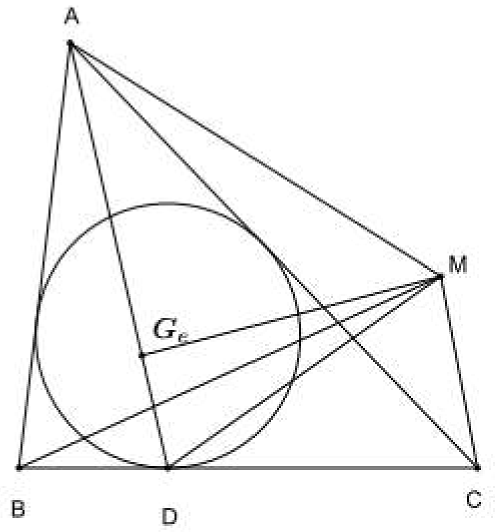

Figure 2: Gergonne's Point.

Proof. We apply Stewart's theorem in to triangle MBC in which MD is a cevian, according to figure 2. We get

$$a\cdot MD^2=BD\cdot MC^2+CD\cdot MB^2-a\cdot BD\cdot DC$$

$$a\cdot MD^2=(s-b)\cdot MC^2+(s-c)\cdot MB^2-a(s-b)(s-c)$$

$$MD^2=\frac{s-b}{a}\cdot MC^2+\frac{s-c}{a}\cdot MB^2-(s-b)(s-c);(5)$$

Now, we apply Steward's theorem in to triangle MAD in which MG_e is a cevian. We get

$$AD\cdot MG_e^2=AG_e\cdot MD^2+G_eD\cdot MA^2-AD\cdot AG_e\cdot G_eD$$

$$MG_e^2=\frac{AG_e}{AD}\cdot MD^2+\frac{G_eD}{AD}\cdot MA^2-AG_e\cdot G_eD$$

Using the expressions (3),(4) and (5), we obtain

$$MG_e^2=\frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot MA^2+(s-a)(s-c)\cdot MB^2+(s-a)(s-b)\cdot MC^2\,]-$$

$$=\frac{(s-a)(s-b)(s-c)}{[bc-(s-a)^2]^2}\cdot[4(s-a)(s-b)(s-c)+abc]$$

Now, observe that

$$a(s-a)+(s-b)(s-c)=bc-(s-a)^2=\frac{1}{4}(-a^2-b^2-c^2+2ab+2bc+2ca)=$$

$$=\frac{1}{4}(4r^2+16Rr)=r^2+4Rr \text{ and } abc=4RF=4Rrs$$

After simplifying a few steps we obtain,

$$MG_e^2 = \frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot MA^2+(s-a)(s-c)\cdot MB^2+(s-a)(s-b)\cdot MC^2]-$$

$$-\frac{4rs^2(R+r)}{(4R+r)^2}$$

4. Main result

Corollary 4.1

Be G the centroid of the triangle ABC and G_e the Gergonne's point, then

$$\boldsymbol{GG_e^2 = \frac{2}{9}(a^2+b^2+c^2)-\frac{8s^2(r^2+2R^2)}{3(r+4R)^2}}$$

Proof. In Theorem 3.6, replace M by the incenter G. We get

$$GG_e^2 = \frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot GA^2+(s-a)(s-c)\cdot GB^2+(s-a)(s-b)\cdot GC^2]-$$

$$-\frac{4rs^2(R+r)}{(4R+r)^2}$$

We know that

$$GA^2=\frac{1}{9}(2b^2+2c^2-a^2), GB^2=\frac{1}{9}(2a^2+2c^2-b^2) \text{ and } GC^2=\frac{1}{9}(2a^2+2b^2-c^2)$$

Then

$$GG_e^2 = \frac{1}{bc-(s-a)^2} \cdot\left[(s-b)(s-c)\cdot\frac{1}{9}(2b^2+2c^2-a^2)+(s-a)(s-c)\cdot\frac{1}{9}(2a^2+2c^2-b^2) + (s-a)(s-b)\cdot\frac{1}{9}(2a^2+2b^2-c^2)\right]-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$GG_e^2 = \frac{1}{9(bc-(s-a)^2}[(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)](2a^2+2b^2+2c^2)-$$

$$-\frac{1}{3(bc-(s-a)^2)}[a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b)]-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$GG_e^2 = \frac{2a^2+2b^2+2c^2}{9(bc-(s-a)^2)}[bc-(s-a)^2][a^2(-s^2+as+bc)+b^2(-s^2+bs+ac) + c^2(-s^2+cs+ab)]-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$GG_e^2=\frac{2}{9}(a^2+b^2+c^2)-\frac{1}{3(bc-(s-a)^2)}[a^2(-s^2+as+bc)+b^2(-s^2+bs+ac)+c^2(-s^2+cs+ab)]-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$GG_e^2=\frac{2}{9}(a^2+b^2+c^2)-\frac{1}{3r(r+4R)[-s^2(a^2+b^2+c^2)+s(a^3+b^3+c^3)+abc(a+b+c)]}-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$GG_e^2=\frac{2}{9}(a^2+b^2+c^2)-\frac{1}{3r(r+4R)}[-s^2(2s^2-2r^2-8Rr)+s(2s^3-6r^2s-12Rrs)+8Rrs^2]-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$GG_e^2=\frac{2}{9}(a^2+b^2+c^2)-\frac{4s^2(r-R)}{3(r+4R)}-\frac{4rs^2(R+r)}{(4R+r)^2}$$

Therefore,

$$GG_e^2=\frac{2}{9}(a^2+b^2+c^2)-\frac{8s^2(r^2+2R^2)}{3(r+4R)^2}$$

Corollary 4.2

Be I the incenter of the triangle ABC and G_e the Gergonne's point, then

$$\boldsymbol{IG_e^2=r^2-\frac{3r^2s^2}{(r+4R)^2}}$$

Proof. In Theorem 3.6, replace M by the incenter I. We get

$$IG_e^2=\frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot IA^2+(s-a)(s-c)\cdot IB^2+(s-a)(s-b)\cdot IC^2]-$$

$$-\frac{4rs^2(R+r)}{(4R+r)^2}$$

Of the triangle whose vertices are A, I and the point of contact of the incircle with side AB, we obtain

$$AI^2=\frac{(s-a)^2}{\cos^2\frac{A}{2}}$$

Now, we know that

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}\text{ and }\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}\text{ then}$$

$$AI^2=\frac{bc(s-a)}{s}$$

Similarly, we can prove $BI^2=\frac{ac(s-b)}{s}$ and $CI^2=\frac{ab(s-c)}{s}$.

$$IG_e^2=\frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot IA^2+(s-a)(s-c)\cdot IB^2+(s-a)(s-b)\cdot IC^2]$$
$$-$$

$$-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$IG_e^2=\frac{1}{bc-(s-a)^2}\cdot\frac{F^2}{s^2}[bc+ca+ab]-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$IG_e^2=\frac{1}{r(r+4R)}\cdot\frac{s^2r^2}{s^2}[s^2+r^2+4Rr]-\frac{4rs^2(R+r)}{(4R+r)^2}$$

$$IG_e^2=r^2+\frac{rs^2}{r+4R}-\frac{4rs^2(R+r)}{(4R+r)^2}$$

Hence,

$$IG_e^2=r^2-\frac{4rs^2(R+r)}{(4R+r)^2}$$

Corollary 4.3

For any acute triangle ABC: $\boldsymbol{r+4R\geq s\sqrt{3}}$

with equality when the triangle is equilateral.

Proof. This follows from Corollary 4.3. We know that $IG_e^2\geq 0$, then

$$IG_e^2=\frac{r^2(r+4R)^2-3r^2s^2}{(r+4R)^2}=\frac{r^2}{(r+4R)^2}\left[(r+4R)-s\sqrt{3}\right]\left[(r+4R)+s\sqrt{3}\right]$$

$$(r+4R)-s\sqrt{3}\geq 0,\qquad r+4R\geq s\sqrt{3}$$

Hence proved.

Corollary 4.4

Be O the circumcenter of the triangle ABC and G_e the Gergonne's point, then

$$\boldsymbol{OG_e^2 = R^2 - \frac{4rs^2(r+R)}{(r+4R)^2}}$$

Proof. In Theorem 3.6, replace M by the circumcenter O, and consider that $OA = OB = OC = R$, we get

$$OG_e^2 = \frac{R^2}{bc-(s-a)^2}\cdot[(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$OG_e^2 = \frac{R^2}{bc-(s-a)^2}\cdot[2ab+2bc+2ca-a^2-b^2-c^2] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$OG_e^2 = \frac{R^2}{bc-(s-a)^2}\cdot[bc-(s-a)^2] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$OG_e^2 = R^2 - \frac{4rs^2(r+R)}{(r+4R)^2}$$

Corollary 4.5

Be H the orthocenter of the triangle ABC and G_e, the Gergonne's point, then

$$\boldsymbol{HG_e^2 = 4R^2 + \frac{8s^2(Rr-2R^2)}{(r+4R)^2}}$$

Proof. In Theorem 3.6, replace M by the orthocenter H, so

$$HG_e^2 = \frac{R^2}{bc-(s-a)^2}\cdot[(s-b)(s-c)HA^2+(s-c)(s-a)HB^2+(s-a)(s-b)HC^2] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

Using the half angle formulas for the cosine function of an internal angle, we have

$$2\cos^2\frac{A}{2} = 1+\cos A, \qquad \cos A = 2\cos^2\frac{A}{2} - 1$$

$$\cos A = \frac{2s(s-a)}{bc} - 1$$

Similarly, $\cos B = \frac{2s(s-b)}{ac} - 1$ and $\cos C = \frac{2s(s-c)}{ab} - 1$, then

$$HG_e^2 = \frac{R^2}{bc-(s-a)^2}\cdot\left[(s-b)(s-c)\left(\frac{2s(s-a)}{bc}-1\right)^2 + (s-c)(s-a)\left(\frac{2s(s-b)}{ac}-1\right)^2 + (s-a)(s-b)\left(\frac{2s(s-c)}{ab}-1\right)^2\right] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

After simplifying a few steps we obtain

$$HG_e^2 = 4R^2 + \frac{16sF^2R^2}{r(r+4R)}\left(\frac{s-a}{b^2c^2} + \frac{s-b}{a^2c^2} + \frac{s-c}{a^2b^2}\right) - \frac{16F^2R^2}{r(r+4R)}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$HG_e^2 = 4R^2 + \frac{16sF^2R^2}{r(r+4R)} \cdot \frac{a^2(s-a) + b^2(s-b) + c^2(s-c)}{a^2b^2c^2} - \frac{16F^2R^2}{r(r+R)} \cdot \frac{a+b+c}{abc} - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$HG_e^2 = 4R^2 + \frac{4s}{r+4R} \cdot [s(a^2+b^2+c^2) - (a^3+b^3+c^3)] - \frac{8Rs^2}{r+4R} - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$HG_e^2 = 4R^2 + \frac{4s}{r+4R} \cdot [(2s^3 - 2r^2s - 8Rrs) - (2s^3 - 6r^2s - 12Rrs)] - \frac{8Rs^2}{r+4R} - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$HG_e^2 = 4R^2 + \frac{4s^2(r+R)}{r+4R} - \frac{8Rs^2}{r+4R} - \frac{4rs^2(r+R)}{(r+4R)^2}$$

Further simplification gives

$$HG_e^2 = 4R^2 + \frac{8s^2(Rr - 2R^2)}{(r+4R)^2}$$

Hence proved.

Corollary 4.6

If N_a is Nagel's point of the triangle ABC and G_e is the Gergonne's point, then

$$\boldsymbol{N_aG_e^2 = -16Rr + \frac{16s^2(Rr+R^2)}{(r+4R)^2}}$$

Proof. We know that

$$N_aA = \frac{a}{s}\sqrt{s^2 - \frac{4F^2}{a(s-a)}} \Rightarrow N_aA^2 = a^2 - \frac{4a}{s}(s-b)(s-c)$$

Similarly $N_aB^2 = b^2 - \frac{4b}{s}(s-a)(s-c)$ and $N_aC^2 = c^2 - \frac{4c}{s}(s-a)(s-b)$.

Using the Theorem 3.6, replace M by Nagel's point N_a, then

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot[(s-b)(s-c)\cdot N_aA^2+(s-a)(s-c)\cdot N_aB^2+(s-a)(s-b)\cdot N_aC^2] - \frac{4rs^2(R+r)}{(4R+r)^2}$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}\cdot\left[(s-b)(s-c)\cdot\left(a^2-\frac{4a}{s}(s-b)(s-c)\right)+(s-a)(s-c)\cdot\left(b^2-\frac{4b}{s}(s-a)(s-c)\right)+(s-a)(s-b)\cdot\left(c^2-\frac{4c}{s}(s-a)(s-b)\right)\right] - \frac{4rs^2(R+r)}{(4R+r)^2}$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}[a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b)] - $$

$$-\frac{4}{s}[a(s-b)^2(s-c)^2+b(s-a)^2(s-c)^2+c(s-a)^2(s-b)^2]-\frac{4rs^2(r+R)}{(r+4R)^2}$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}[a^2(-s^2+as+bc)+b^2(-s^2+bs+ac)+c^2(-s^2+cs+ab)] - $$

$$-\frac{4}{s(bc-(s-a)^2)}[a(-s^2+as+bc)^2+b(-s^2+bs+ac)^2+c(-s^2+cs+ab)^2] - $$

$$-\frac{4rs^2(r+R)}{(r+4R)^2}$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}[-s^2(a^2+b^2+c^2)+s(a^3+b^3+c^3)+abc(a+b+c)] - $$

$$-\frac{4}{s(bc-(s-a)^2)}[a(s^4+a^2s^2+b^2c^2-2as^3-2bcs^2+2abcs) + b(s^4+b^2s^2+a^2c^2-2bs^3-2acs^2+2abcs) +$$

$$+c(s^4+c^2s^2+a^2b^2-2cs^3-2abs^2+2abcs)]-\frac{4rs^2(r+R)}{(r+4R)^2}$$

$$N_aG_e^2 = \frac{1}{bc-(s-a)^2}[-s^2(2s^2-2r^2-8Rr)+s(2s^3-6r^2s-12Rrs)+4Rrs\cdot 2s] - $$

$$-\frac{4}{s(bc-(s-a)^2)}[s^4(a+b+c)+s^2(a^3+b^3+c^3)+abc(a+b+c) - $$

$$-2s^3(2s^3-6r^2s-12Rrs)-24Rrs^3+16Rr^3]-\frac{4rs^2(r+R)}{(r+4R)^2}$$

$$N_aG_e^2 = -\frac{1}{r(r+4R)}(-4r^2s^2 + 4Rrs^2) - \frac{4}{sr(r+4R)}[2s^5 + s^2(2s^3 - 6r^2s - 12Rrs) +$$

$$+4Rrs(s^2 + r^2 + 4Rr) - 2s^3 - 6r^2s - 12Rrs) - 24Rrs^3 + 16Rr^3\,] - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$N_aG_e^2 = \frac{4s^2(R-r)}{r+4R} - \frac{4(16R^2r + 4Rr^2 - 2rs^2)}{r+4R} - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$N_aG_e^2 = -16Rr + \frac{8rs^2}{r+4R} + \frac{4s^2(R-r)}{r+4R} - \frac{4rs^2(r+R)}{(r+4R)^2}$$

$$N_aG_e^2 = -16Rr + \frac{4s^2}{(r+4R)^2}$$

Hence,

$$N_aG_e^2 = -16Rr + \frac{16Rs^2(r+R)}{(r+4R)^2}$$

5 Conclusion

In this article we find a metric relationship for the Gergonne's point. With this relationship we can find the distance between the Gergonne's point and other notable centers of the triangle. The proofs presented here only require basic knowledge of Geometry and its manipulation and application.

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LA GACETA DE LA RSME CHALLENGES-(II)

By Daniel Sitaru

315. Si $a,b\in\mathbb{R}$ y $a<b$, probar que

$$\frac{e^b-e^a}{b-a}\geq\frac{e^a}{24}(b-a)^2+\sqrt{e^{a+b}}.$$

Proposed by Daniel Sitaru – Romania

Solution by Kee – Wai Lau, Hong Kong, China.

Tomando la nueva variable positive $x=b-a$ y dividiendo a ambos lados la desigualdad propuesta por e^{-a}, esta resulta ser equivalente a

$$\frac{e^x-1}{x}-\frac{x^2}{24}-e^{\frac{x}{2}}\geq 0. \qquad (1)$$

Ahora, usando el desarrollo en serie de potencias de la function exponencial,

$$\frac{e^x-1}{x}-\frac{x^2}{24}-e^{\frac{x}{2}}=\sum_{n=3}^{\infty}\left(\frac{1}{(n+1)!}-\frac{1}{2^n n!}\right)x^n=\sum_{n=3}^{\infty}\frac{2^n-n-1}{2^n(n+1)!}x^n,$$

lo que implica (1) por ser $x>0$ y complirse que $2^n-n-1>0$ para cada $n\geq 3$.
Nota. La mayor parte de las soluciones reecibidas utilizan un argumento similar al de la publicada, y A. Plaza observa que dicho argumento puede utilizarse para probar la desigualdad mas general

$$\frac{e^b-e^a}{b-a}>\sum_{k=2}^{n}\frac{e^a(2^k-k-1)}{2^k(k+1)!}(b-a)^k+\sqrt{e^{a+b}},b>a.$$

338. Sean m_a,m_b,m_c y a,b,c las longitudes de las medianas y los lados, respectivamente, de un triangulo ABC. Si S denota el doble del area de dicho triangulo probar que

$$\left(\frac{m_a^2}{m_b^2}+\frac{m_b^2}{m_c^2}+\frac{m_c^2}{m_a^2}\right)\left(\frac{m_a^4}{m_b^4}+\frac{m_b^4}{m_c^4}+\frac{m_c^4}{m_a^4}\right)\left(\frac{m_a^8}{m_b^8}+\frac{m_b^8}{m_c^8}+\frac{m_c^8}{m_a^8}\right)\geq$$

$$\geq S^3\left(\frac{1}{am_b}+\frac{1}{bm_c}+\frac{1}{cm_a}\right)^3.$$

Propuesto por Daniel Sitaru – Rumania

Solution enviada por Bruno Salgueiro Fanego, Viveiro, Lugo.-Spain

Para $n\geq 1$, procederemos a probar la cadena de desigualdades

$$\prod_{k=1}^{n}\left(\frac{m_a^{2^k}}{m_b^{2^k}}+\frac{m_b^{2^k}}{m_c^{2^k}}+\frac{m_c^{2^k}}{m_a^{2^k}}\right)\geq\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)^n\geq S^n\left(\frac{1}{am_b}+\frac{1}{bm_c}+\frac{1}{cm_a}\right)^n \qquad (1)$$

donde las igualdades se cumplen si y solo si ABC es un triangulo equlatero.
La prueba de la primera desigualdad en (1) se deduce usando que

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\geq\frac{x}{y}+\frac{y}{x}+\frac{z}{x} \qquad (2)$$

con igualdad si y solo si $x=y=z$, que es una snecilla aplicacion de la desigualdad entre las medias cuadratica y aritmetica y la de las medias aritmetica y geometrica. En efecto,

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\ge\frac{1}{3}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2\ge$$
$$\ge\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\sqrt[3]{\frac{x}{y}\frac{y}{z}\frac{z}{x}}=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}.$$

Veamos ahora, por induccion en k, que para cada entero $k\ge 1$ se cumple la desigualdad

$$\frac{m_a^{2^k}}{m_b^{2^k}}+\frac{m_b^{2^k}}{m_c^{2^k}}+\frac{m_c^{2^k}}{m_a^{2^k}}\ge\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\qquad(3)$$

con igualdad si y solo si ABC es un triangulo equilatero. El caso $k=1$ es evidente tomando $x=m_a, y=m_b$ y $z=m_c$ en (2). Ademas, en ese caso la igualdad se cumple si y solo si ABC es un triangulo equilatero. Supongamos que (3) se cumple para un cierto valor k y veamos que tambien se verifica para $k+1$. Aplicando, sucesivamente, (2) con $x=m_a^{2^k}, y=m_b^{2^k}$ y $z=m_c^{2^k}$ y (3), temenos que

$$\frac{m_a^{2^{k+1}}}{m_b^{2^{k+1}}}+\frac{m_b^{2^{k+1}}}{m_c^{2^{k+1}}}+\frac{m_c^{2^{k+1}}}{m_a^{2^{k+1}}}=\frac{\left(m_a^{2^k}\right)^2}{\left(m_b^{2^k}\right)^2}+\frac{\left(m_b^{2^k}\right)^2}{\left(m_c^{2^k}\right)^2}+\frac{\left(m_c^{2^k}\right)^2}{\left(m_a^{2^k}\right)^2}\ge$$
$$\ge\frac{m_a^{2^k}}{m_b^{2^k}}+\frac{m_b^{2^k}}{m_c^{2^k}}+\frac{m_c^{2^k}}{m_a^{2^k}}\ge\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a},$$

donde la igualdad se alcanza si y solo si ABC es un triangulo equilatero. Esto concluye la prueba de (3). De este modo, la primera de las desigualdades de (1) se obtiene multiplicando las desigualdades en (3) para $k=1,\dots,n$.
Ahora, como $S=ah_a$, donde h_a es la altura de triangulo ABC desde el vertice A, y se cumple $h_a\le m_a$, con igualdad si y solo si $b=c$, se tiene

$$\frac{S}{am_b}=\frac{h_a}{m_b}\le\frac{m_a}{m_b}$$

y, por analogia,

$$S\left(\frac{1}{am_b}+\frac{1}{bm_c}+\frac{1}{cm_a}\right)\le\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a},$$

con igualdad si y solo si ABC es un triangulo equilatero, lo que prueba la segunda desigualdad de (1).
Es claro que la desigualdad propuesta se corresponde con el caso $n=3$ de (1).

357. Si a, b y c son numeros reales positivos y $x\in\left(0,\frac{\pi}{2}\right)$, probar que

$$\frac{a^2\sin^6 x}{x^6}+\frac{b^2\sin^4 x}{x^4}+\frac{c^2\sin^2 x}{x^2}+3\sqrt[3]{(abc)^2}\,\frac{\tan^2 x}{x^2}>6\sqrt[3]{(abc)^2}$$

Propuesto por D.M. Bătinețu-Giurgiu y Daniel Sitaru – Romania

Solucion enviada, independientemente, por Kee-Wai Lau, Hong Kong, China, y Daniel Văcaru, Pitești, Rumania.
Por la desigualdad entre la media aritmetica y la geometrica, temenos que

$$\frac{a^2\sin^6 x}{x^6}+\frac{b^2\sin^4 x}{x^4}+\frac{c^2\sin^2 x}{x^2}\ge3\sqrt[3]{(abc)^2}\,\frac{\sin^4 x}{x^4}$$

y, por tanto, probando que

$$\frac{\sin^4 x}{x^4}+\frac{\tan^2 x}{x^2}>2, \quad x\in\left(0,\frac{\pi}{2}\right) \qquad (1)$$

habremos concluido.
A partir de la conocidas desigualdades

$$\sin x > x-\frac{x^3}{6} \quad \text{y} \quad \tan x > x+\frac{x^3}{3}, \qquad x\in\left(0,\frac{\pi}{2}\right),$$

usando que

$$\left(1-\frac{x}{6}\right)^4+\left(1+\frac{x}{3}\right)^2=2+\frac{x^4((x-12)^2+216)}{1296},$$

podemos deducir (1). En efecto,

$$\frac{\sin^4 x}{x^4}+\frac{\tan^2 x}{x^2}>\left(1-\frac{x}{6}\right)^4+\left(1+\frac{x}{3}\right)^2>2.$$

366. Si denotamos $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$, probar que, para cada $a>1$,

$$\frac{1}{H_n}\int_1^a\sum_{k=1}^n\frac{1}{k+x^{2k}}dx<1-\frac{1}{a}.$$

Propuesto por Daniel Sitaru – Romania

Solucion enviada por Alberto Stadler, Herrliberg, Suiza.

En primer lugar debemos observer que $k+x^{2k}\geq kx^2$, para $k\geq 1$. En efecto, el caso $k=1$ es obvio y si $k>1$, usando la desigualdad entre la media aritmetica y la media geometrica, se tiene que

$$k+x^{2k}=(k-1)\frac{k}{k-1}+x^{2k}\geq k\sqrt[k]{\left(\frac{k}{k-1}\right)^{k-1}x^{2k}}>kx^2.$$

Por tanto,

$$\int_1^a\sum_{k=1}^n\frac{1}{k+x^{2k}}dx<\sum_{k=1}^n\frac{1}{k}\int_1^a\frac{dx}{x^2}=H_n\left(1-\frac{1}{a}\right)$$

y la resultado se sigue de manera inmediata.

381. Si $1<a\leq b$, probar que

$$4\int_a^b\int_a^b(x^y+y^x)\,dx\,dy\geq(b-a)^2(4+(b-a)^2).$$

Propuesto por Daniel Sitaru – Romania

Solucion enviada pro Sean M. Stewart, Bomaderry, NSW, Australia.

A partir de la desigualdad de Bernoulli

$$(1+t)^r\geq 1+rt, \quad r,t\in\mathbb{R}, \qquad r\geq 1, t\geq -1,$$

tomando, respectivamente, $(t,r)=(x-1,y)$ y $(t,r)=(y-1,x)$, deducimos las desigualdades

$$x^y\geq 1+xy-y \quad \text{y} \quad y^x\geq 1+xy-x,$$

validas ambas para $x, y \geq 1$. Asi, puesto que $(x-1)(y-1) > 0$ cuando $x, y > 1$, llegamos a que

$$x^y + y^x \geq 1 + xy + (x-1)(y-1) > 1 + xy, \quad x, y > 1,$$

y, por tanto,

$$\int_a^b \int_a^b (x^y + y^x)\, dx\, dy > \int_a^b \int_a^b (1 + xy)\, dx\, dy = \frac{(b-a)^2}{4}(4 + (b+a)^2).$$

Finalmente, la desigualdad propuesta se sigue inmediatamente usando que $b + a > b - a$.

396. Sean $x, y, z, t \in (0,1)$ tales que $3\sqrt{3}(xyz + yzt + ztx + txy) = 4$. Probar que

$$\frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \geq 2.$$

Propuesto por Daniel Sitaru – Romania

Solution by Daniel Văcaru, Pitești, Romania.

Consideremos la function $f(x) = x(1-x^2)$ con $x \in (0,1)$. Como $f'(x) = 1 - 3x^2 = 0$ si y solo si $x = \frac{\sqrt{3}}{3}$, es claro que $f(x) \leq f\left(\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{9}$ para $x \in (0,1)$. Entonces $\frac{1}{x(1-x^2)} \geq \frac{9}{2\sqrt{3}}$ y, por tanto,

$$\frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \geq \frac{9}{2\sqrt{3}}(xyz + yzt + ztx + txy) = 2,$$

donde en el último paso se ha usado la relación dada para los valores x, y, z y t.

ABOUT PROBLEM NUMBER 377 BY MATHEMATICAL JOURNAL LA GACETA NR.3/2020

By Marian Dincă-Romania

On the sides of any triangle ABC are built outside or three equilateral triangles $\Delta BCA', \Delta CAB', \Delta ABC'$. Prove that:

$F(a_1, b_1, c_1) \geq 4F(a, b, c)$, where $F(a, b, c)$ –area of ΔABC.

Proof. Let $A'B' = c_1; B'C' = a_1; C'A' = b_1$. In $\Delta A'B'C'$, we have:

$$A'B'^2 = c_1^2 = a^2 + b^2 - 2ab\cos(C + 120°) =$$

$$= a^2 + b^2 - 2ab\cos C\cos 120° + 2ab\sin C\sin 120° =$$

$$= a^2 + b^2 + ab\cos C + 2\sqrt{3}F = a^2 + b^2 + \frac{a^2 + b^2 - c^2}{2} =$$

$$= \frac{3(a^2 + b^2 + c^2)}{2} + 2\sqrt{3}F - 2c^2 = x - 2c^2, \text{where}$$

$$\frac{3(a^2+b^2+c^2)}{2}+2\sqrt{3}F=x$$

Similarly, we get: $b_1^2=x-2b^2; a_1^2=x-2a^2$.

$$16[F(a_1,b_1,c_1)]^2=\sum_{cyc}a_1^2(b_1^2+c_1^2-a_1^2)=\sum_{cyc}(x-2a^2)(x+2a^2-2b^2-2c^2)=$$

$$=3x^2-4x(a^2+b^2+c^2)+4\sum_{cyc}a^2(b^2+c^2-a^2)=$$

$$=3x^2-4x(a^2+b^2+c^2)+64F^2(a,b,c)=$$

$$=\frac{27}{4}(a^2+b^2+c^2)^2+18\sqrt{3}(a^2+b^2+c^2)F(a,b,c)+36F^2(a,b,c)-$$

$$-6(a^2+b^2+c^2)^2-8\sqrt{3}(a^2+b^2+c^2)F(a,b,c)=$$

$$=\frac{3}{4}(a^2+b^2+c^2)^2+10\sqrt{3}(a^2+b^2+c^2)F(a,b,c)+100F^2(a,b,c)$$

Because: $a^2+b^2+c^2\geq 4\sqrt{3}F(a,b,c); (Ionescu-Weitzenboock)$

$$\frac{3}{4}(a^2+b^2+c^2)^2+10\sqrt{3}(a^2+b^2+c^2)F(a,b,c)+100F^2(a,b,c)\geq$$

$$\geq\frac{3}{4}\left[4\sqrt{3}F(a,b,c)\right]^2+10\sqrt{3}\cdot 4\sqrt{3}F(a,b,c)\cdot F(a,b,c)+100F^2(a,b,c)=$$

$$=36F^2(a,b,c)+120F^2(a,b,c)+100F^2(a,b,c)=256F^2(a,b,c)$$

Hence, $16[F(a_1,b_1,c_1)]^2\geq 256F^2(a,b,c)$.Therefore,

$$F(a_1,b_1,c_1)\geq 4F(a,b,c).$$

PROBLEMS FOR JUNIORS

J.1414 If $a,b,c>0, a^2+b^2+c^2+abc=4$ then

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{27}{a^2+b^2+c^2}\geq 12$$

Proposed by Le Ngo Duc-Vietnam

J.1415 In ΔABC holds:

$$\left(2-\frac{r}{R}\right)^2\leq\frac{r_br_c}{a^2}+\frac{r_cr_a}{b^2}+\frac{r_ar_b}{c^2}\leq\left(\frac{R}{r}\right)^2-\frac{R}{r}+\left(\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu-Romania

J.1416 In ΔABC the following relationship holds:

$$R^2\cos 3A-2R(R+r)\cos 2A+(s^2+r^2-R^2)\cos A+2R(R+r)+(r-s)(r+s)=0.$$

Proposed by George Florin Şerban, Neculai Stanciu-Romania

J.1417 If $0<a,b,c,d<2$ then

$$\frac{(a^2+4)(b^2+4)(c^2+4)(d^2+4)}{(a+1)(b+1)(c+1)(d+1)}>\frac{1+abcd}{4}$$

Proposed by Nikos Ntorvas-Greece

J.1418 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f\big(x+f(y)\big)=2f\big(y+f(x)\big)+3xyf(xy),\forall x,y\in\mathbb{R}$$

Proposed by Nguyen Van Canh- Vietnam

J.1419 Prove:

$$\sin 39^\circ=\sin\left(\frac{13\pi}{60}\right)=\frac{\frac{1}{4}\sqrt{\frac{1}{2}(5-\sqrt{5})}-\frac{1}{8}\sqrt{3}(-1-\sqrt{5})}{\sqrt{2}}-\frac{\frac{1}{8}(-1-\sqrt{5})+\frac{1}{4}\sqrt{\frac{3}{2}(5-\sqrt{5})}}{\sqrt{2}}.$$

Proposed by Ngulmun George Baite-India

J.1420 Let $a,b,c\in[1,2], ab+bc+ca=5$. Prove:

$$\frac{a}{3-a^2}+\frac{b}{3-b^2}+\frac{c}{3-c^2}\geq\frac{5}{2}$$

Proposed by Olimjon Jalilov-Uzbekistan

J.1421 If $x, y, z, u, v, w > 0$, then in ΔABC the following relationship holds:

$$\left(\frac{u}{v+w}\cdot xa+\frac{v}{w+u}\cdot yb+\frac{w}{u+v}\cdot zc\right)\left(\frac{v+w}{u}\cdot xa^3+\frac{w+u}{v}\cdot yb^3+\frac{u+v}{w}\cdot zc^3\right)\ge$$
$$\ge 16(xy+yz+zx)\cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu,Claudia Nănuți-Romania

J.1422 In ΔABC the following relationship holds:

$$\frac{a^5}{m_b+m_c}+\frac{b^5}{m_c+m_a}+\frac{c^5}{m_a+m_b}\ge 3\sqrt{3}\cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu,Claudia Nănuți –Romania

J.1423 In ΔABC the following relationship holds:

$$\frac{a^3+b^3+c^3}{3}+\frac{3}{a+b+c}+abc+\frac{a^2+b^2+c^2}{3abc}\ge\frac{8s}{3}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.1424 If $x, y, z \in (0,1)$, then in ΔABC the following relationship holds:

$$\frac{a^4}{(1-x^2)(y+z)}+\frac{b^4}{(1-y^2)(z+x)}+\frac{c^4}{(1-z^2)(x+y)}\ge 12\sqrt{3}\cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.1425 If $m \ge 0; x, y, z \in (0,1)$, then in ΔABC holds:

$$\frac{\tan^{2m+2}\frac{A}{2}}{x(1-x^2)}+\frac{\tan^{2m+2}\frac{B}{2}}{y(1-y^2)}+\frac{\tan^{2m+2}\frac{C}{2}}{z(1-z^2)}\ge\frac{1}{2}\left(\sqrt{3}\right)^{3-2m}$$

Proposed by D.M. Bătineţu-Giurgiu,Claudia Nănuți –Romania

J.1426 If $x, y, z \in (0,1)$, then:

$$\frac{1}{(1-x^2)(y+z)}+\frac{1}{(1-y^2)(z+x)}+\frac{1}{(1-z^2)(x+y)}\ge\frac{9\sqrt{3}}{4}$$

Proposed by D.M. Bătineţu-Giurgiu,Claudia Nănuți –Romania

J.1427 Find all functions $f, g: \mathbb{R} \to \mathbb{R}$ such that $g(0) = g(1) = 1$ and

$$f(xy)g(y)+f(xz)g(z)\ge f(x)f(yz)g(y)g(z)+1; \forall x, y, z \in \mathbb{R}$$

Proposed by D.M. Bătineţu-Giurgiu,Claudia Nănuți –Romania

J.1428 If $t \ge 0; x, y, z \in (0,1)$, then in ΔABC holds:

$$\frac{m_a^{2t+2}}{x(1-x^2)m_b^t m_c^t}+\frac{m_b^{2t+2}}{y(1-y^2)m_c^t m_a^t}+\frac{m_c^{2t+2}}{z(1-z^2)m_a^t m_b^t}\ge\frac{27}{2}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.1429 Solve for real numbers:

$$\begin{cases} 2\sin x + 2\sin y = 1 \\ 2\cos x + 2\cos y = \sqrt{3} \end{cases}.$$

Proposed by Daniel Sitaru,Claudia Nănuți-Romania

J.1430 Solve for real numbers:

$$\frac{1}{5}\binom{4}{4}x^5 + \frac{1}{4}\binom{4}{3}x^4 + \frac{1}{3}\binom{4}{2}x^3 + \frac{1}{2}\binom{4}{1}x^2 + x + \frac{1}{5} = 0$$

Proposed by Daniel Sitaru,Claudia Nănuți -Romania

J.1431 Solve for real numbers:

$$\sqrt{x^7 \cdot x^{7x-7}}(x + 2^{x-1}) = x^8 + 2^{8x-8}$$

Proposed by Daniel Sitaru,Claudia Nănuți -Romania

J.1432 In ΔABC holds: $4ab\cos^2\frac{C}{2} = 3c^2$ then $5c^2 \geq 4\sqrt{3}F + 2ab$.

