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CLASSICAL CONTEST PROBLEMS WITH THE CHARACTERISTIC POLYNOMIAL

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Definition. **Let A be a $n \times n$ matrix. The characteristic polynomial of the matrix A is the function $f_A(X)$, given by: $f_A(X) = \det(XI_n - A)$.**

Definition. **Let A be a $n \times n$ matrix. The equation $f_A(X) = 0$ is the characteristic equation of the matrix A.**

Theorem 1. **Let A be a $n \times n$ matrix and let $f_A(X) = \det(XI_n - A)$ be its characteristic polynomial. Then, a number λ_0 is an eigenvalue of A if and only if $f(\lambda_0) = 0$.**

Theorem 2. **Let A be a $n \times n$ matrix and let $f_A(X) = \det(XI_n - A)$ be its characteristic polynomial. Then, $f_A(X)$ is a polynomial of degree n with complex coefficients. Moreover, $f_A(X)$ has the form: $f_A(X) = X^n - c_1X^{n-1} + c_2X^{n-2} - c_3X^{n-3} + \cdots + (-1)^n c_n$.**

Observation 1. **Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A. Then:** $\begin{cases} c_1 = \mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ c_2 = \mathrm{Tr}(A^*) = \sum_{1 \le i < j \le n} \lambda_i \lambda_j \\ c_n = \det(A) = \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n \end{cases}$.

Observation 2. **Let A be a $n \times n$ matrix and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues.**

1) **If $k \in \mathbb{N}^*$, the only eigenvalues of A^k are λ_i^k, $where\ i = \overline{1,n}$.**
2) **If A^{-1} is the inverse matrix of A, the only eigenvalues of A^{-1} are $\frac{1}{\lambda_i}$, where $i = \overline{1,n}$.**
3) **If $p \in \mathbb{C}[X]$, the only eigenvalues of $p(A)$ are $p(\lambda_i)$, $i = \overline{1,n}$.**

Theorem 4. (Cayley-Hamilton) Every square matrix satisfies its own characteristic equation.

Observation 3. **The Cayley-Hamilton's theorem states that if A is a $n \times n$ matrix, then $f_A(A) = O_n$.**

Problem 1.

Let $A \in M_3(\mathbb{C})$ such that $A^3 = I_3$. Prove that $\mathrm{Tr}(A) = 0$ if and only if $\mathrm{Tr}(A^2) = 0$.

Solution. Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A. Then, λ_1^2, λ_2^2 and λ_3^2 are the eigenvalues of the matrix A^2 and λ_1^3, λ_2^3 and λ_3^3 are the eigenvalues of the matrix A^3.

If $\mathrm{Tr}(A) = 0$, then $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Because $A^3 = I_3 \Longrightarrow \mathrm{Tr}(A^3) = \mathrm{Tr}(I_3) \Leftrightarrow \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3$. From $\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3 = (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3) \Longrightarrow \lambda_1\lambda_2\lambda_3 = 1$, but because $\det(A) = \lambda_1\lambda_2\lambda_3 \Longrightarrow \det(A) = 1$. From Cayley-Hamilton's theorem:

$A^3 - \mathrm{Tr}(A)A^2 + + \mathrm{Tr}(A^*)A - \det(A)I_3 = O_3 \Leftrightarrow A^3 + \mathrm{Tr}(A^*)A - I_3 = O_3 \Leftrightarrow \mathrm{Tr}(A^*)A = O_3$. From $\det(A) = 1$, it is obvious that $A \neq O_3 \implies \mathrm{Tr}(A^*) = 0 \Leftrightarrow \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 0$. Now, because $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = = (\lambda_1 + \lambda_2 + \lambda_3)^2 - 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = 0 \Longrightarrow \mathrm{Tr}(A^2) = 0$, q.e.d.

If $\mathrm{Tr}(A^2) = 0$, then $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$. Because $A^3 = I_3 \implies$ the eigenvalues of A^3 are the same eigenvalues of I_3, but the eigenvalues of I_3 are 1, 1 and 1 $\implies \lambda_1^3 = \lambda_2^3 = \lambda_3^3 = 1$. From $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$ and $\lambda_1^3 = \lambda_2^3 = \lambda_3^3 = 1 \implies$ the only possibility is: $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, \mathcal{E}, \mathcal{E}^2\}$, where $\mathcal{E}^3 = 1, \mathcal{E} \notin \mathbb{R}$. Then, it is obvious that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0 \Leftrightarrow \mathrm{Tr}(A^2) = 0$, q.e.d.

Problem 2.

Let $A \in M_3(\mathbb{R})$ such that $\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = 0$. Show that $\det(A^2 + I_3) = \det(A^2) + 1$.

Solution. Let λ_1, λ_2 and λ_3 be the eigenvalues of A. Then, the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \lambda_3^2$ and the eigenvalues of $A^2 + I_3$ are $\lambda_1^2 + 1, \lambda_2^2 + 1, \lambda_3^2 + 1$. From $\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = 0 \implies \lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0 \Longrightarrow \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{(\lambda_1+\lambda_2+\lambda_3)^2 - (\lambda_1^2+\lambda_2^2+\lambda_3^2)}{2} = 0$.

