Two generalizations for Jordan's inequality

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In this note, two generalizations are presented and demonstrated for a relatively well-known trigonometric inequality - Jordan's inequality. Some particular cases and an application are also presented.

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Is well known and frequently used the inequality $\sin x < x$, **(1)**

for any $x \in (0,\pi)$ · Perhaps less used, but equally important is reverse inequality ,

$$\sin x \geq \frac{2}{\pi}\cdot x \ , \qquad (2)$$

valid but only for $x \in [0,\pi/2]$. Inequality (2) constitutes *Jordan's inequality* .

Together, inequalities (1) and (2) provide a framing with liniar functions of the trigonometric function $\sin x$ in the interval $[0,\pi/2]$:

$$\frac{2}{\pi}\cdot x \leq \sin x < x \ , \qquad (3)$$

Since we still want to use the *convexity / concavity* of some functions, we remind you here the usual definition of *convexity* , as well as two other equivalent forms :

1.*Definition* The function $f : \mathbf{I} \subset (0,\infty) \longrightarrow \mathbb{R}$ is called a *convex function* on the interval $\mathbf{I}$, if

$$f[(1-\lambda)x_1 + \lambda x_2] \leq (1-\lambda)f(x_1) + \lambda f(x_2) \ , \qquad (4)$$

for any $x_1 , x_2 \in \mathbf{I}$ and any $\lambda \in [0,1]$.

2. *Remark*

Taking , $x = (1-\lambda)x_1 + \lambda x_2$ (that is , x is between x_1 and x_2) , relation (4) is rewritten :

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1}\cdot f(x_1) + \frac{x - x_1}{x_2 - x_1}\cdot f(x_2) \ , \qquad (5)$$

or still , equivalently ,

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x} \ , \qquad (6)$$

For a *concave function* , there are relations similar to those in (4), (5), (6), but with the inequality sign " ≤ " replaced by the sign " ≥ " · More about *convexity / concavity* , in [1] , [3] – [5] .

The constant $\frac{2}{\pi}$ from Jordan's inequality (2) is dependent on the interval $\mathbf{I} = [0,\pi/2]$ ·

If we change the interval $\mathbf{I}$ - domain of definition of the ***sine*** function , then this constant changes as in the following ,

3. Proposition (*generalization of Jordan's inequality for intervals of the form* $[0, \alpha]$)

For the angles x, α, such that $0 \le x \le \alpha \le \pi$, we have the inequality :

$$\sin x \ge \frac{\sin\alpha}{\alpha}\cdot x\ , \quad (\forall)\, x \in [0, \alpha] \quad . \tag{7}$$

having equality if $x = 0$ or $x = \alpha$.

Proof

Function $\sin : [0, \pi] \longrightarrow \mathbb{R}$ is concave , so using for example relation (6) (- but with the inverted inequality sign), with the choices : $f(x) = \sin x\, ,\ x_1 = 0\, ,\ x_2 = \alpha$ the inequality **(7)** is obtained.

For $\alpha = \pi / 2$ *Jordan's inequality* is obtained.

For other values of α , interesting inequalities are obtained , as in the following,

4. Corollary

The following inequalities occur :

(*a*) $$\sin x \ge \frac{3}{\pi}\cdot x\ , \quad (\forall)\, x \in [0,\ \pi/6] \quad ; \tag{8}$$

(*b*) $$\sin x \ge \frac{2\sqrt{2}}{\pi}\cdot x\ , \quad (\forall)\, x \in [0,\ \pi/4] \quad ; \tag{9}$$

(*c*) $$\sin x \ge \frac{3\sqrt{3}}{2\pi}\cdot x\ , \quad (\forall)\, x \in [0,\ \pi/3] \quad ; \tag{10}$$

(*d*) $$\sin x \ge \frac{3\sqrt{3}}{4\pi}\cdot x\ , \quad (\forall)\, x \in [0,\ 2\pi/3] \quad ; \tag{11}$$

(*e*) $$\sin x \ge \frac{2\sqrt{2}}{3\pi}\cdot x\ , \quad (\forall)\, x \in [0,\ 3\pi/4] \quad ; \tag{12}$$

(*f*) $$\sin x \ge \frac{3}{5\pi}\cdot x\ , \quad (\forall)\, x \in [0,\ 5\pi/6] \quad . \tag{13}$$

Proof

In inequality (7) the angle α is replaced , in turn with : (*a*) $\alpha = \pi/6$, (*b*) $\alpha = \pi/4$, (*c*) $\alpha = \pi/3$, (*d*) $\alpha = 2\pi/3$, (*e*) $\alpha = 3\pi/4$, (*f*) $\alpha = 5\pi/6$.