Proposed by Daniel Sitaru,Claudia Nănuți -Romania

J.1433 Solve for real numbers:

$$1 + 2\cos x \cdot \cos 2x \cdot \cos 5x = \cos^2 x + \cos^2 2x + \cos^2 5x$$

Proposed by Daniel Sitaru,Claudia Nănuți –Romania

J.1434 In $\Delta ABC, r_a = 3, r_b = 4, r_c = 5$. Find F.

Proposed by Daniel Sitaru,Claudia Nănuți –Romania

J.1435 If $a, b, c, d > 0, ad > bc$ then:

$$\frac{(5a+3b)(7a+5b)(9a+7b)}{(5c+3d)(7c+5d)(9c+7d)} > \left(\frac{a+b}{c+d}\right)^3$$

Proposed by Daniel Sitaru-Romania

J.1436 Solve for real numbers:

$$\sin 2x \cdot \sin 3x \cdot \sin 11x + \sin^2 5x \cdot \sin 6x = \sin x \cdot \sin 3x \cdot \sin 8x$$

Proposed by Daniel Sitaru-Romania

J.1437 If $x, y, z \geq 0, \frac{x+1}{z+1} + \frac{z+1}{y+1} + \frac{y+1}{x+1} = 3$ then

$$\frac{x+1}{z^2 + \sqrt[3]{1+3z}} + \frac{y+1}{x^2 + \sqrt[3]{1+3x}} + \frac{z+1}{y^2 + \sqrt[3]{1+3y}} \leq 3$$

Proposed by Daniel Sitaru-Romania

J.1438 If $\Delta ABC \sim \Delta A'B'C'$ then:

$$4(m_a m_{a'} + m_b m_{b'} + m_c m_{c'}) \geq 9\sqrt[3]{abca'b'c'}$$

Proposed by Daniel Sitaru-Romania

J.1439 If $a, b, c > 0, abc = \frac{1}{\sqrt[4]{3}}$ then:

$$\frac{a^{120} + b^{120} + c^{120}}{a^{40} + b^{40} + c^{40}} \geq \frac{1}{(a^4 + b^4 + c^4)^{10}}$$

Proposed by Daniel Sitaru-Romania

J.1440 Solve for real numbers:

$$\begin{cases} x + 3y + 5z = 9 \\ x, y, z > 0 \\ \prod_{cyc}(x^3 + y^3) = \prod_{cyc} z^3\left(\frac{x}{y} + \frac{y}{x}\right) \end{cases}$$

Proposed by Daniel Sitaru-Romania

J.1441 If $x_1, x_2, \dots, x_n > 0, n \in \mathbb{N}^*$ then:

$$1 + \sum_{cyc} \frac{1}{\sqrt[3]{1 + x_i}} \leq n + \frac{1}{\sqrt[3]{1 + x_1 + x_2 + \dots + x_n}}$$

Proposed by Daniel Sitaru-Romania

J.1442 If $a, b > 0$ then:

$$\left(\sqrt{\frac{a^2+b^2}{2}}\right)^5 + \left(\frac{a^2+b^2}{a+b} - \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\right)^5 \leq \left(\frac{a^2+b^2}{a+b}\right)^5 + \left(\sqrt{ab}\right)^5$$

Proposed by Daniel Sitaru-Romania

J.1443 $\bar{a} = \sqrt{x} \cdot \bar{\imath} + \sqrt{x-1} \cdot \bar{\jmath}, \bar{b} = 2\bar{\imath} + 3\bar{\jmath}, x \geq 1$. Solve for real numbers:

$$18 + \tan\left(\measuredangle(\bar{a}, \bar{b})\right) = 13\sqrt{2}$$

Proposed by Daniel Sitaru-Romania

J.1444 Solve for real numbers:

$$\begin{cases} 0 \leq x, y \leq \pi \\ \cos^2(x-y) = 4(1 - \sin x \sin y)(1 - \cos x \cos y) \\ 6x + 2y = \pi \end{cases}$$

Proposed by Daniel Sitaru-Romania

J.1445 Solve for real numbers:

$$\frac{(x-1)^2}{2} + \frac{(y-2)^2}{4} + \frac{(z-3)^2}{6} + 3 = |x-1| + |y-2| + |z-3|$$

Proposed by Daniel Sitaru,Claudia Nănuți-Romania

J.1446 In $\Delta ABC, K$ −Lemoine's point holds:

$$\frac{KA^3}{a^3b^2}+\frac{KB^3}{b^3c^2}+\frac{KC^3}{c^3a^2}\geq\frac{\sqrt{3}}{36r^2}$$

Proposed by Daniel Sitaru, Claudia Nănuți –Romania

J.1447 Solve for real numbers:

$$\frac{x^2}{5}+\frac{y^2}{6}=\frac{(x+y)^2}{11}+\frac{5}{2}\left(\frac{x}{5}-\frac{y}{6}\right)^2$$

Proposed by Daniel Sitaru, Dan Nănuți-Romania

J.1448 Given $a,b\in\mathbb{R}_+$ such that $a^2+b^2=2$. Prove that:

$$3-ab\geq(a+b)\sqrt{ab}+(a-b)^2\geq 2ab$$

Proposed by Dang Le Gia Khanh-Vietnam

J.1449 In $\Delta ABC, \alpha\geq 15$, the following relationship holds:

$$a)\ \min\left\{\sum m_am_b+\alpha(R^2-4r^2);\sum h_a^2+\alpha(R^2-4r^2)\right\}\geq s^2$$

$$b)\ \max\left\{\sum w_aw_b+\alpha(R^2-4r^2);\sum r_ar-b\right\}\geq s^2$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1450 In ΔABC the following relationship holds:

$$1+\frac{R}{r}\geq\sum_{cyc}\sqrt{\frac{1+\tan^2\frac{A}{2}}{4\tan\frac{B}{2}\tan\frac{C}{2}}}\geq 3$$

Proposed by Alex Szoros-Romania

J.1451 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_aw_a}{m_aw_a+n_ag_a}+\frac{3\cdot\sqrt[3]{\prod m_aw_a}}{\sqrt[3]{\prod m_aw_a}+\sqrt[3]{\prod n_ag_a}}\geq 2\sum_{cyc}\frac{\sqrt{m_am_bw_aw_b}}{\sqrt{m_am_bw_aw_b}+\sqrt{g_ag_bn_an_b}}$$

Proposed by Bogdan Fuștei-Romania

J.1452 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(n_a^2-g_a^2)\sin A}{(\sin B-\sin C)^2}=4R(a+b+c)$$

Proposed by Ertan Yildirim-Turkiye

J.1453 If $a, b, c > 0$ such that $3a^2 + b^2 = 16c^2$ then

$$\frac{3}{a} + \frac{1}{b} \geq \frac{2}{c}$$

Proposed by Marin Chirciu-Romania

J.1454 Determine all pairs of real numbers (x, y) which satisfy the following system of equations

$$\begin{cases} \log\left(x + \sqrt{x^2 + 1}\right) + \log\left(y + \sqrt{y^2 + 1}\right) = 0 \\ 2^{y-x}(1 - 3^{x-y+1}) - 2^{x-y+1} + 1 = 0 \end{cases}$$

Proposed by Neculai Stanciu-Romania

J.1455 If $n \in \mathbb{N}^* - \{1\}, a, b, x_k \in \mathbb{R}^*, X_n = \sum_{k=1}^{n} x_k$, then:

$$\sum_{k=1}^{n} \frac{1}{\left(ax_k + b \cdot \sqrt[n]{x_1 x_2 \dots x_n}\right)^m} \geq \frac{n^{m+1}}{(a+b)^n X_n^m}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.1456 If $m \in \mathbb{R}_+, n \in \mathbb{N}^* - \{1\}; a, b, c, x_k \in \mathbb{R}_+^*, X_n = \sum_{k=1}^{n} x_k$ with $aX_n > b \cdot \max\limits_{1 \leq k \leq n} x_k$, then

$$\sum_{k=1}^{n} \frac{x_1 x_2 \dots x_n + c x_k^{2m+1}}{x_k^m (aX_n - bx_k)^m} \geq \frac{n^{2m+1} x_1 x_2 \dots x_n + c X_n^{2m+1}}{(an - b)^m X_n^{2m}}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți –Romania

J.1457 If ABC is a nonisosceles triangle, then:

$$\frac{a^6}{(a-b)^2(a-c)^2} + \frac{b^6}{(b-a)^2(b-c)^2} + \frac{c^6}{(c-b)^2(c-a)^2} > 4\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți –Romania

J.1458 If $n \in \mathbb{N}, n \geq 3; a, b, c, d, x_k \in \mathbb{R}_+^*, k = \overline{1, n}, x_{n+1} = x_1$ such that $\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \prod_{k=1}^{n} x_k \leq d$, then

$$\sum_{k=1}^{n} (ax_k^{n-1} + bx_k^{n-1} + c) \frac{x_k^3 + x_{k+1}^3}{x_k^2 + x_k x_{k+1} + x_{k+1}^2} \geq \frac{2n}{3d}\big((a+b)d + cn\big) \prod_{k=1}^{n} x_k$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți –Romania

J.1459 If $0 < c \leq b \leq a$ and $3(a - b) = b - c$, then prove that:

$$\frac{a^2 - b^2}{c} \geq \frac{b^2 - c^2}{a} + \frac{c^2 - a^2}{b}$$

Proposed by Neculai Stanciu-Romania

J.1460 In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\cot\frac{B}{2}+\cot\frac{C}{2}-\frac{2|b-c|+s_a+g_a}{h_a}\right)\le\frac{64r_ar_br_c}{(s+n_a)(s+n_b)(s+n_c)}$$

Proposed by Bogdan Fuștei-Romania

J.1461 In ΔABC the following relationship holds:

$$a^2+b^2+c^2\ge\frac{4}{3}\sum_{cyc}m_am_b+\frac{4}{3\sqrt{3}}\sum_{cyc}|(m_a-m_b)(m_a-m_c)|$$

Proposed by Bogdan Fuștei-Romania

J.1462 In ΔABC the following relationship holds:

$$2(4R+r)=\sum_{cyc}\frac{a\cdot n_a}{r_a}+\sum_{cyc}\frac{4F}{s+n_a}$$

Proposed by Bogdan Fuștei-Romania

J.1463 In ΔABC the following relationship holds:

$$3(R-2r)\ge\frac{(m_b-m_c)^2}{a}$$

Proposed by Bogdan Fuștei-Romania

J.1464 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{n_a}\le\sum_{cyc}\frac{g_a}{s_a}$$

Proposed by Bogdan Fuștei-Romania

J.1465 In $\Delta ABC, P\in Int(\Delta ABC)$ the following relationship holds:

$$\sum_{cyc}\sqrt{n_b+n_c}\cdot PA\ge\sqrt{\sum_{cyc}a^2n_a+4Fs}$$

Proposed by Bogdan Fuștei-Romania

J.1466 In ΔABC the following relationship holds:

$$a^2+b^2+c^2\ge 4F\cdot\sqrt{\left(\frac{m_a}{m_b}\right)^2+\left(\frac{m_b}{m_c}\right)^2+\left(\frac{m_c}{m_a}\right)^2}$$

Proposed by Bogdan Fuștei-Romania

J.1467 If $xy+yz+zx\ge 0; x+y, y+z, z+x>0$ then prove:

$$x+y+z\ge\sqrt{(xy+yz+zx)\left(\frac{y+z}{x+z}+\frac{x+z}{x+y}+\frac{x+y}{y+z}\right)}$$

Proposed by Bogdan Fuștei-Romania

J.1468 In ΔABC the following relationship holds:

$$6r \le \sum_{cyc} \frac{a^2}{m_b + m_c} \le \frac{3R^2}{2r}$$

Proposed by Marin Chirciu-Romania

J.1469 If $x, y, z > 0$ and $n \in \mathbb{N}, n \ge 2$ then:

$$\sum_{cyc} \left(\frac{x}{y}\right)^n + \sum_{cyc} \left(\frac{y}{x}\right)^n \ge \frac{2(x^{n-3}y^{2n-3} + y^{n-3}z^{2n-3} + z^{n-3}x^{2n-3})}{x^{n-2}y^{n-2}z^{n-2}} \ge 6$$

Proposed by Marin Chirciu-Romania

J.1470 In ΔABC the following relationship holds:

$$\frac{8r^3}{R^2}(4R + r)^2 \le \sum_{cyc} h_b h_c (h_b + h_c) \le 2r(4R + r)^2$$

Proposed by Marin Chirciu-Romania

J.1471 In ΔABC the following relationship holds:

$$6\left(\frac{r}{4R}\right)^{\frac{2}{3}} \le \sum_{cyc} \frac{s_a}{s_a + m_a} \le \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

J.1472 In ΔABC the following relationship holds:

$$1 \le \sum_{cyc} \frac{rh_a}{h_b h_c} \le \frac{R}{2r}$$

Proposed by Marin Chirciu-Romania

J.1473 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{h_a^3} \ge \sum_{cyc} \frac{1}{r_a^3}$$

Proposed by Marin Chirciu-Romania

J.1474 If $a, b, c > 0$ such that $abc = 1$ and $\lambda \ge 0, \mu \ge 0$ then:

$$\frac{a^4}{b^2 + \lambda b + \mu bc} + \frac{b^4}{c^2 + \lambda c + \mu ca} + \frac{c^4}{a^2 + \lambda a + \mu ab} \ge \frac{3}{\lambda + \mu + 1}$$

Proposed by Marin Chirciu-Romania

J.1475 Let $a > 0, b > 0$ fixed. Solve for real numbers:

$$\sqrt{a^2 + b^2 - x^2} + x\sqrt{a^2 + b^2 - x^2} + x = a + ab + b$$

Proposed by Marin Chirciu-Romania

J.1476 In ΔABC the following relationship holds:

$$\frac{(na+b)(nb+c)(nc+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \leq \left(\frac{n+1}{4}\right)^3 \cdot \frac{1}{r\sqrt{s}}, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.1477 If $x, y, z > 0$ such that $\frac{y}{x} + \frac{z}{y} + \frac{x}{z} = 3$ then:

$$\sqrt{\frac{y}{x}} + \sqrt{\frac{z}{y}} + \sqrt{\frac{x}{z}} \geq \frac{9(x^2+y^2+z^2)}{(x+y+z)^2}$$

Proposed by Marin Chirciu-Romania

J.1478 If $x, y, z > 0$ then:

$$\left(1+\frac{x}{y+\lambda z}\right)^2 + \left(1+\frac{y}{z+\lambda x}\right)^2 + \left(1+\frac{z}{x+\lambda y}\right)^2 \geq 3\left(\frac{\lambda+2}{\lambda+1}\right)^2$$

Proposed by Marin Chirciu-Romania

J.1479 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x+y) = 2f(x) + 3f(y) - 4xyf(2x-3y), \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1480 Find all $\alpha \in \mathbb{R}$ such that:

$$\max\left\{2021^{-\alpha x} + 2021^{-\alpha x^2}\right\} \leq 1, \forall x \in [-1,1]$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1481 In ΔABC the following relationship holds:

$$\sum_{cyc} h_a\sqrt{w_a m_a} \leq \sum_{cyc} h_a^2 + \frac{1}{r^2}\sum_{cyc} w_a m_a (R^2 - 4r^2)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1482 Let $a, b, c > 0, \alpha \geq 3$ such that $a + b + c = 3$. Prove that:

$$\sqrt{a^3+\alpha b} + \sqrt{b^3+\alpha c} + \sqrt{c^3+\alpha a} \geq 3\sqrt{1+\alpha}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1483 Let $a, b, c > 0$ such that $ab + bc + ca = 1$. Prove that:

$$(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab) \geq 1$$

$$\sum_{cyc} a^4b^4 + \left(\sum_{cyc} a\right)^2 \geq \frac{82}{3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1484 If $x, y > 0$, then in ΔABC the following relationship holds:

$$\frac{a^5}{xm_b + ym_c} + \frac{b^5}{xm_c + ym_a} + \frac{c^5}{xm_a + ym_b} \geq \frac{6\sqrt{3}}{x+y} \cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu,Claudia Nănuți-Romania

J.1485 If $a, b, c, x, y > 0$, then:

$$\frac{a}{(bx+cy)^2} + \frac{b}{(cx+ay)^2} + \frac{c}{(ax+by)^2} \geq \frac{9}{(a+b+c)(x+y)^2}$$

Proposed by D.M. Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1486 If $t \geq 0$, then in ΔABC the following relationship holds:

$$\frac{a^{3t+2}}{(m_b+m_c)^t} + \frac{b^{3t+2}}{(m_c+m_a)^t} + \frac{c^{3t+2}}{(m_a+m_b)^t} \geq \frac{3^t\sqrt{3}}{4^{t-1}} \cdot F^{t+1}$$

Proposed by D.M. Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1487 If $m \geq 0$ and $x, y, z \in (0,1)$, then:

$$\left(\sum_{cyc} \frac{x^{2m+1}}{1-x^2}\right)\left(\sum_{cyc} \frac{1}{(x+y)^{2m+2}}\right) \geq \frac{27\sqrt{3}}{2^{2m+3}}$$

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți-Romania

J.1488 If $x, y, z \in (0,1)$, then in ΔABC the following relationship holds:

$$\left(\frac{xa^4}{1-x^2} + \frac{yb^4}{1-y^2} + \frac{zc^4}{1-z^2}\right)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq 18\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți -Romania

J.1489 If $m, n > 0, 2m > n$ and $a, b, c > 0, m(a+b) > nc, m(b+c) > na, m(c+a) > nb$, then:

$$\frac{a^3}{m(b+c)-na} + \frac{b^3}{m(c+a)-nb} + \frac{c^3}{m(a+b)-nc} \geq \frac{a^2+b^2+c^2}{2m-n}$$

Proposed by D.M. Bătinețu-Giurgiu,Nicolae Mușuroia-Romania

J.1490 If $x, y, z \in \mathbb{R}$, then in ΔABC the following relationship holds:

$$(xa+yb+zc)(xa^2+yb^2+zc^2) \geq 8\sqrt{3}(xy+yz+zx)\left(\sqrt{F}\right)^3$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu –Romania

J.1491 If $a, b, c > 0$ such that $a+b+c=3$ and $\lambda \leq 2$ then:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \lambda(ab+bc+ca) - 3(\lambda-1)abc$$

Proposed by Marin Chirciu – Romania

J.1492 In ΔABC the following relationship holds:

$$\left(1+\frac{1}{a}\cot\frac{A}{2}\right)\left(1+\frac{1}{b}\cot\frac{B}{2}\right)\left(1+\frac{1}{c}\cot\frac{C}{2}\right)\ge\left(1+\frac{27}{2}\frac{r}{p^2}\right)^3$$

Proposed by Marin Chirciu – Romania

J.1493 In ΔABC the following relationship holds:

$$\frac{R}{2r}\sum h_a^2\ge(3r)^4\sum\frac{1}{r_a^2}$$

Proposed by Marin Chirciu – Romania

J.1494 If $a,b,c>0$ such that $abc=1$ and $\lambda\ge 0$ then:

$$\frac{b^2+\lambda c^2}{a}+\frac{c^2+\lambda a^2}{b}+\frac{a^2+\lambda b^2}{c}\ge\lambda(a+b+c)+3$$

Proposed by Marin Chirciu – Romania

J.1495 In ΔABC the following relationship holds:

$$pr^2(4R+r)^2\cdot\frac{32R}{3}\le\sum a^6\cot\frac{A}{2}\le p(4R+r)^2\cdot\frac{4R^4}{3r}$$

Proposed by Marin Chirciu – Romania

J.1496 In ΔABC the following relationship holds:

$$\frac{1}{S}\left(2-\frac{r}{R}\right)^2\le\sum\frac{1}{a^2}\cot\frac{A}{2}=\frac{1}{S}\left(\frac{R^2}{r^2}-\frac{7r^2}{R^2}\right)$$

Proposed by Marin Chirciu – Romania

J.1497 Let n be a non-zero fixed integer. Solve in integers:

$$\sqrt{x^2+4-(9n)^2}+\sqrt{(9n)^2+64-x^2}=\sqrt{\left(\frac{x}{n}\right)^2+19}$$

Proposed by Marin Chirciu – Romania

J.1498 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, prove that:

$$(n_a^3+n_b^3+n_c^3)(m_a^3+m_b^3+m_c^3)\ge\frac{\left(g_a^2+g_b^2+g_c^2+\frac{r(R-2r)}{3}\right)^3}{3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1499 Find all functions $d:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^{2020})=f(x^{2022})+2021x^{2021}y^{2022};\forall x,y\in\mathbb{R}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1500 Solve for real numbers:

$$\log\sqrt{x}+x=\log x+x^2,\quad \sqrt{1-x^2}\le 1+x,\quad \sin(\log_{2020}x)\ge \log_{2020}x$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1501 Solve for real numbers:

$$\cos^2 x+2x=\cos^2 4x+8x, \sqrt{x^2+1}\ge 2-9x,$$
$$e^{\tan x}+2021^{\tan x}=e^{-\tan x}+2021^{-\tan x}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1502 Solve for real numbers:

$$2021^{e^{x^2}}+e^{x^2}=2021^{x^2+1}+x^2+1,\quad \left|\frac{x-2}{x+3}\right|=\frac{x-3}{x+2},\quad 2x^2-x+3\ge |x-4|$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1503 In ΔABC the following relationship holds:

$$\sum_{cyc}(w_a-h_a)(w_a+h_a)\ge 3(R^2-4r^2)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1504 Solve for real numbers:

$$\sin^3\sqrt[3]{x}+\sqrt[3]{x}=\sin^3\sqrt{x}+\sqrt{x},\quad \sqrt{x+1}\le 2-x,\quad \sqrt{\frac{2-x}{2+x}}\ge\sqrt{\frac{5+x}{5-x}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

J.1505 Prove that for all $n\in\mathbb{Z}_+$:

$$[n+\sqrt{n^2+1}+\sqrt{n^2+1}+\cdots+\sqrt{n^2+2n}+n+1]=2n^2+3n+1$$

where $[\cdot]$ is floor function.

Proposed by Amrit Awasthi-India

J.1506 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\tan^2\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}+\tan^2\frac{B}{2}}\ge 3$$

Proposed by Marian Ursărescu-Romania

J.1507 In ΔABC the following relationship holds:

$$6\sum_{cyc}\frac{r_a^2}{r_a^2+r_br_c}\le 1+\frac{4R}{r}$$

Proposed by Marian Ursărescu-Romania

J.1508 In ΔABC the following relationship holds:

$$\sum \frac{\sin\frac{A}{2}}{h_a} \ge \frac{1}{2r}.$$

Proposed by Bogdan Fuștei-Romania

J.1509 In ΔABC the following relationship holds:

$$\frac{3s - n_a - n_n - n_c}{6r} \le \sum \frac{r_a}{s + n_a}.$$

Proposed by Bogdan Fuștei-Romania

J.1510 In ΔABC holds:

$$\sum \sqrt{R(m_a - w_a)\cos\frac{B-C}{2}} \le \max\{a,b,c\} - \min\{a,b,c\}.$$

Proposed by Bogdan Fuștei-Romania

J.1511 In $\Delta ABC, T$ –Toricelli's point, holds:

$$\sum \sqrt{\frac{h_b h_c}{r r_a}} \le \frac{2}{\sqrt{3}} \cdot \sum \frac{w_b + w_c}{BT + TC}.$$

Proposed by Bogdan Fuștei-Romania

J.1512 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{g_a + a - s} \le \frac{1}{6r} \cdot \sum (3g_a + n_a) + \sum \frac{r_a}{s + n_a}.$$

Proposed by Bogdan Fuștei-Romania

J.1513 In ΔABC the following relationship holds:

$$\frac{R}{r} \ge 1 + \frac{1}{12} \cdot \sum \frac{b+c}{m_a} \cdot \sqrt{\frac{n_b n_c}{r r_a}}.$$

Proposed by Bogdan Fuștei-Romania

J.1514 In ΔABC the following relationship holds:

$$\left(\frac{R}{r} - 1\right)\sum m_a \ge \sqrt{\frac{1}{8r^2}\sum m_a^2(b^2 + c^2 - a^2) + \frac{3}{2}s^2}.$$

Proposed by Bogdan Fuștei-Romania

J.1515 In ΔABC the following relationship holds:

$$\sum \tan\frac{A}{4} \ge \sum \sqrt{\frac{2(n_a + h_a)}{r_a}} - \frac{s}{r}.$$

Proposed by Bogdan Fuștei-Romania

J.1516 In ΔABC the following relationship holds:

$$\sum \sqrt{\frac{s^2 - r_b r_c - h_b h_c}{n_a}} \ge s \cdot \sqrt{\frac{2}{R}}.$$

Proposed by Bogdan Fuștei-Romania

J.1517 In ΔABC the following relationship holds:

$$\sum \frac{\sqrt{|4(m_b m_c + r^2) - r_b r_c - 5rr_a|}}{n_a} \leq 2\sqrt{2}.$$

Proposed by Bogdan Fuștei-Romania

J.1518 In ΔABC the following relationship holds:

$$\sum \frac{m_a}{h_a} \geq \sum \frac{a}{b+c} + \sum \frac{m_a}{m_b + m_c}.$$

Proposed by Bogdan Fuștei-Romania

J.1519 In ΔABC the following relationship holds:

$$\sum n_a(r_b + r_c) \sin \frac{A}{2} \geq s^2.$$

Proposed by Bogdan Fuștei-Romania

J.1520 In ΔABC the following relationship holds:

$$\sum \frac{n_a}{r_b + r_c} \geq \frac{h_a}{g_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}.$$

Proposed by Bogdan Fuștei-Romania

J.1521 In ΔABC the following relationship holds:

$$\frac{1}{2}\sum (n_a + g_a) \geq \sqrt{\frac{2r}{R}} \cdot \sum \frac{m_a w_a}{h_a}.$$

Proposed by Bogdan Fuștei-Romania

J.1522 In ΔABC, holds:

$$\min\left((\Sigma n_a g_a)\left(\sum h_a h_a\right)^{-1}, \left(\sum m_a w_a\right)\left(\sum h_a h_b\right)^{-1} \right) \geq \frac{R}{2r}.$$

Proposed by Bogdan Fuștei-Romania

J.1523 In ΔABC the following relationship holds:

$$\sum m_a \sqrt{\frac{r_a}{h_a}} \geq \frac{1}{2}\sqrt{\frac{R}{r}} \sum \left(h_a + |b-c| \sin^2 \frac{A}{2} \right).$$

Proposed by Bogdan Fuștei-Romania

J.1524 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{g_a + s - a} \leq \frac{1}{2}\left(\frac{g_a + g_b + g_c}{r} - \frac{1}{2\sqrt{2}} \sum \frac{n_a}{r_a} - \sum \sqrt{\frac{h_a}{r_a}} \right).$$

Proposed by Bogdan Fuștei-Romania

J.1525 In ΔABC the following relationship holds:

$$\sum\sqrt{2(r_b^2+r_c^2)}\geq\sum(n_b+n_c).$$

Proposed by Bogdan Fuștei-Romania

J.1526 In ΔABC the following relationship holds:

$$\left(\frac{R}{r}\right)^3\cdot\prod(1-\cos A)\leq\frac{\sqrt{n_a n_b n_c g_a g_b g_c}}{h_a h_b h_c}.$$

Proposed by Bogdan Fuștei-Romania

J.1527 In ΔABC holds:

$$\frac{|a-b|+|b-c|+|c-a|}{2r}\geq\max\left(\sum\frac{\sqrt{(2m_a-g_a)^2-h_a^2}}{h_b},\sum\sqrt{\left(\frac{m_a w_a}{g_a h_a}\right)^2-1}\right).$$

Proposed by Bogdan Fuștei-Romania

J.1528 In ΔABC the following relationship holds:

$$\sum(b+c)w_a\geq 2s^2+6\left(3-\sqrt{2}\right)F.$$

Proposed by Bogdan Fuștei-Romania

J.1529 If $t,u\geq 0$, then in ΔABC the following relationship holds:

$$\frac{m_a^t w_a^u}{h_b^t+h_c^t}\cdot a^{u+2}+\frac{m_b^t w_b^u}{h_c^t+h_a^t}\cdot b^{u+2}+\frac{m_c^t w_c^u}{h_a^t+h_b^t}\cdot c^{u+2}\geq 2^{u+1}\sqrt{3}F^{u+1}$$

Proposed by D.M. Bătinețu-Giurgiu,Claudia Nănuți-Romania

J.1530 If in tetrahedron $ABCD, h_A, r_A$ —altitude from A, radius of exinscribed sphere, respectively, and r —radius of inscribed sphere, $m\geq 0$, then:

$$\sum_{cyc}\frac{1}{h_A^{m+1}}\geq\frac{1}{4^m\cdot r^{m+1}}\text{ and }\sum_{cyc}\frac{1}{r_A^{m+1}}\geq\frac{1}{2^{m-1}\cdot r^{m+1}}$$

Proposed by D.M. Bătinețu-Giurgiu,Daniel Sitaru -Romania

J.1531 If $M\in Int(\Delta ABC), x=MA, y=MB, z=MC, u=d(M,BC), v=d(M,CA)$

$w=d(M,AB)$, then:

$$\frac{x^2}{u^2+vw+wu}+\frac{y^2}{v^2+wu+uv}+\frac{z^2}{w^2+uv+vw}\geq 4$$

Proposed by D.M. Bătinețu-Giurgiu ,Claudia Nănuți -Romania

J.1532 If $m\geq 0$ and $t,u,x,y,z>0$, then in ΔABC the following relationship holds:

$$\left(\frac{ty+uz}{x}\right)^{m+1}\cdot\frac{a^{2m}}{h_a^2}+\left(\frac{tz+ux}{y}\right)^{m+1}\cdot\frac{b^{2m}}{h_b^2}+\left(\frac{tx+uy}{z}\right)^{m+!}\cdot\frac{c^{2m}}{h_c^2}\geq$$

$$\geq 2^{3m+1}\left(\sqrt{tu}\right)^{m+1}\left(\sqrt{3}\right)^{1-m}F^{1-m}$$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru-Romania

