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JENSEN'S INEQUALITY IN A ROMANIAN APPROACH

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Definition. Consider a function $f: I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$. We say that f is a convex function if, for any two points x and y in I and any $t \in [0,1]$, we have:

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Definition. Consider a function $f: I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$. We say that f is a concave function if, for any two points x and y in I and any $t \in [0,1]$, we have:

$$f(tx + (1-t)y) \ge tf(x) + (1-t)f(y).$$

Theorem 1.

(a) A twice-differentiable function $f: I \to \mathbb{R}$ is convex if and only if $f''(x) \ge 0$, for all $x \in I$.
(b) A twice-differentiable function $f: I \to \mathbb{R}$ is concave if and only if $f''(x) \le 0$, for all $x \in I$.

Example.

(a) Let $f: (0,\infty) \to \mathbb{R}$, $f(x) = x^7$. Then, it's obvious that f is a twice-differentiable function and $f''(x) = 42x^5 \ge 0$, for all $x \in (0,\infty)$. Therefore, f is a convex function.
(b) Let $f: (0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Then, it's obvious that f is a twice-differentiable function and $f''(x) = -\frac{1}{4x^{\frac{3}{2}}} \le 0$, for all $x \in (0,\infty)$. Therefore, f is a concave function.

Jensen's inequality.

(a) If $p_1, p_2, \dots, p_n$ are positive reals such that $p_1 + p_2 + \dots + p_n = 1$ and f is a real continuous function that is convex on $I \subset \mathbb{R}$, then: $f(p_1x_1 + p_2x_2 + \dots + p_nx_n) \le p_1f(x_1) + p_2f(x_2) + \dots + + p_nf(x_n)$, for every $x_1, x_2, \dots, x_n \in I$.

(b) If $p_1, p_2, \dots, p_n$ are positive reals such that $p_1 + p_2 + \cdots + p_n = 1$ and f is a real continuous function that is concave on $I \subset \mathbb{R}$, then:

(c) $f(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \geq p_1f(x_1) + p_2f(x_2) + + \cdots + p_nf(x_n)$, for every $x_1, x_2, \dots, x_n \in I$.

Jensen's inequality $\left(for\ p_1 = \cdots = p_n = \frac{1}{n}\right)$.

(a) If f is a real continuous function that is convex on $I \subset \mathbb{R}$, then

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+\cdots f(x_n)}{n}, \text{ for every } x_1, x_2, \dots, x_n \in I.$$

(b) If f is a real continuous function that is concave on $I \subset \mathbb{R}$, then

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \geq \frac{f(x_1)+\cdots f(x_n)}{n}, \text{ for every } x_1, x_2, \dots, x_n \in I.$$

Observation. If f is strictly convex or strictly concave, the equality in Jensen's Inequality holds $\Leftrightarrow x_1 = \cdots = x_n$.

Problem 1. If A, B and C are 3 angles of a triangle such that $A, B, C \in \left(0, \frac{\pi}{2}\right)$, prove that:

(a) $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$;

(b) $\cos A + \cos B + \cos C \leq \frac{1}{2}$;

Solution. Let $f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, f(x) = \sin x$ and $g: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, g(x) = \cos x$. It's obvious that f and g are 2 continuous functions and f and g are strictly concave functions on $\left(0, \frac{\pi}{2}\right)$. From Jensen's Inequality for $p_1 = p_2 == p_3 = \frac{1}{3} \Longrightarrow \begin{cases} \frac{\sin A+\sin B+\sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}; \\ \frac{\cos A+\cos B+\cos C}{3} \leq \cos\left(\frac{A+B+C}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}; \end{cases}$

In both cases, the equality holds $\Leftrightarrow A = B = C = \frac{\pi}{3}$.

