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THE SPECIAL TRIANGLE WITH SIDES $\sqrt{a(b+c)},\sqrt{b(c+a)},\sqrt{c(a+b)}$

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Abstract. This paper presents some geometric and algebraic inequalities starting from a certain triangle.

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Introduction

It is well-known that if u,v,w are the sides of a triangle, then $\sqrt{u},\sqrt{v},\sqrt{w}$ are the sides of a triangle. If we take $u=a(b+c),v=b(c+a),w=c(a+b)$, where a,b,c are the sides of a triangle, then we obtain that $\sqrt{a(b+c)},\sqrt{b(c+a)},\sqrt{c(a+b)}$ are the sides of a triangle.

If we denote $a'=\sqrt{a(b+c)},b'=\sqrt{b(c+a)},c'=\sqrt{c(a+b)}$, $x=ab+bc+ca$ and we compute area S', of triangle with sides a',b',c', we obtain

$$16S'^2=2\sum(a'b')^2-\sum a'^4=2[ab(c^2+x)+bc(a^2+x)+ca(b^2+x)]-a^2(b^2+c^2+2bc)-$$

$$-b^2(c^2+a^2+2ac)-c^2(a^2+b^2+2ab)=2abc^2+2a^2bc+2ab^2c+2x(ab+bc+ca)-2a^2b^2-$$

$$-2b^2c^2-2c^2a^2-2ab^2c-2a^2bc-2abc^2=2[(ab+bc+ca)^2-a^2b^2-b^2c^2-c^2a^2]=$$

$$=4abc(a+b+c)=16Rrp\cdot 2p\Rightarrow S'=\sqrt{2Rrp^2}\Rightarrow S'=p\sqrt{2Rr}.$$

Also we obtain that

$$p'=\frac{1}{2}\sum\sqrt{a(b+c)};\ r'=\frac{S'}{p'}=\frac{2p\sqrt{2}\sqrt{Rr}}{\sum\sqrt{a(b+c)}};$$

$$R'=\frac{a'b'c'}{4S'}=\frac{\sqrt{abc\prod(a+b)}}{4p\sqrt{2Rr}}=\frac{\sqrt{4Rrp\prod(a+b)}}{4p\sqrt{2Rr}}=\frac{\sqrt{p\prod(a+b)}}{2\sqrt{2}p}=\sqrt{\frac{\prod(a+b)}{8p}};$$

$$\cos A'=\frac{b'^2+c'^2-a'^2}{2b'c'}=\frac{b(a+c)+c(a+b)-a(b+c)}{2\sqrt{bc(a+b)(a+c)}}=\sqrt{\frac{bc}{(a+b)(a+c)}};$$

$$\sin A'=\sqrt{1-\frac{bc}{(a+b)(a+c)}}=\sqrt{\frac{a(a+b+c)}{(a+b)(a+c)}}=\sqrt{\frac{2ap}{(a+b)(a+c)}};$$

$$tgA'=\sqrt{\frac{2ap}{bc}}=a\cdot\sqrt{\frac{2p}{abc}}=a\cdot\sqrt{\frac{2p}{4Rrp}}=\frac{a}{\sqrt{2Rr}};$$

$$m_a'=\sqrt{\frac{2(b'^2+c'^2)-a'^2}{4}}=\sqrt{\frac{2[b(a+c)+c(a+b)]-a(b+c)}{4}}=\sqrt{\frac{ab+ac+4bc}{4}}=$$

$$=\frac{1}{2}\sqrt{ab+ac+4bc};\ h_a'=\frac{2S'}{a'}=\frac{2\sqrt{2}p\sqrt{Rr}}{a'}=\frac{2\sqrt{2}p\sqrt{Rr}}{\sqrt{a(b+c)}}.$$

In the next we obtain some results related to triangle with sides a',b',c' **.**

Main Results

Proposition 1. A refinement of *Ionescu-Weitzenböck* inequality.

