Generalizations for Köber's inequality and Stečkin's inequality

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In this work some generalizations are presented, demonstrated and studied for two trigonometric inequalities : Köber's inequality and Stečkin's inequality. Applications and particular cases are presented

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It is already classic and intensively used , the double inequality ,

$$\frac{2}{\pi}\cdot x \le \sin x \le x \,, \qquad , \tag{1}$$

for any $x \in [0\,,\pi/2]$ ·

The inequality on the left is also called *Jordan's inequality* . (see for example [2], [5]) **.**

Inequality (1) provide a framing with liniar functions of the trigonometric function $\sin x$ in the interval $[0\,,\pi/2]$.

With the substitution $x \to \frac{\pi}{2} - x$, we also obtain a framing with with functions of the first degree for function $\cos x$ in the interval $[0\,,\pi/2]$:

$$1 - \frac{2}{\pi}\cdot x \le \cos x \le \frac{\pi}{2} - x \,, \qquad , \tag{2}$$

In specialized literature - the inequality on the left is also called *Köber's inequality*.

1. ***Remark***

Inequalities (1) and (2) (implicitly *Jordan's inequality* and *Köber's inequality*) are equivalent - in the sense that each is deduced from the other - thanks to substitution $x \to \frac{\pi}{2} - x$ **.**

Jordan's and *Köber's inequalities* can be called *complementary inequalities* (like the functions ***sin*** and ***cos*** and the arguments x and $\frac{\pi}{2} - x$ that participate in these formulas).

In what follows, the following elementary result will be very useful

2. ***Lemma***

If $a\,,b\,,c\,,d > 0$ and $a \le x \le b$, $c \le y \le d$, then $\dfrac{a}{d} \le \dfrac{x}{y} \le \dfrac{b}{c}$ · **(3)**

Proof

Indeed, everything results from the series of inequalities :

$$\frac{a}{d} \overset{(y \le d)}{\le} \frac{a}{y} \overset{(a \le x)}{\le} \frac{x}{y} \overset{(c \le y)}{\le} \frac{x}{c} \overset{(x \le b)}{\le} \frac{b}{c} \,.$$

3. *Proposition*

For any $x \in (0, \pi/2)$, we have : $$\frac{4x}{\pi(\pi-2x)} < \tan x < \frac{\pi x}{\pi - 2x} \quad (4)$$

Proof

Using the framings (1) , (2) and *Lemma* 2 , with the choices : $a = \frac{2}{\pi}\cdot x$, $b = x$, $c = 1 - \frac{2}{\pi}\cdot x$, $d = \frac{\pi}{2} - x$, $x \to \sin x$, $y \to \cos x$, we get ,

$$\frac{\frac{2}{\pi}\cdot x}{\frac{\pi}{2} - x} \le \frac{\sin x}{\cos x} \le \frac{x}{1 - \frac{2}{\pi}\cdot x} \quad \Leftrightarrow \quad \frac{4x}{\pi(\pi-2x)} \le \tan x \le \frac{\pi x}{\pi - 2x} .$$

4. *Remark*

The left inequality in *Proposition* 3 is known as ***Stečkin's inequality*** , mentioned by *Mitrinoviċ* [3] , p.246 – without proof . In addition , was also obtained the inequality on the right of (3) - inequality that can be considered an inequality *converse to Stečkin's inequality* .

Since we still want to use the *convexity / concavity* of some functions, we remind you here the usual definition of *convexity* , as well as two other equivalent forms :

5. *Definition* The function $f : \mathbf{I} \subset (0, \infty) \longrightarrow \mathbb{R}$ is called a *convex function* on the interval $\mathbf{I}$, if

$$f[(1-\lambda)x_1 + \lambda x_2] \le (1-\lambda) f(x_1) + \lambda f(x_2) , \quad (5)$$

for any $x_1 , x_2 \in \mathbf{I}$ and any $\lambda \in [0, 1]$.