J.1533 If $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$, then:

$$(x^2+y^2+z^2)(a^2+b^2+b^2) \geq 4F^2\left(\frac{xy}{ab\sin^2\frac{C}{2}}+\frac{yz}{bc\sin^2\frac{B}{2}}+\frac{zx}{ca\sin^2\frac{C}{2}}\right) \geq 4F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.1534 If $m \geq 0, x, y, z \in (0,1)$, then in ΔABC the following relationship holds:

$$\frac{a^{2m+2}}{(1-x^2)(y+z)^m}+\frac{b^{2m+2}}{(1-y^2)(z+x)^m}+\frac{c^{2m+2}}{(1-z^2)(x+y)^m} \geq$$

$$\geq 6\cdot 2^m\sqrt{3}(xy+yz+zx)^{\frac{1-m}{2}}\cdot F^{m+1}$$

Proposed by D.M. Bătineţu-Giurgiu, Claudia Nănuți -Romania

J.1535 In ΔABC the following relationship holds:

$$\frac{a^2}{(Rb+rc)^4}+\frac{b^2}{(Rc+ra)^4}+\frac{c^2}{(Ra+rb)^4}+a^2+b^2+c^2 \geq \frac{3}{R^2+r^2}$$

Proposed by D.M. Bătineţu-Giurgiu,Dan Nanuți-Romania

J.1536 If $t \geq 0; x, y > 0$, then in ΔABC the following relationship holds:

$$\frac{a^{3t+2}}{(xm_b+ym_c)^t}+\frac{b^{3t+2}}{(xm_c+ym_a)^t}+\frac{c^{3t+2}}{(xm_a+ym_b)^t} \geq \frac{3^t\sqrt{3}}{2^{t-2}(x+y)^t}\cdot F^{t+1}$$

Proposed by D.M. Bătineţu-Giurgiu, Claudia Nănuți –Romania

J.1537 If $t \geq 0$, then in ΔABC the following relationship holds:

$$(m_b^t+w_c^t)\cdot a^{t+2}+(m_c^t+w_a^t)\cdot b^{t+2}+(m_a^t+w_b^t)\cdot c^{t+2} \geq 2^{t+3}\sqrt{3}\cdot F^{t+1}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.1538 If $m, n, x, y > 0$, then in ΔABC the following relationship holds:

$$\frac{m^2a^4+n^2b^4}{(ax+by)c}+\frac{m^2b^4+n^2c^4}{(bx+cy)a}+\frac{m^2c^4+n^2a^4}{(cx+ay)b} \geq \frac{2(m+n)^2\sqrt{3}}{x+y}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu,Dan Nănuți-Romania

J.1539 If $m \geq 0$ and $a, b, c > 0$, then:

$$\frac{a}{(bx+cy)^m}+\frac{b}{(cx+ay)^m}+\frac{c}{(ax+by)^m}\geq\frac{3^m}{(a+b+c)^{m-1}(x+y)^m}$$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu -Romania

J.1540 In ΔABC the following relationship holds:

$$12r\leq\sum\frac{bc}{\sqrt{m_bm_c}}\leq 2R\left(\frac{2R}{r}-1\right)$$

Proposed by Marin Chirciu-Romania

J.1541 If $a,b,c,d>0$ such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\leq 4$ then

$$\frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+d}}+\frac{1}{\sqrt{d+a}}\leq\frac{4}{\sqrt{2}}$$

Proposed by Marin Chirciu-Romania

J.1542 In acute ΔABC the following relationship holds:

$$\sum\frac{\tan^{2n+1}A}{\tan^{2n-1}B}\geq\frac{3(R+r)^2}{s^2},n\in\mathbb{N}^*$$

Proposed by Marin Chirciu-Romania

J.1543 In ΔABC the following relationship holds:

$$\frac{2}{r}\left(\frac{R}{r}-1\right)\leq\sum\frac{r_b+r_c}{r_a^2}\leq\frac{4}{R}\left(\frac{R}{r}-1\right)^2$$

Proposed by Marin Chirciu-Romania

J.1544 In ΔABC the following relationship holds:

$$\frac{2R-r}{3r}\left(\frac{s}{2R}+\frac{3\sqrt{3}}{4}\right)\leq\sum\frac{h_a}{r_a}\cos\frac{A}{2}\leq\frac{Rs}{4r^2}$$

Proposed by Marin Chirciu-Romania

J.1545 In ΔABC the following relationship holds:

$$\sum\frac{m_a}{h_a}\geq\frac{\sum\frac{b+c}{a}+\lambda\sum\frac{m_b+m_c}{m_a}}{2(1+\lambda)},\lambda\geq 0$$

Proposed by Marin Chirciu-Romania

J.1546 In ΔABC the following relationship holds:

$$\frac{2}{r}\leq\sum\frac{h_b+h_c}{r_a^2}\leq\frac{R}{r^2}$$

Proposed by Marin Chirciu-Romania

J.1547 In ΔABC the following relationship holds:

$$\frac{9r}{2(2R^2+r^2)}\le\sum\frac{\sqrt{r_br_c}}{m_b^2+m_c^2}\le\frac{1}{2r}$$

Proposed by Marin Chirciu-Romania

J.1548 In ΔABC the following relationship holds:

$$\frac{9}{4(n+1)(2R^2+r^2)}\le\sum\frac{1}{a^2+nb^2}\le\frac{1}{4(n+1)r^2},n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.1549 In ΔABC the following relationship holds:

$$\sum\frac{rh_a}{h_bh_c}\le\frac{R}{2r}\sum\frac{rr_a}{r_br_c}$$

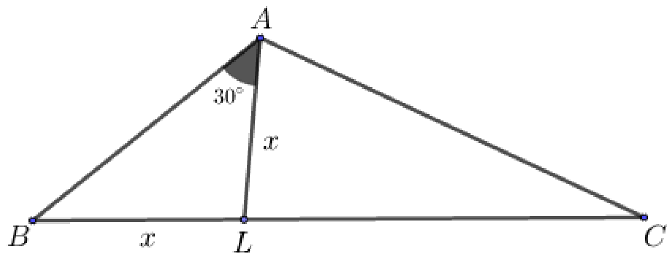
Proposed by Marin Chirciu-Romania

J.1550 In ΔABC the following relationship holds:

$$\left(\sum\frac{r_a}{r_b}\right)\left(\frac{2r}{R}+\sum\frac{r_a}{r_b}\right)\left(\frac{4r}{R}+\sum\frac{r_a}{r_b}\right)\ge 60.$$

Proposed by Soumava Chakraborty-India

J.1551 $\widehat{BAL}=30^\circ$ and $AL=BL=x$. Find $\widehat{LAC}$.



Proposed by Seyran Ibrahimov-Azerbaijan

J.1552 Solve for real numbers:

$$\begin{cases}\log_a\left(\sqrt[c]{x^2}-b^2\right)^{abc}-\log_a\left(\sqrt[c]{y}+b\right)\log_a\left(\sqrt[c]{z}-b\right)=a^2b^2c^2\\ \log_a\left(\sqrt[c]{y^2}-b^2\right)^{abc}-\log_a\left(\sqrt[c]{z}+b\right)\log_a\left(\sqrt[c]{x}-b\right)=a^2b^2c^2,\\ \log_a\left(\sqrt[c]{z^2}-b^2\right)^{abc}-\log_a\left(\sqrt[c]{x}+b\right)\log_a\left(\sqrt[c]{y}-b\right)=a^2b^2c^2\end{cases}$$

$$x,y,z\ge 0;a,b,c>0,a\ne 1.$$

Proposed by Jhoaw Carlos-Bolivia

J.1553 In ΔABC the following relationship holds:

$$(1+m_a)(1+m_b)(1+m_c)\ge(1+3r)^3.$$

Proposed by Ertan Yildirim-Turkiye

J.1554 If $a,b,c,d>0$ then prove:

$$\frac{ab}{a+b+n}+\frac{cd}{c+d+n}<\frac{(a+c)(b+d)}{a+b+c+d+n}.$$

Proposed by Jalil Hajimir-Canada

J.1555 O −circumcenter of acute ΔABC. Let ΔEFK be the medial triangle, $E\in(BC)$, $F\in(CA), K\in(AB), EO\cap AC=\{P\}, FO\cap AB=\{D\}, KO\cap BC=\{Q\}$. Prove that: $[PDQ]\geq\frac{3\sqrt{3}Rr}{2}$.

Proposed by Mehmet Șahin-Turkiye

J.1556 In ΔABC the following relationship holds:

$$2\left(\frac{h_a^2}{w_a^2}+\frac{h_b^2}{w_b^2}+\frac{h_c^2}{w_c^2}\right)\leq\frac{h_a}{m_a}+\frac{h_b}{m_b}+\frac{h_c}{m_c}+3.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1557 In ΔABC the following relationship holds:

$$2+\left(1+\frac{4m_am_bm_c(m_a+m_b+m_c)}{9F^2}\right)^2\geq\frac{(m_a+m_b+m_c)^4}{9F^2}.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1558 In $\Delta ABC, I$ −incentre, N_a −Nagel's point, $\Omega_1=[BN_aI]+[CN_aI], \Omega_2=[AN_aI]$. Prove that:

$$a)\ \Omega_1<\Omega_2\quad b)\ \Omega_1=\Omega_2\quad c)\Omega_1>\Omega_2.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1559 In ΔABC the following relationship holds:

$$\frac{(a+b+c)^4}{16F^2}\leq 2+\left(1+\frac{2R}{r}\right)^2.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1560 In ΔABC the following relationship holds:

$$\frac{m_a}{m_b+m_c}+\frac{m_b}{m_c+m_a}+\frac{m_c}{m_a+m_b}-\frac{1}{2}\leq\frac{R}{2r}.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1561 In ΔABC the following relationship holds:

$$\frac{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}{m_a^2m_b^2m_c^2}\leq\left(\frac{R}{r}\right)^3.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1562 Find:

$$\Omega = \cos^{12} 1° + \cos^{12} 2° + \cos^{12} 3° + \cdots + \cos^{12} 89°.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1563 In ΔABC the following relationship holds:

$$\sin\frac{3A}{2} + \sin\frac{3B}{2} + \sin\frac{3C}{2} \le \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c}.$$

Proposed by Adil Abdullayev-Azerbaijan

J.1564 In ΔABC the following relationship holds:

$$\sum \frac{|h_b - h_c|}{b + c} \ge \frac{1}{R}\sum (m_a - w_a).$$

Proposed by Bogdan Fuștei-Romania

J.1565 In ΔABC the following relationship holds:

$$\sum g_a(r_b + r_c) \cdot \cos\frac{A}{2} \ge (4R + r)s.$$

Proposed by Bogdan Fuștei-Romania

J.1566 In ΔABC holds:

$$\sum \left(\frac{m_b}{h_b} + \frac{m_c}{h_c} - \frac{m_a}{h_a}\right)\cos\frac{A}{2} \ge \sqrt{2\sum \cos\frac{A}{2}\cos\frac{B}{2} - \sum \frac{r_a}{2R}} \cdot \sqrt{2\sum \frac{m_a m_b}{h_a h_b} - \sum \frac{m_a^2}{h_a^2}}$$

Proposed by Bogdan Fuștei-Romania

J.1567 In $\Delta ABC, P \in Int(\Delta ABC), x, y, z > 0$ holds:

$$\sum \sqrt{y + z} \cdot PA \ge \sqrt{\sum xa^2 + 4F\sqrt{xy + yz + zx}}.$$

Proposed by Bogdan Fuștei-Romania

J.1568 In ΔABC holds:

$$\frac{R}{2r}(n_a + n_b + n_c) \ge \sqrt{\frac{1}{8r^2}\sum a^2(m_b^2 + m_c^2 - m_a^2) + \frac{3}{2}s^2}.$$

Proposed by Bogdan Fuștei-Romania

J.1569 In ΔABC the following relationship holds:

$$\sum n_a(r_b + r_c)\sqrt{\frac{r_a}{h_a}} \ge \frac{s^2}{3}\sum \sqrt{\frac{2m_a + n_a + h_a}{h_a}}.$$

Proposed by Bogdan Fuștei-Romania

J.1570 In $\Delta ABC, x, y, z > 0$ holds:

$$\sum xa^2n_a \geq \sqrt{xy+yz+zx}\sqrt{2\sum a^2b^2n_an_b - \sum a^4n_a^2}.$$

Proposed by Bogdan Fuștei-Romania

J.1571 In ΔABC the following relationship holds:

$$\left(\frac{h_a}{r_a}\right)^2 + \left(\frac{h_b}{r_b}\right)^2 + \left(\frac{h_c}{r_c}\right)^2 + \frac{10r}{R} \geq 8.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1572 Find:

$$\Omega = \frac{1}{\sin^2 2^\circ} + \frac{1}{\sin^2 6^\circ} + \frac{1}{\sin^2 10^\circ} + \cdots + \frac{1}{\sin^2 86^\circ}.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1573 In ΔABC the following relationship holds:

$$\frac{r_a^3}{r_b^3} + \frac{r_b^3}{r_c^3} + \frac{r_c^3}{r_a^3} + \frac{54r}{4R+r} \geq 9.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1574 In ΔABC the following relationship holds:

$$\sum \frac{a(\cos B + \cos C)}{b+c} \geq \frac{3}{2}.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1575 If $a, b, c > 0$ then:

$$\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right)\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \geq \frac{3}{2}.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1576 If $a, b, c > 0, abc = 1, n \in \mathbb{N} - \{0\}$ then:

$$\sum \frac{a}{a^{n+2} + b^{n+1} + c^n + 3} \leq \frac{1}{2}.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1577 If $x_1, x_2, \ldots, x_n > 0, n \in \mathbb{N}, n \geq 2, \sum_{k=1}^{n} \frac{x_k}{2019x_k + n - 1} = \frac{1}{2019}$ then:

$$\sqrt[n]{\prod_{k=1}^{n} x_k} \leq \frac{1}{2019}.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1578 If $a, b, c, d, e, f > 0, a + b + c + d + e + f = 6$ then:

$$abcdef\left(\sum a^2\right)\le 6.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1579 If $x_i>0, i\in\overline{1,n}, x_1x_2\cdot\ldots\cdot x_n=1$ then:

$$\sum_{k=1}^{n}\frac{x_k^2+1}{x_k^{n+1}+n-1)}\le 2.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1580 If $a,b,c>0, a+b+c=ab+bc+ca$ then:

$$\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}\ge 3.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1581 In ΔABC the following relationship holds:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{27r^2}{s^2}\ge 4.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1582 In $\Delta ABC, m(\sphericalangle BAC)=90^\circ, AD\perp BC, D\in(BC), I, I_1, I_2$ −incenters in $\Delta ABC, \Delta ABD$ respectively ΔACD. Prove that:

$$[II_1I_2]=\frac{r^3}{a}.$$

Proposed by Mehmet Șahin-Turkiye

J.1583 In $\Delta ABC, m(\sphericalangle BAC)=90^\circ, AD\perp BC, D\in(BC), I, I_1I_2$ −incenters, r_1, r_2, r_2 −inradii of $\Delta ABC, \Delta ABD$ −respectively $\Delta ACD, E, F, G\in(I_1I_2), IE\perp I_1I_2, BF\perp I_1I_2, CK\perp I_1I_2$.

If $IE=x, BF=y, CK=z$ then

$$r\left(\frac{1}{r_1}+\frac{1}{r_2}\right)=2+x\left(\frac{1}{y}+\frac{1}{z}\right).$$

Proposed by Mehmet Șahin-Turkiye

J.1584 ΔABC −equilateral, $D\in[BC], R^*$ −circumradii in ΔABD. Find in terms of a:

$$\Omega=\frac{R^*}{R}.$$

Proposed by Mehmet Șahin-Turkiye

J.1585 In acute $\Delta ABC, AD\perp BC, r_1, r_2, r$ −inradii in $\Delta ABD, \Delta ACD, \Delta ABC$ −respectively. Prove that:

$$\frac{a}{r}+\frac{b}{r_2}+\frac{c}{r_1}+2=\frac{2s}{a}\left(\frac{s-b}{r_1}+\frac{s-c}{r_2}\right).$$

Proposed by Mehmet Șahin-Turkiye

J.1586 $ABCD$ – cyclic quadrilateral, $AB = a, BC = b, CD = c, DA = d, b \neq d$,

$$\{P\} = AB \cap CD, (ad + bc)^2 + (ab + cd)^2 = (b^2 - d^2)^2$$

Find:

$$\Omega = m(\sphericalangle BPC)$$

Proposed by Ravi Prakash-India

J.1587 If $a, b, c \geq 0$, then:

$$\sum_{cyc}\left(a^3 + \frac{ab(a+b)}{2}\right) \geq \frac{3}{4}(a+b)(a+c)(b+c)$$

Proposed by Asmat Qatea-Afghanistan

J.1588 In acute ΔABC holds:

$$\sum_{cyc}\tan^2\frac{A}{2}\cdot\cot^5\frac{B}{2} \geq 3\sqrt{3}\left(\frac{4R}{r} - 5\right)$$

Proposed by Marian Ursărescu – Romania

J.1589 Let $a, b > 1$, prove that for all $x \in \mathbb{R}$ we have:

$$a^x + \frac{a^2}{a^x} + b^x + \frac{b^2}{b^x} \leq (a+b)^x + \frac{(a+b)^2}{(a+b)^x}$$

If $a, b \in (0,1)$ then inequality in reversal.

Proposed by Minh Vu-Vietnam

J.1590 Solve for real numbers:

$$\frac{x+\sqrt{3}}{\sqrt{x}+\sqrt{x+\sqrt{3}}} + \frac{x-\sqrt{3}}{\sqrt{x}-\sqrt{x-\sqrt{3}}} = \sqrt{x}$$

Proposed by Mohammad Nasery-Afghanistan

J.1591 AE, BF, CD – cevians in ΔABC. Prove that:

$$AE \cap BF \cap CD = \{T\} \Leftrightarrow \frac{AT}{DT}\cdot\frac{BT}{ET}\cdot\frac{CT}{FT} = 2 + \frac{AT}{DT} + \frac{BT}{ET} + \frac{CT}{FT}$$

Proposed by Amerul Hassan-Myanmar

J.1592 Without software. Find: P

$$123456789 \cdot (987654321 - 5) = 12193263P495351224$$

Proposed by Amrit Awasthi-India

J.1593 $ABCD$ – convexe quadrilateral, $P \in Int(ABCD), PG \perp DC, PF \perp BD, PE \perp AB$, $PH \perp AC$. Prove that:

$$AE^2 + BF^2 + CH^2 + DG^2 = AH^2 + BE^2 + CG^2 + DF^2$$

Proposed by Amerul Hassan-Myanmar

J.1594 Solve for real numbers:

$$\begin{cases} x^3 - 3x^2y = 3x - y \\ y^3 - 3y^2z = 3y - z \\ z^3 - 3z^2x = 3z - x \end{cases}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1595 ANALOGOUS CEVA'S THEOREM RELOADED FOR HEXAGON

$ABCDEF$ – cyclic hexagon. Prove that:

$$AD \cap BE \cap CF \neq \emptyset \Leftrightarrow \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$$

Proposed by Amerul Hassan-Myanmar

J.1596

$$\begin{cases} \sqrt{\sqrt{a} + \sqrt{b} + \sqrt{c}} = 3 \\ \sqrt[3]{\sqrt{a} - \sqrt{b} - \sqrt{c}} = \frac{1}{3} \\ \sqrt[4]{bc\sqrt{a}} = \sqrt{3} \end{cases}$$

Prove that: $\left(\frac{1}{c\sqrt{b}} + \frac{1}{b\sqrt{c}}\right) = \frac{14762}{6561}$

Proposed by Jay Jay Oweifa-Nigeria

J.1597 A circle is inscribed in a ΔABC if r is the radius of the circle prove that:

$$r = \frac{b + c - a}{2} \tan^{-1}\left(\frac{1}{2}\cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)\right)$$

Proposed by Ghazaly Abiodun-Nigeria

J.1598 For $\alpha > 0$ fixed find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha x) \cdot f\left(\frac{y}{\alpha}\right) = f(\alpha x + y), \forall x, y \in \mathbb{R}.$$

Proposed by Nguyen Van Canh-Vietnam

J.1599 If $a, b, c > 0, a + b + c = 1$ then

$$2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq \frac{1 + a}{1 - a} + \frac{1 + b}{1 - b} + \frac{1 + c}{1 - c}.$$

Proposed by Rajeev Rastogi-India

J.1600 In ΔABC the following relationship holds

$$\cot^2 A + \cot^2 B + cot^2 C + \frac{2r}{R} \geq 2.$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1601 If $a_i, b_i > 0, i \in \overline{1,n}$ then

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)^2 \ge 4\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{a_i^2+b_i^2}{2}}\right).$$

Proposed by Seyran Ibrahimov-Azerbaijan

J.1602 Given $a,b,c > 0$ prove that

$$\frac{1}{a^2+ab+ac}+\frac{1}{b^2+bc+ba}+\frac{1}{c^2+ca+cb} \ge \frac{3}{ab+bc+ca}.$$

Proposed by Rajeev Rastogi-India

J.1603 Solve for real numbers:

$$\begin{cases} x+y+z=3 \\ 2^x+2^y+2^z=7 \\ 2^{-x}+2^{-y}+2^{-z}=\frac{7}{4} \end{cases}.$$

Proposed by Mohammad Nasery-Afghanistan

J.1604 Solve:

$$\begin{cases} \frac{4(x+2y+3z)^3}{27(x+2y)^2}=3z \\ x+y+z=11 \\ x,y,z>0 \end{cases}$$

Proposed by Jalil Hajimir-Canada

J.1605 Let x, y and z be positive real numbers. Prove that:

$$\left(\frac{x^4}{x^3+2y^3}\right)^3+\left(\frac{y^4}{y^3+2z^3}\right)^3+\left(\frac{z^3}{z^3+2x^3}\right)^3 \ge \frac{x^3+y^3+z^3}{27}$$

Proposed by Jalil Hajimir-Canada

J.1606 In ΔABC denote:

$$X_1=\sum_{cyc}s_a, X_2=\sum_{cyc}m_a, X_3=\sum_{cyc}w_a, X_4=\sum_{cyc}k_a,$$

k_a, k_b, k_c – concurrent cevians. Prove that: $X_i \ge 9r, i \in \overline{1,4}$

Proposed by Marian Voinea – Romania

J.1607 If $a,b,c,x,y,z > 0, \{x\}=x-[x], [*]$ - great integer function, then:

$$a\left(\{x\}+\left\{\frac{1}{z}\right\}\right)+b\left(\{y\}+\left\{\frac{1}{x}\right\}\right)+c\left(\{z\}+\left\{\frac{1}{y}\right\}\right) < \frac{3(a+b+c)}{2}$$

Proposed by Ionuț Florin Voinea – Romania

J.1608 If $a,b,c>0, a\le 1, a+b\le 5, a+b+c\le 14$ then:

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\le 6$$

Proposed by Ionuț Florin Voinea – Romania

J.1609 Solve for real numbers:

$$\begin{cases} x+\sqrt{y+3}=\sqrt{y+8} \\ y+\sqrt{z+3}=\sqrt{z+8} \\ z+\sqrt{t+3}=\sqrt{t+8} \\ t+\sqrt{x+3}=\sqrt{x+8} \end{cases}$$

Proposed by Ionuț Florin Voinea – Romania

J.1610 In ΔABC the following relationship holds:

$$\cos A\cos\frac{B}{2}\cos\frac{C}{2}+\cos B\cos\frac{C}{2}\cos\frac{A}{2}+\cos C\cos\frac{A}{2}\cos\frac{B}{2}\le\frac{9}{8}$$

Proposed by Ionuț Florin Voinea – Romania

J.1611 If $a,b,c,x,y,z>0, \{x\}=x-[x], [x]$ – great integer function, then:

$$\frac{1}{b\{x\}+c\left\{\frac{1}{y}\right\}}+\frac{1}{c\{y\}+a\left\{\frac{1}{z}\right\}}+\frac{1}{a\{z\}+b\left\{\frac{1}{x}\right\}}>\frac{6}{a+b+c}$$

Proposed by Ionuț Florin Voinea – Romania

J.1612 If $x,y,z\in\mathbb{R}$ then:

$$\max(3x^2-xy+xz, 3y^2-yz-yx, 3z^2-zx-zy)\ge 0$$

Proposed by Ionuț Florin Voinea – Romania

J.1613 Solve for natural numbers:

$$x^2+y^2=200{,}000{,}000$$

Proposed by Ionuț Florin Voinea – Romania

J.1614 In ΔABC the following relationship holds:

$$(m_a+m_b+m_c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\le 5+\left(\frac{R}{r}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

J.1615 In ΔABC the following relationship holds:

$$\sqrt{1+\frac{27(a^2-b^2)^2}{(a+b+c)^4}}\le\frac{R}{2r}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1616 In ΔABC the following relationship holds:

$$\frac{R}{2r} \geq \left(1+\frac{(4R+r)^2-3s^2}{(4R+r)^2+s^2}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

J.1617 In ΔABC the following relationship holds:

$$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}+\frac{2(r_a^2+r_b^2+r_c^2)}{r_ar_b+r_br_c+r_cr_a} \leq 1+\frac{2R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1618 If $x,y,z>0$ then:

$$(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)+\frac{4xyz}{(x+y)(y+z)(z+x)} \geq 5$$

Proposed by Adil Abdullayev-Azerbaijan

J.1619 In ΔABC the following relationship holds:

$$\frac{4r_ar_br_c}{w_aw_bw_c}=1+\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1620 In ΔABC the following relationship holds:

$$\frac{2m_a}{R} \leq 1+\frac{1}{\sin\frac{A}{2}}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1621 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{r_a}{h_a}-\frac{h_a}{r_a}\right) \geq \frac{(R-2r)(2R-3r)}{2R(R-r)}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1622 In ΔABC the following relationship holds:

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \leq \frac{3m_am_bm_c}{h_ah_bh_c}$$

Proposed by Adil Abdullayev-Azerbaijan

J.1623 If $m \geq 0; x,y,z \in \left(0,\frac{\pi}{2}\right)$ and $x+y+z=2\pi$, then:

$$\frac{1}{(\sin x+\sin y)^{m+1}\sin^m z}+\frac{1}{(\sin y+\sin z)^{m+1}\sin^m x}+\frac{1}{(\sin z+\sin x)^{m+1}\sin^m y} \geq$$

$$\geq \frac{9^{m+1}}{2^{m+1}\cdot\pi^{m+2}}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1624 In ΔABC the following relationship holds:

$$\frac{R}{r}+\frac{r}{R}\geq\frac{5}{2}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1625 If $m\geq 0$ and $x,y>0$, then in ΔABC the following relationship holds:

$$\frac{a^{2m+2}}{(xr_a+yr_b)^m r_c^m}+\frac{b^{2m+2}}{(xr_b+yr_c)^m r_a^m}+\frac{c^{2m+2}}{(xr_c+yr_a)^m r_b^m}\geq\frac{4^{m+1}\cdot s^2}{(x+y)^m}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1626 If $m\geq 0$, then in ΔABC the following relationship holds:

$$\frac{r_a^{m+2}}{(xr_b+yr_c)^m}+\frac{r_b^{m+2}}{(xr_c+yr_a)^m}+\frac{r_c^{m+2}}{(xr_a+yr_b)^m}\geq\frac{3\sqrt{3}F}{(x+y)^m}$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți -Romania

J.1627 If $u,v>0$ and $M\in Int(\Delta ABC), x=MA, y=MB, z=MC$, then:

$$\sum_{cyc}\left(\frac{uxy}{ab}+\frac{vay^2z}{b^2cx}\right)^2\geq\frac{(u+v)^2}{3}$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți –Romania

J.1628 If $u,v>0$ and $M\in Int(\Delta ABC), x=MA, y=MB, z=MC$, then:

$$\sum_{cyc}\left(\frac{x}{a}\left(\frac{uv}{b}+\frac{vz}{c}\right)\right)^4\geq\frac{(u+v)^4}{27}$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți-Romania

J.1629 If $x,y,z>0$, then in ΔABC the following relationship holds:

$$\frac{xa}{y+z}+\frac{yb}{z+x}+\frac{zc}{x+y}\geq\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M.Bătineţu-Giurgiu,Nicolae Mușuroia-Romania

J.1630 In ΔABC the following relationship holds:

$$\frac{1}{\sqrt{h_ah_b}}+\frac{1}{\sqrt{h_bh_c}}+\frac{1}{\sqrt{h_ch_a}}\geq\frac{\sqrt[4]{27}}{\sqrt{F}}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1631 If $u,v\geq 0,u+v>0$ and $m,x,y,z\in[1,\infty),3m=x+y+z$, then in ΔABC holds:

$$((ux+vy)^x+(uy+vz)^x+(uz+vx)^x)ab$$
$$+((ux+vy)^y+(uy+vz)^y+(uz+vx)^y)bc+$$
$$+((ux+vy)^z+(uy+vz)^z+(uz+vx)^z)ca\geq 12\sqrt{3}(u+v)^m\cdot m^m\cdot F$$

Proposed by D.M.Bătineţu-Giurgiu,Gheorghe Boroica-Romania

J.1632 If $m,u,x,y,z\in[1,\infty),3m=x+y+z$, then:

$$(x+y)^u+(y+z)^u+(z+x)^u\geq 2^u\cdot 3\cdot m^m$$

Proposed by D.M.Bătineţu-Giurgiu,Neculai Stanciu-Romania

J.1633 If $x,y,z\in\left(0,\frac{\pi}{2}\right)$, then in ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\tan x+\tan y}{\mathrm{sinn}\, z}+\frac{\tan z}{\mathrm{sinn}\, x+\sin y}\right)a\geq 5\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți –Romania

J.1634 If $x,y,z\in(0,\infty)$, then in ΔABC the following relationship holds:

$$\frac{a^2}{yz}+\frac{b^2}{zx}+\frac{c^2}{xy}+4(x^2a^2+y^2b^2+z^2c^2)\geq 16\sqrt{3}\cdot F$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

J.1635 If $m,x,y,z\in[1,\infty),3m=x+y+z$, then in ΔABC holds:

$$(x^x+y^x+z^x)a^2+(x^y+y^y+z^y)b^2+(x^z+y^z+z^z)c^2\geq 12\sqrt{3}\cdot m^m\cdot F$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți –Romania

J.1636 If $g,m,x,y,z\in[1,\infty),g=\sqrt[3]{xyz},3m=x+y+z$, then in ΔABC holds:

$$(x^x+y^x+z^x)\left(a-\sqrt{ab}+b\right)^2+(x^y+y^y+z^y)\left(b-\sqrt{bc}+b\right)^2+$$
$$+(x^z+y^z+z^z)\left(c-\sqrt{ca}+c\right)^2\geq 12\sqrt{3}\cdot g^m\cdot F$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1637 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\geq 12$$

Proposed by D.M.Bătineţu-Giurgiu,Florică Anastase-Romania

J.1638 If ΔABC the following relationship holds:

$$\frac{x+y}{zh_a^2}+\frac{y+z}{xh_b}+\frac{z+x}{y}\cdot bc\geq 4\sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.1639 If $m > 0$ and $x, y > 0$, then in ΔABC the following relationship holds:

$$(a^{m+1} + b^{m+1} + c^{m+1})\left(\frac{a^{m+1}}{(ax+by)^m} + \frac{b^{m+1}}{(bx+cy)^m} + \frac{c^{m+1}}{(cx+ay)^m}\right) \geq$$

$$\geq \frac{2^{m+2} \cdot \left(\sqrt[4]{3}\right)^{6-m}}{(x+y)^m} \cdot F\left(\frac{m+2}{2}\right)$$

Proposed by D.M.Bătinețu-Giurgiu,Neculai Stanciu -Romania

J.1640 If $x, y > 0$, then in ΔABC the following relationship holds:

$$\frac{a^3}{bRx + cry} + \frac{b^3}{cRx + ary} + \frac{c^3}{aRx + bry} \geq \frac{4\sqrt{3}}{Rx + ry} \cdot F$$

Proposed by D.M.Bătinețu-Giurgiu,Florică Anastase-Romania

J.1641 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{ayz}{x^2} + \frac{bzx}{y^2} + \frac{cxy}{z^2} \geq 2 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1642 Solve for real numbers:

$$\left[x[x]\right] + \left[\frac{[x]}{x}\right] = 2x, [*] - GIF.$$

Proposed by Jalil Hajimir-Canada

J.1643 If $x \in \mathbb{R}$, $[*]$ −great integer function, then:

$$[30x] + [x] \geq [15x] + [10x] + [6x]$$

Proposed by Jalil Hajimir-Canada

J.1644 Solve:

$$1 - \frac{[x^2]}{3} > \cos x, [*] - GIF.$$

Proposed by Jalil Hajimir-Canada

J.1645 Solve:

$$x[\sin x] + [x]\sin x = 0, [*] - GIF.$$

Proposed by Jalil Hajimir-Canada

J.1646 In ΔABC the following relationship holds:

$$h_a^2 + h_a^2 + h_a^2 + \lambda r(R - 2r) \leq p^2, \lambda \leq 4$$

Proposed by Marin Chirciu – Romania

J.1647 In ΔABC the following relationship holds:

$$48pr^3 \leq \sum a^4 \tan\frac{A}{2} \leq 16(R^3 - 5r^3)$$

Proposed by Marin Chirciu – Romania

J.1648 If $a, b, c > 0$ such that $a + b + c \leq 3$ and $n \in \mathbb{N}^*$ then:

