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AN AMAZING CHAIN OF INTEGRALS

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***PROBLEM* (1)_*PROVE*:**

$$I=\int_0^1 \frac{\log^2(1+x)\log(1+x^2)}{1+x}dx$$
$$=\frac{5}{2}Li_4\left(\frac{1}{2}\right)-\frac{35}{16}\zeta(3)\log\left(\frac{1}{2}\right)-\frac{7\pi^2}{64}\log^2(2)+\frac{41}{96}\log^4(2)-\frac{209\pi^4}{7680}$$

⟨Proposed by Narendra Bhandari⟩

$$*\ I=\int_0^1 \frac{\log^2(1+x)\log(1+x^2)}{1+x}dx$$
$$=\int_0^1 \log^2(1+x)\log(1+x^2)\,d(\log(1+x)),\ \begin{cases}u=\log(1+x^2)\\ dv=\log^2(1+x)\,d(\log(1+x))\end{cases}$$
$$\Rightarrow\begin{cases}du=\dfrac{2x}{1+x^2}\\ v=\dfrac{1}{3}\log^3(1+x)\end{cases}$$
$$\Rightarrow I=\frac{1}{3}\log^3(1+x)\log(1+x^2)\Big|_0^1-\frac{2}{3}\int_0^1\frac{x\log^3(1+x)}{1+x^2}dx=\frac{1}{3}\log^4(2)-\frac{2}{3}J$$
$$*\ J=\int_0^1\frac{x\log^3(1+x)}{1+x^2}dx,\ let: x\to\frac{1}{1+x}\Rightarrow J$$
$$=\int_{\frac{1}{2}}^1\frac{(x-1)\log^3(x)}{x(2x^2-2x+1)}dx$$
$$=\int_{\frac{1}{2}}^1\frac{(x-1)\log^3(x)}{x((1-x)^2+x^2)}dx=\int_{\frac{1}{2}}^1\log^3(x)\left(\frac{2x-1}{(1-x)^2+x^2}-\frac{1}{x}\right)dx$$

Use: $\frac{2x-1}{(1-x)^2+x^2} = -\Re\left(\frac{1+i}{1-(1+i)x}\right) \Rightarrow J$

$$= \int_{\frac{1}{2}}^{1} \log^3(x)\left(-\Re\left(\frac{1+i}{1-(1+i)x}\right) - \frac{1}{x}\right)dx$$

$$= \frac{1}{4}\log^4(2) - \Re\int_{\frac{1}{2}}^{1}\log^3(x)\frac{1+i}{1-(1+i)x}dx$$

$$= \frac{1}{4}\log^4(2) - \Re\int_{0}^{1}\log^3(x)\frac{1+i}{1-(1+i)x}dx$$

$$+ \Re\int_{0}^{\frac{1}{2}}\log^3(x)\frac{1+i}{1-(1+i)x}dx$$

$$= \frac{1}{4}\log^4(2) + 6\Re Li_4(1+i) + \Re\int_0^{\frac{1}{2}}\log^3(x)\frac{1+i}{1-(1+i)x}dx \xrightarrow{x=\frac{x}{2}}$$

$$= \frac{1}{4}\log^4(2) + 6\Re Li_4(1+i)$$

$$+ \Re\int_0^1 \frac{1}{2}(\log(x)$$

$$- \log(2))^3\frac{1+i}{1-\frac{1+i}{2}x}dx$$ .Expand out and integrate yields:

$$J = \frac{1}{4}\log^4(2) + 6\Re Li_4(1+i) - 6\Re Li_4\left(\frac{1+i}{2}\right) - 6\log(2)\Re Li_3\left(\frac{1+i}{2}\right)$$

$$- 3\log^2(2)\Re Li_2\left(\frac{1+i}{2}\right) - \frac{1}{2}\log^4(2)$$

$$= 6\Re\left(Li_4(1+i) - Li_4\left(\frac{1+i}{2}\right)\right) - 6\log(2)\Re Li_3\left(\frac{1+i}{2}\right)$$

$$- 3\log^2(2)\Re Li_2\left(\frac{1+i}{2}\right) - \frac{1}{4}\log^4(2)$$

* Use: $\Re Li_3\left(\frac{1+i}{2}\right) = \frac{35}{64}\zeta(3) - \frac{5\pi^2}{192}\log(2) + \frac{1}{48}\log^3(2)$, $\Re Li_2\left(\frac{1+i}{2}\right)$