6. *Remark*

Taking , $x = (1-\lambda)x_1 + \lambda x_2$ (that is , x is between x_1 and x_2) , relation (4) is rewritten :

$$f(x) \le \frac{x_2 - x}{x_2 - x_1}\cdot f(x_1) + \frac{x - x_1}{x_2 - x_1}\cdot f(x_2) , \quad (6)$$

or otherwise arranged ,

$$f(x) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1}\cdot x + \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1} , \quad (7)$$

For a *concave function* , there are relations similar to those in (5), (6), (7), but with the inequality sign " $\le$ " replaced by the sign " $\ge$ " .

More about *convexity / concavity* , in [1] , [3] – [6] .

We will have the opportunity to use such inequalities several times to prove *Köber's inequality*,

$$\cos x \ge 1 - \frac{2}{\pi}\cdot x \quad (8)$$

The constant $\frac{2}{\pi}$ from *Köber 's inequality* is dependent on the interval $\mathbf{I} = [0, \pi/2]$.

If we change the interval $\mathbf{I}$ - domain of definition of the ***cosine*** function , then this constant changes as in the following ,

7. Proposition (*generalization of Köber's inequality for intervals of the form* $[0,\alpha]$)

For the angles x, α, such that $0 \le x \le \alpha \le \pi/2$, we have the inequality :

$$\cos x \ge 1 - \frac{1-\cos\alpha}{\alpha}\cdot x \;, \quad (\forall)\, x \in [0,\alpha] \;. \tag{9}$$

with equality if $x = 0$ or $x = \alpha$.

Proof

Function $\cos : [0,\pi/2] \longrightarrow \mathbb{R}$ is concave , so using for example relation (6) (- but with the inverted inequality sign), with the choices : $f(x) = \cos x$, $x_1 = 0$, $x_2 = \alpha$ the inequality **(8)** is obtained.

For $\alpha = \pi/2$, *Köber's inequality* is obtained .

For other values of α , interesting inequalities are also obtained , as in the following ,

8. Corollary

The following inequalities occur :

(*a*) $$\cos x \ge 1 - \frac{3(2-\sqrt{3})}{\pi}\cdot x \;, \quad (\forall)\, x \in [0, \pi/6] \;; \tag{10}$$

with equality if $x = 0$ or $x = \pi/6$.

(*b*) $$\cos x \ge 1 - \frac{2(2-\sqrt{2})}{\pi}\cdot x \;, \quad (\forall)\, x \in [0, \pi/4] \;; \tag{11}$$

with equality if $x = 0$ or $x = \pi/4$.

(*c*) $$\cos x \ge 1 - \frac{3}{2\pi}\cdot x \;, \quad (\forall)\, x \in [0, \pi/3] \;; \tag{12}$$

with equality if $x = 0$ or $x = \pi/3$.

Proof

In inequality (9) the angle α is replaced , in turn with : (*a*) $\alpha = \pi/6$, (*b*) $\alpha = \pi/4$, (*c*) $\alpha = \pi/3$, si se efectueaza calcule elementare .

9. Remark

The inequalities in *Proposition* 7 and *Corollary* 8 have a simple geometric interpretation : the graph of the ***cosine*** function is above the graph of the chord of extremities A(0,1) and B(α , $\cos\alpha$) - on the considered interval .

10. Remark

Since , for example in inequality (12) , for the interval $[0,\pi/3] \subset [0,\pi/2]$, we have

$1 - \frac{3}{2\pi}\cdot x > 1 - \frac{2}{\pi}\cdot x \iff 4 > 3$, it turns out that the inequality $\cos x \ge 1 - \frac{3}{2\pi}\cdot x$

is stronger than *Köber's inequality* . so that inequality (12) refines *Köber's inequality* for the (sub)interval $x \in [0,\pi/3]$.

All inequalities (10)-(12) refine *Köber's inequality* - on the respective subintervals .

At the same time, *Proposition* 7 also provides the following monotonicity result .

11. *Corollary*

The function $\varphi : (0\,,\pi/2\,] \longrightarrow \mathbb{R}$, $\varphi(t) = \dfrac{1-\cos t}{t}$ is monotonically increasing on $(0\,,\pi/2\,]$.

