A Simple Diophantine Equation

Introduction

The history of the theory of numbers and Diophantine analysis is rich and immense [1]. Since antiquity, people have contributed tremendously to finding the solutions to higher degree Diophantine equations and to near misses of those equations for which no integer solutions exists such as $a^n + b^n = c^n$ where $a, b, c \in \mathbb{Z}$, $n \in \mathbb{N}$ and $n \geq 3$ [2]. In this article, we will find some parametric solutions to such a Diophantine equation and will derive a simple proof of the universality of that equation.

The Diophantine Equation

The equation [3] which we will examine in this article is,

$$p^2 + q^2 = r^2 + n \tag{1}$$

where, $p, q, r, n \in \mathbb{N}$. The solution to (1) when $n = 0$ is known since antiquity. If we vary the value of n in the above equation, we can obtain a family of "near-miss" equations.

The Parametric Solution

Let us divide the solution to (1) in the following two cases:

Case 1: When $n = 2k$, $k \in \mathbb{N}$
Let $p = Ak + B$, $q = Ck + D$ and $r = Ek + F$ be a possible parametric solution to (1), where, $A, B, C, D, E, F \in \mathbb{Z}$. Then, we have the following equality,

$$(Ak + B)^2 + (Ck + D)^2 = (Ek + F)^2 + 2k \tag{2}$$

since (2) is an identity for all k, therefore, comparing the coefficients of all the powers of k, we obtain,

$$A^2 + C^2 = E^2, AB + CD = EF + 1, B^2 + D^2 = F^2$$

selecting the two smallest Pythagorean triplets (A, C, E) and (B, D, F) which satisfies the condition $AB + CD = EF + 1$, we obtain the following identity,

$$(3k-8)^2 + (4k-15)^2 = (5k-17)^2 + 2k \tag{3}$$

where, $k \in \mathbb{N}$.

<u>**Case 2**</u>: When $n = 2k-1$, $k \in \mathbb{N}$
Observe that, (1) can be written as,

$$p^2 - (2k-1) = r^2 - q^2$$

it is known from the identity $2a+1 = (a+1)^2 - a^2$ that any odd integer $2a-1$ can be written as the difference of two integer squares, therefore by letting p be any even number $2m$, we obtain the following identity,

$$(2m)^2 - (2k-1) = 2(2m^2-k) + 1 = (2m^2-k+1)^2 - (2m^2-k)^2$$

rearranging the above terms, we finally have,

$$(2m)^2 + (2m^2-k)^2 = (2m^2-k+1)^2 + (2k-1)$$

where, $m \in \mathbb{N}$.

Conclusion

Continuing the procedure used in "Case-1", some similar identities follows,

$$(5k-8)^2 + (12k-15)^2 = (13k-17)^2 + 2k$$

$$(5k-12)^2 + (12k-35)^2 = (13k-37)^2 + 2k$$
$$(7k-12)^2 + (24k-35)^2 = (25k-37)^2 + 2k$$
$$(7k-16)^2 + (24k-63)^2 = (25k-65)^2 + 2k$$
$$(9k-16)^2 + (40k-63)^2 = (41k-65)^2 + 2k$$
$$(9k-20)^2 + (40k-99)^2 = (41k-101)^2 + 2k$$
$$(11k-20)^2 + (60k-99)^2 = (61k-101)^2 + 2k$$
$$(11k-24)^2 + (60k-143)^2 = (61k-145)^2 + 2k$$

and so on.
Generalizing the above identities, we obtain the following pair of symmetric identities,

$$(\left|a^2-b^2\right|k-4a)^2+((2ab)k-(4a^2-1))^2=((a^2+b^2)k-(4a^2+1))^2+2k$$

and

$$(\left|a^2-b^2\right|k-4b)^2+((2ab)k-(4b^2-1))^2=((a^2+b^2)k-(4b^2+1))^2+2k$$

where, $|a-b|=1$ and $a,b\in\mathbb{N}$.
Similarly, a nice parametric solution to (1) for "Case-2" could be given by subtracting 1 from both the sides of (3), i.e.,

$$(3k-8)^2+(4k-15)^2-1=(5k-17)^2+2k-1=25k^2-168k+288=(3k-12)^2+(4k-12)^2$$

from which we obtain the following identity,

$$(3k-12)^2+(4k-12)^2=(5k-17)^2+2k-1$$

where, $k\in\mathbb{N}-\{3,4\}$.
The above parametric solutions shows that for every natural number n there exists infinitely many integers p,q and r such that (1) is true. Hence, any natural number n can be written in the form $x^2+y^2-z^2$ which proves its universality. This is a useful result/lemma in the field of Diophantine analysis, additive number theory and combinatorics.

References

[1] Dickson, Leonard Eugene, History of the Theory of Numbers, Volume II : Diophantine Analysis. Carnegie Institution of Washington, Washington (1920)
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[3] Andreescu, T. and Andrica, D. and Cucurezeanu, I., An Introduction to Diophantine Equations: A Problem-Based Approach. Birkhäuser Boston (2010)

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