$$= \frac{5\pi^2}{96} - \frac{1}{8}\log^2(2), \Re\left(Li_4(1+i) - Li_4\left(\frac{1+i}{2}\right)\right)$$

$$= -\frac{5}{8}Li_4\left(\frac{1}{2}\right) + \frac{209\pi^4}{30720} + \frac{7\pi^2}{256}\log^2(2) - \frac{3}{128}\log^4(2)$$

$$\Rightarrow J = 6\left(-\frac{5}{8}Li_4\left(\frac{1}{2}\right)+\frac{209\pi^4}{30720}+\frac{7\pi^2}{256}log^2(2)-\frac{3}{128}log^4(2)\right)$$

$$-6\,log(2)\left(\frac{35}{64}\zeta(3)-\frac{5\pi^2}{192}log(2)+\frac{1}{48}log^3(2)\right)$$

$$-3\,log^2(2)\left(\frac{5\pi^2}{96}-\frac{1}{8}log^2(2)\right)-\frac{1}{4}log^4(2)$$

$$=-\frac{15}{4}Li_4\left(\frac{1}{2}\right)+\frac{209\pi^4}{5120}+\frac{21\pi^2}{128}log^2(2)-\frac{9}{64}log^4(2)-\frac{105}{32}log(2)\,\zeta(3)$$

$$\Rightarrow I = \frac{1}{3}log^4(2)-\frac{2}{3}J$$

$$=\frac{1}{3}log^4(2)$$

$$-\frac{2}{3}\left(-\frac{15}{4}Li_4\left(\frac{1}{2}\right)+\frac{209\pi^4}{5120}+\frac{21\pi^2}{128}log^2(2)-\frac{9}{64}log^4(2)\right.$$

$$\left.-\frac{105}{32}log(2)\,\zeta(3)\right)$$

$$=\frac{5}{2}Li_4\left(\frac{1}{2}\right)+\frac{35}{16}log(2)\,\zeta(3)-\frac{7\pi^2}{64}log^2(2)+\frac{41}{96}log^4(2)-\frac{209\pi^4}{7680}$$

$$=\frac{5}{2}Li_4\left(\frac{1}{2}\right)-\frac{35}{16}\zeta(3)\,log\left(\frac{1}{2}\right)-\frac{7\pi^2}{64}log^2(2)+\frac{41}{96}log^4(2)-\frac{209\pi^4}{7680}\,(Q.E.D)$$

PROBLEM(3)_PROVE:

$$I=\int_0^1\frac{x\,log^2(x)\,log(1-x)}{1+x^2}dx=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}log^4(2)+\frac{5\pi^2}{96}log^2(2)$$

⟨Proposed by Narendra Bhandari⟩

$$I=\int_0^1\frac{x\,log^2(x)\,log(1-x)}{1+x^2}dx=\frac{1}{2}\int_0^1 log^2(x)\,log(1-x)\,d(log(x^2+1))\xrightarrow{IBP}$$

$$=\underbrace{\frac{1}{2}log(x^2+1)\,log^2(x)\,log(1-x)\Big|_0^1}_{=0}$$

$$-\int_0^1\frac{log(x)\,log(1-x)\,log(x^2+1)}{x}dx+\frac{1}{2}\int_0^1\frac{log^2(x)\,log(x^2+1)}{1-x}dx$$

$$=\frac{1}{2}J-K$$

∗ **Use** : Cornel Ioan Valean's generalized alternating harmonic series:

$$I(m)=\frac{(-1)^m}{(m-1)!}\int_0^1 \frac{\log^{m-1}(x)\log\left(\frac{1+x^2}{2}\right)}{1-x}dx$$
$$=m\zeta(m+1)-2^{-m}(1-2^{-m+1})\log(2)\zeta(m)$$
$$-\sum_{k=0}^{m-1}\beta(k+1)\beta(m-k)$$
$$-\sum_{k=1}^{m-2}2^{-m-1}(1-2^{-k})(1-2^{-m+k+1})\zeta(k+1)\zeta(m-k)$$

