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METHODS FOR COMPUTE THE DISTANCE BETWEEN TWO NON-COPLANAR LINES

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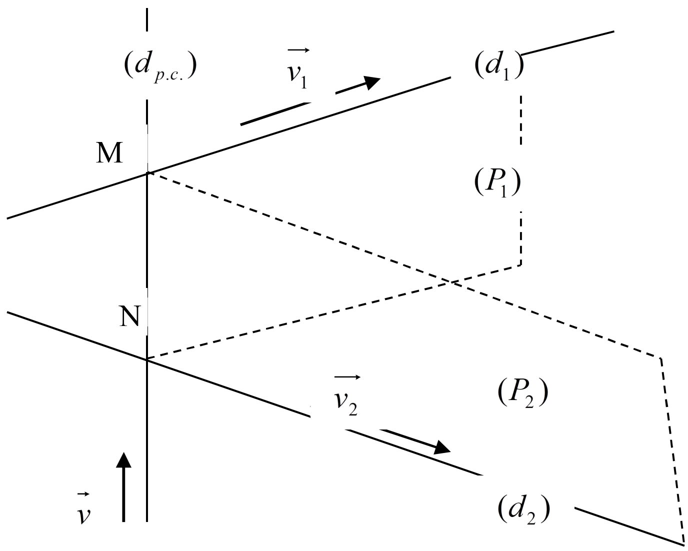
Method 1. The distance between two non-coplanar lines can be calculated without determining the position of the segment that defines the distance, using a calculation formula.
Thus, to calculate the distance between the non-coplanar lines AB and CD d(AB,CD), we can use the formula

(1) **d(AB,CD)=** $\dfrac{6\cdot \mathrm{V(ABCD)}}{\mathrm{AB}\cdot\mathrm{CD}\cdot\sin\angle(\mathrm{AB},\mathrm{CD})}$,where V(ABCD) is the volume of the tetrahedron ABCD and $\angle(AB,CD)$ is the measure of the angle between the lines AB and CD . This relationship is known as Chasles' formula.

Method 2. Calculating the distance by determining the position of the segment that defines it, using the mixed product. The mixed product of three vectors $\vec{v}_1,\vec{v}_2$ and $\vec{v}_3$ is the number $\vec{v}_1\cdot(\vec{v}_2\times\vec{v}_3)=(\vec{v}_1,\vec{v}_2,\vec{v}_3)$ that is determined by calculating the determinant formed with the coordinates of the three vectors written on the lines of the determinant.



In the figure above, the non-coplanar lines are (d_1) and (d_2) with the director vectors $\vec{v_1}$ and $\vec{v_2}$. The perpendicular line common to the two lines is $(d_{p.c})$ which has as its director vector on $\vec{v}$. By $d_{p.c} \perp d_1$ and $d_{p.c} \perp d_2$, yields that $\vec{v} = \vec{v_1} \times \vec{v_2}$. If $M_1 \in (d_1)$ and $M_2 \in (d_2)$ then the two planes (P_1) and (P_2) have the equations:

$$(P_1): (\vec{r} - \vec{r_1}, \vec{v_1}, \vec{v}) = 0, \text{ respectively } (P_2): (\vec{r} - \vec{r_2}, \vec{v_2}, \vec{v}) = 0,$$

where $\vec{r_1}$ (respectively $\vec{r_2}$) is the position vector of the point M_1 (respectively M_2).

The equation of the common perpendicular line is given as the intersection of the two planes (P_1) respectively (P_2). So,

$$(d_{p.c.})\begin{cases}(P_1)\\(P_2)\end{cases} \Rightarrow (d_{p.c.})\begin{cases}(\vec{r} - \vec{r_1}, \vec{v_1}, \vec{v}) = 0\\(\vec{r} - \vec{r_2}, \vec{v_2}, \vec{v}) = 0\end{cases}.$$

Next, the coordinates of point M (respectively N) are determined as the intersection of two lines:

$$\{M\}\begin{cases}(d_1)\\(d_{p.c.})\end{cases}; \qquad \{N\}\begin{cases}(d_2)\\(d_{p.c.})\end{cases}.$$

