A Complex Result

Introduction

It is known from complex analysis that many curious and non-trivial identities could be derived from the method of separating a function into real and imaginary parts. In this article, we will derive a simple integral technique using the same method.

Prerequisites

Definition 1

A function f of the complex variable z is analytic in an open set if it has a derivative at each point in that set [1].

Cauchy's Integral Theorem

Let C denote a simple closed contour and f is analytic at each point interior to and on C, then [1],

$$\oint_C f(z)\, dz = 0$$

The Integral Technique

Theorem: Let Γ be a closed rectangular contour in the complex plane having vertices $(0, 0)$, $(R, 0)$, (R, i) and $(0, i)$ traversed counterclockwise, where R tends to infinity. Let f be a function of the complex variable z satisfying the below conditions,

1. f is analytic inside and on Γ
2. $\exists M \in \mathbb{R}^+ - \{0\} \mid \left|f(x+iy)\right| \leq M \;\; \forall x \in \mathbb{R}^+, 0 < y < 1$
3. $f(ix) = f(x)$ for all $0 < x < 1$

then,

$$\int_0^\infty e^{-ax^2} f(x)dx = \Re\left\{\int_0^\infty e^{-a(x+i)^2} f(x+i)dx\right\}$$

and

$$\int_0^1 e^{-ax^2} f(x)dx = -\Im\left\{\int_0^\infty e^{-a(x+i)^2} f(x+i)dx\right\}$$

where, $a \in \mathbb{R}^+ - \{0\}$.

Proof: Let

$$J = \oint_\Gamma e^{-az^2} f(z)\, dz$$

using methods from elementary contour integration, we can divide the above integral (J) as,

$$J = \lim_{R\to\infty}\left(\int_0^R e^{-az^2} f(z)dz + \int_0^i e^{-az^2} f(z)dz + \int_{R+i}^i e^{-az^2} f(z)dz + \int_i^0 e^{-az^2} f(z)dz\right)$$

using definition 1 and Cauchy's integral theorem along with condition (1), we have the following result,

$$J = 0$$

which can be written as,

$$\lim_{R\to\infty}(I_1 + I_2 + I_3 + I_4) = 0$$

where,

$$I_1 = \int_0^R e^{-ax^2} f(x)dx$$

$$I_2 = i\int_0^1 e^{-a(R+iy)^2} f(R+iy)dy$$

$$I_3 = -\int_0^R e^{-a(x+i)^2} f(x+i)dx$$

$$I_4 = -i\int_0^1 e^{-a(iy)^2} f(iy)dy$$

Using condition (2), we have,

$$|I_2| \leq \int_0^1 \left| e^{-a(R+iy)^2} f(R+iy) \right| dy \leq e^{-a(R^2-1)} \int_0^1 \left| f(R+iy) \right| dy \leq M e^{-a(R^2-1)}$$

which can be written as,

$$\lim_{R\to\infty} |I_2| \leq \lim_{R\to\infty} M e^{-a(R^2-1)} = 0$$

Simplifying I_4 and by using condition (3), we obtain,

$$I_4 = -i \int_0^1 e^{-a(iy)^2} f(iy) dy = -i \int_0^1 e^{ax^2} f(x) dx$$

since $\Im\{I_1\} = \Re\{I_4\} = 0$, the final step is to equate the real and imaginary parts of I_2 with that of I_1 and I_4, which completes the proof.

Conclusion

The clever selection of f (f satisfying all the three conditions) can be used to evaluate various new as well as already known integrals related to Weierstrass transform, Gaussian-type integrals, error function and other integral transforms.

References

[1] Brown JW Churchill RV. Complex Variables and Applications. 6th ed. New York: McGraw-Hill; 1996, 70-71, 142-144.

Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
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