Proof

Indeed, for $0 \le x \le \alpha \le \pi/2$, from Prroposition 7 we have ,

$$\cos x > 1 - \frac{1-\cos\alpha}{\alpha}\cdot x \;,\; (\forall)\, x \in (\,0\,,\alpha\,] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{1-\cos x}{x} < \frac{1-\cos\alpha}{\alpha} \;,\; (\forall)\, x \in (\,0\,,\alpha\,] \quad \Leftrightarrow \quad \varphi(x) < \varphi(\alpha) \;,\; (\forall)\, x \in (\,0\,,\alpha\,] \;.$$

12. *Corollary* (***generalization of Stečkin's inequality for intervals of the form*** $(0\,,\alpha)$)

For the angles $x\,,\alpha$, such that $0 < x \le \alpha < \pi/2$, we have the inequality :

$$\frac{2\sin\alpha}{\alpha(\pi-2x)} < \tan x < \frac{\alpha x}{\alpha-(1-\cos\alpha)x} \;. \qquad (13)$$

Proof

Completing the *inequalities* of *Jordan* (*see* [2]) and *Köber* - generalized to intervals of the form $(0\,,\alpha)$, up to a double inequality (a framing) , in the following way :

$$\frac{\sin\alpha}{\alpha}\cdot x < \sin x < x \;,\; 1-\frac{1-\cos\alpha}{\alpha}\cdot x < \cos x < \frac{\pi}{2}-x \;,\; (\forall)\, x \in (\,0\,,\alpha)$$

and using Lemma 2 , we get

$$\frac{\dfrac{\sin\alpha}{\alpha}\cdot x}{\dfrac{\pi}{2}-x} < \frac{\sin x}{\cos x} < \frac{x}{1-\dfrac{1-\cos\alpha}{\alpha}\cdot x} \quad \Leftrightarrow \quad \frac{2\sin\alpha}{\alpha(\pi-2x)}\cdot x < \tan x < \frac{\alpha x}{\alpha-(1-\cos\alpha)x} \;.$$

For $\alpha = \pi/2$, the inequality from Proposition 2 is obtained .

In the double inequality (13), the inequality on the left is the *generalization of Stečkin's inequality*, and the one on right is the *generalization of the converse of Stečkin's inequality*.

13. *Proposition* (***generalization of Köber's inequality for intervals of the form*** $[\alpha\,,\pi/2\,]$)

For the angles $x\,,\alpha$, such that $0 \le x \le \alpha \le \pi/2$, we have the inequality :

$$\cos x \ge \frac{\cos\alpha}{\pi-2\alpha}\cdot(\pi-2x) \quad , \quad (\forall)\, x \in [\,\alpha\,,\,\pi/2\,] \qquad (14)$$

with equality if $x = \alpha$ or $x = \pi/2$.

Proof

Again we use the fact that function $\cos : [\,0\,,\pi/2\,] \longrightarrow \mathbb{R}$ is concave , so using (6) (- but with the inverted inequality sign) , with the choices : $f(x) = \cos x\,,\, x_1 = \alpha\,,\, x_2 = \pi/2$, the inequality **(14)** is obtained**.**

For $\alpha = 0$, *Köber's inequality* is obtained .

Here are some specifications for the important angles in the range $[\,0\,,\pi/2\,]$ **:**

14. *Corollary*

The following inequalities occur :

(*a*) $$\cos x \geq \frac{3\sqrt{3}}{4\pi}\cdot(\pi - 2x) \ , \quad (\forall)\, x \in [\pi/6,\ \pi/2] \ ; \qquad (15)$$

with equality if $x = \pi/6$ or $x = \pi/2$.

(*b*) $$\cos x \geq \frac{\sqrt{2}}{\pi}\cdot(\pi - 2x) \ , \quad (\forall)\, x \in [\pi/4,\ \pi/2] \ ; \qquad (16)$$

with equality if $x = \pi/4$ or $x = \pi/2$.