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} \geq 3$$

Proposed by Marin Chirciu – Romania

J.1649 In ΔABC the following relationship holds:

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} \leq \left(\frac{2R}{r}\right)^2 - 13$$

Proposed by Marin Chirciu – Romania

J.1650 In ΔABC the following relationship holds:

$$\sum \cos\frac{A}{2} \geq \frac{r}{R}\sum \frac{r_b + r_c}{a}$$

Proposed by Marin Chirciu – Romania

J.1651 In ΔABC the following relationship holds:

$$\frac{5R - 2r}{4R^2} \leq \frac{1}{r_a + r_b} + \frac{1}{r_b + r_c} + \frac{1}{r_c + r_a} \leq \frac{R + 2r}{4Rr}$$

Proposed by Marin Chirciu – Romania

J.1652 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a + r}{r_a - r} \geq \sum_{cyc} \frac{4F + (b - c)^2}{a \cdot s_a}$$

Proposed by Marin Chirciu – Romania

J.1653 In ΔABC the following relationship holds:

$$\frac{16r}{R} \leq \frac{(h_a + r_a)(h_b + r_b)(h_c + r_c)}{m_a m_b m_c} \leq \frac{4R}{r}$$

Proposed by Marin Chirciu – Romania

J.1654 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{bc}\cot^2\frac{A}{2} \geq \sum \frac{r_a}{bc}\cot^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.1655 In ΔABC the following relationship holds:

$$\frac{6}{R} \le \sum \frac{r_a}{bc}\csc^2\frac{A}{2} \le \frac{3}{r}$$

Proposed by Marin Chirciu – Romania

J.1656 In ΔABC the following relationship holds:

$$\frac{1}{4r}\left(2-\frac{2r}{R}-\frac{r^2}{R^2}\right) \le \sum \frac{r_a}{bc}\sin^2\frac{A}{2} \le \frac{1}{4r}\left(4-\frac{8r}{R}+\frac{3r^2}{R^2}\right)$$

Proposed by Marin Chirciu – Romania

J.1657 In ΔABC the following relationship holds:

$$a^3b^3+b^3c^3+c^3a^3 \ge 648R^3r^3$$

Proposed by Marin Chirciu – Romania

J.1658 In ΔABC the following relationship holds:

$$\sum \frac{r_b+r_c}{h_a} \ge \sum \frac{h_b+h_c}{r_a}$$

Proposed by Marin Chirciu – Romania

J.1659 If $a,b,c>0$ and $n \in \mathbb{N}^*, k \in \mathbb{N}$ then:

$$\sum \frac{a^{2n}}{b^{2n}} \sum \frac{a^k}{b^k} \ge \left(\sum \frac{a^n}{b^n}\right)^2$$

Proposed by Marin Chirciu – Romania

J.1660 If $a,b,c>0$ such that $a+b+c=3$ and $\lambda \ge 0$ then:

$$\frac{a^3+b^3}{a^2+\lambda ab+b^2}+\frac{b^3+c^3}{b^2+\lambda bc+c^2}+\frac{c^3+a^3}{c^2+\lambda ca+a^2} \ge \frac{6}{\lambda+2}$$

Proposed by Marin Chirciu – Romania

J.1661 Solve:

$$\begin{cases} |x|+4|y|+6|z|=27 \\ x^2+4y^2+9z^2=81 \\ [x^3]+[y^3]+[z^3]=62 \end{cases}$$

Proposed by Jalil Hajimir-Canada

J.1662 Let a,b,c be positive real numbers such that $abc=1$. Prove that:

$$\frac{7-6a}{2+a^2}+\frac{7-6b}{2+b^2}+\frac{7-6c}{2+c^2} \ge 1$$

Proposed by Jalil Hajimir-Canada

J.1663 Let a,b,c be positive real numbers. Prove that:

$$\sqrt[15]{8192(a^5+b^5)^2(b^5+c^5)^2(c^5+a^5)^2} \le a^2+b^2+c^2$$

Proposed by Jalil Hajimir-Canada

J.1664 Let a, b, c be positive real numbers. Prove that:

$$\frac{b^4+c^2}{a^3}+\frac{c^4+a^2}{b^3}+\frac{a^4+b^2}{c^3}\geq 6$$

Proposed by Jalil Hajimir-Canada

J.1665 In ΔABC the following relationship holds:

$$\frac{w_a^2+w_ah_a+h_a^2}{h_a+w_a}\leq\frac{3}{2}w_a$$

Proposed by Daniel Sitaru-Romania

J.1666 In ΔABC the following relationship holds:

$$\left(\sum_{cyc}am_a\right)^3\left(\sum_{cyc}am_a^2\right)^{-\frac{1}{2}}\left(\sum_{cyc}am_a^3\right)^{-\frac{1}{3}}\left(\sum_{cyc}am_a^6\right)^{-\frac{1}{6}}\leq 4s^2$$

Proposed by Daniel Sitaru-Romania

J.1667 If $a, b, c > 0$ then:

$$3\left(a+b+c+\frac{3abc}{ab+bc+ca}\right)^4\geq 256abc(a+b+c)$$

Proposed by Daniel Sitaru-Romania

J.1668 In ΔABC the following relationship holds:

$$1+2\prod_{cyc}\frac{m_a}{h_a}\geq\sqrt{2}\sum_{cyc}\frac{\frac{m_am_b}{h_ah_b}-\cos\frac{A}{2}}{\sqrt{1+\sin\frac{A}{2}}}$$

Proposed by Bogdan Fuștei-Romania

J.1669 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{x}{y+z}\left(1+\sin\frac{A}{2}\right)}\leq\sqrt{\frac{2(x+y+z)^2}{(x+y)(y+z)(z+x)}}$$

Proposed by Bogdan Fuștei-Romania

J.1670 In ΔABC the following relationship holds:

$$m_am_bm_c\geq\frac{a^2b^2+b^2c^2+c^2a^2}{8R}$$

Proposed by Bogdan Fuștei-Romania

J.1671 In ΔABC the following relationship holds:

$$2Rs\geq\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

Proposed by Bogdan Fuștei-Romania

J.1672 Let $M \in Int(\Delta ABC)$ the following relationship holds:

$$\sum_{cyc} \frac{g_a(r_b + r_c)}{h_a} \cdot AM \geq 2R(4R + r)$$

Proposed by Bogdan Fuștei-Romania

J.1673 In ΔABC the following relationship holds:

$$\frac{R}{r} \geq \max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\} + \min\left\{\frac{m_a m_b}{h_a h_b}, \frac{m_b m_c}{h_b h_c}, \frac{m_c m_a}{h_c h_a}\right\}$$

Proposed by Bogdan Fuștei-Romania

J.1674 In ΔABC the following relationship holds:

$$(m_a + m_b + m_c)\sqrt{m_a m_b m_c} \geq 9F\sqrt{\frac{R}{2}}$$

Proposed by Bogdan Fuștei-Romania

J.1675 In $\Delta ABC, (r_i)_{i=\overline{1,3}}$ –Malfatti's radies, the following relationship holds:

$$\sum_{cyc} \sqrt{r_2 r_3} \geq \frac{3(3 - \sqrt{3})r}{2}$$

Proposed by Bogdan Fuștei-Romania

J.1676 In $\Delta ABC, (r_i)_{i=\overline{1,3}}$ –Malfatti's radies, the following relationship holds:

$$\sum_{cyc} \sqrt{r_2 r_3} \geq \frac{1}{2}(g_a + g_b + g_c - s)$$

Proposed by Bogdan Fuștei-Romania

J.1677 In ΔABC the following relationship holds:

$$\frac{27(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{abc} \geq (10m_a - 3a)(10m_b - 3b)(10m_c - 3c)$$

Proposed by Bogdan Fuștei-Romania

J.1678 If $M \in Int(\Delta ABC), \omega = \max\left\{\left(\sqrt{n_a} - \sqrt{n_b}\right)^2, \left(\sqrt{n_b} - \sqrt{n_c}\right)^2, \left(\sqrt{n_c} - \sqrt{n_a}\right)^2\right\}$ the following relationship holds:

$$\sum_{cyc} AM \cdot \cos\frac{A}{2} \geq \sqrt[3]{n_a n_b n_c} + \frac{2}{3}\sum_{cyc} \frac{r_a h_a}{s + n_a} + \frac{1}{3}\omega$$

Proposed by Bogdan Fuștei-Romania

J.1679 Solve for real numbers:

$$\left[x[x]\right]+\left[\frac{[x]}{x}\right]=2x,[*]-GIF.$$

Proposed by Jalil Hajimir-Canada

J.1680 If $x\in\mathbb{R}$, $[*]$ −great integer function, then:

$$[30x]+[x]\geq[15x]+[10x]+[6x]$$

Proposed by Jalil Hajimir-Canada

J.1681 Solve:

$$1-\frac{[x^2]}{3}>\cos x\,,[*]-GIF.$$

Proposed by Jalil Hajimir-Canada

J.1682 Solve:

$$x[\sin x]+[x]\sin x=0,[*]-GIF.$$

Proposed by Jalil Hajimir-Canada

J.1683 If $x,y,z>0$, then in ΔABC the following relationship holds:

$$\frac{y+z}{xh_a}+\frac{z+x}{yh_b}+\frac{x+y}{zh_c}\geq\frac{6\cdot\sqrt[4]{27}}{s}$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți-Romania

J.1684 If $m\geq 0;x,y,z,u,v>0$ and $x+y+z=a$, then:

$$\frac{x}{(u+vy)^m}+\frac{y}{(u+vz)^m}+\frac{z}{(u+vx)^m}\geq\frac{3^m\cdot a}{(3u+av)^m}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți-Romania

J.1685 If $t,u,v,x,y,z>0$ and $tuv=w^3>0$, then in ΔABC holds:

$$\left(ty+u+\frac{v}{x}\right)a+\left(uz+v+\frac{t}{y}\right)b+\left(vx+t+\frac{u}{z}\right)c\geq 6w\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți –Romania

J.1686 Let ΔABC and $\Delta A_1B_1C_1$ with sides $a_1=\sqrt{a},b_1=\sqrt{b},c_1=\sqrt{c},R_1$ −circumradiu, then:

$$R_1^2=\frac{Rs}{4R+r}$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți -Romania

J.1687 If $m\geq 0;x,y>0$, then in ΔABC the following relationship holds:

$$\frac{r_a^{2m+2}}{(xb^2+yc^2)^m}+\frac{r_b^{2m+2}}{(xc^2+ya^2)^m}+\frac{r_c^{2m+2}}{(xa^2+yb^2)^m}\ge\frac{3^{2m+2}\left(\sqrt{3}\right)^{m+1}\cdot F^{m+1}}{2^m(x+y)^m(s^2-r^2-4Rr)^m}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

J.1688 If $x,y,z>0$ and $x+y+z=a$, then:

$$\frac{x}{u+vy}+\frac{y}{u+vz}+\frac{z}{u+vx}\ge\frac{3a}{3u+av};\forall u,v>0$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

J.1689 In ΔABC the following relationship holds:

$$\frac{ar_a}{r_b+r_c}+\frac{br_b}{r_c+r_a}+\frac{cr_c}{r_a+r_b}\ge\sqrt[4]{3}\cdot\sqrt{F}$$

Proposed by D.M.Bătineţu-Giurgiu,Neculai Stanciu-Romania

J.1690 In ΔABC the following relationship holds:

$$\frac{bc}{(s-a)^2}+\frac{ca}{(s-b)^2}+\frac{ab}{(s-c)^2}\ge 12$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1691 If $x,y,z>0$, then in ΔABC the following relationship holds:

$$\frac{y+z}{xh_a}+\frac{z+x}{yh_b}+\frac{x+y}{zh_c}\ge\frac{2\cdot\sqrt[4]{27}}{\sqrt{F}}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1692 If $a,b,x,,y,z>0$ and $xyz=1$, then:

$$\frac{xy}{ax^2+by^2}+\frac{yz}{ay^2+bz^2}+\frac{zx}{az^2+bx^2}\ge\frac{18}{(a+b)(x^3+y^3+z^3+x+y+z)}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

J.1693 In ΔABC the following relationship holds:

$$\frac{r_a}{(r_b+r_c)h_a}+\frac{r_b}{(r_c+r_a)h_b}+\frac{r_c}{(r_a+r_b)h_c}\ge\frac{\sqrt[4]{F}}{2\sqrt{F}}$$

Proposed by D.M.Bătineţu-Giurgiu,Florică Anastase –Romania

J.1694 If $m\ge 0$ and $a,b,c,x,y>0$, then:

$$\frac{(a^4+2b^2c^2)^{m+1}}{(bx+cy)^m}+\frac{(b^4+2c^2a^2)^{m+1}}{(cx+ay)^m}+\frac{(c^4+2a^2b^2)^{m+1}}{(ax+by)^m}\ge\frac{(a+b+c)^{3m+4}}{3^{2m+2}(x+y)^m}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți-Romania

J.1695 If $m,n>0$, then in ΔABC the following relationship holds:

$$(m^2+n^2)(a^2+b^2+c^2)\geq 8mn\sqrt{3}F+\sum_{cyc}(ma-nb)^2$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1696 In ΔABC the following relationship holds:

$$\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\geq\frac{\sqrt{3}}{F}+\frac{1}{8F^2}\cdot\sum_{cyc}(a-b)^2$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1697 If $m,n>0$, then in ΔABC the following relationship holds:

$$(m^2+n^2)\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)\geq\frac{2mn\sqrt{3}}{F}+\frac{1}{4F^2}\cdot\sum_{cyc}(ma-nb)^2$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1698 In $\Delta ABC, n_a$ —Nagel's cevian, g_a —Gerggone's cevian, the following relationship holds:

$$(n_a+g_b)c^3+(n_b+g_c)a^3+(n_c+g_a)b^3\geq 16\sqrt{3}F^2$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1699 If $m\geq 0$ and $x,y,z>0$, then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{x^{m+1}a^m}{(y+z)^{m+1}h_a^{m+2}}\geq\frac{\left(\sqrt{3}\right)^{1-m}}{2F}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

J.1700 If $m\geq 0, x\in\mathbb{R}$, then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{(b\sin^2 x+c\cos^2 x)^m h_a^{m+2}}\geq\frac{\sqrt{3}}{2^m\cdot F^{m+1}}$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți –Romania

J.1701 If $m\geq 0$ and $x\in\mathbb{R}$, then in ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^{m+2}}{(b\cos^2 x+c\sin^2 x)^m}\geq 4\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.1702 If $x,y,z>0$, then in ΔABC the following relationship holds:

$$\frac{xa^2}{my+nz}+\frac{yb^2}{mz+nx}+\frac{zc^2}{mx+ny}\geq\frac{4\sqrt{3}F}{m+n};\forall m,n\in(0,\infty)$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

J.1703 If $m,n>0, M\in Int(\Delta ABC), x=MA, y=MB, z=MC$, then:

$$(m^2+n^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\geq 2mn+\sum_{cyc}\left(\frac{mx}{a}-\frac{ny}{b}\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți -Romania

J.1704 In ΔABC the following relationship holds:

$$\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{\sin^2 A+\sin^2 B+\sin^2 C}\leq\frac{R}{2r}$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1705 In ΔABC the following relationship holds:

$$\left(\frac{r_ar_b}{r_c}\right)^\mu+\left(\frac{r_br_c}{r_a}\right)^\mu+\left(\frac{r_cr_a}{r_b}\right)^\mu\geq\left(\frac{h_ah_b}{h_c}\right)^\mu+\left(\frac{h_bh_c}{h_a}\right)^\mu+\left(\frac{h_ch_a}{h_b}\right)^\mu,\mu\geq\frac{1}{2}$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1706 In ΔABC the following relationship holds:

$$\frac{w_a}{w_b}+\frac{w_b}{w_c}+\frac{w_c}{w_a}\leq\frac{s}{r\sqrt{3}}$$

Proposed by Rahim Shahbazov-Azerbaijan

J.1707 In $\Delta ABC, AD, BE, CF$ – Nagel's cevians. Prove that:

$$FE\cdot ED\cdot DF\geq\frac{2r^2F}{R}$$

Proposed by Ertan Yildirim-Turkiye

J.1708 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin B+\sin C-\sin A}{\tan A}\geq 4-\frac{5r}{R}$$

Proposed by Ertan Yildirim-Turkiye

J.1709 In ΔABC the following relationship holds:

$$\frac{9r^2}{R}\leq\sum_{cyc}\frac{bc}{a(\tan A+\cot A)}\leq\frac{9R}{4}$$

Proposed by Ertan Yildirim-Turkiye

J.1710 Let ΔDEF be the intouch triangle of ΔABC Prove that:

$$\frac{[ADE]+[BFD]+[CEF]}{[ABC]}\geq\frac{3r}{2R}$$

Proposed by Ertan Yildirim-Turkiye

J.1711 If a right triangle has the legs a, respectively b, and c be the hypothenuse, and s the semiperimeter, then prove that $s(s-c)=(s-a)(s-b)$.

Proposed by Neculai Stanciu-Romania

J.1712 In ΔABC the following relationship holds:

$$4R\sum_{cyc} r_a^2+\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}+\sum_{cyc}\frac{a}{\tan A}=\sum_{cyc}\frac{h_a}{2r_a+h_a}+\sum_{cyc}(r_a^2+s^2)(r_a-r)+2(R+r)$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

J.1713 In ΔABC the following relationship holds:

$$\sum\frac{1}{2\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\leq\frac{1}{6\sqrt{3}}\left(\frac{5R}{r}+\frac{r}{R}+3\right).$$

Proposed by Marian Ursărescu-Romania

J.1714 In ΔABC the following relationship holds:

$$r_a^5r_b+r_b^5r_c+r_c^5r_a\geq 3(4Rr+r^2)^3.$$

Proposed by Marian Ursărescu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.1484 Find:

$$\Omega(a)=\int\frac{2\sqrt{2}(a^2+1)\sin\left(\frac{\pi}{4}+x\right)+2a(2+\sin 2x)}{2\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)+\sin 2x+2a^2}dx\,,a>0$$

Proposed by Olabintan Bolu-Nigeria

S.1485 Let $(u_n)_{n\geq 1}$ be a positive sequence so that $\lim\limits_{n\to\infty}(u_{n+1}-u_n)=l>0$. Find:

$$\lim_{n\to\infty}\left(\frac{u_1^n+u_2^n+\cdots+u_n^n}{1+2^n+\cdots+n^n}\right)^{\frac{1}{n}}$$

Proposed by Ali Jaffal-Lebanon

S.1486 For $\frac{1}{e}<a<b$ then prove:

$$\log\left(\frac{a}{b}\right)^2\log ab^{ab}>\log\left(\frac{\left(\frac{a^2}{e}\right)^{a^2}}{\left(\frac{b^2}{e}\right)^{b^2}}\right)$$

Proposed by Nikos Ntorvas-Greece

S.1487 If $a,b>1$ then

$$\left(\frac{a-b}{a+b}\right)^2<\log\left(\frac{(a+b)^2}{4ab}\right)^{4(a+b)}<\frac{(a^2-b^2)^2}{ab}$$

Proposed by Nikos Ntorvas-Greece

S.1488 If $a,b,c>0,e$ −Euler's number, then:

$$\frac{ae^{a-1}}{bc}+\frac{be^{b-1}}{ca}+\frac{ce^{c-1}}{ab}\geq\frac{(a^2+b^2+c^2)^2}{abc(a+b+c)}$$

Proposed by Nikos Ntorvas-Greece

S.1489 Solve for real numbers: $\begin{cases}a-2b+3c=abc\\3a^2+2b-3ab=1.\\4b^2-3a-a^2=6b\end{cases}$

Proposed by Soumava Chakraborty-India

S.1490 If $a>0$ and $x,y,z\in(0,a)$, then:

$$\left(\frac{x}{a^2-y^2}+\frac{y}{a^2-z^2}+\frac{z}{a^2-x^2}\right)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{27\sqrt{3}}{8a^3}$$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

S.1491 If $t>0;x,y,z>0$, then in ΔABC holds:

$$\frac{a^{3t+2}}{(xh_b+yh_c)^t}+\frac{b^{3t+2}}{(xh_c+yh_a)^t}+\frac{c^{3t+2}}{(xh_a+yh_b)^t}\geq\frac{3^t\sqrt{3}}{2^{t-2}(x+y)^t}\cdot F^{t+1}$$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

S.1492 If $x,y>0;y=1$, find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^x-\left(\sqrt[n]{n!}\right)^x\right)\left(\sqrt[n]{(2n-1)!!}\right)^y$$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru-Romania

S.1493 In ΔABC the following relationship holds:

$$\frac{a}{(Rb+rc)^2}+\frac{b}{(Rc+ra)^2}+\frac{c}{(Ra+rb)^2}+2s\geq\frac{6}{R+r}$$

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți –Romania

S.1494 If $t\geq 0$, then in ΔABC the following relationship holds:

$$\frac{a^{3t+2}}{(h_b+h_c)^t}+\frac{b^{3t+2}}{(h_c+h_a)^t}+\frac{c^{3t+2}}{(h_a+h_b)^t}\geq\frac{3^t\sqrt{3}}{4^{t-1}}\cdot F^{t+1}$$

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți –Romania

S.1495 If $x>0$, find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{\big((n+1)!\big)^x}-\sqrt[n]{(n!)^x}\right)n^{1-x}$$

Proposed by D.M. Bătinețu-Giurgiu,Daniel Sitaru-Romania

S.1496 If $x,y,z\in(0,1)$, then in ΔABC the following relationship holds:

$$\frac{h_a-r}{x(h_a+r)(1-x^2)}+\frac{h_b-r}{y(h_b+r)(1-y^2)}+\frac{h_c-r}{z(h_c+r)(1-z^2)}\geq\frac{9\sqrt{3}}{4}$$

Proposed by D.M. Bătinețu-Giurgiu,Florică Anastase –Romania

S.1497 If $x,y,z\in(0,1)$, then in ΔABC the following relationship holds:

$$\frac{a^2}{(1-x^2)(y+z)}+\frac{b^2}{(1-y^2)(z+x)}+\frac{c^2}{(1-z^2)(x+y)}\geq 9F$$

Proposed by D.M. Bătinețu-Giurgiu,Florică Anastase –Romania

S.1498 If $n\in\mathbb{N},n\geq 2$ and $a,b_k,x_k>0,\forall k=\overline{1,n},\frac{1}{x_k+a}+\frac{1}{x_{k+1}+a}\geq b_k;\forall k=\overline{1,n}$, then:

$$\sum_{cyc}\left(\frac{1}{x_k+1}+\frac{1}{x_{k+1}+a}\right)\geq 2a\left(2\sum_{k=1}^{n}b_k-n\right),x_{n+1}=x_1$$

Proposed by D.M. Bătinețu-Giurgiu,Daniel Sitaru-Romania

S.1499 If $x,y>0$, then in ΔABC the following relationship holds:

$$\frac{a^4+b^4}{(xa+yb)c}+\frac{b^4+c^4}{(xb+yc)a}+\frac{c^4+a^4}{(xc+ya)b}\geq\frac{8\sqrt{3}}{m+n}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu,Daniel Sitaru –Romania

S.1500 If $a,b,c,m,n,x,y>0$, then:

$$\frac{m^2a^4+n^2b^4}{(ax+by)c}+\frac{m^2b^4+n^2c^4}{(bx+cy)a}+\frac{m^2c^4+n^2a^4}{(cx+ay)b}\geq\frac{(m+n)^2}{2(x+y)}(a^2+b^2+c^2)$$

Proposed by D.M. Bătinețu-Giurgiu,Daniel Sitaru -Romania

S.1501 If $0 < a \le b < \frac{\pi}{2}$ then prove:

$$\int_a^b \frac{\tan\left(\frac{\pi-2x}{4}\right)(1+\sin x)}{\sin x}dx \le \frac{\sin b - \sin a}{\sin a}$$

Proposed by Daniel Sitaru,Claudia Nănuți -Romania

S.1502 If $0 < a \le b$ then prove:

$$\int_a^b \frac{1}{\sqrt{2[x]+1}} \cdot \prod_{k=1}^{[x]} \sin\left(\frac{k\pi}{2n+1}\right)dx \ge \frac{1}{2^a} - \frac{1}{2^b}$$

Proposed by Daniel Sitaru-Romania

S.1503 If $z_1, z_2, z_3 \in \mathbb{C}, z_1 + z_2 + z_3 = 3 + 4i$ then

$$2\sum |z_1| \le 5 + \sum (|z_1 - z_2| + |3 + 4i - 2z_3|)$$

Proposed by Daniel Sitaru-Romania

S.1504 Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sin^{2n}\frac{\pi}{7} - 2^{1-2n}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos\frac{(2n-2k)\pi}{7}}$$

Proposed by Daniel Sitaru-Romania

S.1505 If $0 < a \le b < \frac{\pi}{6}$ then find:

$$\Omega(a,b) = \int_a^b \frac{(1+\tan^2 x)^2}{\cos^2 x - 3\sin^2 x}dx$$

Proposed by Daniel Sitaru,Claudia Nănuți -Romania

S.1506 Solve for real numbers:

$$\begin{vmatrix} \cos x & \cos x & \cos 2x \\ \cos 3x & \cos 5x & \cos 4x \\ \sin 3x & \sin 5x & \sin 4x \end{vmatrix} = 0$$

Proposed by Daniel Sitaru-Romania

S.1507 Find:

$$\omega = \int \cosh(3)\cosh\left(\frac{3x}{2}\right)\cosh\left(\frac{13x}{2}\right)dx$$

Proposed by Daniel Sitaru-Romania

S.1508 If $x \in \mathbb{R}$ then:

$$e^{e^{-x}+x} + e^{e^{-x}-x} \ge 2\cosh x \cdot e^{\operatorname{sech} x}$$

Proposed by Daniel Sitaru-Romania

S.1509 Find:

$$\Omega = \lim_{n\to\infty} \frac{n(1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + n\cdot 1)}{1\cdot n^2 + 2\cdot(n-1)^2 + 3\cdot(n-2)^2 + \cdots + n\cdot 1^2}$$

Proposed by Daniel Sitaru-Romania

S.1510

$$\Omega_1(n) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|(i-j)\left(\frac{1}{2n-i+1} - \frac{1}{2n-j+1}\right)\right|, \Omega_2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{1}{2n-i+1} - \frac{1}{2n-j+1}\right|$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{\Omega_1(n)}{\Omega_2(n)}$$

Proposed by Daniel Sitaru-Romania

S.1511 Solve for real numbers:

$$5\left(\sqrt[5]{1-x} + \sqrt[5]{1+x}\right) = 2 + 4\left(\sqrt[4]{1-x} + \sqrt[4]{1+x}\right)$$

Proposed by Daniel Sitaru-Romania

S.1512 If $a, b, c \in \mathbb{R}; \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$, then solve for real numbers:

$$\sin x \cdot \sin y \cdot \sin z = \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}$$

Proposed by Daniel Sitaru-Romania

S.1513

$$f, g: \mathbb{R} \to \mathbb{R}, f(1) = 3, g(1) = 2, xf(y) + yf(x) = 2f(xy)$$

$xg(y) + yg(x) = 2g(xy), \forall x, y \in \mathbb{R}$. Find:

$$\Omega = \int_0^1 \frac{f\left(\frac{\sinh x}{3}\right)}{f\left(\frac{\sinh x}{3}\right) + g\left(\frac{\cosh x}{2}\right)} dx$$

Proposed by Daniel Sitaru-Romania

S.1514 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{(n-k+1)H_k}{k(n-k+1)^2 + k}$$

Proposed by Daniel Sitaru,Claudia Nănuți –Romania

S.1515 If $x \geq 0$ then:

$$\frac{e^x}{\sqrt[3]{1+e^{-x}}}+\frac{e^{-x}}{\sqrt[3]{1+e^x}} \geq \frac{2}{\sqrt[3]{1+\operatorname{sech} x}}$$

Proposed by Daniel Sitaru,Dan Nănuți -Romania

S.1516 If the sequence $(x_n)_{n\geq 0}$ is defined by $x_0 = a > 0$ and $x_{n+1} = \frac{4x_n+1}{2x_n+3}, \forall n \in \mathbb{N}^*$, then find x_n and $\lim_{n\to\infty} x_n$

Proposed by Neculai Stanciu-Romania

S.1517 In ΔABC prove or disprove the following relationships:

$$\sum_{cyc} m_a m_b \geq s^2 \text{ or } \sum_{cyc} m_a m_b + 14(R^2 - 4r^2) \geq s^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

S.1518 If $f, g, h: \mathbb{R} \to \mathbb{R}$ are derivable with $f(x_0) = a, g(x_0) = b, h(x_0) = c$ and $(fg)'_{(x_0)} = c_1,\ (gh)'_{(x_0)} = a_1, (hf)'_{(x_0)} = b_1$, then compute $(fgh)'_{(x_0)}$.

Proposed by Neculai Stanciu-Romania

S.1519 If $a \in \mathbb{R}_+^*$ and $a, b, c, d, x, y, z \in \mathbb{R}_+^*, X = x + y + z, cX > d \cdot \max\{x, y, z\}$, then:

$$\sum_{cyc} \frac{aX + bx}{cX - dx} \geq \frac{3(3a+b)}{3c-d}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți –Romania

S.1520 If $a, b \in \mathbb{R}, a < b$ and $f: [a, b] \to \mathbb{R}$ is a continuous function such that $\int_a^b f(x)\,dx = 2021$, then find $\int_a^b x(f(x) + f(a+b-x))\,dx$.

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți –Romania

S.1521 If $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in \mathbb{N}$ is the Lucas's sequence and $a, b, c \in \mathbb{R}_+^*$, then prove that:

$$\frac{L_n}{\sqrt{L_n^2 + aL_{n+1}L_{n+2}}} + \frac{L_{n+1}}{\sqrt{L_{n+1}^2 + bL_{n+2}L_n}} + \frac{L_{n+2}}{\sqrt{L_{n+2}^2 + cL_nL_{n+1}}} \geq 1, \forall n \in \mathbb{N}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1522 If $a, b, c \in (1, \infty)$ and $bc < a^2$ then prove that:

$$\frac{\left(\log_a \frac{a}{b}\right)^2}{\log_a ab} + \frac{\left(\log_a \frac{a}{c}\right)^2}{\log_a ac} + \frac{(\log_a bc)^2}{\log_a \frac{a^2}{bc}} \geq 1$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1523 In ΔABC the following relationship holds:

$$\frac{m_a^2}{b}+\frac{m_b^2}{c}+\frac{m_c^2}{a}\geq 6s\left(\frac{r}{R}\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu, Flaviu Cristian Verde-Romania

S.1524 If $x,y,z\in\mathbb{R}_+^*$ then prove that

$$\frac{xy}{x^3+y^3+xyz}+\frac{yz}{y^3+z^3+xyz}+\frac{zx}{z^3+x^3+xyz}\leq\frac{3}{x+y+z}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1525 If $a,b\in\mathbb{R}_+^*,c,u,v\in\mathbb{R}_+^*,n\in\mathbb{N},n\geq 2,x_k,y_k\in\mathbb{R}_+^*,k=\overline{1,n},X_n=\sum_{k=1}^n x_k$, $Y_n=\sum_{k=1}^n y_k$, where $bY_n+cX_n>v\cdot\max_{1\leq k\leq n}x_k$, then prove the following inequality

$$\sum_{k=1}^n\frac{aX_n+ux_k}{bY_n+cX_n-vx_k}\geq\frac{(an+u)nX_n}{bnY_n+(cn-v)X_n}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.1526 If $(F_n)_{n\geq 0},F_0=F_1=1,F_{n+2}=F_{n+1}+F_n$ and $T_n=\frac{n(n+1)}{2}$ for $n\geq 1$ prove that

$$\sum_{k=1}^n\frac{F_k^4}{T_k}\geq\frac{3F_n^2F_{n+1}^2}{nT_{n+1}}$$

Proposed by D.M.Bătinețu-Giurgiu –Romania

S.1527 If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive real sequences defined by $a_n=\sum_{k=1}^n\frac{1}{k}$ and $\frac{b_{n+3}}{b_n}\left(\frac{b_{n+1}}{b_{n+2}}\right)^3=\left(\frac{e^{a_n}}{n}\right)^3$, $\forall n\in\mathbb{N}^*$ then compute $\lim_{n\to\infty}\sqrt[n^3]{b_n}$.