5. Remark

The inequalities in *Proposition* 3 and *Corollary* 4 have a simple geometric interpretation : the graph of the ***sine*** function is above the graph of the chord starting from the origin - on the considered interval.

6. Remark

Since , for example in inequality (10) , for the interval $[0, \pi/3] \subset [0, \pi/2]$, we have $\frac{3\sqrt{3}}{2\pi}\cdot x > \frac{2}{\pi}\cdot x$, it turns out that the inequality $\sin x \ge \frac{3\sqrt{3}}{4\pi}\cdot x$ is stronger than *Jordan's inequality*

so that inequality (10) refines *Jordan's inequality* for the (sub)interval $x \in [0, \pi/3]$.

All inequalities (8)-(10) refine *Jordan's inequality* - on the respective subintervals . How much the angle α is smaller, then the more inequality (7) is 'stronger' ! Inequalities (11)-(13) are 'weaker' than *Jordan's inequality* (on their common domain) .

At the same time, *Proposition* 3 also provides the following monotonicity result .

7. Corollary

The function $\varphi : (0\,,\pi] \longrightarrow \mathbb{R}$, $\varphi(t) = \dfrac{\sin t}{t}$ is monotonically decreasing on $(0\,,\pi]$.

Proof

Indeed, for $0 \le x \le \alpha \le \pi$, we have ,

$$\sin x \ge \frac{\sin\alpha}{\alpha}\cdot x\ ,\quad (\forall)\, x\in[0\,,\alpha] \quad\Leftrightarrow\quad \frac{\sin x}{x} \ge \frac{\sin\alpha}{\alpha}\ ,\quad (\forall)\, x \le \alpha\ .$$

8. Proposition ***(generalization of Jordan's inequality for intervals of the form*** $[\alpha,\beta]$ ***)***

For the angles x , α , β , such that $0 \le \alpha \le x \le \beta \le \pi$, we have the inequality :

$$\sin x \ge \frac{\sin\beta - \sin\alpha}{\beta-\alpha}\cdot x + \frac{\beta\sin\alpha - \alpha\sin\beta}{\beta-\alpha}\ ,\quad (\forall)\, x\in[\alpha,\beta]\ , \qquad \textbf{(14)}$$

having equality if $x = \alpha$ or $x = \beta$.

Proof 1

Again we use the concavity of the function $\sin : [0\,,\pi] \longrightarrow \mathbb{R}$, for which using relation (6) (- but with the inequality sign reversed), with the choices: $f(x) = \sin x$, $x_1 = 0$, $x_2 = \alpha$ is obtained ,

$$\frac{\sin x - \sin\alpha}{x-\alpha} \ge \frac{\sin\beta - \sin x}{\beta - x} \quad\Leftrightarrow\quad (\beta-\alpha)\cdot\sin x \ge (\sin\beta - \sin x)\cdot x + \beta\sin\alpha - \alpha\sin\beta$$

hence the inequality **(14)** .

Proof 2

For any $x \in [\alpha\,,\beta]$, there is $t\in[0\,,1]$, such that $x = (1-t)\,\alpha + t\,\beta$ $\left(\Leftrightarrow\ t = \dfrac{\beta - x}{\beta-\alpha}\right)$

With $f(x) = \sin x$, *concave* on $[\alpha\,,\beta] \subset [0\,,\pi]$, we get **:**

$$\sin x = \sin\left[t\alpha + (1-t)\beta\right] \ge t\sin\alpha + (1-t)\sin\beta = \frac{\beta-x}{\beta-\alpha}\cdot\sin\alpha + \left(1 - \frac{\beta-x}{\beta-\alpha}\right)\cdot\sin\beta =$$

$$= \frac{(\beta-x)\cdot\sin\alpha + (x-\alpha)\cdot\sin\beta}{\beta-\alpha} = \frac{\sin\beta - \sin\alpha}{\beta-\alpha}\cdot x + \frac{\beta\sin\alpha - \alpha\sin\beta}{\beta-\alpha}\ .$$

For $\alpha = 0$ and $\beta = \pi/2$, *Jordan's inequality* is obtained .