Problem 2. If $x_1, x_2, \dots, x_n \in (0, \infty)$, prove that:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1x_2 \dots x_n}, \text{for every } n \geq 2.$$

Solution. Let $f:(0,\infty)\to\mathbb{R}, f(x)=\ln x$. Then, we know that f is a continuous function and f is strictly concave on $(0,\infty)$. From Jensen's Inequality for $p_1=p_2=\dots=p_n=\frac{1}{n}\Rightarrow \frac{\ln x_1+\dots+\ln x_n}{n}\le \ln\left(\frac{x_1+\dots+x_n}{n}\right)\Leftrightarrow \ln(\sqrt[n]{x_1\dots x_n})\le \ln\left(\frac{x_1+\dots+x_n}{n}\right)\Leftrightarrow \sqrt[n]{x_1\dots x_n}\le\frac{x_1+\dots+x_n}{n}$.

The equality holds $\Leftrightarrow x_1=x_2=\dots=x_n$.

Problem 3. Let x and y be 2 positive reals. Prove that:

$$\frac{1}{(1+\sqrt{x})^2}+\frac{1}{(1+\sqrt{y})^2}\ge\frac{2}{x+y+2}.$$

Solution. Let $f:(0,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{(1+\sqrt{x})^2}$. Then, $f'(x)=-\frac{1}{(\sqrt{x}+1)^3\sqrt{x}}$ and $f''(x)=\frac{4x^{\frac{3}{2}}+x}{2x^{\frac{5}{2}}(\sqrt{x}+1)^4}\ge 0$ for every x $\in \in (0,\infty)$. Therefore, f is a continuous convex function. From Jensen's Inequality for $p_1=p_2=\frac{1}{2}\Rightarrow\frac{f(x)+f(y)}{2}\ge\ge f\left(\frac{x+y}{2}\right)\Leftrightarrow\frac{1}{(1+\sqrt{x})^2}+\frac{1}{(1+\sqrt{y})^2}\ge\frac{2}{\left(1+\sqrt{\frac{x+y}{2}}\right)^2}$.

We must show that $\frac{2}{\left(1+\sqrt{\frac{x+y}{2}}\right)^2}\ge\frac{2}{x+y+2}\Leftrightarrow\frac{1}{\left(1+\sqrt{\frac{x+y}{2}}\right)^2}\ge\frac{1}{x+y+2}\Leftrightarrow\Leftrightarrow x+y+2\ge\left(1+\sqrt{\frac{x+y}{2}}\right)^2\Leftrightarrow x+y+2\ge 1+\frac{x+y}{2}+\sqrt{2}\sqrt{x+y}\Leftrightarrow x+y+2\ge 2\sqrt{2(x+y)}$, which is true from $AM-GM$ inequality for the numbers $(x+y)$ and 2.

Therefore, $\frac{1}{(1+\sqrt{x})^2}+\frac{1}{(1+\sqrt{y})^2}\ge\frac{2}{x+y+2}$ for every x and y positive reals. The equality holds if we have equality in all inequalities, $x=y=1$.

Problem 4. Let $x_1,x_2,\dots,x_n>0$. Prove that:

$$\left(\frac{x_1+\dots+x_n}{n}\right)^{x_1+\dots+x_n}\le x_1^{x_1}x_2^{x_2}\dots x_n^{x_n}, \text{ for every } n\ge 2.$$

Solution. $\left(\frac{x_1+\dots+x_n}{n}\right)^{x_1+\dots+x_n}\le x_1^{x_1}x_2^{x_2}\dots x_n^{x_n}\Leftrightarrow (x_1+\dots+x_n)\ln\left(\frac{x_1+\dots+x_n}{n}\right)\le x_1\ln x_1+\dots+x_n\ln x_n$. Now, let $f:(0,\infty)\to\mathbb{R}, f(x)=x\ln x$. It's obvious that f is a continuous function and $f'(x)=\ln x+1$ and $f''(x)=\frac{1}{x}$. Therefore, f is a twice-differentiable function on $(0,\infty)$ and because $f''(x)=\frac{1}{x}\ge 0\Rightarrow f$ is a convex function. From Jensen's Inequality $\Rightarrow$

$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \Leftrightarrow \frac{x_1+\cdots+x_n}{n}\ln\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{x_1\ln x_1+\cdots+x_n\ln x_n}{n} \Leftrightarrow$ $(x_1+\cdots+x_n)\ln\left(\frac{x_1+\cdots+x_n}{n}\right) \le x_1\ln x_1+\cdots+x_n\ln x_n \Leftrightarrow \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^{x_1+x_2+\cdots+x_n} \le$ $x_1^{x_1}x_2^{x_2}\ldots x_n^{x_n}$.