In any triangle ABC **is true the inequality:**

$$ab+bc+ca\ge 4\sqrt{3}\sqrt{\frac{R}{2r}}S.$$

Proof. By *Ionescu-Weitzenböck* inequality we have successively that

$$a'^2+b'^2+c'^2\ge 4\sqrt{3}S'\Leftrightarrow$$

$$\Leftrightarrow a(b+c)+b(c+a)+c(a+b)\ge 4\sqrt{3}p\sqrt{2Rr}\Leftrightarrow ab+bc+ca\ge 4\sqrt{3}S\sqrt{\frac{R}{2r}}\overset{Euler}{\ge}4\sqrt{3}S$$ **, q.e.d.**

Proposition 2. In any triangle ABC is true the inequality:

$$\sum\sqrt{ab(a+c)(b+c)}\geq 2\sqrt{6}\sqrt{Rr}\,p+p^2+r^2+4Rr$$

Proof. By *Hadwiger-Finsler* inequality we obtain successively

$$a'^2+b'^2+c'^2\geq 4\sqrt{3}S'+\sum(a'-b')^2\Leftrightarrow$$

$$\Leftrightarrow\sum a'b'\geq 2\sqrt{3}S'+\frac{1}{2}\sum a'^2\Leftrightarrow\sum\sqrt{ab(a+c)(b+c)}\geq 2\sqrt{6}\sqrt{Rr}\,p+\frac{1}{2}\sum a(b+c)=$$

$$=2\sqrt{6}\sqrt{Rr}\,p+\sum ab=2\sqrt{6}\sqrt{Rr}\,p+p^2+r^2+4Rr\text{, q.e.d.}$$

Proposition 3. In any triangle ABC is true the inequality:

$$\sum\sqrt{ab(a+c)(b+c)}\geq 4\sqrt{6}\sqrt{Rr}\,p$$

Proof. This inequality results from *Gordon* inequality $a'b'+b'c'+c'a'\geq 4\sqrt{3}S'\Leftrightarrow$

$$\Leftrightarrow\sum\sqrt{ab(a+c)(b+c)}\geq 4\sqrt{6}\sqrt{Rr}\,p\text{, q.e.d.}$$

Proposition 4. In any triangle ABC is true the inequality:

$$\sum\sqrt{ab(a+c)(b+c)}\leq\frac{2\sqrt{6}}{3}\sqrt{Rr}\,p+\frac{5}{3}\sum ab$$

Proof. By reverse *Hadwiger-Finsler* inequality we deduce successively

$$\sum a'^2\leq 4\sqrt{3}S'+3\sum(a'-b')^2\Leftrightarrow$$

$$\Leftrightarrow 6\sum a'b'\leq 4\sqrt{3}S'+5\sum a'^2\Leftrightarrow\sum a'b'\leq\frac{2\sqrt{3}}{3}S'+\frac{5}{6}\sum a'^2$$

$$\Leftrightarrow\sum\sqrt{ab(a+c)(b+c)}\leq\frac{2\sqrt{3}}{3}p\sqrt{2Rr}+\frac{5}{3}\sum ab\text{, q.e.d.}$$

Proposition 5. In any triangle ABC is true the inequality:

$$\sqrt{a(b+c)}+\sqrt{b(c+a)}+\sqrt{c(a+b)}\ge\sqrt[4]{864Rrp^2}$$

Proof. From *Mitrinović* inequality, i.e. $p'\ge3\sqrt{3}r'$ **we obtain**

$$\frac{1}{2}\sum\sqrt{a(b+c)}\ge3\sqrt{3}\cdot\frac{2p\sqrt{2}\sqrt{Rr}}{\sum a(b+c)}\Leftrightarrow(\sum\sqrt{a(b+c)})^2\ge12p\sqrt{6}\sqrt{Rr}\Leftrightarrow$$

$$\Leftrightarrow\sum\sqrt{a(b+c)}\ge\sqrt[4]{864Rrp^2}$$ **, q.e.d.**

Proposition 6. In any triangle ABC **is true the inequality:**

$$\sum\sqrt{a(b+c)}\ge\sqrt[3]{54}\cdot\sqrt[6]{Rrp\prod(a+b)}\ge2^{\frac{7}{6}}\cdot3\cdot\sqrt[3]{Rrp}$$