Put : $m=3\Rightarrow I(3)=-\frac{1}{2}\int_0^1 \frac{\log^2(x)\log\left(\frac{1+x^2}{2}\right)}{1-x}dx$

$$=3\zeta(4)-\frac{3}{32}\log(2)\zeta(3)-\beta(1)\beta(3)-\beta(2)\beta(2)-\beta(3)\beta(1)$$
$$-\frac{1}{64}\zeta(2)\zeta(2)$$

$$=3\zeta(4)-\frac{3}{32}\log(2)\zeta(3)-\frac{\pi}{2}\cdot\frac{\pi^3}{32}-G^2-\frac{1}{64}\zeta^2(2)\Rightarrow\int_0^1\frac{\log^2(x)\log\left(\frac{1+x^2}{2}\right)}{1-x}dx$$
$$=-6\zeta(4)+\frac{3}{16}\log(2)\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)$$

$$\Leftrightarrow\int_0^1\frac{\log^2(x)\log(x^2+1)}{1-x}dx-\int_0^1\frac{\log^2(x)\log(2)}{1-x}dx$$
$$=-6\zeta(4)+\frac{3}{16}\log(2)\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)$$

$$\Leftrightarrow\int_0^1\frac{\log^2(x)\log(x^2+1)}{1-x}dx-2\log(2)\zeta(3)$$
$$=-6\zeta(4)+\frac{3}{16}\log(2)\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)\Rightarrow J$$
$$=-6\zeta(4)+\frac{3}{16}\log(2)\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{1}{32}\zeta^2(2)+2\log(2)\zeta(3)$$

$$=-\frac{\pi^4}{15}+\frac{3}{16}\log(2)\zeta(3)+\frac{\pi^4}{32}+2G^2+\frac{\pi^4}{1152}+2\log(2)\zeta(3)$$
$$=-\frac{199\pi^4}{5760}+2G^2+\frac{35}{16}\log(2)\zeta(3)$$

$$*\ K=\int_0^1\frac{\log(x)\log(1-x)\log(x^2+1)}{x}dx$$
$$=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{2k-1}\log(x)\log(1-x)\,dx$$

From : $\int_0^1 x^{k-1}\log(1-x)\,dx = -\frac{H_k}{k} \xrightarrow{\text{Derivative w.r.t } k} \int_0^1 x^{k-1}\log(x)\log(1-x)\,dx$

$$= \frac{H_k}{k^2} + \frac{{H_k}^{(2)}}{k} - \frac{\zeta(2)}{k}$$

$$\Rightarrow K = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{H_{2k}}{4k^2} + \frac{{H_{2k}}^{(2)}}{2k} - \frac{\zeta(2)}{2k}\right)$$

$$= \frac{1}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}H_{2k}}{k^3} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}{H_{2k}}^{(2)}}{k^2} - \frac{\zeta(2)}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$$

Use : $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}, \sum_{k=1}^{\infty} \frac{(-1)^{k-1}H_{2k}}{k^3}$

$$= \frac{195}{32}\zeta(4) + \frac{5}{4}\log^2(2)\zeta(2) - \frac{35}{8}\log(2)\zeta(3) - \frac{5}{24}\log^4(2)$$

$$- 5Li_4\left(\frac{1}{2}\right)(*), \sum_{k=1}^{\infty} \frac{(-1)^{k-1}{H_{2k}}^{(2)}}{k^2}$$

$$= 2G^2 - \frac{353}{64}\zeta(4) - \frac{5}{4}\log^2(2)\zeta(2) + \frac{35}{8}\log(2)\zeta(3) + \frac{5}{24}\log^4(2)$$

$$+ 5Li_4\left(\frac{1}{2}\right)(**)$$

$$\Rightarrow K = \frac{1}{4}\left(\frac{195}{32}\zeta(4) + \frac{5}{4}\log^2(2)\zeta(2) - \frac{35}{8}\log(2)\zeta(3) - \frac{5}{24}\log^4(2) - 5Li_4\left(\frac{1}{2}\right)\right)$$

$$+ \frac{1}{2}\left(2G^2 - \frac{353}{64}\zeta(4) - \frac{5}{4}\log^2(2)\zeta(2) + \frac{35}{8}\log(2)\zeta(3) + \frac{5}{24}\log^4(2)\right.$$

$$\left. + 5Li_4\left(\frac{1}{2}\right)\right) - \frac{\zeta(2)}{2}\cdot\frac{\pi^2}{12}$$

$$= G^2 - \frac{5}{16}\log^2(2)\zeta(2) + \frac{35}{32}\log(2)\zeta(3) - \frac{119}{64}\zeta(4) + \frac{5}{96}\log^4(2) + \frac{5}{4}Li_4\left(\frac{1}{2}\right)$$

$$= G^2 - \frac{5\pi^2}{96}\log^2(2) + \frac{5}{96}\log^4(2) + \frac{5}{4}Li_4\left(\frac{1}{2}\right) - \frac{119\pi^4}{5760}$$