Problem 5. Let $x, y, z > 0$. Prove that:

$$\frac{x}{\sqrt{x^2+8yz}} + \frac{y}{\sqrt{y^2+8xz}} + \frac{z}{\sqrt{z^2+8xy}} \ge 1.$$

(IMO 2001)

Solution. Let $x+y+z=S$ and let $a=\frac{x}{S}, b=\frac{y}{S}, c=\frac{z}{S}$. Then, $a+b+c=1$ and the inequality is equivalent to: $\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}} \ge 1$. Let $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}$. It's obvious that f is a continuous function and $f'(x)=-\frac{1}{2x^{\frac{3}{2}}}$ and $f''(x)=\frac{3}{4x^{\frac{5}{2}}} \ge 0$. Then, f is a convex function on $(0,\infty)$. From Jensen's Inequality for $p_1=a, p_2=b$ and $p_3=c \Longrightarrow$ $f\big(a(a^2+8bc)+b(b^2+8ac)+c(c^2+8ab)\big) \le af(a^2+8bc)+bf(b^2++8ac)+$ $cf(c^2+8ab) \Leftrightarrow \frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}} \ge \frac{1}{\sqrt{a^3+b^3+c^3+24abc}}$. Now, we must show that: $\frac{1}{\sqrt{a^3+b^3+c^3+24abc}} \ge 1 \Leftrightarrow a^3+b^3+c^3+24abc \le 1 \Leftrightarrow a^3+b^3+c^3+24abc \le$ $(a+b+c)^3 \Leftrightarrow a^3+b^3++c^3+24abc \le a^3+b^3+c^3+3(a+b)(b+c)(c+a) \Leftrightarrow$ $8abc \le (a+b)(b+c)(c+a)$.

Now, from $AM-GM$ inequality $\Longrightarrow \begin{cases} a+b \ge 2\sqrt{ab} \\ b+c \ge 2\sqrt{bc} \\ c+a \ge 2\sqrt{ca} \end{cases} \Longrightarrow (a+b)(b+c)(c+a) \ge 8abc$.

Therefore, $\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}} \ge 1 \Longrightarrow \frac{x}{\sqrt{x^2+8yz}}+\frac{y}{\sqrt{y^2+8xz}}+\frac{z}{\sqrt{z^2+8xy}} \ge 1$.

Problem 6. Let a, b, c be positive real numbers such that $abc=1$. Prove that:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

(IMO 1995)

Solution. Let $\frac{1}{x}=a,\frac{1}{y}=b,\frac{1}{z}=c \Rightarrow abc=\frac{1}{xyz}=1$. The inequality is equivalent to: $\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\geq\frac{3}{2}$. Now, let's consider the function $f:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{x}$, which is obvious a continuous convex function on $(0,\infty)$. From Jensen's Inequality for $p_1=\frac{x}{x+y+z}, p_2=\frac{y}{x+y+z}, p_3=\frac{z}{x+y+z} \Rightarrow \frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}=xf\left(\frac{y+z}{x}\right)+yf\left(\frac{x+z}{y}\right)++zf\left(\frac{x+y}{z}\right)\geq(x+y+z)f\left(\frac{(x+y)+(y+z)+(z+x)}{x+y+z}\right)=\frac{x+y+z}{2}$.

Now, we must show that $\frac{x+y+z}{2}\geq\frac{3}{2}$, which is true from $AM-GM$ inequality because $\frac{x+y+z}{2}\geq\frac{3\sqrt[3]{xyz}}{2}=\frac{3}{2}$.

Therefore, $\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$.

