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SOLVING CAUCHY PROBLEM FOR THE HEAT EQUATION USING MAPLE

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Abstract:

In this paper, we use Maple to investigate different solutions of the heat equation, starting with initial heat profile in one dimensional media under suitable assumptions. We used two different methodologies to solve the equation and to compare the results with numerical solutions. Moreover, the well-known Cauchy problem of heat equation was explored in details to verify consistency in the solutions.
Keywords: heat equation, Fourier transform, Cauchy problem, maple.

Introduction .1

Partial differential equations (PDEs) are important in many fields of science, .2 because they can be formulated whenever a change in a system is happened. Since the time of Newton, differential equations were increasingly used to help understanding the physical, chemical and biological phenomena. In recent years, their uses extended in the industrial, engineering, economic and social sciences. In fact, most of the natural laws in physics such as Maxwell's equations, Newton's laws of cooling and motion, Navier-Stokes's equations and Schrödinger equation in quantum mechanics can be written in terms of partial differential equations. In other words, these laws describe physical phenomena by finding relationships between space and partial derivatives with respect of time. The partial appears in these equations because they represent natural things (such as velocity, acceleration, force, friction, flux and current), and accordingly we get equations that link partial derivatives of unknown quantities that we want to know.

Several methods have been employed in order to solve PDFs. The common method is the reduction of PDFs into an ordinary differential equation (ODFs), which can be viewed as a special case of a symmetry reduction [1, 2]. Fourier transform and Fourier series are also widely used in this field. [3]. Recently, the numerical methods were extensively used to approximate the solutions of PDEs. This includes the finite element method (FEM), finite volume methods (FVM) and finite difference methods (FDM)[4, 5]. Most of these methods are adopted in educational different software and can easily be employed in personal computers.

Recently, remarkable progress has been achieved in computer algebra software and graphical drawing utilities. Mathematical, maple and Matlab are the mostly used software in this field. While Matlab is appropriate for large-scale numerical calculations, Maple was demonstrated to be the best for mathematical science education. This is probably due to simplicity in manipulating the mathematical formulas. The use of such advanced computer software can improve forms and methods of educational and learning activities through efficient combination of traditional methods of teaching higher mathematics and modern achievements computer mathematics.

The heat equation is an essential partial differential equation in physics, which describes the distribution of heat (or variation in temperature) in a given region over time. Engineers are always encountered with heat equation when designing heat exchanger, heat sinks and cooling systems. Moreover, heat equation is widely used to determine the temperature in inaccessible areas, such as surface of fuel rod in nuclear reactor or core of the sun.

In this paper, we will study the heat flow where the conduction of thermal energy is much more important than convection (That is primarily the heat flow though solids) [6, 7]. We shall simulate the heat equation for one dimensional media. Two different methods will be employed as implemented in the computer program Maple. Results from simulation will be compared and evaluated against the known solutions.

Project design .3

As we mentioned above, the heat equation is a PDE describing the distribution of temperature (denoted as *u(x,t)*) over time. In one spatial dimension the temperature obeys the relation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial^2 x} \qquad (1)$$

where c^2 is called the speed of the wave ($c^2 = K/_{\sigma.\rho}$: thermal conductivity / (specific heat × density)) and it depends solely on the medium. Typically, projects related to partial differential equations are supplemented with initial and boundary conditions.
Since the equation (1) contains the first derivative in time, the initial value problem consists of solving the differential equation with one prerequisite. In other words, the initial temperature (usually at t= 0) should be given as initial condition. It is possible that the initial temperature is not constant, but depends on the spatial variable (x). Hence, the temperature distribution should be known:

$$u(x,0) = \Phi(x) \qquad (2)$$

Various types of boundary conditions can be considered depending on the situation we are dealing with. For instance, when the end of the rod is in contact with a heat bath (heat reservoir), the temperature at this end can be approximated by the following specified temperature:

$$u(0,t) = u_B(t), \qquad (3)$$

where the $u_B(t)$ is the heat of the path. Other boundary condition when the end of the rod is completely isolated can be written as:

$$u(L,t) = 0 \qquad (4)$$

Where L is the length of the rod

One should mention that when analytical solution of heat equation cannot be obtained, numerical methods can be useful in such cases.