Proposed by D.M.Bătinețu-Giurgiu –Romania

S.1528 Prove that if $0<a\leq b$ and $x_k\in\left(0,\frac{\pi}{2}\right),\forall k=\overline{1,n}$ then

$$\sum_{k=1}^n(a\cdot\sin x_k+b\cdot\tan x_k)>(a+b)\cdot\sum_{k=1}^n x_k$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1529 If $A_1A_2\dots A_n,n\geq 3$ is a convex polygon, $M\in Int(\Delta ABC)$ is the projection of the interior bisector of angle $\angle A_kMA_{k+1}$ in triangle $A_kMA_{k+1},M_k=pr_{A_kA_{k+1}}M$ and $C_k\in[M_kB_k],k=\overline{1,n}$ then prove that

$$\sum_{k=1}^n MC_k\leq\left(\sum_{k=1}^n MA_k\right)\cos\frac{\pi}{n}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1530 If ABC is a right angled triangle with $\angle A = 90^\circ$ and $(F_n)_{n\geq 0}, F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for any positive integer n, i.e. the Fibonacci sequence, then prove that:

$$\frac{F_m^2}{\left(bF_n + cF_p\right)^2} + \frac{F_n^2}{\left(bF_p + cF_m\right)^2} + \frac{F_p^2}{(bF_m + cF_n)^2} \geq \frac{3}{2a^2}, \forall m, n, p \in \mathbb{N}^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1531 Let $(A, +, \cdot)$ be a ring with $0 \neq 1$ and $x, y \in A$. If $x^2 y = 1 - y$, then prove that

$$x \cdot y = y \cdot x.$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1532 Find:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!\,(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!\,(2n-1)!!}}{n}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.1533 In $\Delta ABC, P \in Int(ABC)$ the following relationship holds:

$$PA \cdot \cos\frac{A}{2} + PB \cdot \cos\frac{B}{2} + PC \cdot \cos\frac{C}{2} \geq \frac{1}{3}\left(n_a + n_b + n_c + 2 \cdot \sum_{cyc} \frac{r_a h_a}{(s + n_a)}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1534 In ΔABC the following relationship holds:

$$\frac{1}{4} \cdot \sum_{cyc} \frac{s}{h_a + r_a} \leq \sqrt{\frac{R}{2r}} \cdot \cos\frac{\pi - A}{4} \cos\frac{\pi - B}{4} \cos\frac{\pi - C}{4}$$

Proposed by Bogdan Fuștei-Romania

S.1535 In ΔABC the following relationship holds:

$$\frac{3s - n_a - n_b - n_c}{6r} \geq \sum_{cyc} \frac{h_a}{s + n_a}$$

Proposed by Bogdan Fuștei-Romania

S.1536 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$\frac{y+z}{a} + \frac{z+x}{b} + \frac{y+z}{c} \geq \sum_{cyc} a^2 x + 4F\sqrt{xy + yz + zx}$$

Proposed by Bogdan Fuștei-Romania

S.1537 If $xy + yz + zx > 0$ and $x + y, y + z, z + x > 0$ then prove:

$$\sum_{cyc} ax(b + c - a) \geq 4F\sqrt{xy + yz + zx}$$

Proposed by Bogdan Fuștei-Romania

S.1538 In $\Delta ABC, n_a$ —Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{n_a m_a}{h_a} \geq \sqrt{\frac{1}{8r^2}\sum_{cyc} a^2(m_b^2+m_c^2-m_a^2)+\frac{3}{2}s^2}$$

Proposed by Bogdan Fuștei-Romania

S.1539 In ΔABC the following relationship holds:

$$\sum_{cyc} m_a\sqrt{a(b+c-a)} \geq \sqrt{\frac{9}{8}\sum_{cyc} a(b+c-a)(b^2+c^2-a^2)+18F^2}$$

Proposed by Bogdan Fuștei-Romania

S.1540 If $xy+yz+zx>0$ and $x+y, y+z, z+x>0$ then prove:

$$ax(b+c)+by(c+a)+cz(a+b) \geq 8F\sqrt{xy+yz+zx}$$

Proposed by Bogdan Fuștei-Romania

S.1541 In ΔABC the following relationship holds:

$$2\left(1-\frac{r}{R}\right) \leq \sum_{cyc}\frac{rr_a}{r_b r_c} \leq \frac{R}{r}-1$$

Proposed by Marin Chirciu-Romania

S.1542 In ΔABC the following relationship holds:

$$\frac{s}{R} \leq \sum_{cyc}\frac{h_b+h_c}{b+c} \leq \frac{s}{2r}$$

Proposed by Marin Chirciu-Romania

S.1543 Solve for real numbers:

$$a^{4(x^2+y)}+a^{4(y^2+x)}=\frac{2}{a}, a>1.$$

Proposed by Marin Chirciu-Romania

S.1544 In ΔABC the following relationship holds:

$$\frac{(4R+r)^2}{8(2R^2+r^2)} \leq \sum_{cyc}\frac{h_a^2}{b^2+c^2} \leq \frac{r^2(5R-4r)^2}{2R^2(2R^2+r^2)}$$

Proposed by Marin Chirciu-Romania

S.1545 In ΔABC the following relationship holds:

$$\frac{9}{8s} \leq \sum_{cyc}\frac{b^2+c^2}{(b+c)^3} \leq \frac{s}{12Rr}$$

Proposed by Marin Chirciu-Romania

S.1546 If $a, b, c > 0$ such that $abc = 1$ and $n \in \mathbb{N}, \lambda \geq 1$ then:

$$\frac{1}{a^n + \lambda} + \frac{1}{b^n + \lambda} + \frac{1}{c^n + \lambda} \leq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

S.1547 In ΔABC the following relationship holds:

$$\frac{s}{R} \leq \sum_{cyc} \frac{\sin^4 B + \sin^4 C}{\sin^3 B + \sin^3 C} \leq \frac{s}{2R}\left[\left(\frac{R}{r}\right)^2 - 2\right]$$

Proposed by Marin Chirciu-Romania

S.1548 Let $a, b, c \in \mathbb{R}$ such that $\left||a|x^3 + |b|x^2 + |c|\right| \leq \pi^2; \forall |x| \leq 1$. Prove that:

$$3|a| + 2|b| \leq 90$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1549 Let $0 < x < 1$. Without softwares, prove that:

$$5x^4 - 8x^3 + 18x^2 - 8x + 1 > 0$$

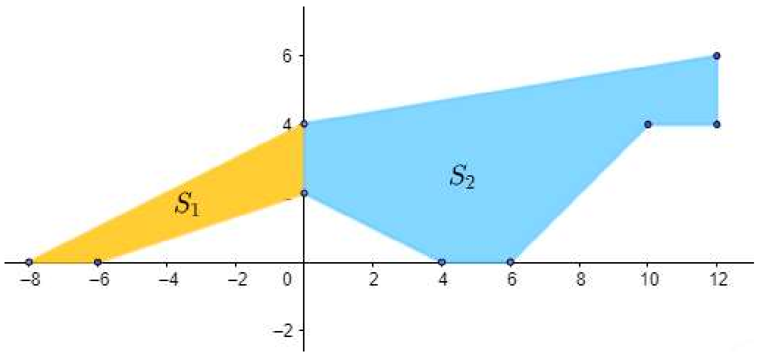
Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1550 Let $a \geq b \geq c$ and $2t = a + b$. Prove that:

$$\frac{1}{\sqrt{4a^2 + bc}} + \frac{1}{\sqrt{4b^2 + ca}} + \frac{1}{\sqrt{4c^2 + ab}} \geq \frac{2}{\sqrt{4t^2 + tc}} + \frac{1}{\sqrt{4c^2 + t^2}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1551 Find area of S_1 and area of S_2.



Let $\alpha \geq 1$. Find the maximum and minimum of the expression $P = x^\alpha + y^\alpha + |x^\alpha - y^\alpha|$;

$\forall (x, y) \in S_1$

Let $\alpha \geq 1$. Find the maximum and minimum of the expression $P = x^\alpha + y^\alpha + |x^\alpha - y^\alpha|$,

$\forall (x, y) \in S_2$.

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1552 On xOy, let $A(0,3), B(-4,0), C(0,-3), D(4,0)$. $S: Int(ABCD)$. Find the maximum and minimum of the expression $Q = x + y + |x - y|; \forall(x,y) \in S$.

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1553 In ΔABC, prove that:

$$\frac{1}{2}\sum_{cyc}\frac{m_a}{m_b} \geq \frac{\sum s_a + \sum g_a + \sum w_a}{\sum n_a + \sum g_a}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1554 If $u, v > 0; x, y, z \in (0,1)$, then in ΔABC the following relationship holds:

$$\frac{a^4}{(1-x^2)(uy+vz)} + \frac{b^4}{(1-y^2)(uz+vx)} + \frac{c^4}{(1-z^2)(ux+vy)} \geq \frac{12\sqrt{3}}{u+v} \cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu,Claudia Nănuți -Romania

S.1555 If $(A, +, \cdot)$ is commutative ring, $a \in A^* = A - \{0\}$ and $f: A \to A$ odd function such that exist $b \in A, f(a^2 + ax + b) = f(x), \forall x \in A$, then f is periodic function on A.

Proposed by D.M. Bătinețu-Giurgiu,Claudia Nănuți –Romania

S.1556 If $a \in (1, \frac{\pi}{2})$ and $m, n \in (0, \infty)$, then find:

$$\int_{\frac{1}{a}}^{a} \frac{\sin^m x \cdot \tan^n\left(\frac{1}{x}\right)}{(1+x^2)\left(\sin^m\left(\frac{1}{x}\right)\tan^n x + \sin^m\left(\frac{1}{x}\right)\tan^n\left(\frac{1}{x}\right)\right)} dx$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.1557 If $(A, +, \cdot)$ is commutative ring and $f: A \to A$ is even function such that exist $a \in A^* = A - \{0\}, b \in A$ with property $f'(x) = f(x^2 + ax + b); \forall x \in A$, then f is periodic function on A.

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.1558 If $(w_n)_{n\geq 1}$ is Wallis sequence, $w_n = \frac{2^2 \cdot 4^2 \cdot \ldots \cdot (2n)^2}{1^2 \cdot 3^2 \cdot \ldots \cdot (2n-1)^2} \cdot \frac{1}{2n+1}$ and $\lim_{n\to\infty} w_n = \frac{\pi}{2}$, find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{(2n-1)!!} \cdot \left(\frac{\pi}{2} - w_n\right)$$

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți –Romania

S.1559 If $(w_n)_{n\ge1}$ is Wallis sequence, $w_n=\frac{2^2\cdot4^2\cdot\ldots\cdot(2n)^2}{1^2\cdot3^2\cdot\ldots\cdot(2n-1)^2}\cdot\frac{1}{2n+1}$ and $\lim_{n\to\infty}w_n=\frac{\pi}{2}$, find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{n!}\cdot\left(\frac{\pi}{2}-w_n\right)$$

Proposed by D.M. Bătineţu-Giurgiu,Gheorghe Boroica-Romania

S.1560 If $x\ge0$ and $(a_n)_{n\ge1}$ is a sequence of positive real numbers such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$, find

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x}-\sqrt[n]{a_n^x}\right)\left(\sqrt[n]{n!}\right)^{1-x}$$

Proposed by D.M. Bătineţu-Giurgiu,Gheorghe Boroica –Romania

S.1561 If $u,v\ge0,u+v=1$ and $f,g:\mathbb{R}_+^*\to\mathbb{R}_+^*$ continuous function such that

$$\lim_{x\to\infty}\frac{f(x+1)}{x\cdot f(x)}=a>0$$

$\lim_{x\to\infty}\frac{g(x+1)}{x\cdot g(x)}=b>0$ and exists $\lim_{x\to\infty}\frac{1}{x}\big(f(x)\big)^{\frac{1}{x}}$, $\lim_{x\to\infty}\frac{1}{x}\big(g(x)\big)^{\frac{1}{x}}$, then find:

$$\Omega=\lim_{x\to\infty}\left(\big(f(x+1)\big)^{\frac{u}{x+1}}-\big(f(x)\big)^{\frac{u}{x}}\right)\big(g(x)\big)^{\frac{v}{x}}$$

Proposed by D.M. Bătineţu-Giurgiu,Nicolae Mușuroia –Romania

S.1562 If $(a_n)_{n\ge1}$ is a sequence of strictly positive real numbers such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$, $(x_n)_{n\ge1},x_n=\sum_{k=1}^n\frac{1}{n+k}$, then find:

$$\Omega=\lim_{n\to\infty}\left(e^{x_{n+1}}-e^{x_n}\right)\sqrt[n]{a_n^2}$$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

S.1563 If $x,y,z\ge0$ and $x+y+z=1$, then find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{(2n+1)!!}\right)^x\left(\sqrt{(n+1)!}\right)^y-\left(\sqrt[n]{(2n-1)!!}\right)^x\left(\sqrt[n]{n!}\right)^y\right)n^z$$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru –Romania

S.1564 If $t\ge0,(a_n)_{n\ge1},(b_n)_{n\ge1}$ are sequences of real numbers such that

$\lim_{n\to\infty}\frac{a_{n+1}}{n^{t+1}\cdot a_n}=a>0$ and $\lim_{n\to\infty}\frac{b_{n+1}}{n^t\cdot b_n}=b>0$, then find: $\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{b_{n+1}}}-\sqrt[n]{\frac{a_n}{b_n}}\right)$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru –Romania

S.1565 If $x,y\ge0,x+y=1,(a_n)_{n\ge1},(b_n)_{n\ge1}$ are sequences of real numbers such that

$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$ and $\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0$, then find: $\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x}-\sqrt[n]{b_n^y}\right)$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru -Romania

S.1566 In ΔABC the following relationship holds:

$$\sum \csc^3 A \csc B \geq \frac{16}{3}$$

Proposed by Marin Chirciu – Romania

S.1567 In ΔABC the following relationship holds:

$$(h_a + h_b + h_c)\left(\frac{1}{h_a + h_b} + \frac{1}{h_b + h_c} + \frac{1}{h_c + h_a}\right) \geq \frac{2r}{R}\left(5 - \frac{r}{R}\right)$$

Proposed by Marin Chirciu – Romania

S.1568 In ΔABC the following relationship holds:

$$8 \leq \frac{(h_a + r_a)(h_b + r_b)(h_c + r_c)}{r_a r_b r_c} \leq \frac{4R}{r}$$

Proposed by Marin Chirciu – Romania

S.1569 In ΔABC the following relationship holds:

$$1 \leq \frac{m_a + m_b + m_c + \lambda(r_a + r_b + r_c)}{(\lambda + 1)(h_a + h_b + h_c)} \leq \frac{R}{2r}$$

where $\lambda \geq 1$.

Proposed by Marin Chirciu – Romania

S.1570 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{bc}\cos^2\frac{A}{2} \leq \sum \frac{r_a}{bc}\cos^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.1571 In non-obtuse ΔABC the following relationship holds:

$$3 \leq \sum \frac{m_a h_a}{h_b h_c} \leq \frac{R^2}{r^2} - \frac{R}{r} + 1$$

Proposed by Marin Chirciu – Romania

S.1572 In ΔABC the following relationship holds:

$$\frac{R - r}{R^2} \leq \sum \frac{h_a}{bc}\tan^2\frac{A}{2} \leq \frac{R - r}{2Rr}$$

Proposed by Marin Chirciu – Romania

S.1573 If $a, b, c > 0$ such that $a^3 + b^3 + c^3 + 3abc = 6$ and $\lambda \geq 0$ then:

$$\sum \frac{a^2 + b^2}{a + \lambda b} \geq \frac{6}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

S.1574 In ΔABC the following relationship holds:

$$\sum r_a^3 \geq (3r)^6 \sum \frac{1}{r_a^3}$$

Proposed by Marin Chirciu – Romania

S.1575 If $a, b, c > 0$ such that $abc = 1$ and $\lambda \geq 0$ then:

$$\frac{b^3 + \lambda c^3}{a^2} + \frac{c^3 + \lambda a^3}{b^2} + \frac{a^3 + \lambda b^3}{c^2} \geq \lambda(a + b + c) + 3$$

Proposed by Marin Chirciu – Romania

S.1576 In ΔABC the following relationship holds:

$$3 \sum a^6 \tan \frac{A}{2} \geq \sum a^6 \cot \frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.1577 In ΔABC the following relationship holds:

$$3 \sum \frac{1}{a^2} \tan \frac{A}{2} \leq \sum \frac{1}{a^2} \cot \frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.1578 Find all $m \in \mathbb{R}$ such that:

$$max\left\{e^{-mx^2} + e^{mx^2}\right\} \leq 2; \forall x \in [-1,1]$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1579 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian. Prove that:

$$2\sum_{cyc} |b - c| m_a \leq \sum_{cyc} n_a^2 + \sum_{cyc} g_a^2 \leq 2\sum_{cyc} m_a^2 + \frac{22s^2(R^4 - 16r^4)}{r^4}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1580 If $x, y, z > 0$ then

$$\sum_{cyc} \left(\frac{x}{y}\right)^3 + \sum_{cyc} \left(\frac{y}{x}\right)^3 \geq \frac{2}{xyz} \cdot \sum_{cyc} x^3 \geq 6$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1581 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian. Prove that:

$$2\sum_{cyc} n_a^2 \leq \sum_{cyc} g_a^2 + \sum_{cyc} m_a^2 + \frac{11s^2(R^4 - 16r^4)}{r^4}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1582 In ΔABC the following relationship holds:

$$\sum_{cyc}\min\{h_a^3, 3h_a-2\}, = \frac{3(s^2+r^2+4Rr)-12R}{2R}$$

$$\min\left\{\sum_{cyc}h_a^2, \sum_{cyc}r_a^2, s^2\right\}+\max\left\{\sum_{cyc}m_a^2, \sum_{cyc}w_a^2, s^2\right\}\le\frac{27R^2}{2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1583 In $\Delta ABC, p_a$ –Spieker's cevian, $\lambda\ge 6$, prove that:

$$\sum_{cyc}p_a^2\le\sum_{cyc}m_a^2+\lambda(R^2-4r^2),\qquad \sum_{cyc}m_a^2+4r(R-2r)\le\sum_{cyc}s_a^2$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1584 In ΔABC the following relationship holds:

$$\sum_{cyc}w_a+\frac{r(R-2r)}{R}\le\sum_{cyc}m_a,\qquad \sum_{cyc}h_a+\frac{2r(R-2r)}{3R}\le\sum_{cyc}r_a$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1585 In $\Delta ABC, v_a$ –Bevan's cevian, prove that:

$$\sum_{cyc}p_a^2+r(R-2r)\le\sum_{cyc}n_a^2\le\min\left\{\sum_{cyc}v_a^2, \sum_{cyc}r_a^2\right\}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1586 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{h_bh_c}+\frac{r(R-2r)}{3R}\le\sum_{cyc}r_a,\qquad \sum_{cyc}\sqrt{r_br_c}+\frac{r(R-2r)}{5R}\le\sum_{cyc}r_a$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1587 In ΔABC the following relationship holds:

$$\sum_{cyc}w_a+\sum_{cyc}|b-c|\ge\sum_{cyc}m_a,\qquad \sum_{cyc}g_a+\sum_{cyc}\frac{|b-c|(s-a)}{b+c}\ge\sum_{cyc}w_a$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1588 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$x^{2021}f(x^{2021})+f(-x^{2021})=x^{2021}+2022, \forall x\in\mathbb{R}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1589 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}}\leq\frac{2}{3}\left(\frac{4R}{r}+1\right)$$

Proposed by Marian Ursărescu-Romania

S.1590 Find:

$$\Omega=\lim_{n\to\infty}n\cdot\int_0^n\frac{\tan^{-1}x\left(\tan^{-1}\left(\frac{x}{n}\right)\right)^2}{x\left(n+x\cdot\tan^{-1}\left(\frac{x}{n}\right)\right)^2}dx$$

Proposed by Florică Anastase-Romania

S.1591 Find:

$$\Omega=\lim_{p\to\infty}\frac{1}{p}\cdot\sum_{m=0}^{p-1}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}-\sqrt{k^2-k+1}\right]\right)^{-1}$$

where $[\cdot]$ denotes great integer function.

Proposed by Florică Anastase-Romania

S.1592 Let $x,y,z>0$ such that $x+y+z+2=5x^3y^3z^3$. Prove that:

$$81(x^5+y^5+z^5)+4x^5y^5z^5\geq(5xyz-2)^5+\frac{4(x+y+z+2)}{5}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1593 If $a_k,b_k\in(1,\infty);k=\overline{1,n}$ such that $\sum_{k=1}^{n}(a_k+b_k)=4n$, then:

$$\sum_{k=1}^{n}\sqrt{n(\log^2a_k+\log^2b_k)+(n^2+1)\log a_k\cdot\log b_k}\leq n(n+1)$$

Proposed by Florică Anastase-Romania

S.1594 In acute $\Delta ABC, AD\perp BC, DE\perp CA, DF\perp AB$. Let φ_a be the radii of circle through B,C,E,F. Similarly are defined φ_b,φ_c. Prove that:

$$\frac{a\varphi_a+b\varphi_b+c\varphi_c}{\tan A+\tan B+\tan C}=F\ \text{ and }\ \frac{1}{h_a\varphi_a}+\frac{1}{h_b\varphi_b}+\frac{1}{h_c\varphi_c}=\frac{2}{F}$$

Proposed by Mehmet Şahin-Turkiye

S.1595 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin B\sin C}{\sqrt{6(\sin^2A+\sin^2B)}}\geq\frac{27}{4}\left(\frac{r}{R+r}\right)^2$$

Proposed by Marian Ursărescu-Romania

S.1596 In acute $\Delta ABC, AD \perp BC, DE \perp CA, DF \perp AB$. Let φ_a be the radii of circle through B, C, E, F. Similarly are defined φ_b, φ_c. Prove that:

$$\varphi_a\varphi_b\varphi_c \geq \left(\frac{F}{R}\right)^3 \text{ and } \frac{1}{r_a\varphi_a}+\frac{1}{r_b\varphi_b}+\frac{1}{r_c\varphi_c}=\frac{2s(R-r)}{F^2}$$

Proposed by Mehmet Şahin-Turkiye

S.1597 Let a, b, c be non-negative real numbers. Prove that:

$$\left(a+\frac{2b+1}{a+b+1}\right)\left(b+\frac{2c+1}{b+c+1}\right)\left(c+\frac{2a+1}{c+a+1}\right) \geq (a+1)(b+1)(c+1)$$

Proposed by Choy Fai Lam-Hong Kong

S.1598 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_bh_c}{\sqrt{2(h_a^2+h_bh_c)}} \geq \frac{9r}{2}$$

Proposed by Marian Ursărescu-Romania

S.1599 Let $a, b, c > 0$ and $\alpha, \beta, \gamma > 0$ (fixed) such that $\alpha x + \beta y + \gamma c = \alpha + \beta + \gamma$. Find the minimum value of: $P = x^4 + y^4 + z^4$.

Proposed by Nguyen Van Canh-BenTre-Vietnam

S.1600 If $a_1, a_2, \dots, a_n > 0$ such that $\sum_{i=1}^{n}\sqrt{a_i} = \sqrt{n}$, then prove that

$$\left(1+\frac{1}{a_2}\right)^{a_1}\cdot\left(1+\frac{1}{a_3}\right)^{a_2}\cdot\ldots\cdot\left(1+\frac{1}{a_1}\right)^{a_n} \geq 1+\frac{n}{\sum_{i=1}^{n}a_i}$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

S.1601 Let $a_i \in (0,\infty), i=\overline{1,n}, n \in \mathbb{N}, a_1+a_2+\cdots+a_n = ne^4$, then:

$$\sqrt{\log a_1^{a_2}+\log a_1^{a_3}+\cdots+\log a_1^{a_n}}+\cdots+\sqrt{\log a_n^{a_1}+\log a_n^{a_2}+\cdots+\log a_n^{a_{n-1}}} \leq 9n^2$$

Proposed by Florică Anastase-Romania

S.1602 Find:

$$\Omega = \lim_{n\to\infty}\left[\sum_{k=1}^{n^2}\frac{1}{\sqrt{k}}\right]\cdot\int_0^{\frac{\pi}{4}}\tan^n x\, dx$$

where $[\cdot]$ is floor function.

Proposed by Florică Anastase-Romania

S.1603 For $f(x)=\int_0^{\tan^{-1}x} e^{\tan^2 t}\,dt$ let be the expression:

$$\Omega=\int_2^3 \sin\frac{\pi}{x}dx+\int_0^1 \frac{xf(x)}{e^{x^2}}dx+\frac{1}{2e}\int_0^1 \frac{e^{x^2}}{1+x^2}dx$$

Find $\alpha,\beta\in\mathbb{R}$ such that $\max\alpha\le\Omega\le\min\beta$.

Proposed by Florică Anastase-Romania

S.1604 Let be the function $f:[0,1]\to\mathbb{R}$ integrable such that $f(1)=1$ and

$$\int_x^y f(t)\,dt=\frac{1}{2}\big(yf(y)-xf(x)\big);\forall x,y\in[0,1].$$

Find: $\Omega=\int_0^{\frac{\pi}{4}} f(x)\cdot\tan^2 x\,dx.$

Proposed by Florică Anastase-Romania

S.1605 Prove that:

$$\sqrt[3]{\cos\frac{\pi}{19}\cos\frac{7\pi}{19}\cos\frac{8\pi}{19}}+\sqrt[3]{\cos\frac{2\pi}{19}\cos\frac{3\pi}{19}\cos\frac{5\pi}{19}}-\sqrt[3]{\cos\frac{4\pi}{19}\cos\frac{6\pi}{19}\cos\frac{9\pi}{19}}=\frac{1}{2}\sqrt[3]{3\sqrt[3]{19}-1}$$

Proposed by Vasile Mircea Popa-Romania

S.1606 Prove that:

$$\cos\frac{\pi}{19}\cos\frac{7\pi}{19}\cos\frac{8\pi}{19}=\frac{\sqrt{19}}{12}\cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{7}{2\sqrt{19}}\right)+\frac{\pi}{3}\right]+\frac{5}{24}$$

Proposed by Vasile Mircea Popa-Romania

S.1607 Find:

$$\Omega=\lim_{n\to\infty}\left[\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}\right]^{-1}\cdot\int_2^4 n^2\left(e^{\frac{x}{n+x}}-1\right)dx$$

where $[\cdot]$ denotes greatest integer function.

Proposed by Florică Anastase-Romania

S.1608 If $x>0, r=pq, 1\le p\le q$ then:

$$1+rx\le(1+qx)^p\le(1+px)^q\le(1+x)^r$$

Proposed by Seyran Ibrahimov-Azerbaijan

S.1609 If $u,v\ge 0, u+v=1$ and $f:(0,\infty)\to(0,\infty)$ continuous function such that

$$\lim_{x\to\infty}\frac{f(x+1)}{x\cdot f(x)}=a>0 \text{ and exist } \lim_{x\to\infty}\frac{1}{x}\big(f(x)\big)^{\frac{1}{x}}, \text{then find:}$$

$$\Omega = \lim_{x\to\infty}\left(\left(f(x+1)\right)^{\frac{u}{x+1}} - \left(f(x)\right)^{\frac{u}{x}}\right)\cdot x^v$$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru -Romania

S.1610 If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0$,

find: $\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x} - \sqrt[n]{a_n^x}\right)n^{1-x}, x \in [0,\infty)$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru -Romania

S.1611 If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that

$\lim_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n} = a > 0$, find:

$$\Omega = \lim_{n\to\infty}\left((n+1)^2\cdot\sqrt[n+1]{a_{n+1}} - n^2\cdot\sqrt[n]{a_n}\right)n^{1-x}, x \in [0,\infty)$$

Proposed by D.M. Bătineţu-Giurgiu,Neculai Stanciu-Romania

S.1612 If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that

$\lim_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n} = a > 0$ and $(w_n)_{n\geq 1}$ Wallis sequence, $w_n = \frac{2^2\cdot 4^2\cdot\ldots\cdot(2n)^2}{1^2\cdot 3^2\cdot\ldots\cdot(2n-1)^2}\cdot\frac{1}{2n+1}$, $\lim_{n\to\infty} w_n = \frac{\pi}{2}$,

then find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{a_n}\left(\frac{\pi}{2} - w_n\right)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.1613 If $x,y,z \geq 0$ such that $x+y+z = 1$ and $(a_n)_{n\geq 1}$ sequence of real numbers strictly positive with $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0$, then find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x\left((n+1)!\right)^y} - \sqrt[n]{a_n^x(n!)^y}\right)\left(\sqrt[n]{(2n-1)!!}\right)^z$$

Proposed by D.M. Bătineţu-Giurgiu,Daniel Sitaru -Romania

S.1614 If $x,y \geq 0, x+y = 1$ and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real numbers strictly positive such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0$, $\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = b > 0$, then find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x} - \sqrt[n]{a_n^x}\right)\cdot\sqrt[n]{b_n^y}$$

Proposed by D.M. Bătineţu-Giurgiu, Claudia Nănuți –Romania

S.1615 If $m,n \geq 0; m+n, x, y > 0$, then in ΔABC the following relationship holds:

$$\frac{a^3}{mxb+nyc} + \frac{b^3}{mxc+nya} + \frac{c^3}{mxa+nyb} \geq \frac{4\sqrt{3}}{mx} \geq \frac{4\sqrt{3}}{mx+ny}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu,Daniel Sitaru –Romania

S.1616 If $x,y\geq 0$ and $(a_n)_{n\geq 1}$ sequence of real numbers strictly positive such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0, \text{then find: } \Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x}-\sqrt[n]{a_n^x}\right)\left(\sqrt[n]{(2n-1)!!}\right)^y$$

Proposed by D.M. Bătinețu-Giurgiu,Daniel Sitaru -Romania

S.1617 Let be positive numbers $x_1\leq x_2\leq\cdots\leq x_n$. If $x_1x_2\cdot\ldots\cdot x_n=n^n$ then:

$$(1+e^{x_1})(1+e^{2x_2})(1+e^{3x_3})\cdot\ldots\cdot(1+e^{nx_n})\geq\left(1+\sqrt{e^{n(n+1)}}\right)^n$$

Proposed by Marin Chirciu-Romania

S.1618 Solve for real numbers:

$$6+\log_{80}\frac{x}{x^5+128}=x+\frac{3}{\sqrt[3]{x-1}}$$

Proposed by Marin Chirciu-Romania

S.1619 In ΔABC the following relationship holds:

$$12r\leq\sum\frac{a^2}{\sqrt{m_bm_c}}\leq 3R\sqrt{\frac{2R}{r}}$$

Proposed by Marin Chirciu-Romania

S.1620 In ΔABC the following relationship holds:

$$\sum\frac{r_a}{\sin^2 A}\geq 6R$$

Proposed by Marin Chirciu-Romania

S.1621 Let be $m,n\in\mathbb{N}, n\geq 2, m\geq 2, n\neq m$. Solve for real numbers:

$$\frac{\sqrt[n]{x-1}+x}{\sqrt[n]{x-1}-x}=\frac{\sqrt[m]{x-1}+x}{\sqrt[m]{x-1}-x}$$

Proposed by Marin Chirciu-Romania

S.1622 If $x,y,z>0$ such that $x+y+z=3$ and $\lambda\geq 0$ then:

$$\frac{1}{\sqrt{x}+\lambda\sqrt{y}}+\frac{1}{\sqrt{y}+\lambda\sqrt{z}}+\frac{1}{\sqrt{z}+\lambda\sqrt{x}}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

S.1623 If $a,b,c>0$ then:

$$\sum\frac{a}{\sqrt[4]{a^2+80bc}}\geq 1$$

Proposed by Marin Chirciu-Romania

S.1624 If $a, b, c > 0$ such that $(a+b)^{n+1} + (b+c)^{n+1} + (c+a)^{n+1} = 6 \cdot 2^n, n \in \mathbb{N}$,

$n \geq 2$ then:

$$\sum (b+c)(b^n + c^n) \geq 3 \cdot 2^n$$

Proposed by Marin Chirciu-Romania

S.1625 In ΔABC the following relationship holds:

$$\sum \frac{\sin^{2n+1}\frac{A}{2}}{\sin^{2n-1}\frac{B}{2}} \geq 1 - \frac{r}{2R}, n \in \mathbb{N}^*$$

Proposed by Marin Chirciu-Romania

S.1626 If $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$ and $\lambda \geq 0$ then

$$\sum \frac{a^2}{bc(a+\lambda b)} \geq \frac{3}{(\lambda+1)abc}$$

Proposed by Marin Chirciu-Romania

S.1627 In ΔABC the following relationship holds:

$$\frac{1}{2R^3} \leq \sum \frac{h_a}{a^4} \leq \frac{1}{16r^3}.$$

Proposed by Marin Chirciu-Romania

S.1628 Find the value of $\alpha > 2$ for which the area bounded by

$$y = \frac{1}{x}, y = \frac{1}{2x-1}, x = 2, x = \alpha \text{ is } \log\left(\frac{4}{\sqrt{5}}\right).$$

Proposed by Sridhar Rao-India

S.1629 If $n \in \mathbb{N}$ and $I(n) = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} \frac{2k+1}{(2k+1)^2+1}$ then $\frac{I(n)}{I(n-1)} = \frac{n(2n+1)}{n^2+(n+1)^2}$.