Proof. From $p'^2\ge\frac{27}{2}R'r'$ **we have successively:**

$$\frac{1}{4}(\sum\sqrt{a(b+c)})^2\ge\frac{27}{2}\sqrt{\frac{\prod(a+b)}{8p}}\frac{2\sqrt{2}\sqrt{Rr}p}{\sum\sqrt{a(b+c)}}\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{4}(\sum\sqrt{a(b+c)})^3\ge\frac{27}{2}\sqrt{Rrp\prod(a+b)}\Leftrightarrow(\sum\sqrt{a(b+c)})^3\ge54\sqrt{Rrp\prod(a+b)},$$

where by *Césaro* inequality, i.e. $\prod(a+b)\ge8abc=32Rrp$ **we obtain the desired inequality.**

Proposition 7. In any triangle ABC **is true the inequality:**

$$\sum\sqrt{a(b+c)}\ge\sqrt[3]{54}\cdot\sqrt[6]{Rrp\prod(a+b)}\ge\sqrt[4]{864Rrp^2}\ge2^{\frac{7}{6}}\cdot3\cdot\sqrt[3]{Rrp}$$

Proof. The first two inequalities from the above yields immediately from

$p'^2\ge\frac{27}{2}R'r'\ge27r'^2$**, and the last is equivalent to**

$$(\sqrt[4]{864Rrp^2})^{12} \ge \left(2^{\frac{7}{6}} \cdot 3 \cdot \sqrt[3]{Rrp}\right)^{12} \Leftrightarrow 2^{15} \cdot 3^9 \cdot p^2 \ge 2^{14} \cdot 3^{12} \cdot Rr \Leftrightarrow p^2 \ge \frac{27Rr}{2}$$ **, true.**

Proposition 8. In any triangle ABC **is true the inequality:**

$$8p\left(\sum\sqrt{a(b+c)}\right)^2 \le 27\prod(a+b)$$

Proof. From $p' \le \frac{3\sqrt{3}}{2}R'$ **we obtain** $\frac{1}{2}\sum a(b+c) \le \frac{3\sqrt{3}}{2}\sqrt{\frac{\prod(a+b)}{8p}} \Leftrightarrow$

$$\Leftrightarrow \left(\sum\sqrt{a(b+c)}\right)^2 \le \frac{27}{8}\frac{\prod(a+b)}{p}$$ **, q.e.d.**

Proposition 9. In any triangle ABC **is true the inequality:**

If $x, y, z > 0$**, such that** $x+y+z=1$**, then**

$$\sum\sqrt{1-x^2} \ge \sqrt[3]{54} \cdot \sqrt[6]{\frac{\prod(1-x^2)}{4}} \ge \sqrt[4]{216\prod(1-x)} \ge 2^{\frac{7}{6}} \cdot 3 \cdot \sqrt[3]{\frac{\prod(1-x)}{4}}.$$

Proof. Using *Ravi* substitutions $a=y+z, b=x+z, c=x+y, p=x+y+z, r=\sqrt{\frac{xyz}{x+y+z}}$**,**

$R=\frac{\prod(x+y)}{4\sqrt{xyz(x+y+z)}}$**, from 7. we obtain**

$$\sum\sqrt{(y+z)(2x+y+z)} \ge \sqrt[3]{54} \cdot \sqrt[6]{\frac{\prod(x+y)\prod(2x+y+z)}{4}} \ge$$

$$\ge \sqrt[4]{216(x+y+z)\prod(x+y)} \ge 2^{\frac{7}{6}} \cdot 3 \cdot \sqrt[3]{\frac{\prod(x+y)}{4}},$$

and by $x+y+z=1$**, q.e.d.**

Proposition 10. In any triangle ABC is true the inequality:

If $x,y,z>0$, such that $x+y+z=1$, then $8\left(\sum\sqrt{1-x^2}\right)^2\le 27\prod(1+x)$.