$$+ \frac{35}{32}\log(2)\zeta(3)$$

$$*\Rightarrow I=\frac{1}{2}J-K$$

$$=\frac{1}{2}\left(-\frac{199\pi^4}{5760}+2G^2+\frac{35}{16}log(2)\zeta(3)\right)$$

$$-\left(G^2-\frac{5\pi^2}{96}log^2(2)+\frac{5}{96}log^4(2)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)-\frac{119\pi^4}{5760}\right.$$

$$\left.+\frac{35}{32}log(2)\zeta(3)\right)$$

$$=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}log^4(2)+\frac{5\pi^2}{96}log^2(2)\ (Q.E.D)$$

NOTE: (∗), (∗
∗) From the book: (Almost) Impossible Integrals, Sums, and Series by Cornel Ioan Valean

PROBLEM(4)_PROVE:

$$I=\int_0^1\frac{x\,log(x)\,log^2(1-x)}{1+x^2}dx=-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}log^4(2)+\frac{\pi^2}{32}log^2(2)$$

⟨Proposed by Narendra Bhandari⟩

∗ Power series of: $\frac{x}{x^2+1}=\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-1}$,and: $\int_0^1 x^{n-1}\,log^2(1-x)\,dx=\frac{H_n^2+H_n^{(2)}}{n}$

⇒ Devirative with respect to n

$$\Rightarrow\int_0^1 x^{n-1}\,log(x)\,log^2(1-x)\,dx=\frac{2\zeta(3)}{n}+\frac{2\zeta(2)H_n}{n}-\frac{H_n^{(2)}}{n^2}-\frac{H_n^2}{n^2}$$

$$-\frac{2H_nH_n^{(2)}}{n}-\frac{2H_n^{(3)}}{n}$$

$$\Rightarrow I=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-1}\,log(x)\,log^2(1-x)\,dx$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2\zeta(3)}{2n}+\frac{2\zeta(2)H_{2n}}{2n}-\frac{H_{2n}^{(2)}}{4n^2}-\frac{H_{2n}^2}{4n^2}-\frac{2H_{2n}H_{2n}^{(2)}}{2n}-\frac{2H_{2n}^{(3)}}{2n}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\zeta(3)}{n}+\frac{\zeta(2)H_{2n}}{n}-\frac{H_{2n}^{(2)}}{4n^2}-\frac{H_{2n}^2}{4n^2}-\frac{H_{2n}H_{2n}^{(2)}}{n}-\frac{H_{2n}^{(3)}}{n}\right)$$

$$*\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(3)}{n}=\zeta(3)\,log(2)$$

$$*\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(2)H_{2n}}{n}=\zeta(2)\left(\frac{5}{48}\pi^2-\frac{1}{4}\log^2(2)\right)$$

$*\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(3)}}{n}$,use: Cornel Ioan Valean's generalized alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(m)}}{n}$$

$$=\frac{(-1)^m}{(m-1)!}\int_0^1\frac{\log^{m-1}(x)\log\left(\frac{x^2+1}{2}\right)}{1-x}dx$$

$$=m\zeta(m+1)-2^{-m}(1-2^{-m+1})\log(2)\zeta(m)$$

$$-\sum_{k=0}^{m-1}\beta(k+1)\beta(m-k)$$

$$-\sum_{k=1}^{m-1}2^{-m-1}(1-2^{-k})(1-2^{-m+k+1})\zeta(k+1)\zeta(m-k)$$

Let: $m=3\Rightarrow\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(3)}}{n}=\frac{199\pi^4}{11520}-\frac{3}{32}\log(2)\zeta(3)-G^2$

$*$ From : generating function of: $\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}x^n$

$$=-\frac{1}{3}\log^3(1-x)\log(x)-\log^2(1-x)Li_2(1-x)+\frac{1}{2}Li_2^2(x)$$

$+2\log(1-x)Li_3(1-x)+Li_4(x)-2Li_4(1-x)+2\zeta(4)$, then:

$$\sum_{n=1}^{\infty}\frac{H_{2n}^2}{n^2}(-1)^{n-1}=2G^2-\frac{5}{48}\log^4(2)+\log^2(2)\zeta(2)-\frac{35}{16}\log(2)\zeta(3)+\frac{231}{32}\zeta(4)$$

$$-\pi G\log(2)-2\pi\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)-\frac{5}{2}Li_4\left(\frac{1}{2}\right)$$