(*c*) $$\cos x \geq \frac{3}{2\pi}\cdot(\pi - 2x) \ , \quad (\forall)\, x \in [\pi/3,\ \pi/2] \ ; \qquad (17)$$

with equality if $x = \pi/3$ or $x = \pi/2$.

Proof

In inequality (13) the angle α is replaced , in turn with : (*a*) $\alpha = \pi/6$, (*b*) $\alpha = \pi/4$, (*c*) $\alpha = \pi/3$, and elementary calculations are then performed.

15. *Proposition* ***(generalization of Köber's inequality for intervals of the form*** $[\alpha, \beta]$ ***)***

For the angles x , α , β , such that $0 \leq \alpha \leq x \leq \beta \leq \pi$, we have the inequality :

$$\cos x \geq \frac{\cos\beta - \cos\alpha}{\beta - \alpha}\cdot x + \frac{\beta\cos\alpha - \alpha\cos\beta}{\beta - \alpha} \ , \quad (\forall)\, x \in [\alpha, \beta] \ , \qquad (18)$$

having equality if $x = \alpha$ or $x = \beta$.

Proof 1

Again we use the concavity of the function $\cos : [0, \pi/2] \longrightarrow \mathbb{R}$, for which using relation (6) (- but with the inequality sign reversed), with the choices: $f(x) = \cos x$, $x_1 = \alpha$, $x_2 = \beta$ is obtained ,

$$f(x) \geq \frac{f(\beta) - f(\alpha)}{\beta - \alpha}\cdot x + \frac{\beta f(\alpha) - \alpha f(\beta)}{\beta - \alpha} \Leftrightarrow \cos x \geq \frac{\cos\beta - \cos\alpha}{\beta - \alpha}\cdot x + \frac{\beta\cos\alpha - \alpha\cos\beta}{\beta - \alpha} \ ,$$

hence the inequality **(18)** .

Proof 2

For any $x \in [\alpha, \beta]$, there is $t \in [0, 1]$, such that $x = t\alpha + (1-t)\beta$ $\left(\Leftrightarrow t = \frac{\beta - x}{\beta - \alpha} \right)$

With $f(x) = \cos x$, *concave* on $[\alpha, \beta] \subset [0, \pi]$, we get :

$$\cos x = \cos\left[t\alpha + (1-t)\beta\right] \geq t\cos\alpha + (1-t)\cos\beta = \frac{\beta - x}{\beta - \alpha}\cdot\cos\alpha + \left(1 - \frac{\beta - x}{\beta - \alpha}\right)\cdot\cos\beta =$$

$$= \frac{(\beta - x)\cdot\cos\alpha + (x - \alpha)\cdot\cos\beta}{\beta - \alpha} = \frac{\cos\beta - \cos\alpha}{\beta - \alpha}\cdot x + \frac{\beta\cos\alpha - \alpha\cos\beta}{\beta - \alpha} \ .$$

For $\alpha = 0$ and $\beta = \pi/2$, ***Köber's*** *inequality* is obtained .

For $\alpha = 0$ and $\beta = \alpha$, the *generalization of* ***Köber's*** *'s inequality* from *Proposition* 7 is obtained.

For $\alpha = \alpha$ and $\beta = \pi/2$, the *generalization of* ***Köber's*** *'s inequality* from *Proposition* 13 is obtained.

16. *Remark*

The inequality in *Proposition* 15 has a simple geometric interpretation : the graph of the ***sine*** function on the interval $[\alpha , \beta]$ is above the graph of the chord of extremities $A(\alpha , \cos\alpha)$, $B(\beta , \cos\beta)$, which has the equation : $(AB):\ y = \dfrac{\cos\beta - \cos\alpha}{\beta-\alpha}\cdot x + \dfrac{\beta\cos\alpha - \alpha\cos\beta}{\beta-\alpha}$.