Proposed by Rohan Shinde-India

S.1630 Prove that $n \in \mathbb{N}$ and $x \in \mathbb{R}$: $\exists! P(x) \in \mathbb{Q}[x], \deg P(x) = 2$ such that

$$P(x) + P(x+1) + \cdots + P(x+n) = x^2.$$

Proposed by Gantumur Choijilsuren-Mongolia

S.1631 Prove that:

$$\lim_{n\to\infty} \left(\frac{\sqrt{n^2-n}}{n} + \frac{1}{4}\sin\left(\frac{2}{n}\right)\right)^{n^2+\sin(3n)} = e^{-\frac{1}{8}}.$$

Proposed by Abdul Mukhtar-Nigeria

S.1632 Find:

$$\Omega = \lim_{n\to\infty} n^2\left(\int_0^1 \frac{dx}{x^2+2x\cos\left(\frac{\pi}{n}\right)+1} - \lim_{n\to\infty}\int_0^1 \frac{dx}{x^2+2x\cos\left(\frac{\pi}{n}\right)+1}\right).$$

Proposed by Max Wong-Hong Kong

S.1633 Prove that:

$$\sum_{i+j+k+s=m}\left(\binom{n}{i}\binom{n}{j}\binom{n}{k}\binom{n}{s}\right) = \binom{4n}{m}.$$

Proposed by Adil Abdullayev-Azerbaijan

S.1634 In acute ΔABC the following relationship holds:

$$\frac{2h_a^2}{w_a^2} \le \frac{h_a}{m_a} + 1.$$

Proposed by Adil Abdullayev-Azerbaijan

S.1635 In ΔABC the following relationship holds:

$$\left(\frac{m_a}{a}\right)^4 + \left(\frac{m_b}{b}\right)^4 + \left(\frac{m_c}{c}\right)^4 \ge \frac{27}{16}.$$

Proposed by Adil Abdullayev-Azerbaijan

S.1636 In $\Delta ABC, x, y, z > 0$ holds:

$$\sum \frac{n_a}{h_a^2}\cdot x \ge \sqrt{xy+yz+zx}\sqrt{2\sum \frac{n_a n_b}{h_a^2 h_b^2} - \sum \frac{n_a^2}{h_a^4}}.$$

Proposed by Bogdan Fuștei-Romania

S.1637 In $\Delta ABC, \Delta A_1B_1C_1, P \in Int(\Delta A_1B_1C_1)$ holds:

$$m_a h_b h_c \cdot PB_1PC_1 + m_b h_a h_c \cdot PA_1PC_1 + m_c h_a h_b \cdot PA_1PB_1 \ge m_a m_b m_c h_a h_b h_c$$

Proposed by Bogdan Fuștei-Romania

S.1638 In ΔABC the following relationship holds:

$$\sum \cos^2\frac{A}{2}\left(\frac{\sqrt{n_b}}{h_b} + \frac{\sqrt{n_c}}{h_c} - \frac{\sqrt{n_a}}{h_a}\right) \ge$$

$$\ge \sqrt{2\sum \cos^2\frac{A}{2}\cos^2\frac{B}{2} - \sum \cos^4\frac{A}{4}}\cdot\sqrt{2\sum \frac{\sqrt{n_b n_c}}{h_b h_c} - \sum \frac{n_a}{h_a^2}}$$

Proposed by Bogdan Fuștei-Romania

S.1639 In $\Delta ABC, x, y, z > 0$ holds:

$$\sum\left(\frac{m_a}{h_a}\right)^2 \geq \sqrt{xy+yz+zx}\sqrt{2\sum\frac{m_a^2 m_b^2}{h_a^2 h_b^2} - \sum\left(\frac{m_a}{h_a}\right)^4}$$

Proposed by Bogdan Fuștei-Romania

S.1640 If $x, y, z > 0, x + y + z = 3$ then:

$$\frac{1}{x^9} + \frac{1}{y^9} + \frac{1}{z^9} \geq x^3 + y^3 + z^3.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1641 If $x^9 + 1 = (x+1)(x^2+ax+1)(x^2+bx+1)(x^2+cx+1)(x^2+dx+1)$, $\forall x \in \mathbb{C}$. Find:

$$\Omega = a^6 + b^6 + c^6 + d^6.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1642 In ΔABC the following relationship holds:

$$\frac{2}{3\sqrt{3}}\left(\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c}\right) \leq \frac{R}{2r}.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1643 In ΔABC the following relationship holds:

$$\left(\frac{h_a}{r_b} + \frac{h_b}{r_c} + \frac{h_c}{r_a}\right)\left(\frac{h_b}{r_a} + \frac{h_c}{r_b} + \frac{h_a}{r_c}\right) \leq 9.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1644 If $x, y, z > 0$ then

$$\frac{x+y}{z+y} + \frac{y+z}{x+z} + \frac{z+x}{y+x} \leq \frac{3(x+y)(y+z)(z+x)}{8xyz}.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1645 If $x, y, z > 0$ then:

$$\frac{x}{y}\sqrt{\frac{z+x}{z+y}} + \frac{y}{z}\sqrt{\frac{x+y}{x+z}} + \frac{z}{x}\sqrt{\frac{z+y}{x+y}} + \frac{8xyz}{(x+y)(y+z)(z+x)} \geq 4.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1646 If $x, y, z > 0$ then:

$$\prod(3x^2 + 10xy + 3y^2) \geq 512xyz(x+y)(y+z)(z+x).$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1647 In ΔABC the following relationship holds:

$$\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\ge\frac{3R}{2r}.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1648 If $a,b>0$ then:

$$\frac{a}{b}+\frac{b}{a}\ge\frac{2\sqrt[4]{8(a^4+b^4)}}{a+b}.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1649 In ΔABC the following relationship holds:

$$\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\le\frac{R}{r}.$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1650 In acute $\Delta ABC, P\in Int(\Delta ABC), D\in(BC), E\in(CA), F\in(AB), AP=m,$ $BP=n, CP=p, m(\sphericalangle BPD)=m(\sphericalangle CPE)=m(\sphericalangle APF)>90^\circ$. Prove that:

$$ma^2+nb^2+pc^2\le\frac{(m+n+p)^2}{9}+\frac{(m+n)^3+(n+p)^3+(p+m)^3}{4}.$$

Proposed by Mehmet Șahin-Turkiye

S.1651 O −circumcenter of acute ΔABC. Let ΔEFK be the medial triangle, $E\in(BC)$, $F\in(CA), K\in(AB0, EO\cap AC=\{P\}, FO\cap AB=\{D\}, KO\cap BC=\{Q\}$. Prove:

$$OD\cdot OP\cdot OQ=R^3.$$

Proposed by Mehmet Șahin-Turkiye

S.1652 If $a,b,c\ge 0, a+b+c=2$ then:

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\ge 2.$$

Proposed by Mehmet Șahin-Turkiye

S.1653 In $\Delta ABC, AD, BE, CF$ −altitudes, I_1,I_2,I_3,I_4,I_5,I_6 −incenters of $\Delta ABD,\Delta ACD,\Delta BCE,\Delta BAE,\Delta CAF$, respectively ΔCBF. Prove that I_1I_2, I_3I_4, I_5I_6 can be sides in a triangle.

Proposed by Mehmet Șahin-Turkiye

S.1654 In acute $\Delta ABC, AD, BE, CF$ −altitudes, r_1,r_2,r_3,r_4,r_5,r_6 −inradii of $\Delta ABD,\Delta ACD,\Delta BCE,\Delta BAE,\Delta ACF$ respectively ΔBCF. Prove that:

$$r_1+r_2+r_3+r_4+r_5+r_6+s=h_a+h_b+h_c.$$

Proposed by Mehmet Șahin-Turkiye

S.1655 Find:

$$\Omega = \lim_{p\to\infty}\left[\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)\right]^{H_p}$$

where H_p – harmonic number

Proposed by Florică Anastase – Romania

S.1656 Solve for real numbers:

$$\sum_{k=1}^{n}\frac{x+\sin k}{x+\sin\frac{n}{2}\cos\left(\frac{n}{2}-k\right)}+n-3=0, n\in\mathbb{N}^*$$

Proposed by Florică Anastase – Romania

S.1657 In ΔABC the following relationship holds:

$$\frac{c(h_a^2+h_b^2)}{r_a^2+r_b^2}+\frac{a(h_b^2+h_c^2)}{r_b^2+r_c^2}+\frac{b(h_c^2+h_a^2)}{r_c^2+r_a^2}\le\frac{3\sqrt{3}R(R-r)}{r}$$

Proposed by Ertan Yildirim-Turkiye

S.1658

$$x_1=1, x_2=2, (n-1)x_n+nx_{n-1}=n(n-1)\log\left(\frac{x_{n-1}^2+(n-1)^2}{x_{n-2}^2+(n-2)^2}\right), n\ge 3$$

Find:

$$\Omega=\lim_{n\to\infty}\frac{x_n(\sqrt[3]{\gamma}-1)^2}{n}$$

Proposed by Ruxandra Daniela Tonilă – Romania

S.1659 Let $\lambda\in\mathbb{R}$ fixed. Solve for real numbers:

$$\begin{cases} ax+by=(\lambda+1)^2+\lambda^2\\ ax^2+by^2=(\lambda+1)^3+\lambda^3\\ ax^3+by^3=(\lambda+1)^4+\lambda^4\\ ax^4+by^4=(\lambda+1)^5+\lambda^5\end{cases}$$

Proposed by Marin Chirciu – Romania

S.1660 Given $a,x,y,z\in\mathbb{R}$ such that $xy+yz+zx-z-2a^2=x+y+z-2a=0$.Prove $|axyz|<\sqrt{3}$

Proposed by Dang Le Gia Khanh – Vietnam

S.1661 Given (u_n): $\begin{cases} u_1=4\\ u_{n+1}=\sqrt{u_n+6}+\frac{3-u_n}{2+\sqrt{u_n+1}} \quad (n\ge 1)\end{cases}$

Let $y_n = \sum_{k=1}^{n} u_k$. Find $\lim\limits_{n\to\infty} \frac{y_n}{n}$

Proposed by Dang Le Gia Khanh – Vietnam

S.1662 Prove the inequality:

$$\ln\frac{\pi}{7} > \cos e > \ln\frac{1}{\pi}$$

Proposed by Olimjon Jalilov – Uzbekistan

S.1663 Find:

$$\Omega = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{(n-i)(n-j)} \cdot \log\frac{n}{i} \cdot \log\frac{n}{j}$$

Proposed by Mohammad Nasery-Afghanistan

S.1664 Find:

$$\Omega(x) = \lim_{n\to\infty} n \cdot \tan^{-1}\left[\left\{\frac{1}{(x^2+1)n+1}\right\}\tan\left(\frac{\pi}{4} - \frac{x}{2n}\right)\right]^n$$

Proposed by Mohammad Nasery-Afghanistan

S.1665 Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt{n + \sqrt{n-1+\sqrt{n-2}+\cdots+\sqrt{2+\sqrt{1}}}} - \sqrt{n}\right)$$

Proposed by Mohammad Nasery-Afghanistan

S.1666 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{e^{\frac{1}{n}} + e^{-\frac{1}{n}}}{2^{\frac{1}{n}} + 3^{\frac{1}{n}}}\right)^{n\left(\frac{4}{3}+\frac{10}{9}+\frac{82}{81}+\cdots+\frac{1+3^{2^n}}{3^{2^n}}\right)}$$

Proposed by Mohammad Nasery-Afghanistan

S.1667 $D \in (BC), O$ – circumcentre in ΔABC. Prove that:

$$b^2 + c^2 + OD^2 + BD \cdot DC + as + n_a w_a \geq s^2 + a^2$$

Proposed Radu Diaconu – Romania

S.1668 Let $f(x) = \sum_{n=1}^{2019} x^n$ and $\alpha = f(2020)(mod\ 1000); \eta = f(2019)(mod\ 1000)$

i) Evaluate $\mu(\eta) \cdot \tau(\alpha)$

ii) Find the number of digit in η^α

iii) Determine which of $(\mathbb{Z}_\alpha,\oplus,\otimes)$ and $(\mathbb{Z}_\eta,\oplus,\otimes)$ is an integral domain

Where μ is Mobius function and $\tau(n)$ is the number of divisors of

Proposed by Fawole Abdulrasheed-Nigeria

S.1669 Find without softs $(n \in \mathbb{N})$:

$$\Omega_1(n) = \int \sin^n x \sin\big((n+2)x\big)\,dx, \Omega_2(n) = \int \sin^n x \cos\big((n+2)x\big)\,dx$$

$$\Omega_3(n) = \int \cos^n x \sin\big((n+2)x\big)\,dx, \Omega_4(n) = \int \cos^n x \cos\big((n+2)x\big)\,dx$$

Proposed by Mustapha Isaah-Ghana

S.1670 In ΔABC the following relationship holds:

$$w_a > s_a \Leftrightarrow 1 + \cos A > \frac{a^2}{2(b^2+c^2)}$$

Proposed by Adil Abdullayev, Rahim Shahbazov-Azerbaijan

S.1671 Solve for real numbers:

$$2\left(e^x - e^{\frac{1}{x}}\right) = \left(x - \frac{1}{x}\right)\left(e^x + e^{\frac{1}{x}}\right)$$

Proposed by Ionuț Florin Voinea – Romania

S.1672 Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\log(n+k) - \log n}{n+k}$$

Proposed by Adil Abdullayev-Azerbaijan

S.1673 In acute $\Delta ABC, o_a$ –circumcevian, the following relationship holds:

$$\frac{o_a}{w_a} \ge \frac{w_a}{h_a}$$

Proposed by Adil Abdullayev-Azerbaijan

S.1674 Find:

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\cos 2021x}{\cos^{2023} x}\,dx$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

S.1675 Let $a \ge 0$ be a real number and $n \in \mathbb{N}^*$. Prove that:

$$\sqrt{a - 2\sqrt{a} + 2} + \sqrt{a - 2\sqrt{2}a + 6} + \cdots + \sqrt{a - 2\sqrt{na} + n(n+1)} \ge \frac{n(n+1)}{2}$$

Proposed by Marin Chirciu – Romania

S.1676 If $a,b,c,\lambda>0$ such that $a+b+c=6\lambda abc$ then:

$$\sqrt{\lambda+\frac{1}{a^2}}+\sqrt{\lambda+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}\geq\sqrt{27\lambda}$$

Proposed by Marin Chirciu – Romania

S.1677 In ΔABC the following relationship holds:

$$\sum\frac{1}{a}\cot^2\frac{A}{2}\geq 9\sum\frac{1}{a}\tan^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

S.1678 If $a,b,c>0$ such that $abc=1$ and $\lambda\geq 0$ then:

$$\left(a+\frac{\lambda}{b}\right)^2+\left(b+\frac{\lambda}{c}\right)^2+\left(c+\lambda\frac{1}{a}\right)^2\geq(2\lambda+1)(a+b+c)+3\lambda^2$$

Proposed by Marin Chirciu – Romania

S.1679 If $a,b>0$ then:

$$\left(a^{\sqrt{ab}}+b^{\sqrt{ab}}\right)\sqrt{(a+b)^{a+b}}\geq\left(\sqrt{a^{a+b}}+\sqrt{b^{a+b}}\right)(a+b)^{\sqrt{ab}}$$

Proposed by Daniel Sitaru-Romania

S.1680 Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{(2n+1)!}{(2n+1-k)!}\right)\left(\sum_{k=0}^{n}\frac{\left((2n+1)!\right)^2}{k!\left((2n+1-k)!\right)^2}\right)^{-1}$$

Proposed by Daniel Sitaru-Romania

S.1681 Solve for real numbers:

$$\begin{cases}4x^4=y(x^6+x^4+x^2+1)\\5y^5=z(y^8+y^6+y^4+y^2+1)\\6z^6=x(z^{10}+z^8+z^6+z^2+1)\end{cases}$$

Proposed by Daniel Sitaru-Romania

S.1682 In ΔABC the following relationship holds:

$$\frac{m_a^2+m_ah_a+h_a^2}{m_a+h_a}\geq\frac{3}{2}h_a$$

Proposed by Daniel Sitaru-Romania

S.1683 If $a,b,c,d,e>0$ then:

$$a+\frac{ad}{b\cot^2\frac{\pi}{20}}+\frac{bd}{c\cot^2\frac{3\pi}{20}}+\frac{d^2}{e\cot^2\frac{7\pi}{20}}+\frac{ed}{a\cot^2\frac{9\pi}{20}}\geq 5d$$

Proposed by Daniel Sitaru-Romania

S.1684 Find:

$$\Omega = \lim_{n\to\infty}(1+n)^{-n}\sum_{k=1}^{n-1}\left(\binom{n}{k}\sum_{m=1}^{n}\frac{m^n}{m^k}\right)$$

Proposed by Daniel Sitaru-Romania

S.1685 Find all $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x)f(yz)+9\le f(xy)+5f(xz);\forall x,y,z\in\mathbb{R}$$

Proposed by Daniel Sitaru-Romania

S.1686 Find:

$$\Omega = \lim_{n\to\infty}\binom{2n-2}{n-1}^{-1}\sum_{k=0}^{n}\left(1-\frac{2k}{n}\binom{n}{k}\right)^2$$

Proposed by Daniel Sitaru-Romania

S.1687 If $a,b,c,x,y>0$ then:

$$\frac{a^{\sqrt{xy}}+b^{\sqrt{xy}}+c^{\sqrt{xy}}}{\sqrt{a^{x+y}}+\sqrt{b^{x+y}}+\sqrt{c^{x+y}}}\le\frac{\frac{1}{\sqrt{a^{x+y}}}+\frac{1}{\sqrt{b^{x+y}}}+\frac{1}{\sqrt{c^{x+y}}}}{\frac{1}{a^{\sqrt{xy}}}+\frac{1}{b^{\sqrt{xy}}}+\frac{1}{c^{\sqrt{xy}}}}$$

Proposed by Daniel Sitaru-Romania

S.1688 In $\Delta ABC,\omega_1=\max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}+\min\left\{\frac{m_am_b}{h_ah_b},\frac{m_bm_c}{h_bh_c},\frac{m_cm_a}{h_ch_a}\right\},\omega_2=$ $\max\left\{\frac{n_a}{h_a},\frac{n_b}{h_b},\frac{n_c}{h_c}\right\}-\min\left\{\frac{n_a}{h_a},\frac{n_b}{h_b},\frac{n_c}{h_c}\right\}$ the following relationship holds:

$$\frac{s}{r}\ge\sqrt{3(4\omega_1-1)+2\sum_{cyc}\frac{n_an_b}{h_ah_b}+\frac{3}{2}\omega_2^2}$$

Proposed by Bogdan Fuștei-Romania

S.1689 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{m_bm_c}{m_aw_a}}\ge\sum_{cyc}\frac{(b+c)\sqrt{(s-b)(s-c)}}{bc}$$

Proposed by Bogdan Fuștei-Romania

S.1690 In ΔABC the following relationship holds:

$$2\sum_{cyc}m_ah_a\ge\sum_{cyc}h_bh_c\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)$$

Proposed by Bogdan Fuștei-Romania

S.1691

Let $q=\prod_{cyc}\frac{m_aw_a}{n_ag_a},\omega_1=\sum_{cyc}\left(\frac{m_a}{h_a}\right)^q,\omega_2=\left(\frac{m_a}{h_a}\right)^q+\left(\frac{m_b}{h_b}\right)^q-\left(\frac{m_c}{h_c}\right)^q,$

$\omega_3=\left(\frac{m_a}{h_a}\right)^q-\left(\frac{m_b}{h_b}\right)^q+\left(\frac{m_c}{h_c}\right)^q,\omega_4=-\left(\frac{m_a}{h_a}\right)^q+\left(\frac{m_b}{h_b}\right)^q+\left(\frac{m_c}{h_c}\right)^q,\alpha_1=\sum_{cyc}\frac{m_a}{h_a}$

$\alpha_2=\frac{m_a}{h_a}+\frac{m_b}{h_b}-\frac{m_c}{h_c},\alpha_3=\frac{m_a}{h_a}-\frac{m_b}{h_b}+\frac{m_c}{h_c},\alpha_4=-\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}$, then in ΔABC holds:

$$\sqrt{\omega_1\omega_2\omega_3\omega_4}\ge\left(\frac{\sqrt{3}}{4}\right)^{1-q}\cdot\left(\frac{1}{4}\right)^{q-1}\cdot\left(\sqrt{\alpha_1\alpha_{22}\alpha_3\alpha_4}\right)^q$$

Proposed by Bogdan Fuștei-Romania

S.1692 In $\Delta ABC, I$ —incenter and $\omega=\max\left\{\left(\sqrt{\frac{n_a}{h_a}}-\sqrt{\frac{n_b}{h_b}}\right)^2,\left(\sqrt{\frac{n_b}{h_b}}-\sqrt{\frac{n_c}{h_c}}\right)^2,\left(\sqrt{\frac{n_c}{h_a}}-\sqrt{\frac{n_a}{h_a}}\right)^2\right\}$

the following relationship holds: $\frac{a}{AI}+\frac{b}{BI}+\frac{c}{CI}\ge 3\cdot\sqrt[3]{\prod_{cyc}\frac{n_a}{h_a}}+2\sum_{cyc}\frac{r_a}{s+n_a}+\omega$

Proposed by Bogdan Fuștei-Romania

S.1693 In $\Delta ABC,\omega_1=\min\left\{\frac{m_am_b}{h_ah_b},\frac{m_bm_c}{h_bh_c},\frac{m_cm_a}{h_ch_a}\right\},\omega_2=\max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}$ holds:

$$\frac{m_am_bm_c(m_a+m_b+m_c)}{9F^2}\ge\frac{1}{2}(\omega_1+\omega_2)$$

Proposed by Bogdan Fuștei-Romania

S.1694 Find:

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos 2013}{\cos^{2025}x}dx$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți -Romania

S.1695 If $t,u,v,x,y,z>0$ and $tuv\ge 1$, then in ΔABC holds:

$$\left(ty+u+\frac{v}{x}\right)a^4+\left(uz+v+\frac{t}{y}\right)b^4+\left(vx+t+\frac{u}{z}\right)c^4\ge 1296r^2$$

Proposed by D.M.Bătinețu-Giurgiu,Florică Anastase –Romania

S.1696 Let $a>0$ and $f:\mathbb{R}\to\mathbb{R}_+^*=[0,\infty)$ continuous such that $f(x)\cdot f(-x)=1$, then find:

$$\Omega=\int_{-a}^{a}\frac{1}{(x^2+2021)(1+f(x))}dx$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.1697 If $x, y \in [1, \infty)$, then:

$$2\sqrt{x} \le x^{\sin^2 y} + x^{\cos^2 y} \le x + 1$$

Proposed by D.M.Bătineţu-Giurgiu,Florică Anastase-Romania

S.1698 If $t, u, v, x, y, z > 0$ and $tuv = w^3 \ge 1$, then in ΔABC holds:

$$\left(ty + u + \frac{v}{x}\right)a^2 + \left(uz + v + \frac{t}{y}\right)b^2 + \left(vx + t + \frac{u}{z}\right)c^2 \ge 12w\sqrt{3} \cdot F$$

Proposed by D.M.Bătineţu-Giurgiu,Neculai Stanciu-Romania

S.1699 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x}{(y+z)h_a} + \frac{y}{(z+x)h_b} + \frac{z}{(x+y)h_c} \ge \frac{3\sqrt{3}}{2s}$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți Romania

S.1700 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x}{(y+z)h_a} + \frac{y}{(z+x)h_b} + \frac{z}{(x+y)h_c} \ge \frac{\sqrt[4]{27}}{2\sqrt{F}}$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

S.1701 If $m \ge 0$ and $xyz = 1$, then in ΔABC the following relationship holds:

$$\frac{(x+y)^m}{z} \cdot a^4 + \frac{(y+z)^m}{x} \cdot b^4 + \frac{(z+x)^m}{y} \cdot c^4 \ge 2^{m+4} \cdot F^2$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți -Romania

S.1702 If $m \ge 0; x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{x^{m+1}a^{m+2}}{(y+z)^{m+1}h_a^m} + \frac{y^{m+1}b^{m+2}}{(z+x)^{m+1}h_b^m} + \frac{z^{m+1}c^{m+2}}{(x+y)^{m+1}h_c^m} \ge 2\left(\sqrt{3}\right)^{1-m} \cdot F$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți –Romania

S.1703 If $m \ge 0, M \in Int(\Delta ABC), x = MA, y = MB, z = MC$, then:

$$\frac{(xy)^{m+1}(m_a h_a)^m}{ab} + \frac{(yz)^{m+1}(m_b h_b)^m}{bc} + \frac{(zx)^{m+1}(m_c h_c)^m}{ca} \ge \frac{4^m}{3^m} \cdot F^{2m}$$

Proposed by D.M.Bătineţu-Giurgiu,Claudia Nănuți -Romania

S.1704 If $x, y, z > 0$, then in ΔABC the following relationship holds:

$$(x^2a^4 + y^2b^4 + z^2c^4)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \ge 12F^2$$

Proposed by D.M.Bătineţu-Giurgiu,Neculai Stanciu-Romania

S.1705 If $x \in \left(0, \frac{\pi}{2}\right), M \in Int(\Delta ABC), u = MA, v = MB, w = MC$, then:

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2} \geq \sin 2x + \sum_{cyc} \left(\frac{u}{a}\sin x - \frac{v}{b}\cos x\right)^2$$

Proposed by D.M.Bătineţu-Giurgiu,Florică Anastase-Romania

S.1706 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$(n_a + m_b)c^3 + (w_b + h_c)b^3 + (n_c + n_a)a^3 \geq 16\sqrt{3} \cdot F^2$$

Proposed by D.M.Bătineţu-Giurgiu,Florică Anastase –Romania

S.1707 In ΔABC, the following relationship holds:

$$3961(a^2 + b^2 + c^2) \geq 8 \cdot 44 \cdot 45 \cdot \sqrt{3} \cdot F + \sum_{cyc}(45a - 44b)^2$$

Proposed by D.M.Bătineţu-Giurgiu,Neculai Stanciu -Romania

S.1708 If $m, n \geq 0$ and $x, y, z > 0$, then in ΔABC the following relationship holds:

$$x^m \cdot a^{2n} + y^m \cdot b^{2n} + z^m \cdot c^{2n} \geq 2^{2n}\left(\sqrt{3}\right)^{2-n} \cdot \sqrt[3]{(xyz)^m} \cdot F^n$$

Proposed by D.M.Bătineţu-Giurgiu-Romania

S.1709 If $u, v, w, x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{(x+y)(w+u)}{vz} \cdot a^4 + \frac{(y+z)(u+v)}{wx} \cdot b^4 + \frac{(z+x)(v+w)}{uy} \cdot c^4 \geq 64F^2$$

Proposed by D.M.Bătineţu-Giurgiu,Florică Anastase -Romania

S.1710 If $u, v, w, x, y, z > 0$, then in ΔABC the following relationship holds:

$$\frac{(x+y)(w+u)}{vz} \cdot a^2 + \frac{(y+z)(u+v)}{wx} \cdot b^2 + \frac{(z+x)(v+w)}{uy} \cdot c^2 \geq 16\sqrt{3} \cdot F$$

Proposed by D.M.Bătineţu-Giurgiu,Dan Nănuți -Romania

S.1711 If $a_1, a_2, \dots, a_n > 0, n \in \mathbb{N} - \{0\}, a_1 + a_2 + \dots + a_n = 1$ then:

$$\lim_{x \to 0} \frac{1 - \cos(xa_1)\cos(xa_2) \cdot \dots \cdot \cos(xa_n)}{x^2} \geq \frac{1}{2n}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1712 If in $\Delta ABC, AX, BY, CZ$ are Nagel's cevians, AD, BE, CF are Gergonne's cevians then:

$$[XYZ] = [DEF]$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1713 In ΔABC the following relationship holds:

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\ge\frac{h_b}{r_a}+\frac{h_c}{r_b}+\frac{h_a}{r_c}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1714 If $x,y,z>0$ then:

$$\frac{x^3}{yz}+\frac{y^3}{zx}+\frac{z^3}{xy}\ge 3\cdot\sqrt[3]{\frac{x^{12}+y^{12}+z^{12}}{3}}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1715 In ΔABC the following relationship holds:

$$\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}+\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{A}{2}}+\frac{2r}{R}\ge 4$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1716 If $x,y,z>0, x^6+y^6+z^6=x^4+y^4+z^4$ then:

$$\frac{x^4}{y^2+yz+z^2}+\frac{y^4}{z^2+zx+x^2}+\frac{z^4}{x^2+xy+y^2}\ge 1$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1717 In ΔABC the following relationship holds:

$$\frac{w_a^2}{bc}+\frac{w_b^2}{ca}+\frac{w_c^2}{ab}\le 2+\frac{2abc}{(a+b)(b+c)(c+a)}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1718 If $x,y,z>0$ then:

$$\frac{x^3}{y^2+yz+z^2}+\frac{y^3}{z^2+zx+x^2}+\frac{z^3}{x^2+xy+y^2}\ge\sqrt[4]{\frac{x^4+y^4+z^4}{3}}$$

Proposed by Rahim Shahbazov-Azerbaijan

S.1719 In $\Delta ABC, I$ – incenter, I_a, I_b, I_c -excenters, holds:

$$\frac{II_a^2}{\sec^2\frac{A}{2}}+\frac{II_b^2}{\sec^2\frac{B}{2}}+\frac{II_c^2}{\sec^2\frac{C}{2}}=a^2+b^2+c^2$$

Proposed by Ertan Yildirim-Turkiye

S.1720 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin A}{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}=\frac{8r(R+r)}{r}$$

Proposed by Ertan Yildirim-Turkiye

S.1721 In ΔABC the following relationship holds:

$$12R \leq \frac{(a+b)^2}{a\sin B + b\sin A} + \frac{(b+c)^2}{b\sin C + c\sin B} + \frac{(c+a)^2}{c\sin A + a\sin C} \leq \frac{6R^2}{r}$$

Proposed by Ertan Yildirim-Turkiye

S.1722 In ΔABC the following relationship holds:

$$\frac{\tan B + \tan C}{a} + \frac{\tan C + \tan A}{b} + \frac{\tan A + \tan B}{c} \geq \frac{3}{r}$$

Proposed by Ertan Yildirim-Turkiye

S.1723 In ΔABC the following relationship holds:

$$F = r^2\left(\sqrt{\prod_{cyc}\left(1 - \frac{2\sin A}{\sin A + \sin B + \sin C}\right)}\right)^{-1}$$

Proposed by Amerul Hassan-Myanmar

S.1724 If $a, b, c \geq 0$ and $a^2 + b^2 + c^2 = 3$. Find the minimum value of H

$$H = \frac{a}{9b^2 + 13c + 2} + \frac{b}{9c^2 + 13a + 2} + \frac{c}{9a^2 + 13b + 2} + \frac{a^3 + b^3 + c^3}{a^3b^3 + b^3c^3 + a^3c^3}$$

Proposed by Nguyen Nhat Huy-Vietnam

S.1725 If $x_1, x_2, \dots, x_n > -1, n \in \mathbb{N} - \{0\}$ then:

$$\frac{x_1^2 + 1}{x_1 + 1} + \frac{x_2^2 + 1}{x_2 + 1} + \dots + \frac{x_n^2 + 1}{x_n + 1} \geq 2n\left(\sqrt{2} - 1\right)$$

Proposed by Nikos Ntorvas-Greece

S.1726 Solve for real numbers:

$$\tan\frac{\pi}{14} = \frac{-1 + \sqrt{\frac{1}{3}\left(24 + 8\sqrt{21}\cos\left(\frac{1}{3}(2\pi + \tan^{-1}x)\right)\right)}}{\sqrt{\frac{1}{3}\left(21 + 8\sqrt{21}\cos\left(\frac{1}{3}(2\pi + \tan^{-1}x)\right)\right)}}$$

Proposed by Carlos Paiva-Brazil

S.1727 If $0 < x, y, z < 2\pi$ then:

$$|\cos x + \cos y + \cos z + \cos(x + y + z)| < \frac{1}{2}$$

Proposed by Khaled Abd Imouti-Syria

S.1728

$$\lim_{x\to\infty}\left(2x\sqrt{x^2 + 1} - 2x^2 - 1\right)^{\frac{1}{\sqrt[3]{x^3 + x + 1}}}$$

Proposed by Max Wong-Hong Kong

S.1729 If $a, b, x_k \in \left(0, \frac{\pi}{2}\right), k = \overline{1, n}$, such that $a + \sqrt{a^2 + 4ab} > 2ab$ then:

$$\sum_{k=1}^{n} \frac{a + b \cos x_k}{(3 - x_k^2) \cos^2 x_k} \geq (a + b) \sum_{k=1}^{n} x_k^2$$

Proposed by Florică Anastase-Romania

S.1730 If $x_k \in (0,1), k = \overline{1, n}$ then prove:

$$\sum_{k=1}^{n} \frac{\tan x_k}{(3 - x_k^2) \sin^2 x_k \ \sin^{-1} x_k} \geq n - \sum_{k=1}^{n} x_k^2$$

Proposed by Florică Anastase-Romania

S.1731

$$a_k(x) = \sqrt{k^2 x^{2k-2} - k x^{k-1} + \sqrt{k^2 x^{2k-2} - k x^{k-1} + \sqrt{\ldots}}}; x > 0.$$

Prove that the roots of the next equation are in geometric progression.

$$\sum_{k=2}^{n} \int_2^4 a_k(x)\, dx = \frac{3 \cdot 8^n + 20 \cdot 4^n - 114 \cdot 2^n + 124}{3 \cdot 2^n + 3}$$

Proposed by Costel Florea-Romania

S.1732 If $x, y, z \in \mathbb{R}, \sum x^2 = \alpha, \sum x^3 = \beta, \sum x^2 y^2 = \gamma, xyz = \delta$then prove:

$(\sum x)^2 \geq \frac{3}{\beta + \delta}\left(\alpha\delta + \sqrt{f(\alpha, \beta, \gamma, \delta)}\right)$, where $f(\alpha, \beta, \gamma, \delta) = (\beta + \delta)(2\delta\alpha^2 - 3\gamma\delta + \beta\gamma)$.