For $\alpha = 0$ and $\beta = \alpha$, the *generalization of Jordan's inequality* from *Proposition* 3 is obtained.

9. Remark

The inequality in *Proposition* 8 has a simple geometric interpretation : the graph of the ***sine*** function on the interval $[\alpha\,,\beta]$ is above the graph of the chord of extremities $A(\alpha\,,\sin\alpha)$, $B(\beta\,,\sin\beta)$, which has the equation:

$$\textbf{(AB)}:\quad y = \frac{\sin\beta - \sin\alpha}{\beta-\alpha}\cdot x + \frac{\beta\sin\alpha - \alpha\sin\beta}{\beta-\alpha}\ .$$

10. *Corollary*

The following inequalities occur :

$$(a)\quad \sin x \geq \frac{6(\sqrt{2}-1)}{\pi}\cdot x+\frac{3-2\sqrt{2}}{2}, \qquad (\forall)\, x\in[\pi/6,\ \pi/4] \quad , \tag{15}$$

with equality if $x=\pi/6$ or $x=\pi/4$;

$$(b)\quad \sin x \geq \frac{3(\sqrt{3}-1)}{\pi}\cdot x+\frac{2-\sqrt{3}}{2} , \qquad (\forall)\, x\in[\pi/6,\ \pi/3] \quad , \tag{16}$$

with equality if $x=\pi/6$ or $x=\pi/3$;

$$(c)\quad \sin x \geq \frac{3}{2\pi}\cdot x+\frac{1}{4} , \qquad (\forall)\, x\in[\pi/6,\ \pi/2] \quad , \tag{17}$$

with equality if $x=\pi/6$ sau $x=\pi/2$;

$$(d)\quad \sin x \geq \frac{6(\sqrt{3}-\sqrt{2})}{\pi}\cdot x+\frac{4\sqrt{2}-3\sqrt{3}}{2}, \qquad (\forall)\, x\in[\pi/4,\ \pi/3] \quad , \tag{18}$$

with equality if $x=\pi/4$ or $x=\pi/3$;

$$(e)\quad \sin x \geq \frac{2(2-\sqrt{2})}{\pi}\cdot x+\sqrt{2}-1 , \qquad (\forall)\, x\in[\pi/4,\ \pi/2] \quad , \tag{19}$$

with equality if $x=\pi/4$ or $x=\pi/2$;

$$(f)\quad \sin x \geq \frac{3(2-\sqrt{3})}{\pi}\cdot x+\frac{3\sqrt{3}-4}{2} , \qquad (\forall)\, x\in[\pi/3,\ \pi/2] \quad , \tag{20}$$

with equality if $x=\pi/3$ or $x=\pi/2$.

Proof

In inequality (14) the angles α and β are replaced in turn by : ***(a)*** $\alpha=\pi/6$, $\beta=\pi/4$; ***(b)*** $\alpha=\pi/6$, $\beta=\pi/3$; ***(c)*** $\alpha=\pi/6$, $\beta=\pi/2$; ***(d)*** $\alpha=\pi/4$, $\beta=\pi/3$; ***(e)*** $\alpha=\pi/4$, $\beta=\pi/2$; ***(f)*** $\alpha=\pi/3$, $\beta=\pi/2$, and routine calculations are made .

The corollary chose several combinations of angles from quadrant I. Obviously, you can also choose angles from quadrant II or from the first two quadrants ·

11. *Application* [2]

If in triangle ABC we have $A, B, C \in [\pi/6, \pi/2]$, then ,

$$\sin A + \sin B + \sin C > \frac{9}{4} \cdot \tag{21}$$

Proof

Using the inequality (17) with $A, B, C \in [\pi/6, \pi/2]$,

$$\Rightarrow \quad \sum_{cycl} \sin A \geq \sum_{cycl}\left(\frac{3}{2\pi}\cdot A+\frac{1}{4}\right) = \frac{3}{2\pi}\cdot\pi+\frac{3}{4}=\frac{3}{2}+\frac{3}{4}=\frac{9}{4} \quad .$$

Inequality (21) is strict, because the angles of the triangle cannot take only the values $\pi/6$ and $\pi/2$.

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