17. *Corollary*

The following inequalities occur :

$$(a)\quad \cos x \geq -\frac{6(\sqrt{3}-\sqrt{2})}{\pi}\cdot x + \frac{3\sqrt{3}-2\sqrt{2}}{2}\ ,\quad (\forall)\, x\in[\pi/6,\ \pi/4]\ , \tag{19}$$

with equality if $x = \pi/6$ or $x = \pi/4$;

$$(b)\quad \cos x \geq -\frac{3(\sqrt{3}-1)}{\pi}\cdot x + \frac{2\sqrt{3}-1}{2}\ ,\quad (\forall)\, x\in[\pi/6,\ \pi/3]\ , \tag{20}$$

with equality if $x = \pi/6$ or $x = \pi/3$;

$$(c)\quad \cos x \geq \frac{6(\sqrt{2}-1)}{\pi}\cdot x + \frac{4\sqrt{2}-3}{2}\ ,\quad (\forall)\, x\in[\pi/4,\ \pi/3]\ , \tag{21}$$

with equality if $x = \pi/4$ or $x = \pi/3$;

Proof

In inequality (18) the angles α and β are replaced in turn by : **(*a*)** $\alpha = \pi/6$, $\beta = \pi/4$; **(*b*)** $\alpha = \pi/6$, $\beta = \pi/3$; **(*c*)** $\alpha = \pi/4$, $\beta = \pi/3$ - and routine calculations are made .

18. *Proposition* **(*generalization of Stečkin's inequality for intervals of the form* (α,β))**

For the angles x , α , β , such that $0 \leq \alpha \leq x \leq \beta \leq \pi$, we have the inequality :

$$\frac{(\sin\beta - \sin\alpha)\cdot x + \beta\sin\alpha - \alpha\sin\beta}{\beta-\alpha}\cdot\frac{2}{\pi - 2x} < \tan x < \frac{(\beta-\alpha)\,x}{(\cos\beta-\cos\alpha)x + \beta\cos\alpha - \alpha\cos\beta}\ . \tag{22}$$

Proof

Completing *Jordan* and *Köber*'s inequalities – generalized to intervals of the form (α,β) up to a double inequality (a framing) , as follows:

$$\frac{\sin\beta - \sin\alpha}{\beta-\alpha}\cdot x + \frac{\beta\sin\alpha - \alpha\sin\beta}{\beta-\alpha} < \sin x < x\ ,$$

$$\frac{\cos\beta - \cos\alpha}{\beta-\alpha}\cdot x + \frac{\beta\cos\alpha - \alpha\cos\beta}{\beta-\alpha} < \cos x < \frac{\pi}{2} - x\ ,\ (\forall)\, x\in(0,\alpha)$$

and using *Lemma* 2, we get ,

$$\frac{\dfrac{\sin\beta-\sin\alpha}{\beta-\alpha}\cdot x+\dfrac{\beta\sin\alpha-\alpha\sin\beta}{\beta-\alpha}}{\dfrac{\pi}{2}-x}<\frac{\sin x}{\cos x}<\frac{x}{\dfrac{\cos\beta-\cos\alpha}{\beta-\alpha}\cdot x+\dfrac{\beta\cos\alpha-\alpha\cos\beta}{\beta-\alpha}}\Leftrightarrow$$

$$\Leftrightarrow\quad \frac{(\sin\beta-\sin\alpha)\cdot x+\beta\sin\alpha-\alpha\sin\beta}{\beta-\alpha}\cdot\frac{2}{\pi-2x}<\tan x<\frac{(\beta-\alpha)\,x}{(\cos\beta-\cos\alpha)\,x+\beta\cos\alpha-\alpha\cos\beta}\,.$$

For $\alpha=0$, $\beta=\pi/2$ one obtains *Stečkin's inequality* and its converse from the *Proposition* 2

For $\alpha=0$, $\beta=\alpha$, one obtains *Stečkin* 's inequality and its converse (generalized over intervals of form $(0,\alpha)$) from Corollary 12 .

In the double inequality (22) , the inequality on the left is the *generalization of Stečkin's inequality* , and the one on right is the *generalization of the converse of Stečkin's inequality* .

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