Proposed by Soumava Chakraborty-India

S.1733 For $x, y, z > 2$ prove:

$$\log\left(\frac{(3 + x)^2 (3 + y)^2 (3 + z)^2}{8}\right)^3 \leq \sum_{cyc} (x^2 + 8) \sin\frac{\pi}{x}$$

Proposed by Florică Anastase-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.694 Find:

$$\Omega = \int_0^1 \left(\frac{\log x \cdot \log\left(\sqrt{1-x^2}\right) \cdot \left(1 + \log\left(\sqrt{1-x^2}\right)\right)}{x}\right) dx$$

Proposed by Abdul Mukhtar-Nigeria

U.695 If $0 < a \leq b$ then prove:

$$\frac{\int_a^b \int_a^b \frac{x^2}{y} \log\left(\frac{x^2+2by}{x^2+2ay}\right) dx\, dy \geq 2}{3}(b-a)^3$$

Proposed by Asmat Qatea-Afghanistan

U.696 Prove: $\int_0^\infty x^2 e^{-x} \operatorname{erf} x \log x \, dx = \frac{2\pi - \gamma(\pi+2) - 2(1+\pi)\log 2 + 4G}{16\sqrt{\pi}}$, where G −Catalan constant, γ −Euler-Mascheroni constant.

Proposed by Rana Ranino-Algerie

U.697 Prove that:

$$\Omega = \int_0^\infty \frac{\tan^{-1}(ax)}{\sinh(bx)} dx = \frac{\pi}{b}\left(\log\left(\frac{\Gamma\left(\frac{b}{2\pi a}\right)}{\Gamma\left(\frac{1}{2} + \frac{b}{2\pi a}\right)}\right)\right) + \frac{1}{2}\log\left(\frac{b}{2\pi a}\right)$$

Proposed by Ose Favour-Nigeria

U.698 Prove that:

$$\sum_{n=2}^{\infty} \frac{H_n^2(-1)^n}{n-1} = \frac{3\zeta(3)}{4} + \frac{\log^3 2}{3} - 2\log^2 2 + \log 2\left(2 - \frac{\pi^2}{12}\right) + \frac{\pi^2}{4} - 2$$

Proposed by Emmy Exquisite-Nigeria

U.699 Prove that:

$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2} < \gamma$, where γ is Euler-Mascheroni constant.

Proposed by Kaushik Mahanta-India

U.700 Find:

$$\sum_{n=0}^{\infty}\sum_{n_1=0}^{m_1}\sum_{n_2=1}^{m_2}\dots\sum_{n_i=1}^{m_i}(-1)^n\log\left(1+\frac{1}{n+n_1+n_2+\dots+n_i}\right),\forall i\in\mathbb{N}^*$$

Proposed by Serlea Kabay-Liberia

U.701 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n(H_n-1)}\left(\log(n!)-\sum_{k=2}^{n}\frac{\Gamma'(k)}{\Gamma(k)}\right)$$

Proposed by Daniel Sitaru,Claudia Nănuți –Romania

U.702 If $0<a\le b<\frac{\pi}{8}$ then find

$$\Omega(a,b)=\int_a^b\int_a^b\frac{(1+\tan x)(1+\tan y)\left(1+\tan\left(\frac{\pi}{4}-x-y\right)\right)}{1+\tan x\cdot\tan y\cdot\tan\left(\frac{\pi}{4}-x-y\right)}dxdy.$$

Proposed by Daniel Sitaru-Romania

U.703 Prove that for $n\in\mathbb{N}$

$$\int_0^{\infty}\frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\frac{2^{-(2n+1)}\pi^{2n+1}}{2n+1}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.704 Solve the integral for c, if $a+b=c$

$$\int_0^{\infty}\frac{ax^2+bx+c}{x^4+x^3+x^2+x+1}\frac{dx}{\sqrt{x}}=\pi$$

Proposed by Srinivasa Raghava-AIRMC-India

U.705 Let the equation $\frac{\sqrt{5}x^2+1}{\sqrt{5}x^2-1}+\frac{5x-1}{5x+1}=\frac{1}{x}+\frac{1}{2}(1+\sqrt{5})$ with a,b,c,d the roots then find the value of k, such that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+k\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=\sqrt{5}$.

Proposed by Srinivasa Raghava-AIRMC-India

U.706 For any complex number ω prove the relation

$$\frac{\mathcal{F}_x[e^{-nx}\sin(e^{-x})](\omega)}{\mathcal{F}_x[e^{-nx}\cos(e^{-nx})](\omega)}=\tan\left(\frac{\pi}{2}(n-i\omega)\right),\mathcal{F}_x(\cdot)-\text{Fourier Transformation.}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.707 Prove the inequality for $n\ge 1$

$$\tanh\left(\frac{n+x}{\pi}\right)e^{-\frac{x}{\pi}}\le e^{\frac{n}{\pi}}\sqrt{5\varphi-8},\varphi-\text{Golden Ratio.}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.708 Let the sequence $a(n-2)+a(n-1)+a(n+1)+a(n)=\varphi^{-n}$,

$a(0)=\varphi^{-1}, a(1)=\varphi, a(2)=\varphi^2$, then evaluate the sum

$$\sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.709 If we have the function $F(n,k)=\frac{1}{\Gamma(n)}\int_0^{\infty}e^{-x}x^{n-1}(\sin x+\cos x)^k\,dx$ then prove that the sum $\sum_{n=0}^{\infty}F(n,2k-1)$ is a rational number for $k=1,2,3,4,\ldots$

Proposed by Srinivasa Raghava-AIRMC-India

U.710 Count the number of positive integer solutions to the following equation

$$1a^2+2b^2+3c^2+4d^2=1234.$$

Proposed by Srinivasa Raghava-AIRMC-India

U.711 If we have the integral

$$\oint_{|z|=\pi}\frac{\sin z}{\sin(2z)+\cos\left(\frac{z}{2}\right)}dz=\pi\alpha$$

Prove the relation $192\alpha^8-138\alpha^6+186\alpha^4+133\alpha^2+3=0$.

Proposed by Srinivasa Raghava-AIRMC-India

U.712 If we have the equation $A''(z)+A'(z)+A(z)=e^{z-z^2}, A(0)=1, A'(0)=0$ then show that:

$$\int_0^{\infty}A(z)e^{-z}\,dz=\frac{1}{6}\left(4+\sqrt{\pi}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.713 For $k\geq 2$, prove the relation:

$$\frac{\partial}{\partial k}\int_0^{\frac{\pi}{2}}\sin x\log\left(\frac{\sin^2 x}{k-\sqrt{k^2-\sin^2 x}}\right)dx=\coth^{-1}k$$

Proposed by Srinivasa Raghava-AIRMC-India

U.714 Prove the inequality

$$\frac{\tan^{-1}\left(e^{\frac{\pi}{x}}\right)}{n}+e^{-\frac{x}{\pi}}>\frac{\pi}{2n}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.715 For $n>1$, prove the inequality

$$\frac{\sin^{-1}\left(e^{\frac{x}{\pi}}\right)}{n}+e^{-\frac{x}{\pi}}\geq\frac{1}{n}\left(\frac{\sqrt{n\left(\sqrt{n^2+4}+n\right)}}{\sqrt{2}}+\sin^{-1}\left(\frac{\sqrt{n\left(\sqrt{n^2+4}-n\right)}}{\sqrt{2}}\right)\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.716 $n\in\mathbb{N}^*, A$ —Stirling's constant. Prove that:

$$\sqrt[3]{\left(A^3-\sum_{i=1}^{n}\frac{1}{A^{3i}}\right)\left(A\sqrt{A}-\sum_{i=1}^{n}\frac{1}{A^{2i}}\right)}+\sum_{i=1}^{n}\frac{1}{A^{2i}}<A^2$$

Proposed by Daniel Sitaru-Romania

U.717 Prove that: $\int_0^{\frac{\pi}{4}}\tan x\, Li_2(\tan^2 x)\,dx=\frac{\pi^2}{12}\log 2-\frac{5}{16}\zeta(3)$

where $Li_3(x)=\sum_{n\geq 1}\frac{x^n}{n^3}$ is trilogarithmic function and $\zeta(3)=Li_3(1)$.

Proposed by Naren Bhandari-Bajura-Nepal

U.718 Prove that:

$$\int_0^1\frac{x}{1+x^2}\log^2\left(\frac{x}{1-x}\right)dx=\int_0^{\infty}\frac{\log^2 x\,dx}{*\,1+x)(1+(1+x)^2)}=\frac{\log^3 2}{24}+\frac{13\pi^2}{96}\log 2$$

Proposed by Naren Bhandari-Bajura-Nepal

U.719 Prove that:

$$\int_0^1\frac{(\tan^{-1}x)^5dx}{(1+x)(1+x^2)}=$$

$$=\frac{225\pi}{256}\zeta(5)+\frac{\pi^6}{49152}-\frac{15\pi^3}{512}\zeta(3)+\frac{\pi^5}{2048}\log 2+\frac{1}{262144}\left(\psi_5\left(\frac{3}{4}\right)-\psi_5\left(\frac{1}{4}\right)\right)$$

where ζ is Zeta function, $\psi_n(z)$ is n^{th} polygamma function.

Proposed by Naren Bhandari-Bajura-Nepal

U.720 Prove that:

$$\int_0^{\frac{\pi}{4}}Re\left\{Li_2\left(\frac{(1+\tan^4 x)^2}{(1-\tan^4 x)^2}\right)\right\}dx=\frac{1}{24}\pi^3+\frac{1}{2}\pi(\sinh^{-1}1)^2+\pi\log 2\sinh^{-1}1-$$

$$-\frac{5}{2}\pi\log^2 2-\pi Li_2\left(3-2\sqrt{2}\right)-\pi Li_2\left(\frac{1-\sqrt{2}}{2}\right)$$

Proposed by Sujeethan Balendran-SriLanka

U.721 Find:

$$\Omega=\int_0^{\infty}\frac{\log\left(\Gamma^2\left(\frac{\pi+x+\pi x^2}{\pi(1+x^2)}\right)(1-e^{-\frac{x^2}{\sqrt{1+x^2}}}\right)}{1+x^2}dx$$

Proposed by Sujeethan Balendran-SriLanka

U.722 Prove that:

$$\int_0^{\infty}\frac{\cos\left(\frac{x^2}{\sqrt{1+x^2}}\right)}{1+x^2+x^4}dx-\int_0^{\infty}\frac{x^2\sin\left(\frac{x^2}{\sqrt{1+x^2}}\right)}{(1+x^2+x^4)\sqrt{1+x^2}}dx+\frac{\sqrt{3}}{3e}\int_{-1}^{1}e^{-x}\tan^{-1}\left(\sqrt{\frac{1-x}{3(1+x)}}\right)dx$$

Proposed by Sujeethan Balendran-SriLanka

U.723 Prove that:

$$\int_0^{\infty}\frac{Li_2\left(\frac{\left(1-x^4\right)^2}{(1+x^4)^2}\right)}{1+x^4}dx=\sqrt{2}\int_0^{1}\frac{Re\left(Li_2\left(\frac{\left(1-x^4\right)^2}{(1+x^4)^2}\right)\right)}{1+x^2}dx$$

Proposed by Sujeethan Balendran-SriLanka

U.724 Prove that:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n x^{2n+2}}{(2k+1)[2(n-k)+1]}=(\tan^{-1}x)^2$$

Proposed by Emmy Exquisite-Nigeria

U.725 Prove that:

$$\int_0^{\infty}\frac{\log\left(1-e^{-\frac{x^2}{\sqrt{1+x^2}}}\right)}{1+x^2}dx=$$

$$=\frac{\gamma}{2}-\frac{1}{2}-\frac{\log(2^{\pi+2}\pi)}{2}+\sum_{n=1}^{\infty}2\pi\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{1}{n\pi}}\right)-\frac{1}{2(1+n)}$$

Proposed by Sujeethan Balendran-SriLanka

U.726 Prove that:

$$\sum_{k=1}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)}{(k^2+1)\Gamma(k+1)}$$

Proposed by Sujeethan Balendran-SriLanka

U.727 If $S(k)=\sqrt{k^2+k+1}-\sqrt{k^2-k+1}$, then prove that

$$\lim_{j\to\infty}\left(\frac{1}{\log j}\sum_{p=1}^{j}\frac{1}{p}\sum_{m=0}^{p-1}\frac{1}{m}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}[S(k)]\right)^{-1}-\frac{\log j}{2}\right)=\gamma$$

where γ is Euler-Mascheroni constant and $[\cdot]$ denotes greatest integer function.

Proposed by Naren Bhandari-Bajura-Nepal

U.728 Prove that:

$$\sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=k}^{\infty}\frac{1}{n2^n}=Li_4\left(\frac{1}{2}\right)+\frac{7}{8}\zeta(3)\log 2-\frac{\pi^4}{288}+\frac{\log^4 2}{24}-\frac{\pi^2}{24}\log 2$$

$$\sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{k=1}^{\infty}\frac{1}{2^n}=\frac{7}{4}\zeta(3)+\frac{\log^3 2}{3}-\frac{\pi^2}{6}\log 2$$

where $Li_n(z)$ is polylogarithm function of order n.

Proposed by Naren Bhandari-Bajura-Nepal

U.729 For positive integers $m,n>1$, if

$$P_n(m)=\prod_{j=1}^{m+1}\left(\sum_{k=1}^{n^{m+1}-n^m}\frac{1}{1+k^m}\right)\text{ then }\lim_{n\to\infty}\frac{P_m(n)}{\log^{m+1}n}=0$$

Proposed by Naren Bhandari-Bajura-Nepal

U.730 Prove that following result:

$$Li_2(\sqrt{2}-1)+Li_2(\sqrt{5}-2)-\frac{1}{4}\left(Li_2(3-2\sqrt{2})+Li_2(9-4\sqrt{5})\right)=$$

$$=\frac{5\pi^2}{48}-\frac{1}{4}\log^2(\sqrt{2}-1)-\frac{3}{4}\log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

where $Li_2(x)$ is dilogarithm function.

Proposed by Naren Bhandari-Bajura-Nepal

U.731 Prove that:

$$\int_0^1\frac{\log^2(1+x)}{(1+x)(1+x^2)}dx=$$

$$=\frac{7}{48}\log^3 2+\frac{3\pi}{32}\log^2 2-\frac{\pi^2}{192}\log 2+\frac{7\pi^3}{128}-G\log 2-2Im\left(Li_3\left(\frac{1+i}{2}\right)\right)$$

Proposed by Naren Bhandari-Bajura-Nepal

U.732 Prove that:

$$\frac{4}{3\pi}\int_0^{\frac{\pi}{2}}\frac{\sqrt{2}\left(1+\cos x\sqrt{1+\sin^2 x}\right)\sqrt{1+\sqrt{1-\sin^4 x}}-4}{\cos^2 x\,(1+\sin^2 x)-1}dx=\frac{20}{9\pi}+\frac{\Gamma^2\left(\frac{1}{4}\right)}{9\pi\sqrt{2\pi}}+\frac{4\sqrt{2\pi}}{3\Gamma^2\left(\frac{1}{4}\right)}$$

Proposed by Naren Bhandari-Bajura-Nepal

U.733 Prove that:

$$\int_0^\infty \log^3(1-e^{-\pi x})\tanh(\pi x)\,dx=\frac{15}{4\pi}Li_4\left(\frac{1}{2}\right)-\frac{169\pi^3}{7680}+\frac{\log^4 2}{8\pi}-\frac{5\pi}{64}\log^2 2$$

where $Li_n(z)$ is polylogarithm function of order n.

Proposed by Naren Bhandari-Bajura-Nepal

U.734 Prove that:

$$\sum_{n=1}^{\infty}(-1)^{n+1}H_n\left(\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots\right)=\frac{\pi}{16}\log 2+\frac{3}{16}\log^2 2-\frac{\pi^2}{192}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{n+1}\left(\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\cdots\right)=\frac{\zeta(3)}{8}-\frac{\log^3 2}{48}-\frac{\pi^2}{192}\log 2$$

where H_n is n^{th} harmonic number and $\zeta(z)$ is Riemann zeta function.

Proposed by Naren Bhandari-Bajura-Nepal

U.735 Prove that:

$$\sum_{n=1}^{\infty}\frac{H_n-H_{\frac{n}{2}}}{n}\left[\binom{2n}{n}\right]\left(-\frac{1}{4}\right)^n=$$

$$=2Li_2\left(-\frac{1}{\sqrt{2}}\right)-2Li_2\left(-\frac{1}{1+\sqrt{2}}\right)+\frac{\pi^2}{12}+\frac{\log^2 2}{4}+\log\left(\frac{1}{2}\right)\log\left(1+\sqrt{2}\right)$$

Proposed by Naren Bhandari-Bajura-Nepal

U.736 For all $n\geq 1$, if $f_n(x)=\sin\left(g_n(x)\right)$ and $g_n(x)=x^n$ such that

$$\alpha=\lim_{n\to\infty}n\left(\int_0^\infty f_n(x)g_n^{-1}(x)\,dx-1\right)$$

then prove that:

$$\lim_{n\to\infty}n^2\left(-\frac{\alpha}{n}+\int_0^\infty f_n(x)g_n^{-1}(x)dx-1\right)=1-\gamma+\frac{\gamma^2}{2}-\frac{\pi^2}{24}$$

where γ is Euler-Mascheroni constant and $g_n^{-1}(x)$ denotes reciprocal of it.

Proposed by Naren Bhandari-Bajura-Nepal

U.737 For $n>0$ prove the followings:

$$\sum_{k=4n-3}^{\infty}\frac{(-1)^{\frac{k(k+1)}{2}}}{k(k+1)}=\frac{1}{4}\sum_{j=0}^{2}\binom{2}{j}(-1)^j\psi_0\left(\frac{4n-3+2j}{4}\right)$$

$$\sum_{k=4n-1}^{\infty}\frac{(-1)^{\frac{k(k+1)}{2}}}{k(k+1)}=\frac{1}{4}\sum_{j=1}^{2}\binom{2}{j}(-1)^{j+1}\psi_0\left(\frac{4n-1+2j}{4}\right)$$

and find the closed form for $k = 4n - 2, 4n$. Here $\psi_0(z)$ denotes digamma function.

Proposed by Naren Bhandari-Bajura-Nepal

U.738 Prove that sharp inequality:

$$\frac{\partial}{\partial x}\frac{e^{\sin^{-1}x}\sin^{-1}x}{e^{\sin^{-1}x}-1} < 1 - e^{-\frac{2}{\pi}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.739 Find:

$$\Omega = \lim_{\varepsilon\to 0_+}\int_{\varepsilon}^{1}\frac{\sqrt{x}\log x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

U.740 Find:

$$\Omega = \int_{-\infty}^{\infty}\int_{0}^{\infty} x\log\left(\frac{(a-x)^2+(b-y)^2}{(a+x)^2+(b+y)^2}\right)dxdy\,;a,b>0$$

Proposed by Toby Joshua-Nigeria

U.741

If $\lim_{x\to 1_-}\sum_{n=0}^{\infty}\frac{(-1)^n(n^2-3)x^n}{n^2+1} = \frac{1}{2} - \eta\left(\psi\left(\frac{-\eta}{2}\right) - \psi\left(\frac{\eta}{2}\right) - \psi\left(\frac{1}{2}-\frac{\eta}{2}\right) + \psi\left(\frac{1}{2}+\frac{\eta}{2}\right)\right)$.

Then find the value of e^{η}, where $\psi(x)$ is the digamma function.

Proposed by Tobi Joshua-Nigeria

U.742 Find:

$$\Omega = \int_{0}^{\infty}\frac{\log(1+x)}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

U.743 Prove that:

$$\int_0^{\infty}\frac{Li_2\left(\frac{(1-x^4)^2}{(1+x^4)^2}\right)}{1+x^4}dx = \frac{\sqrt{2}}{24}\pi^3 + \frac{\sqrt{2}}{2}\pi(\sinh^{-1}1)^2 + \sqrt{2}\pi\log 2\sinh^{-1}1 -$$

$$-\frac{5\sqrt{2}}{2}\pi\log^2 2 - \sqrt{2}\pi Li_2(3-2\sqrt{2}) - \sqrt{2}\pi Li_2\left(\frac{1-\sqrt{2}}{2}\right)$$

Proposed by Balendran Sujeethan-SriLanka

U.744 Find a closed form:

$$\Omega(a) = \int_0^{\infty}\frac{x^2}{(x^2+1)(x^4+a^4)}dx\,;a\in\mathbb{R}$$

Proposed by Vasile Mircea Popa-Romania

U.745 For $a_1, a_2, \dots, a_n > 0$ prove that:

$$2+\sum_{n=2}^{\infty}\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}\geq\sum_{n=1}^{\infty}\frac{1}{n+1}\left(1+\left(\frac{n}{e}\right)^{n-1}a_1a_2\cdot\ldots\cdot a_n\right)$$

Proposed by Seyran Ibrahimov-Azerbaijan

U.746 Find a closed form:

$$\Omega(a)=\int_0^{\infty}\frac{x\sqrt{x}}{(x^2+1)(1+ax)}dx\,;a>0$$

Proposed by Vasile Mircea Popa-Romania

U.747 Find:

$$\Omega=\int_0^{\infty}\left(\frac{\log\left(\sqrt{\frac{x^2}{1+x^2}}\right)}{1+x^2}\right)dx\int_0^{\infty}\left(\frac{\cot^{-1}\sqrt{\frac{x^2}{1-x^2}}}{\sqrt{1-x^2}}\right)dx.$$

Proposed by Precious Itsuokor-Nigeria

U.748 Find:

$$\Omega_1(p,q)=\int_0^{\frac{\pi}{2}}\cos(2px+q\tan x)dx\,,p,q\in\mathbb{N},\Omega_2=\int_0^{\frac{\pi}{2}}\sin(x+\tan x)\,dx.$$

Proposed by Rohan Shinde-India

U.749 Find:

$$\Omega=\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1}\frac{x\cdot\log x}{1-x^2+x^4}dx\right).$$

Proposed by Vasile Mircea Popa-Romania

U.750 Show that:

$$\left(\int_0^{\infty}\left(\int_0^{x}\frac{60y^{59}(2018^{60}-2019^{60})}{(2018^{60}+y^{60})(2019^{60}+y^{60})}dy-60\log\left(\frac{2019}{2018}\right)\right)dx\right)^{-1}\frac{\sin 3^\circ}{\pi}=$$

$$=\frac{1+\sqrt{3}-\phi\left(\sqrt{3}-1\right)\sqrt{1+\phi^2}}{4\sqrt{2}\phi\pi}.$$

Proposed by Naren Bhandari-Nepal

U.751 Find the closed form for the integral $\Omega=\int_{-\infty}^{\infty}\frac{dx}{\left(1+x^{2^n}\right)^{n+1}}$, where n is a positive.

Proposed by Naren Bhandari-Nepal

U.752 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\int_{\frac{1}{n}}^{\infty} \frac{e^{-x}\log(1+x^2)\tan^{-1}(1+nx^n)}{1+x^n}dx.$$

Proposed by Mokhtar Kassani-Algerie

U.753 If $\Phi_n = \sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}\cos\left(\frac{2\pi(j-i)}{7}\right) - 2\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}\cos\left(\frac{2\pi(i-j)}{7}\right)$. Define

$M = \left\{\sqrt[n]{\Phi_n^{\cos(n\pi)}}\sin\left(\frac{n\pi}{4}\right)\middle| n\in\mathbb{N}\right\}$. Find M' —derived set.

Proposed by Surjeet Singhania-India

U.754 Find:

$$\Omega = \int_0^{\infty}\int_1^{\infty}\int_1^{\infty}\frac{y\log x}{(x+y)^2(1+y^2)(1+z^2)}dx\,dy\,dz.$$

Proposed by Probal Chakraborty-India

U.755 Prove or disprove

$$Re\left(\int_0^{\frac{\pi}{2}} x^n e^{ix}dx\right) \equiv \left(\frac{\pi}{2}\right)^n \sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^i\Gamma(n+1)}{\left(\frac{\pi}{2}\right)^{2i}\Gamma(n+1-2i)}$$

for $n \ge 1$

Proposed by Akerele Olofin – Nigeria

U.756 Show:

$$\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu+1}}{\mu\binom{\mu+\phi}{\phi}} = 2^{\phi}\ln(2) - \sum_{\tau=0}^{\phi-1}\binom{\phi}{\tau}\frac{(-1)^{\phi-\tau-1}}{\tau-1}(2^{\phi}-2^{\tau})$$

Where ϕ denotes the golden ratio.

Proposed by Akerele Olofin – Nigeria

U.757 If $A = \{a_1, a_2, a_3, \dots, a_n\}$ where $a_i \ge 1, 1 \le i \le n$, $\rho(A)$ and $n(A)$ denotes the power set and the cardinality of set A respectively. Show that:

$$n(A) \cong \frac{\ln(\omega)}{\omega\ln(2)}$$

Where $\omega = \left(\rho(A)\right)^{\left(\rho(A)\right)^{\left(\rho(A)\right)}}$

Proposed by Akerele Olofin – Nigeria

U.758 Prove or disprove:

$$Li_2(-\phi)+Li_2\left(-\frac{1}{\phi}\right)+Li_2(\phi)=\frac{17}{30}\pi^2+\frac{1}{2}\left(\ln^2(\phi)+\ln^2\left(-\frac{1}{\phi}\right)\right)-\ln^2(\phi)$$

Where $\phi=\frac{\sqrt{5}+1}{2}$ denotes the Golden ratio.

Proposed by Akerele Olofin – Nigeria

U.759 Show:

$$\mathcal{Im}\left(\int_0^\infty e^{-\omega(5-i4)\frac{\log(\omega)}{\omega}}d\omega\right)=-\left(\frac{\log(41)}{2}-\gamma\right)\arctan\left(\frac{5}{4}\right)$$

Where $a\pm ib\in\mathbb{C}, \mathcal{Im}(.)$ Denotes the imaginary part of an argument and γ denotes the Euler Mascheroni constant.

Proposed by Akerele Olofin – Nigeria

U.760 Show:

$$\int_0^1\frac{(x^3+1)\arctan(x)+x\log(x)+(x^2+1)\log(1+x)}{1+x^2}=$$

$$=\frac{1}{96}\left(\pi^2+12\pi(2+\log(2))+48(\log(16)-G-3)\right)$$

where G denotes the Catalan – constant.

Proposed by Akerele Olofin – Nigeria

U.761 If $\Phi(n)=\int_0^1\frac{Li_n(x)\ln^3(1-x)}{x}dx$

Show that $\Phi(x)$ can be expressed as:

$$\Phi(n)=-\sum_{k=1}^{\infty}\frac{(H_k)^3+3H_kH_k^{(2)}+2H_k^{(2)}}{k^{n+1}}$$

where $H_n^{(m)}$ is the nth generalized harmonic number of weight m

Proposed by Akerele Olofin – Nigeria

U.762 If $I_n=\int_0^1\log^3(x)J_n(x)dx, n\in\mathbb{N}$

Show:

$$I_n=\frac{1}{2^n\Gamma(n+1)}\lim_{a\to 0}\frac{\partial^3}{\partial a^3}\left(\frac{{}_1F_2\left(\frac{a+n+1}{2};\frac{a+n+3}{2},n+1,-\frac{1}{4}\right)}{(a+n+1)}\right)$$

where ${}_pF_q(a_1,a_2,\dots,a_p;b_1,b_2,\dots,b_q;z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{q}(b_i)_n}\frac{z^n}{n!}$

Proposed by Akerele Olofin – Nigeria

U.763 Prove or disprove the inequality below:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(\sqrt{n})^n F_1F_3F_5\dots F_{2n-1}}>\sum_{n=1}^{\infty}\frac{(-1)^n}{(\sqrt{5})^n F_2F_4F_6\dots F_{2n}}$$

where F_n denotes the nth Fibonacci number.

Proposed by Akerele Olofin – Nigeria

U.764 Prove or disprove:

$$\int_0^1 x^3\ln^3(x)\left\{\frac{1}{x}\right\}dx+\frac{32}{128}\sum_{k=1}^{\infty}\frac{k(\log^3(k+1))}{(k+1)^4}+\frac{24}{128}\sum_{k=1}^{\infty}\frac{k(\log^2(k+1))}{(k+1)^4}$$
$$+\frac{12}{128}\sum_{k=1}^{\infty}\frac{k(\log(k+1))}{(k+1)^4}$$
$$=\frac{2}{27}+\frac{1}{128}\left(\frac{\pi^4}{30}-12\zeta'(3)+24\zeta''(3)-32\zeta'''(3)\right)$$

Where $\zeta^n(s)$ denotes the nth derivative of the zeta function and $\{.\}$ denotes the fractional part.

Proposed by Akerele Olofin – Nigeria

U.765 Show:

$$\Psi=\int_0^1\int_0^1\int_0^1\dots\int_0^1\frac{\ln(\prod_{i=1}^n(1+x_i))}{\prod_{i=1}^n(1+x_i^2)}dx_1dx_2dx_3\dots dx_n$$
$$=n\int_0^1\frac{\ln(1+x_1)}{\prod_{i=1}^n(1+x_i^2)}dx_1dx_2\dots dx_n=\frac{n\pi^n\ln(2)}{2^{2n+1}}$$

where n is a natural number.

Proposed by Akerele Olofin – Nigeria

U.766 Show:

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{\binom{r+\varphi}{\varphi}}=\int_0^{\infty}\frac{x^{\varphi}}{(x+1)^{\varphi+1}(x+2)}dx$$

Where φ denotes the golden ratio.

Proposed by Akerele Olofin – Nigeria

U.767 Show: $\frac{e^{\pi}}{15}\,{}_2F_1(2,4;7;1)=e^{\pi}$

Where ${}_2F_1(a,b;c;z)$ denotes the ordinary hypergeometric function.

Proposed by Akerele Olofin – Nigeria

U.768 Show that:

$$\sum_{n=2}^{\infty}(Li_n(1)-1)=1$$

Proposed by Akerele Olofin – Nigeria

U.769 Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{2}} x\cdot\cot x\cdot\log(1-\sin^4 x)\,dx$$

Proposed by Naren Bhandari-Nepal

U.770

$$\int_0^{\frac{\pi}{2}}\left(\sqrt[3]{\cot(x)}+\frac{1}{\sqrt[3]{\cot(x)}}\right)^2\sin^3(x)\,dx=$$

$$=\frac{4}{3}+\frac{2^{\frac{8}{3}}\sqrt{3}\Gamma\left(\frac{4}{3}\right)^3}{\pi}-\frac{2^{\frac{4}{3}}\pi^2}{\Gamma\left(-\frac{2}{3}\right)^3}$$

Proposed by Srinivasa Raghava – India

U.771 Let the integral

$$\alpha=\frac{1}{\pi}\int_0^{\infty}\frac{x^3+x+1}{x^4+x^2+1}\frac{dx}{\sqrt[3]{x}}$$

then find the value of the expression

$$27\alpha^2(7-3\alpha^2)^2$$

Proposed by Srinivasa Raghava – India

U.772 Find:

$$\Omega=\int_1^{\infty}\frac{\sqrt{x}\ln x}{x^2-x+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.773 Prove that:

$$\sum_{k=1}^{n}\frac{\left(H_k^{(p)}\right)^2}{k^p}=\frac{1}{3}\left(\left(H_n^{(p)}\right)^3-H_n^{(3p)}\right)+\sum_{k=1}^{n}\frac{H_k^{(p)}}{k^{2p}}$$

Proposed by Nawar Alasadi-Iraq

U.774 Prove that:

$$\sum_{n=0}^{\infty}\frac{(-1)^{0+1+2+\cdots+\cdots+n}}{(2n+1)^2}=\frac{\sqrt{2}\pi^2}{16}$$

Proposed by Amrit Awasthi-India

U.775 Prove:

$$\sum_{n\geq 1}\frac{(2x)^{2n}(n!)^2}{n^2(2n+1)!}=2(\sin^{-1}(x))^2+\frac{4\sqrt{1-x^2}}{x}\sin^{-1}(x)-4$$

valid for $|x|<1$

Proposed by George Moses-Nigeria

U.776 Find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to 2}\frac{\underbrace{\left(\left(\left((x)!\right)!\right)!\dots\right)!}_{n\ times}-2}{\underbrace{\left(\left(\left((x)!\right)!\right)!\dots\right)!}_{n-1\ times}-2}\right)$$

Proposed by Mohammad Nasery-Afghanistan

U.777 Find:

$$\Omega=\lim_{\substack{x\to 1\\ x>1}}\left(\frac{x^{x^{x^{\cdots}}}}{x-1}\right)^{\frac{1}{\sqrt{x-1}}}$$

Proposed by Mohammad Nasery-Afghanistan

U.778 Prove:

$$\sum_{k=1}^{\infty}\left(\frac{1}{k(k+2)(k+4)\dots(k+2n)}-\frac{1}{(k+1)(k+3)\dots(k+2n+1)}\right)=\frac{\sqrt{\pi}}{2^{n+1}\left(n+\frac{1}{2}\right)!}$$

Proposed by Asmat Qatea-Afghanistan

U.779 If k is a positive real number then prove that: $I=\int_0^{\frac{\pi}{2}}\sin(k\cos^2 x)\,dx=\frac{\pi}{2}\sin\left(\frac{k}{2}\right)J_0\left(\frac{k}{2}\right)$,

where $J_v(z)$ represents Bessel Function of first kind.