Proof. By *Ravi* substitutions and 8. yields $8p\left(\sum\sqrt{a(b+c)}\right)^2\le 27\prod(a+b)\Leftrightarrow$

$\Leftrightarrow 8(x+y+z)\left(\sum\sqrt{(y+z)(2x+y+z)}\right)^2\le 27\prod(2x+y+z)$ **, q.e.d.**

Proposition 11. In any triangle ABC is true the inequality:

$$\left(\sum\frac{ab+ac+2bc}{\sqrt{ab+ac+4bc}}\right)^2\le\frac{9}{2}\frac{\prod(a+b)}{p}$$

Proof. From *Tereshin,*inequality i.e. $\sum\frac{b'^2+c'^2}{m'_a}\le 12R'$ **yields that**

$\sum\frac{b(a+c)+c(a+b)}{\frac{1}{2}\sqrt{ab+ac+4bc}}\le 12\sqrt{\frac{\prod(a+b)}{8p}}\Leftrightarrow\sum\frac{ab+ac+2bc}{\sqrt{ab+ac+4bc}}\le\frac{3\sqrt{2}}{2}\sqrt{\frac{\prod(a+b)}{p}}$ **, q.e.d.**

Proposition 12. In any triangle ABC is true the inequality:

$$16p\sum ab\le 9\prod(a+b)$$

Proof. We have successively

$$\sum a'\sin A'\le\frac{9}{2}R'\Leftrightarrow\sum\sqrt{a(b+c)}\sqrt{\frac{2ap}{(a+b)(a+c)}}\le\frac{9}{2}\sqrt{\frac{\prod(a+b)}{8p}}\Leftrightarrow$$

$$\Leftrightarrow 8p\sum\frac{a\sqrt{b+c}}{\sqrt{(a+b)(a+c)}}\le 9\sqrt{\prod(a+b)}\Leftrightarrow 8p\sum a(b+c)\le 9\prod(a+b)\Leftrightarrow$$

$\Leftrightarrow 16p\sum ab\le 9\prod(a+b)$**, q.e.d.**

Proposition 13. In any triangle ABC is true the inequality:

$$\sum\frac{1}{\sqrt{ab+bc+4bc}}\geq\sqrt{\frac{8p}{\prod(a+b)}}$$

Proof. From $\frac{1}{m'_a}+\frac{1}{m'_b}+\frac{1}{m'_c}\geq\frac{2}{R'}$ **we have** $\sum\frac{1}{\sqrt{ab+bc+4bc}}\geq\frac{1}{\sqrt{\frac{\prod(a+b)}{8p}}}$**, q.e.d.**

Proposition 14. In any triangle ABC is true the inequality:

$$\sum\sqrt{ab+ac+4bc}\leq 9\sqrt{\frac{\prod(a+b)}{8p}}$$

Proof. From $m'_a+m'_b+m'_c\leq\frac{9}{2}R'$**, we deduce** $\sum\sqrt{ab+ac+4bc}\leq 9\sqrt{\frac{\prod(a+b)}{8p}}$**, q.e.d.**

Proposition 15. In any triangle ABC is true the inequality:

$$\sum\frac{bc}{(a+b)(a+c)}\leq 6\cdot\frac{abc}{\prod(b+c)}$$

Proof. Yields immediately from $\cos^2 A'+\cos^2 B'+\cos^2 C'\geq 6\cos A'\cos B'\cos C'$ **.**

Proposition 16. In any triangle ABC is true the inequality:

$$\sum\sqrt{a(b+c)}\geq\sqrt{12\sqrt{6}}\sqrt{Rrp^2}$$

Proof. From $(\sin A'+\sin B'+\sin C')^2\geq 6\sqrt{3}\sin A'\sin B'\sin C'$ **we obtain successively**

$$\left(\sum\sqrt{\frac{2ap}{(a+b)(b+c)}}\right)^2\geq 6\sqrt{3}\cdot\sqrt{\frac{8abcp^3}{\prod(a+b)^2}}\Leftrightarrow\frac{2p\left(\sum\sqrt{a(b+c)}\right)^2}{\prod(a+b)}\geq\frac{6\sqrt{3}p\sqrt{8\cdot 4Rrp^2}}{\prod(a+b)}\Leftrightarrow \text{q.e.d.}$$

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