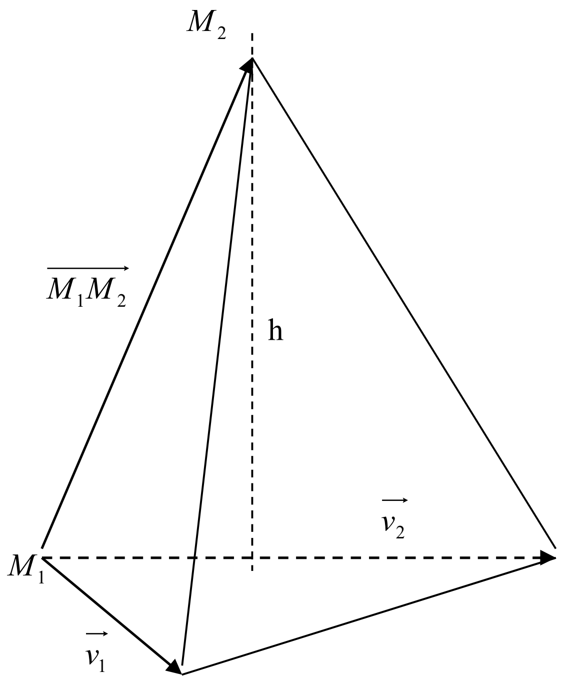
Finally, the distance between the non-coplanar lines (d_1) and (d_2) is determined as the distance between points M and N:

$$d((d_1),(d_2)) = d(M, N).$$

Method 3. Calculation of the distance between two non-coplanar lines without determining the position of the segment that defines it. This method is based on the mixed product.

If the non-coplanar lines are (d_1) and (d_2) with the director vectors $\vec{v_1}$ and $\vec{v_2}$ two points are considered $M_1 \in (d_1)$ and $M_2 \in (d_2)$

We have the figure below:



The vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{M_1M_2}$ determine a tetrahedron whose height is the sought distance. It is known from the geometric interpretation of the mixed product that the volume of the parallelepiped determined by three vectors is the absolute value of the mixed product. We have:

$d((d_1),(d_2)) =$ h, i.e. the height of the tetrahedron formed by the vectors $\{ \overrightarrow{v_1} , \overrightarrow{v_2} , \overrightarrow{M_1M_2} \}$.

We write the volume of the tetrahedron in two ways:

$$(1) \qquad V_{tetrahedron} = \frac{V_{parallelipiped}}{6} = \frac{|\, (\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{M_1M_2}) \,|}{6}$$

$$(2) \qquad V_{tetrahedron} = \frac{A_b \cdot h}{3} = \frac{\left\| \overrightarrow{v_1} \times \overrightarrow{v_2} \right\|}{2} \cdot \frac{h}{3}$$

From (1) and (2) results in the calculation formula for the searched distance:

$$(*) \qquad h=\frac{|(\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{M_1M_2})|}{\|\overrightarrow{v_1}\times\overrightarrow{v_2}\|}.$$

Method 4. Let (d_1) , (d_2),be two non-coplanar lines and let MN be their common perpendicular.
We assume that $A\in(d_1)$ and $B\in(d_2)$ are two points which are known, so that the vector $\overrightarrow{AB}$ is known. If $\overrightarrow{v_1}$ is the direction vector of the line (d_1) and $\overrightarrow{v_2}$ is the direction vector of the line (d_2) then the vector $\overrightarrow{MN}$ is expressed as follows: $\overrightarrow{MN}=\overrightarrow{MA}+\overrightarrow{AB}+\overrightarrow{BN}=\alpha\overrightarrow{v_1}+\overrightarrow{AB}+\beta\overrightarrow{v_2}$ where the real numbers α and β are still undetermined. They will be determined from the orthogonality conditions

$$\begin{cases}\overrightarrow{MN}\cdot\overrightarrow{v_1}=0\\ \overrightarrow{MN}\cdot\overrightarrow{v_2}=0\end{cases}.$$

This constitute a system of linear equations in the unknowns α and β. After determining the unknowns α and β we will find

$$d((d_1),(d_2))=\|\overrightarrow{MN}\|=\|\alpha\overrightarrow{v_1}+\overrightarrow{AB}+\beta\overrightarrow{v_2}\|.$$

Method 5. Determining the distance between (d_1) and (d_2) non-coplanar lines with determining the position of the segment MN that defines the common perpendicular line. We consider the same figure as above.

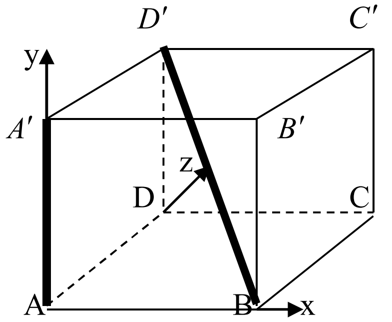
$M\in(d_1)\Rightarrow M(\alpha,f_1(\alpha),f_2(\alpha))$;

$N\in(d_2)\Rightarrow N(\beta,g_1(\beta),g_2(\beta))$, where f_1,f_2,g_1,g_2 are linear functions. Unknowns α and β are determined from the orthogonality conditions

$$\begin{cases}\overrightarrow{MN}\cdot\overrightarrow{v_1}=0\\ \overrightarrow{MN}\cdot\overrightarrow{v_2}=0\end{cases}.$$

Then, we shall determine the coordinates of the points M and N. Hence, $d((d_1),(d_2))=d(M,N)$.