Proposed by Rohan Shinde-India

U.780 Find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sum_{k=1}^{n}\frac{1}{k}\right)^2-2\gamma\cdot\log n-\log^2 n\right).$$

Proposed by Abdul Mukhtar-Nigeria

U.781 Prove that: $\lim_{n\to\infty}\left(\frac{1}{x-1}+\frac{1}{x^2-1}+\dots+\frac{1}{x^n-1}\right)=\sum_{k=1}^{\infty}\frac{\varphi(k)}{x^k}$, $|x|<1$, where $\varphi(n)$ Euler's totient function.

Proposed by Rahul Sethi-India

U.782 If $f(x)=\tanh(n\pi x)$, $g(x)=x^2+1, \forall x\in[0,\infty)$, $\chi_{n,m}=n^m-\Gamma\left(\frac{1}{n^m}\right)$ and $\lambda(k,j)=j!\,(k-j)!$, where m is fixed positive real number, then prove that:

$$\lim_{n\to\infty}\prod_{k=0}^{\infty}\prod_{j=0}^{k}\left(1+\int_0^\infty \frac{xf(x)}{g^2(x)}\frac{dx}{n}\right)^{\frac{2n\chi_{n,m}}{\lambda(k,j)}}=e^{\gamma e^2}$$

where e —Euler number and γ —Euler-Mascheroni constant.

Proposed by Naren Bhandari, Kaushik Mahanta, Surjeet Singhania, Shivam Sharma-India

U.783 If $xf'(x)-\left(1+f(x)\right)\log(x^2)=0, x>0, f(e)=e-1$ then

$$\int_{\min f}^{e} f(x)\,dx<\frac{2^2-3}{4}.$$

Proposed by Lazaros Zachariadis-Greece

U.784 For $n\ge 2$ prove that:

$$\int_0^{\frac{\pi}{n}}\log(\tan x)\,dx=\pi\log\left[\frac{G\left(\frac{n+1}{n}\right)G\left(\frac{3n-2}{2n}\right)}{G\left(\frac{n-1}{n}\right)G\left(\frac{n+2}{n}\right)}\right]-\frac{\pi}{n}\log\left[\frac{\pi^{\frac{n}{2}}}{\sin\left(\frac{\pi}{n}\right)^{\frac{n-2}{2}}\sqrt{1-\sin^2\left(\frac{\pi}{n}\right)}}\right]$$

Proposed by Toby Joshua-Nigeria

U.785 If $a_0=1, a_1=-1, a_n=4a_{n-1}+29a_{n-2}, n\ge 2$. Find the reminder of a_{2019} dividing by 101.

Proposed by Gantumur Choijilsuren-Mongolia

U.786 If $a,b,n,k\in\mathbb{N}$ and $S(a,b,n)=\{k|k\equiv a(mod\ b), k|n\}$ then prove that

$$\sum_{k=1}^{\infty}\frac{x^{ak}}{1-x^{bk}}=\sum_{n=1}^{\infty}|S(a,b,n)|x^n$$

Proposed by Angad Singh-India

U.787 Prove that: $\int_{-\infty}^{\infty}e^{-x^2}H_{2n}(ax)\,dx=\sqrt{\pi}\frac{(2n)!}{n!}(a^2-1)^n, n>0$, where H_{2n} is the Hermite polynomials.

Proposed by Toby Joshua-Nigeria

U.788 Find:

$$\Omega=\lim_{n\to\infty}\left(n\prod_{m=1}^{n}\left(1-\frac{1}{m}+\frac{5}{4m^2}\right)\right).$$

Proposed by Ngulmun George Baite-India

U.789 Find:

$$\Omega=\int_0^\infty\frac{x^b(1+x^c)\log(1+ax)\log x}{(dx^n+e)^m}dx.$$

Proposed by Olabintan Bolu-Nigeria

U.790 Find:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{xyz}{(x+y)(y+z)(z+x)}\,dxdydz.$$

Proposed by Asmat Qatea-Afghanistan

U.791 Let $A=\int_0^\infty\frac{\log x}{1+e^x+e^{2x}+e^{3x}}dx\,, B=\int_0^1\frac{x\log(1+x)}{1+x^2}dx$. Prove that: $A+B=\frac{\pi^2}{96}-\frac{\log^2 2}{2}-\frac{\pi\log R}{8}$

and hence find the value of R.

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

U.792 For all $x,a,b,c\in\mathbb{R}, a\neq 0$ prove that:

$$\sum_{k=0}^{n}\binom{n}{k}\sin\left(x+\frac{k\pi}{2}\right)=\left(\sqrt{2}\right)^n\sin\left(x+\frac{n\pi}{4}\right)$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{b^k}{a^k}\sin\left(bx+c+\frac{k\pi}{2}\right)=\left(\frac{\sqrt{a^2+b^2}}{a}\right)^n\sin\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{b^k}{a^k}\cos\left(bx+c+\frac{k\pi}{2}\right)=\left(\frac{\sqrt{a^2+b^2}}{a}\right)^n\cos\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)$$

Proposed by Amrit Awasthi-India

U.793 Find:

$$\Omega(a,b)=\lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-1}{b}\right)^n, a,b>0$$

Proposed by Jalil Hajimir-Canada

U.794 In acute ΔABC the following relationship holds:

$$\mu(A)e^{\mu(A)+\sec A}+\mu(B)e^{\mu(B)+\sec B}+\mu(C)e^{\mu(C)+\sec C}>e\pi$$

Proposed by Jalil Hajimir-Canada

U.795 Evaluate:

$$\Omega=\int_0^{1398\pi}\left[(\sin^2 x)^{\cos^2 x}+(\cos^2 x)^{\sin^2 x}\right]dx$$

Proposed by Jalil Hajimir-Canada

U.796 Prove that:

$$\frac{1}{4}<\frac{1}{ab}\int_a^{2a}\int_b^{2b}\frac{(e^x+e^y)(e^{2x}+e^{2y})(e^{3x}+e^{3y})}{4(e^{6x}+e^{6y})}dxdy<1$$

Proposed by Jalil Hajimir-Canada

U.797 Prove without softs:

$$\frac{\pi}{2}<\int_0^2\cot^{-1}(\sin x)\,dx<2$$

Proposed by Jalil Hajimir-Canada

U.798 Prove that:

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k}^2 f_k > \frac{4^{\left[\frac{n}{2}\right]}}{\left(\left[\frac{n}{2}\right]+1\right)^2}\left(f_{\left[\frac{n}{2}\right]+1}-1\right), n \in \mathbb{N}$$

f_n is the n^{th} Fibonacci number, $f_0 = f_1 = 1, f_{n+2} = f_{n+1} + f_n$ and $[*]$ is the greatest integer part of $*$.

Proposed by Jalil Hajimir-Canada

U.799 Find:

$$\Omega(a,b,c) = \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{a}-\sqrt[n]{b}}{c}\right)^n, a,b,c > 0$$

Proposed by Jalil Hajimir-Canada

U.800 If $x, y, z \in \mathbb{R}$ then:

$$2^{3^{4x^2+4y^2}} + 2^{3^{4y^2+4z^2}} \geq 4^{3^{x^2+2y^2+z^2}}$$

Proposed by Jalil Hajimir-Canada

U.801 Prove that:

$$1 > \int_0^{\frac{\pi}{2}} (\tan x)^{\cos x} dx < \frac{\pi}{2}$$

Proposed by Jalil Hajimir-Canada

U.802 Find:

$$\Omega = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \sin nx}{4n^2-1}$$

Proposed by Jalil Hajimir-Canada

U.803 Prove without softwares

$$\int_0^{\infty} \frac{dx}{1+x^n} \geq \frac{\pi}{n}, n \geq 2$$

Proposed by Jalil Hajimir-Canada

U.804 Find:

$$\Omega = \int_{-\infty}^{\infty} \frac{\sin x \sin(t-x)}{x(t-x)} dx$$

Proposed by Jalil Hajimir-Canada

U.805 Find the maximum and minimum value of:

$$f(x) = \sin^3 x\sqrt{1+\cos^2 x} + \cos^3 x\sqrt{1+\sin^2 x}$$

Proposed by Jalil Hajimir-Canada

U.806 Find a closed form:

$$\Omega = \int_0^{\infty} 2^{-\sqrt{2}t}\left(t^2 \sin\sqrt{3}t\right) dt$$

Proposed by Jalil Hajimir-Canada

U.807 Evaluate:

$$\lim_{x\to 0}\frac{\sqrt{\frac{1-2x}{1+2x}}\cdot\sqrt[4]{\frac{1-4x}{1+4x}}\cdot\ldots\cdot\sqrt[2n]{\frac{1-2nx}{1+2nx}}-1}{\sqrt[3]{\frac{1-3x}{1+3x}}\cdot\sqrt[5]{\frac{1-5x}{1+5x}}\cdot\ldots\cdot\sqrt[2n+1]{\frac{1-(2n+1)x}{1+(2n+1)x}}-1}$$

Proposed by Jalil Hajimir-Canada

U.808 Find without softs:

$$\Omega=\int_0^{\frac{1}{2}}\log(\Gamma(x))\cos(\pi x)\,dx$$

Proposed by Jalil Hajimir-Canada

U.809 Find the image of the line represented by $z=t+3ti, t\in\mathbb{R}$ on the sphere with equation given by $x^2+y^2+\left(z-\frac{1}{2}\right)^2=4$ in stereographic projection.

Proposed by Jalil Hajimir-Canada

U.810 If $a,b,c>2, a+b+c=9$ then:

$$\Gamma\left(\frac{a}{b}\right)+\Gamma\left(\frac{b}{c}\right)+\Gamma\left(\frac{c}{a}\right)\geq 6$$

Proposed by Jalil Hajimir-Canada

U.811 Find:

$$\int_{\frac{1}{5}}^{5}\frac{x+\log x}{x\left(2+\cosh\left(x-\frac{1}{x}\right)\right)}dx$$

Proposed by Jalil Hajimir-Canada

U.812 Find:

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)(2n+3)}$$

Proposed by Jalil Hajimir-Canada

U.813 Find:

$$\Omega=\lim_{x\to 0}\frac{\sum_{k=1}^{n}\log(\cos kx)}{x\sin nx}$$

Proposed by Jalil Hajimir-Canada

U.814 Find a closed form:

$$\Omega=\int_0^{\infty}\frac{x\sin x}{1+x^2}dx$$

Proposed by Jalil Hajimir-Canada

U.815 If $0<a\leq b\leq 1$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dxdydz}{x^2y^2+z^2+3}\leq\frac{(b-a)^2}{3}\int_a^b\frac{dx}{x+x^x}$$

Proposed by Daniel Sitaru-Romania

U.816 Find:

$$\Omega = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \sin nx}{4n^2 - 1}$$

Proposed by Jalil Hajimir-Canada

U.817 Prove without softwares

$$\int_0^{\infty} \frac{dx}{1+x^n} \geq \frac{\pi}{n}, n \geq 2$$

Proposed by Jalil Hajimir-Canada

U.818 Find:

$$\Omega = \int_{-\infty}^{\infty} \frac{\sin x \sin(t-x)}{x(t-x)} dx$$

Proposed by Jalil Hajimir-Canada

U.819 Find the maximum and minimum value of:

$$f(x) = \sin^3 x \sqrt{1+\cos^2 x} + \cos^3 x \sqrt{1+\sin^2 x}$$

Proposed by Jalil Hajimir-Canada

U.820 Find a closed form:

$$\Omega = \int_0^{\infty} 2^{-\sqrt{2}t} \left(t^2 \sin \sqrt{3} t\right) dt$$

Proposed by Jalil Hajimir-Canada

U.821 Evaluate:

$$\lim_{x \to 0} \frac{\sqrt{\frac{1-2x}{1+2x}} \cdot \sqrt[4]{\frac{1-4x}{1+4x}} \cdot \ldots \cdot \sqrt[2n]{\frac{1-2nx}{1+2nx}} - 1}{\sqrt[3]{\frac{1-3x}{1+3x}} \cdot \sqrt[5]{\frac{1-5x}{1+5x}} \cdot \ldots \cdot \sqrt[2n+1]{\frac{1-(2n+1)x}{1+(2n+1)x}} - 1}$$

Proposed by Jalil Hajimir-Canada

U.822 Find without softs:

$$\Omega = \int_0^{\frac{1}{2}} \log\big(\Gamma(x)\big) \cos(\pi x)\, dx$$

Proposed by Jalil Hajimir-Canada

U.823 Prove that:

$$\int \sin(x^2)\,dx = \frac{3\sqrt{2}\sqrt{\pi}S\left(\frac{\sqrt{2}x}{\sqrt{\pi}}\right)}{8\Gamma\left(\frac{7}{4}\right)}\Gamma\left(\frac{3}{4}\right)$$

where:$\Gamma(.)\rightarrow$ Gamma function.Show that:

$$\int_0^{+\infty}\sin(\alpha x^2)\,dx = \sqrt{\frac{\pi}{7\alpha}}$$

Proposed by Simon Peter-Madagascar

U.824 Let $\varphi = \frac{1}{1025}\int_0^1 x^{\frac{5}{3}}(1+x)\log(x)\left\{8\Phi\left(x^2,2,\frac{5}{4}\right)-\Phi\left(x^2,3,\frac{5}{4}\right)-16\Phi\left(x^2,1,\frac{5}{4}\right)\right\}dx$

Show that:

$$\varphi = \sum_{n=1}^{\infty}\frac{\left[\frac{n-1}{n}\right]^2}{\left(n-\frac{1}{3}\right)^2\left(4\left[\frac{n-1}{n}\right]+1\right)^3}$$

$$= -\frac{2322}{343}-\frac{6219}{686}G+\frac{15}{9604}(7215-2389\sqrt{3})\pi+\frac{324315}{4802}\log(2)-\frac{324675}{9604}\log(3)-$$

$$-\frac{18657}{2744}\zeta(2)+\frac{225}{6272}\pi^3-\frac{3}{4}\psi^{(3)}\left(\frac{4}{3}\right)-\frac{75}{5488}\psi'\left(\frac{11}{6}\right)+\frac{225}{224}\zeta(3)$$

Where:G: Catalan's constant,$\zeta(.)$: zeta Riemann function,ψ: diagamma function

Proposed by Simon Peter-Madagascar

U.825 Find a closed form:

$$\Omega = \int_0^{\infty}\frac{(x\cdot\tan^{-1}x)^2}{x^4-x^2+1}dx$$

Proposed by Timson Azeez Folorunsho-Nigeria

U.826 Find a closed form:

$$\Omega = \int_0^1\left(x\cdot\sqrt[3]{1-x^3}\cdot\log x\cdot\log(1-x^3)\right)dx$$

Proposed by Abdul Mukhtar-Nigeria

U.827 Prove that:

$$\lim_{n\to\infty}\left(\frac{1}{\log n}\sum_{i=1}^{n}\left(\sum_{k=1}^{i}\frac{(-1)^{k+1}}{k}\binom{i}{k}\right)\left(\sum_{k=1}^{i}\frac{(-1)^{k-1}}{(H_k)^{(-1)}}\binom{i}{k}\right)-\frac{\log n}{2}\right)=\gamma$$

γ is Euler – Mascheroni constant and H_k is kth Harmonic number.

Proposed by Naren Bhandari-Nepal

U.828 A rectangular plate $PQRS$ is bounded by the lines $x=0, y=0, x=4, y=2$.

Determine the potential distribution using the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, subject to the boundary conditions:

$$\begin{aligned} u(0,y)&=0 \qquad 0\le y\le 2\\ u(4,y)&=0 \qquad 0\le y\le 2\\ u(x,2)&=0 \qquad 0\le x\le 4\\ u(x,0)&=x(4-x) \qquad 0\le x\le 4 \end{aligned}$$

Proposed by Precious Itsuokor-Nigeria

U.829 Find:

$$\int_0^1\left(\frac{\ln(1+x^2)}{x(1+x^2)}+\frac{\ln(1+x^2)}{x^2}\arctan(x)\right)dx$$

Proposed by Precious Itsuokor-Nigeria

U.830

$$\left.\begin{cases} a,b,c>0\\ m,n,k\in\mathbb{N}^*\\ 2k>m>k\\ a+b+c=\dfrac{n}{2k-m}\end{cases}\right\}\Rightarrow k[ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)]$$
$$\le m(a^4+b^4+c^4)+nabc$$

Proposed by Pavlos Trifon-Greece

U.831 Find the value of

$${}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{1}{3},\frac{2}{3},\frac{256}{27}\left(\frac{5^3}{6^4}\right)\right)$$

Proposed by Probal Chakraborty-India

U.832 Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{2}} xe^{2\pi x}\cdot\log(\tan x)\,dx$$

Proposed by Kaushik Mahanta-India

U.833 Evaluate:

$$\int\frac{x}{e^{x^n}+1}dx, n\in\mathbb{R}$$

Proposed Arslan Ahmed-Yemen

U.834 Find a closed form:

$$\Omega = \int_{-\infty}^{\infty} \left(\frac{x^2 e^{-\left(\emptyset x - \frac{\emptyset}{x}\right)^2}}{x^2\emptyset^2 - 2x^2\emptyset^2 + x^2 + \emptyset^2} \right) dx$$

$\emptyset$ - golden ratio.

Proposed by Ekpo Samuel-Nigeria

U.835 If $2021! \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\prod_{\alpha=0}^{2022}(k+l+\alpha)} = \frac{1}{A}$ then find the value of A.

Proposed by Syed Shahabudeen-India

U.836 Find a closed form:

$$\Omega(a) = \int_0^{\infty} \frac{x\sqrt{x}}{(x^2+1)(1+a^2x^2)} dx, a > 0.$$

Proposed by Vasile Mircea Popa-Romania

U.837 Find:

$$\Omega = \int_0^{\infty} \frac{\left(x^{2\phi} + x^{\phi+1} + 1\right)\log x}{(x^{2\phi}+1)(x^4+x^2+1)} dx.$$

Proposed by Asmat Qatea-Afganistan

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ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-WINTER 2023

PROBLEMS FOR JUNIORS

JP.451 Prove that if $m \geq 0, n \geq 1$, then in any ΔABC with usual notations is true the inequality

$$\sum_{cyc}\frac{b^n+c^n}{a^m}\ge 2^{n+1}3^{1-m-n}\left(5-\frac{r}{R}\right)^m$$

Proposed by Marius Drăgan, Neculai Stanciu-Romania

JP.452 Solve for real numbers:

$$\begin{cases}\sqrt{x}-y^5=3\\ \sqrt[5]{\sqrt{x}-3}-\sqrt[5]{y^5+6}=-1\end{cases}$$

Proposed by George Florin Şerban, Neculai Stanciu-Romania

JP.453 Prove that:

$$\prod_{cyc}\frac{(1+\sin A\sin B)(1+\sin A\sin C)}{1+\sin A\sqrt{\sin B\sin C}}\ge\left(1+\frac{1}{R}\cdot\sqrt[3]{\frac{F^2}{4R}}\right)^3$$

Proposed by Florică Anastase-Romania

JP.454 If $a,b,c>0$ such that $a+b+c=1$, then prove that:

$$\sum_{cyc}ab(3a+2b+c)\le\frac{2}{3}$$

Proposed by Laura and Gheorghe Molea-Romania

JP.455 In ΔABC the following relationship holds:

$$\frac{(a^2+b)(a^2+c)(b^2+a)(b^2+c)(c^2+a)(c^2+b)}{(a+1)^2(b+1)^2(c+1)^2}\ge 1728r^6$$

Proposed by Daniel Sitaru-Romania

JP.456 In ΔABC the following relationship holds:

$$b\left(\frac{a}{b}\right)^{\frac{2\sqrt{3}s}{9R}}+c\left(\frac{b}{c}\right)^{\frac{2\sqrt{3}s}{9R}}+a\left(\frac{c}{a}\right)^{\frac{2\sqrt{3}s}{9R}}\le 3\sqrt{3}R$$

Proposed by Daniel Sitaru-Romania

JP.457 If $a,b,c>0, a+b+c=3$ then:

$$\frac{(a+2b)^2}{2a+b}-\frac{b^2}{a}+\frac{(b+2c)^2}{2b+c}-\frac{c^2}{b}+\frac{(c+2a)^2}{2c+a}-\frac{a^2}{c}\le 6$$

Proposed by Daniel Sitaru-Romania

JP.458 If $a,b,c>0$ such that $a^2+b^2+c^2=1$ then prove:

$$\sum_{cyc}ab(3a^3b+4ab-2c^2+1)\le 2$$

Proposed by Gheorghe Molea-Romania

JP.459 If $x,y,z>0$ then:

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\ge \max\left\{\frac{x}{y}+\frac{y}{z}+\frac{z}{x};\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right\}$$

Proposed by Daniel Sitaru-Romania

JP.460 If $a,b,c,m,n>0$ then:

$$\left(\frac{a}{\sqrt[3]{mb+nc}}+\frac{b}{\sqrt[3]{mc+na}}+\frac{c}{\sqrt[3]{ma+nb}}\right)^3\ge\frac{(a+b+c)^4}{(m+n)(a^2+b^2+c^2)}$$

Proposed by Daniel Sitaru-Romania

JP.461 If $m,n,p,x,y,z>0$ then in ΔABC with area F holds:

$$\frac{(m+n)(z+x)}{py}a^2b^2+\frac{(p+m)(y+z)}{mz}b^2c^2+\frac{(n+p)(x+y)}{nx}c^2a^2\ge 64F^2$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

JP.462 In ΔABC the following relationship holds:

$$\left(\frac{1}{h_ah_b}+\frac{1}{h_bh_c}+\frac{1}{h_ch_a}\right)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\ge\frac{9}{16F^2}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

JP.463 In $\Delta ABC, I$ −incenter and R_a, R_b, R_c circumradius of $\Delta IAB, \Delta IBC, \Delta ICA$, then

$$\frac{R_b^3R_c^3}{R_ah_a^2}+\frac{R_c^3R_a^3}{R_bh_b^3}+\frac{R_a^3R_b^3}{R_ch_c^2}\ge\frac{4R^3}{3F}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

JP.464 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cos^2\frac{A}{2}}{2\cos^6\frac{A}{2}+\cos^6\frac{B}{2}}\le\frac{8R}{9r}$$

Proposed by Marian Ursărescu-Romania

JP.465 In ΔABC the following relationship holds:

$$\sum_{cyc}(a^2+b^2-c^2)^2+8\sum_{cyc}a^2b^2\ge 3888r^4$$

Proposed by Daniel Sitaru-Romania

PROBLEMS FOR SENIORS

SP.451 If $ABCD$ is a convex quadrilateral such that $AC \cap BD = \{O\}, AE = EC, BF = FD$ with order $A - O - E - C$ respectively $B - F - O - D, EF \cap AB = \{J\}, EF \cap CD = \{K\}, CJ \cap BK = \{L\}$ and M the point of KJ, then prove that O, M and L are collinear.

Proposed by Marius Drăgan, Neculai Stanciu-Romania

SP.452 If $a, b, c > 0$ then:

$$(1+a)\left(1+\frac{b}{a}\right)\left(1+\frac{c}{b}\right)\left(1+\frac{81}{c}\right) \geq 256$$

Proposed by Daniel Sitaru-Romania

SP.453 Let $\varepsilon_i, i = \overline{1,n}$ be roots of equation $z^{n+1} = 1, \varepsilon_i \neq 1, \forall i = \overline{1,n}$

Solve for natural numbers

$$n^2 + \sum_{k=1}^{n} \frac{3\varepsilon_k - 2}{1 - \varepsilon_k} + \frac{3}{2} = 0, n \in \mathbb{N}^*$$

Proposed by Florică Anastase-Romania

SP.454 Let $(z_k)_{k=\overline{1,n-1}}$ be roots of the unity by order $n, z_k \neq 1, \forall k = \overline{1,n-1}$, $n \in \mathbb{N}^*, n \geq 3$.Find:

$$\Omega = \sum_{\substack{l=\overline{1-n-2} \\ k=\overline{2,n-1}, l<k}} \frac{z_l z_k}{(1-z_l)(1-z_k)} + \frac{n-2}{3} \cdot \sum_{k=1}^{n-1} \frac{z_k}{1-z_k}$$

Proposed by Florică Anastase-Romania

SP.455 Let $(\varepsilon_i)_{i=\overline{1,n}}$ − be roots of unity by order n, n −even number. Solve for natural numbers:

$$\sum_{1 \leq i < j \leq n} \left(\varepsilon_i - \varepsilon_j\right)^n = 3n - 2, n \in \mathbb{N}^*$$

Proposed by Florică Anastase, Păun Alexandru-Romania

SP.456 Let $\varepsilon_i, i = \overline{1,n}$ be roots of the equation $z^{n+1} = 1, \varepsilon_i \neq 1, \forall i = \overline{1,n}$.

Solve for complex numbers:

$$z^{2n} + \frac{4}{5n} \sum_{k=1}^{n} \frac{3\varepsilon_k - 2}{1 - \varepsilon_k} \cdot z^n + 4i(z^n - 1) = 0$$

Proposed by Florică Anastase, Raluca Caraion-Romania

SP.457 If $a, b, c, k > 0$ such that $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = k$, then:

$$27\prod_{cyc}(a+b)+6k\left(\sum_{cyc}a\right)^2\geq 14k^3$$

Proposed by Gheorghe Molea-Romania

SP.458 Let $\Delta A_1B_1C_1, \Delta A_2B_2C_2$ be triangles with sides a_1, b_1, c_1, circumradii R_1, respectively a_2, b_2, c_2, R_2. Prove that:

$$\left(\frac{1}{a_1^3}+\frac{1}{b_1^3}+\frac{1}{c_1^3}\right)\left(\frac{1}{a_2^5}+\frac{1}{b_2^5}+\frac{1}{c_2^5}\right)\geq\frac{1}{9R_1^3R_2^5}$$

Proposed by Bătinețu-Giurgiu, Daniel Sitaru-Romania

SP.459 If $x, y\in\mathbb{R}_+$ and $x+y=4$ then in ΔABC holds:

$$\frac{a^4+b^4}{c^y}w_c^x+\frac{b^4+c^4}{a^y}w_a^x+\frac{c^4+a^4}{b^y}w_b^x\geq 3\cdot 2^{x+1}F^x$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

SP.460 In $\Delta ABC, M\in(BC), N\in(CA), P\in(AB)$ such that

$$\frac{MB}{MC}=\frac{NC}{NA}=\frac{PA}{PB}=x>0.\text{Prove that:}$$

$$(MN^4+NP^4+PM^4)(x+1)^4\geq 16(x^2-x+1)^2F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

SP.461 In ΔABC the following relationship holds:

$$\frac{m_aw_a-r^2}{h_ah_b+r^2}+\frac{m_bw_c-r^2}{h_bh_c+r^2}+\frac{m_cw_a-r^2}{h_ah_a+r^2}\geq\frac{12}{5}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

SP.462 Let ABC be an triangle, D be a point on side BC and M be the symmetrical of A with respect to D. If $\frac{BM^2}{AB}+\frac{CM^2}{AC}=AB+AC$, then prove that AD is the bisector of the angle $\widehat{A}$, or is the height from the vertex A.

Proposed by Neculai Stanciu-Romania

SP.463 If K –Lemoine's point in ΔABC, then:

$$\sum_{cyc}\frac{aKA+bKB-cKC}{2aKA+cKC}\geq 1$$

Proposed by Daniel Sitaru-Romania

SP.464 In $\Delta ABC, AA_1, BB_1, CC_1$ internal bisector, A_2, B_2, C_2 contact point with circumcircle of triangle ABC. Prove that:

$$A_1A_2 \cdot B_2C_2 + B_1B_2 \cdot A_2C_2 + C_1C_2 \cdot A_2B_2 \geq Rs$$

Proposed by Marian Ursărescu-Romania

SP.465 If $x, y, z > 0, x + y + z = 3\sqrt{3}$ then:

$$\frac{x^3y}{(x^2+1)^2} + \frac{y^3z}{(y^2+1)^2} + \frac{z^3x}{(z^2+1)^2} \leq \frac{27}{16}$$

Proposed by Daniel Sitaru-Romania

UNDERGRADUATE PROBLEMS

UP.451 Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{2n} (-1)^k \cdot \frac{4n+1}{4n-2k+1}\binom{2n}{k}}$$

Proposed by Ruxandra Daniela Tonilă-Romania

UP.452 If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{x+y} \leq \frac{37}{72}(b-a)\log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

UP.453 If $m > 0$ then find:

$$\Omega(m) = \lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{m+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{m+1}{x}}\right)\sin^m\left(\frac{\pi}{x}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.454 Let $A, B \in M_4(\mathbb{R})$ such that $AB + BA = O_4$, then prove:

$$\det(A^4 + A^2 + B^2) \geq 0$$

Proposed by Marian Ursărescu-Romania

UP.455 Let ω be a root of the equation $x^4 + (x-1)^4 + 1 = 0$.

Find: $\Omega = \omega^{300} + \omega^{303}$.

Proposed by Daniel Sitaru-Romania

UP.456 In ΔABC the following relationship holds:

$$\left(m_b^2 + m_c^2\right)\sin A + \left(m_c^2 + m_a^2\right)\sin B + \left(m_a^2 + m_b^2\right)\sin C \geq 54\sqrt{3} \cdot \frac{r^3}{R}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

UP.457 If $a, b, c, d > 0$ such that $ab + bc + cd + da = 1$ and $\lambda \geq 0$ then:

$$a\sqrt{b^2+\lambda} + b\sqrt{c^2+\lambda} + c\sqrt{d^2+\lambda} + d\sqrt{a^2+\lambda} \geq \sqrt{1+4\lambda}$$

Proposed by Marin Chirciu-Romania

UP.458 Find:

$$\Omega = \int_0^\infty \frac{x\log x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

UP.459 Calculate the integral:

$$\Omega = \int_0^\infty \frac{\log(x+1)}{x^3+1}dx$$

Proposed by Vasile Mircea Popa-Romania

UP.460 In $\Delta ABC, I$ −incenter, O −circumcenter, the following relationship holds:

$$\sum_{cyc}\left(\cos\widehat{IOA} - \sin\frac{A}{2}\right)^2 \geq \frac{3r^2}{R^2}$$

Proposed by Daniel Sitaru-Romania

UP.461 Find:

$$\Omega_n = \sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\frac{k^3+k^2-3k-2}{(k+2)!}\right)$$

Proposed by Florică Anastase-Romania

UP.462 Prove that:

$$\prod_{p=1}^{n}\left(1+\left(\sum_{k=1}^{n}\frac{2k}{k^4+(2p-1)k^2+p^2}\right)^2\right) < 4$$

Proposed by Florică Anastase-Romania

UP.463 Let $\lambda \geq \frac{7}{4}$. Solve for real numbers:

$$\begin{cases} \dfrac{x}{\sqrt{x^2-\lambda x+\lambda^2}} = \log_{\lambda^2-\lambda}(\lambda^2-y) \\ \dfrac{y}{\sqrt{y^2-\lambda y+\lambda^2}} = \log_{\lambda^2-\lambda}(\lambda^2-z) \\ \dfrac{z}{\sqrt{z^2-\lambda z+\lambda^2}} = \log_{\lambda^2-\lambda}(\lambda^2-x) \end{cases}$$

Proposed by Marin Chirciu-Romania

UP.464 In ΔABC the following relationship holds:

$$\frac{a^3}{(5a+7b)(7a+5b)}+\frac{b^3}{(5b+7c)(7c+5b)}+\frac{c^3}{(5c+7a)(7c+5a)}\geq\frac{\sqrt{3}r}{24}$$

Proposed by Daniel Sitaru-Romania

UP.465 In ΔABC the following relationship holds:

$$2^4(R+r-d)^3[4(R-r+d)^3+R^3]\leq a^6+b^6+c^6\leq$$

$$\leq 2^4(R+r+d)^3[4(R-r-d)^3+R^3]$$ **where we denote by $d=\sqrt{R^2-2Rr}$**

Proposed by Marius Drăgan, Neculai Stanciu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com