Problem 7. Let $x,y,z>-1$ such that $x+y+z=1$. Prove that:

$$\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+y}}+\frac{1}{\sqrt{1+z}}\geq\frac{3\sqrt{3}}{2}.$$

Solution. Let $f:(-1,\infty)\to\mathbb{R}, f(x)=\frac{1}{\sqrt{1+x}}$. Then, it's obvious that f is a continuous function and $f'(x)=-\frac{1}{2(x+1)^{\frac{3}{2}}}$ and $f''(x)=\frac{3}{4(x+1)^{\frac{5}{2}}}$. Therefore, f is a convex function on$(-1,\infty)$. From Jensen's Inequality, we have: $\frac{1}{\sqrt{1+x}}++\frac{1}{\sqrt{1+y}}+\frac{1}{\sqrt{1+z}}=f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)=3f\left(\frac{1}{3}\right)=\frac{3\sqrt{3}}{2}$. In conclusion, $\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1+y}}+\frac{1}{\sqrt{1+z}}\geq\frac{3\sqrt{3}}{2}$.

Problem 8. Let $0<x_i<\frac{1}{2}, i=\overline{1,n}$. Prove that:

$$\frac{\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}}{\left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}}\leq\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}(1-x_i)}.$$

Solution. Let $f:\left(0,\frac{1}{2}\right)\to\mathbb{R}, f(x)=\ln\left(\frac{x}{x-1}\right)$. Then, f is continuous and $f'(x)=-\frac{1}{(x-1)x}$ and $f''(x)=\frac{2x-1}{(x-1)^2x^2}$. Therefore, $f''(x)\leq 0$ and so f is concave on $\left(0,\frac{1}{2}\right)$.

From Jensen's Inequality $\Rightarrow$

$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \le f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \Leftrightarrow \frac{1}{n}\ln\left(\frac{x_1x_2\ldots x_n}{(1-x_1)(1-x_2)\ldots(1-x_n)}\right) \le \ln\left(\frac{\frac{x_1+x_2+\cdots+x_n}{n}}{1-\frac{x_1+x_2+\cdots+x_n}{n}}\right) \Leftrightarrow$$

$$\frac{1}{n}\ln\left(\frac{x_1x_2\ldots x_n}{(1-x_1)(1-x_2)\ldots(1-x_n)}\right) \le \ln\left(\frac{x_1+x_2+\cdots+x_n}{n-(x_1+x_2+\cdots+x_n)}\right) \Leftrightarrow \ln\left(\frac{(x_1x_2\ldots x_n)^{\frac{1}{n}}}{((1-x_1)(1-x_2)\ldots(1-x_n))^{\frac{1}{n}}}\right) \le$$

$$\ln\left(\frac{x_1+x_2+\cdots+x_n}{n-(x_1+x_2+\cdots+x_n)}\right) \Leftrightarrow \frac{(x_1x_2\ldots x_n)^{\frac{1}{n}}}{((1-x_1)(1-x_2)\ldots(1-x_n))^{\frac{1}{n}}} \le \frac{x_1+x_2+\cdots+x_n}{n-(x_1+x_2+\cdots+x_n)} \Leftrightarrow \frac{(\prod_{i=1}^n x_i)^{\frac{1}{n}}}{(\prod_{i=1}^n (1-x_i))^{\frac{1}{n}}} \le \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)}.$$

Therefore, $\frac{(\prod_{i=1}^n x_i)^{\frac{1}{n}}}{(\prod_{i=1}^n (1-x_i))^{\frac{1}{n}}} \le \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)}$.

References:

(1) https://www.probabilitycourse.com/chapter6/6_2_5_jensen's_inequality.php
(2) https://mathworld.wolfram.com/JensensInequality.html
(3) https://artofproblemsolving.com/community/c6_high_school_olympiads
(4) https://artofproblemsolving.com/community/c4_high_school_math
(5) https://artofproblemsolving.com/community/c7_college_math
(6) http://www.ssmrmh.ro (ROMANIAN MATHEMATICAL MAGAZINE)