Examples and discussions .4

In order to illustrate the efficiency of the previous methods, solutions of heat equation for some represented examples will be shown in this section. We will consider the heat equation in 1D finite bar with length of *l* with initial condition of (free heat transfer at the end):

$$u(x,0) = \Phi(x) = x(l-x)\,, \quad u(0,t) = 0, \qquad u(l,t) = 0 \qquad (5)$$

The fundamental solutions 4.1

The fundamental solution of the heat equation (1) with set of boundary condition given by (5) can be written as: (see appendix A for maple program)

$$u(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)e^{-\lambda_n^2} \qquad (6)$$

where $$\lambda = \frac{cn\pi}{L} \qquad (7)$$

Starting with n=1, *l*= 2π and c=2, the fundamental model has the formula (see figure 1):

$$u(x,t) = \sin\left(\frac{x}{2}\right)e^{-t} \qquad (8)$$

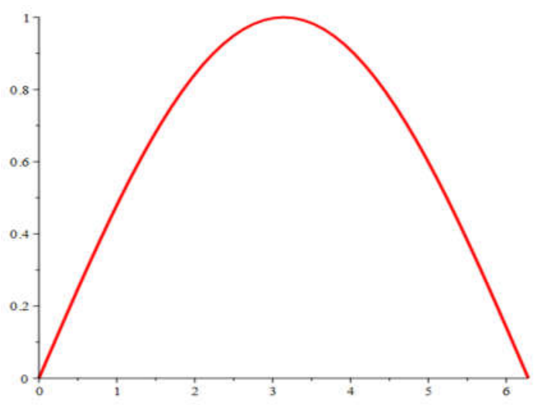

Figure 1. The fundamental solution of heat equation. It can be easily verified that this mode in (8) is indeed a solution of the heat equation.

3.2 Solution Using Fourier Transforms

In order to solve the heat equation using Fourier Transforms, we follow the four following steps: (see appendix B for maple program)

Transform the equation into Fourier space. (1)

Solving the resulting ordinary differential equation. (That is finding Eigenvalues and eigenfunctions). (2)

Transform the results back into real space. (3)

Evaluate the solution (That is inverse Fourier integral). (4)

Applying the previous steps using maple, the final solution for the heat equation using the same boundary conditions in (5) can be written as:

$$solution = \sum_{m=0}^{\infty} \frac{8l^2 e^{-\frac{\pi^2 c^2 (2m+1)^2 t^2}{l^2}} sin\left(\frac{(2m+1)\pi x}{l}\right)}{(2m+1)^3\,\pi^3} \quad (9)$$

Using the same values for the constant as above (*l*= 2π and c=2), we obtain the fundamental mode by substituting m with 0:

$$S_0 = \sin(\frac{x}{2})e^{-t} \quad (10)$$

Which is equal to that obtained in the previous method.

3.3 Solving Cauchy problems for heat equations

The Cauchy problem for the heat equation is the pure initial value problem. It is s quite known to be ill-posed. In other word, errors measurement can be significantly enlarged and ruin the solution. Hence, the solution is no longer continues [8, 9]. The problem arises when scientists attempt to finds the temperature at the surface of a body using inner measurement. Several methods haven been employed in order to tackle the Cauchy problem for heat equation. Nanfuka *et al.* approximate the time derivative by using a cubic smoothing spline. Their method gives goo results at the ends of the measurement interval [10]. Elden studies a modification of the heat equation, where a fourth-order mixed derivative term is added. The error estimation for the modified equation shows that the solution approximate of the solution of the Cauchy problem [11, 12].

Mathematically the problem can be expressed as: (see appendix B for maple program)

$$\begin{Bmatrix} u_t - \Delta u = 0 \qquad t \geq 0, \\ u(x,0) = \Phi(x) \end{Bmatrix} \quad (11)$$

Assume the same formula for $\Phi(x)$ as in the previous sections and considering passion's formula:

$$\frac{\int_{-\infty}^{\infty}\xi(1-\xi)e^{-\frac{(x-\xi)^2}{4t}}d\xi}{2\sqrt{2\pi t}} \qquad (12)$$

We obtaine the solution:

$$\frac{\int_{-\infty}^{\infty}(\xi\, e^{\frac{x^2}{4t}}\, e^{\frac{x\xi}{2t}}\, e^{-\frac{\xi^2}{4t}}\, l-\xi^2 e^{\frac{x^2}{4t}}\, e^{\frac{x\xi}{2t}}\, e^{-\frac{\xi^2}{4t}})d\xi}{2\sqrt{\pi t}} \qquad (13)$$

Which can be simplified to be written as:

$$sol = xl - x^2 - 2t \qquad (14)$$

It is clear that at the solution is continues and satisfy the boundary condition at:

$$u(0,t) = 0, \qquad u(l,t) = 0$$

4. Conclusion

As demonstrated in the results, heat equation was solved very efficiently by two different methods using Maple software. The solution for the studied case were identical to those expected. Moreover, Maple was able to handle the Cauchy problem for heat equation very effectively without any modifications or approximations.

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