$*$ From : generating function of: $\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n}x^n=\frac{1}{6}\log^3(1-x)\log(x)-\frac{1}{24}\log^4(1-x)$

$$+\frac{1}{2}\log^2(1-x)Li_2(1-x)-\log(1-x)Li_3(1-x)+Li_4(1-x)$$

$-Li_4\left(\frac{x}{x-1}\right)-\zeta(4)$, then:

$$\sum_{n=1}^{\infty}\frac{H_{2n}H_{2n}^{(2)}}{n}(-1)^{n-1}=\frac{5}{96}log^4(2)-\frac{3}{8}log^2(2)\,\zeta(2)+\frac{35}{64}log(2)\,\zeta(3)-\frac{137}{128}\zeta(4)+\frac{1}{4}\pi G\,log(2)+\frac{\pi}{2}\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)$$

$$*\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{n^2}(-1)^{n-1}=2G^2-\frac{353}{64}\zeta(4)-\frac{5}{4}log^2(2)\,\zeta(2)+\frac{35}{8}\log(2)\zeta(3)+\frac{5}{24}log^4(2)+5Li_4\left(\frac{1}{2}\right)\,(*)$$

$*$ Combine these results:

$$I=\zeta(3)\,log(2)+\frac{5}{48}\zeta(2)\pi^2-\frac{1}{4}\zeta(2)log^2(2)-\frac{1}{4}\left(2G^2-\frac{353}{64}\zeta(4)-\frac{5}{4}log^2(2)\,\zeta(2)+\frac{35}{8}\log(2)\,\zeta(3)+\frac{5}{24}log^4(2)+5Li_4\left(\frac{1}{2}\right)\right)$$
$$-\frac{1}{4}\left(2G^2-\frac{5}{48}log^4(2)+log^2(2)\,\zeta(2)-\frac{35}{16}log(2)\,\zeta(3)+\frac{231}{32}\zeta(4)-\pi G\,log(2)-2\pi\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)-\frac{5}{2}Li_4\left(\frac{1}{2}\right)\right)$$
$$-\left(\frac{5}{96}log^4(2)-\frac{3}{8}log^2(2)\,\zeta(2)+\frac{35}{64}log(2)\,\zeta(3)-\frac{137}{128}\zeta(4)+\frac{1}{4}\pi G\,log(2)+\frac{\pi}{2}\Im\left(Li_3\left(\frac{1+i}{2}\right)\right)+\frac{5}{4}Li_4\left(\frac{1}{2}\right)\right)$$
$$-\left(\frac{199\pi^4}{11520}-\frac{3}{32}log(2)\,\zeta(3)-G^2\right)$$
$$=-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}log^4(2)+\frac{\pi^2}{32}log^2(2)\,(Q.E.D)$$

$(*)$: $https://math.stackexchange.com/questions/3803424/how-to-find-sum-n-1-infty-frac-1nh-2nn3-and-sum-n-1-infty-f/3803762\#3803762$

PROBLEM(2)_PROVE:

$$I=\int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\log\left(\frac{x}{1-x}\right)dx$$
$$=\frac{5}{8}Li_4\left(\frac{1}{2}\right)-\frac{89\pi^4}{23040}+\frac{5}{192}\log^4(2)+\frac{\pi^2}{48}\log^2(2)$$

⟨Proposed by Narendra Bhandari⟩

From : problem(3), problem(4) :

$$P_3=\int_0^1 \frac{x\log^2(x)\log(1-x)}{1+x^2}dx=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}\log^4(2)+\frac{5\pi^2}{96}\log^2(2)$$

$$P_4=\int_0^1 \frac{x\log(x)\log^2(1-x)}{1+x^2}dx=-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}\log^4(2)+\frac{\pi^2}{32}\log^2(2)$$

$$I=P_3-P_4=-\frac{5}{4}Li_4\left(\frac{1}{2}\right)+\frac{13\pi^4}{3840}-\frac{5}{96}\log^4(2)+\frac{5\pi^2}{96}\log^2(2)$$
$$-\left(-\frac{15}{8}Li_4\left(\frac{1}{2}\right)+\frac{167\pi^4}{23040}-\frac{5}{64}\log^4(2)+\frac{\pi^2}{32}\log^2(2)\right)$$

$$=\frac{5}{8}Li_4\left(\frac{1}{2}\right)-\frac{89\pi^4}{23040}+\frac{5}{192}\log^4(2)+\frac{\pi^2}{48}\log^2(2)\,(Q.E.D)$$