A SIMPLE PROOF FOR ARSLANGIC'S INEQUALITY AND APPLICATIONS

DANIEL SITARU - ROMANIA

Abstract. In this paper we will prove the famous Arslangic's inequality and we will give a few applications.

ARSLANGIC'S INEQUALITY

If $x, y, z > 0$ then:

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}} \tag{1}$$

Proof.

$$\frac{x}{y} + \frac{x}{y} + \frac{y}{z} \geq 3\sqrt[3]{\frac{x^2}{yz}} = \frac{3x}{\sqrt[3]{xyz}} \tag{2}$$

$$\frac{y}{z} + \frac{y}{z} + \frac{z}{x} \geq 3\sqrt[3]{\frac{y^2}{zx}} = \frac{3y}{\sqrt[3]{xyz}} \tag{3}$$

$$\frac{z}{x} + \frac{z}{x} + \frac{x}{y} \geq 3\sqrt[3]{\frac{z^2}{xy}} = \frac{3z}{\sqrt[3]{xyz}} \tag{4}$$

By adding $(2); (3); (4)$:

$$3\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) = \frac{3x}{\sqrt[3]{xyz}} + \frac{3y}{\sqrt[3]{xyz}} + \frac{3z}{\sqrt[3]{xyz}}$$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}$$

Equality holds for $x = y = z$. □

Application 1: If $a, b, c > 0$ then:

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{ab+bc+ca}{\sqrt[3]{a^2b^2c^2}}$$

Proof.

We take in (1):

$$x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}, \qquad \frac{\frac{1}{a}}{\frac{1}{b}} + \frac{\frac{1}{b}}{\frac{1}{c}} + \frac{\frac{1}{c}}{\frac{1}{a}} \geq \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{\sqrt[3]{\frac{1}{abc}}}$$

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{ab+bc+ca}{abc \cdot \sqrt[3]{\frac{1}{abc}}}, \qquad \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{ab+bc+ca}{\sqrt[3]{a^2b^2c^2}}$$

Equality holds for $a = b = c$.

□

Application 2: If $a, b, c > 1$ then:

$$\log_b a + \log_c b + \log_a c \geq \frac{\ln(abc)}{\sqrt[3]{\ln a \cdot \ln b \cdot \ln c}}$$

Proof.
We take in (1):

$$x = \ln a, y = \ln b, z = \ln c, \qquad \frac{\ln a}{\ln b} + \frac{\ln b}{\ln c} + \frac{\ln c}{\ln a} \geq \frac{\ln a + \ln b + \ln c}{\sqrt[3]{\ln a \cdot \ln b \cdot \ln c}}$$

$$\log_b a + \log_c b + \log_a c \geq \frac{\ln(abc)}{\sqrt[3]{\ln a \cdot \ln b \cdot \ln c}}$$

Equality holds for $a = b = c$.

□

Application 3: If $a, b, c \in \mathbb{R}$ then:

$$2^{a-b} + 2^{b-c} + 2^{c-a} \geq \frac{2^a + 2^b + 2^c}{\sqrt[3]{2^{a+b+c}}}$$

Proof.
We take in (1):

$$x = 2^a, y = 2^b, z = 2^c, x, y, z > 0$$

$$\frac{2^a}{2^b} + \frac{2^b}{2^c} + \frac{2^c}{2^a} \geq \frac{2^a + 2^b + 2^c}{\sqrt[3]{2^a \cdot 2^b \cdot 2^c}}, \qquad 2^{a-b} + 2^{b-c} + 2^{c-a} \geq \frac{2^a + 2^b + 2^c}{\sqrt[3]{2^{a+b+c}}}$$

Equality holds for $a = b = c$.

□

Application 4: If $x, y > 0$ then:

$$\frac{x}{y} + \frac{y}{x} + 1 \geq \frac{x + 2y}{\sqrt[3]{x^2 y}}$$

Proof.

We take in $(1) : y = z$.

$$\frac{x}{y} + \frac{y}{y} + \frac{y}{x} \geq \frac{x + y + y}{\sqrt[3]{x \cdot y \cdot y}}, \qquad \frac{x}{y} + \frac{y}{x} + 1 \geq \frac{x + 2y}{\sqrt[3]{x^2 y}}$$

Equality holds for $x = y$.

□

Application 5: If $0 < a \leq b$ then:

$$5(b-a)^2(b+a)\ln\left(\frac{b}{a}\right) \geq 27(b\sqrt[3]{b^2} - a\sqrt[3]{a^2})(\sqrt[3]{b^2} - \sqrt[3]{a^2})^2$$

Proof.

$$\int_a^b \int_a^b \int_a^b \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) dxdydz = 3(b-a)\int_a^b \int_a^b \left(\frac{x}{y}\right) dxdy =$$

$$= 3(b-a)\left(\int_a^b x dx\right)\left(\int_a^b \frac{dy}{y}\right) = 3(b-a) \cdot \frac{b^2 - a^2}{2} \cdot (\ln b - \ln a) =$$

$$= 3(b-a)\cdot\frac{b^2-a^2}{2}\cdot\ln\Big(\frac{b}{a}\Big)$$

$$\int_a^b\int_a^b\int_a^b\Big(\frac{x+y+z}{\sqrt[3]{xyz}}\Big)dxdydz = 3\int_a^b\int_a^b\int_a^b\Big(\frac{x}{\sqrt[3]{xyz}}\Big)dxdydz =$$

$$= 3\left(\int_a^b\frac{x}{\sqrt[3]{x}}dx\right)\left(\int_a^b\frac{1}{\sqrt[3]{y}}dy\right)^2 = 3\left(\int_a^b x^{\frac{2}{3}}dx\right)\left(\int_a^b y^{-\frac{1}{3}}dy\right)^2 =$$

$$= 3\cdot\frac{b^{\frac{5}{3}}-a^{\frac{5}{3}}}{\frac{5}{3}}\cdot\left(\frac{b^{\frac{2}{3}}-a^{\frac{2}{3}}}{\frac{5}{3}}\right)^2 = \frac{81}{10}(b\sqrt[3]{b^2}-a\sqrt[3]{a^2})(\sqrt[3]{b^2}-\sqrt[3]{a^2})^2$$

$$3(b-a)\cdot\frac{b^2-a^2}{2}\cdot\ln\Big(\frac{b}{a}\Big) \geq \frac{81}{10}(b\sqrt[3]{b^2}-a\sqrt[3]{a^2})(\sqrt[3]{b^2}-\sqrt[3]{a^2})^2$$

$$5(b-a)^2(b+a)\ln\Big(\frac{b}{a}\Big) \geq 27(b\sqrt[3]{b^2}-a\sqrt[3]{a^2})(\sqrt[3]{b^2}-\sqrt[3]{a^2})^2$$

Equality holds for $a=b$.

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com