Example. Compute the distance between a diagonal of the cube and an edge that does not intersect it.



To use these five methods, we choose the Cartesian coordinate system $Axyz$ with the origin at the vertex A, and
$B \in (Ax, A' \in (Az, D \in (Ay, A'(0,0,a), B(a,0,0), C(a,a,0)$, $D'(0,a,a)$.

Solution (method 1).

$$d(AA', BD') = \frac{6 \cdot V(AA'D'B)}{AA' \cdot BD' \cdot \sin\angle(AA', BD')} .$$

$$AA' = a,\ BD' = a\sqrt{3},\ \sin\angle(AA', BD') = \sin\angle(DD', BD') = \frac{BD}{BD'} = \frac{a\sqrt{2}}{a\sqrt{3}} \ ;$$

$$V(AA'D'B) = \frac{a^3}{6}\text{ .Hence: } d(AA', BD') = \frac{a\sqrt{2}}{2} .$$

Solution (method 2).

$$d(AA', BD') = d(M, N);\ \vec{v_1} = \overrightarrow{AA'}(0,0,a), \vec{v_2} = \overrightarrow{BD'}(-a,a,a), \vec{v} = \vec{v_1} \times \vec{v_2} = (-a^2, -a^2, 0);$$

$$(P_1): x - y = 0, (P_2): x - y + 2z = 0, (d_{p.c.}): \begin{cases} x - y = 0 \\ x - y + 2z = 0 \end{cases}$$

$$(d_1) \equiv (AA'): \begin{cases} x = 0 \\ y = 0 \end{cases}; (d_2) \equiv (BD'): \begin{cases} x + y = a \\ y - z = 0 \end{cases};$$

$$\{M\} \begin{cases} x = 0 \\ y = 0 \\ x = y \\ x - y + 2z - a = 0 \end{cases} ; \{N\} \begin{cases} x + y = a \\ y - z = 0 \\ x = y \\ x - y + 2z - a = 0 \end{cases}$$

$$M(0,0,\frac{a}{2}), N(\frac{a}{2},\frac{a}{2},\frac{a}{2}), d(AA',BD') = d(M,N) = \frac{a\sqrt{2}}{2}.$$

Solution (method 3).

$$d(AA',BD') = \frac{\left|(\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{AB})\right|}{\left\|\overrightarrow{v_1}\times\overrightarrow{v_2}\right\|};$$

$\overrightarrow{v_1} = \overrightarrow{AA'}(0,0,a), \overrightarrow{v_2} = \overrightarrow{BD'}(-a,a,a), \vec{v} = \overrightarrow{v_1}\times\overrightarrow{v_2} = (-a^2,-a^2,0)$,

$\overrightarrow{AB}(a,0,0)$; $(\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{AB}) = -a^3$, $\left\|\overrightarrow{v_1}\times\overrightarrow{v_2}\right\| = a^2\sqrt{2}$;

hence $d(AA',BD') = \dfrac{a\sqrt{2}}{2}$.

Solution (method 4).

$\overrightarrow{MN} = \alpha\cdot\overrightarrow{v_1} + \beta\cdot\overrightarrow{v_2} + \overrightarrow{AB} = (a-\beta a, \beta a, \alpha a + \beta a).$

$$\begin{cases}\overrightarrow{MN}\cdot\overrightarrow{v_1} = 0\\ \overrightarrow{MN}\cdot\overrightarrow{v_2} = 0\end{cases} \Leftrightarrow \begin{cases}\alpha+\beta = 0\\ \alpha+3\beta = 1\end{cases} \Leftrightarrow \alpha = -\frac{1}{2}, \beta = \frac{1}{2}$$

$\overrightarrow{MN}(\frac{a}{2},\frac{a}{2},0)$; $d(AA',BD') = \left\|\overrightarrow{MN}\right\| = \dfrac{a\sqrt{2}}{2}.$

Solution (method 5).

$M(0,0,\alpha), N(\beta, a-\beta, a-\beta), \overrightarrow{MN}(\beta, a-\beta, a-\beta-\alpha)$.

$$\begin{cases}\overrightarrow{MN}\cdot\overrightarrow{v_1} = 0\\ \overrightarrow{MN}\cdot\overrightarrow{v_2} = 0\end{cases} \Leftrightarrow \begin{cases}\alpha+\beta = a\\ \alpha+3\beta = 2a\end{cases} \Rightarrow \alpha = \beta = \frac{a}{2} \Rightarrow M(0,0,\frac{a}{2}), N(\frac{a}{2},\frac{a}{2},\frac{a}{2}).$$

Hence:

$$d(AA',BD') = d(M,N) = \frac{a\sqrt{2}}{2}.$$

Reference:

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