$\det(A^2 + I_3) = \det(A^2) + 1 \Leftrightarrow (\lambda_1^2 + 1)(\lambda_2^2 + 1)(\lambda_3^2 + 1) = \lambda_1^2\lambda_2^2\lambda_3^2 + 1 \Leftrightarrow \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = 0$. From $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0 \Longrightarrow (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)^2 = 0 \Longrightarrow \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = 0$.

Therefore, $\det(A^2 + I_3) = \det(A^2) + 1$, q.e.d.

Problem 3.

Let X be a positive real number and $A \in M_2(\mathbb{R})$ such that $\det(A^2 + XI_2) = 0$. Show that $\det(A^2 + A + XI_2) = X$.

Solution. Let $f_A(X) = \det(XI_2 - A) = (-1)^2 \det(A - XI_2) = \det(A - XI_2) = X^2 - \mathrm{Tr}(A)X + \det(A)$.

$\det(A^2 + XI_2) = \det(A + i\sqrt{X}I_2) \cdot \det(A - i\sqrt{X}I_2) = f_A(-i\sqrt{X})f_A(i\sqrt{X}) = |f_A(i\sqrt{X})|^2 = 0$. Then, we have: $|-X + \det(A) - i\mathrm{Tr}(A)\sqrt{X}| = 0 \Leftrightarrow (\det(A) - X)^2 + X\mathrm{Tr}(A)^2 = 0 \overset{X>0}{\Longrightarrow} \begin{cases} \det(A) = X \\ \mathrm{Tr}(A) = 0 \end{cases}$ and from Cayley-Hamilton's theorem, we know that: $A^2 + XI_2 = O_2 \Longrightarrow \det(A^2 + A + XI_2) = \det(A) = X$, q.e.d.

Problem 4.

Let $A \in M_3(\mathbb{R})$ such that $\det(A) = 1$. Show that $\det(A^2 - A + I_3) = 0$ if and only if:

$$\begin{cases} \det(A + I_3) = 6 \\ \det(A - I_3) = 0 \end{cases}.$$

Solution. Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix A. Then, we know that $\lambda_1\lambda_2\lambda_3 = 1$ from $\det(A) = 1$, the eigenvalues of the matrix $A^2 - A + I_3$ are $\lambda_1^2 - \lambda_1 + 1$, $\lambda_2^2 - \lambda_2 + 1$ and $\lambda_3^2 - \lambda_3 + 1$, the eigenvalues of the matrix $A + I_3$ are $\lambda_1 + 1$, $\lambda_2 + 1$ and $\lambda_3 + 1$ and the eigenvalues of the matrix $A - I_3$ are $\lambda_1 - 1$, $\lambda_2 - 1$ and $\lambda_3 - 1$.

If $\det(A + I_3) = 6$ and $\det(A - I_3) = 0 \Rightarrow \begin{cases}(\lambda_1+1)(\lambda_2+1)(\lambda_3+1) = 6\\ (\lambda_1-1)(\lambda_2-1)(\lambda_3-1) = 0\end{cases}$.

From $(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) = 0 \overset{WLOG}{\Longrightarrow} \lambda_1 = 1$. From $\lambda_1\lambda_2\lambda_3 = 1 \Rightarrow \lambda_2 = \frac{1}{\lambda_3}$ and from $(\lambda_1 + + 1)(\lambda_2 + 1)(\lambda_3 + 1) = 6 \Rightarrow \lambda_3^2 - \lambda_3 + 1 = 0 \Rightarrow (\lambda_1^2 - \lambda_1 + 1)(\lambda_2^2 - \lambda_2 + 1)(\lambda_3^2 - \lambda_3 + 1) = 0$ and so $\det(A^2 - A + I_3) = 0$.

If $\det(A^2 - A + I_3) = 0 \Rightarrow (\lambda_1^2 - \lambda_1 + 1)(\lambda_2^2 - \lambda_2 + 1)(\lambda_3^2 - \lambda_3 + 1) = 0$. WLOG, let's suppose that $\lambda_1^2 - \lambda_1 + 1 = 0$. Then, $\lambda_{1_{1,2}} = \frac{1 \pm i\sqrt{3}}{2}$.

Let $f_A(X) = \det(XI_3 - A) = X^3 - \mathrm{Tr}(A)X^2 + \mathrm{Tr}(A^*)X - \det(A)$. Because the characteristic equation of the matrix A has one of its three solutions λ_1 and $f_A(X) \in \mathbb{R}[X]$ from $A \in M_3(\mathbb{R})$, then the second solution of the characteristic equation of the matrix A is $\overline{\lambda_1}$. WLOG, let's suppose that $\begin{cases}\lambda_1 = \frac{1+i\sqrt{3}}{2}\\ \lambda_2 = \frac{1-i\sqrt{3}}{2}\end{cases}$ and from $\lambda_1\lambda_2\lambda_3 = 1 \Rightarrow \lambda_3 = 1$. Therefore, we know that: $\{\lambda_1, \lambda_2, \lambda_3\} = \left\{\frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, 1\right\}$. Now, it is easy to see that $(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1) = 6$ and $(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1) = 0 \Leftrightarrow \det(A + I_3) = 6$ and $\det(A - I_3) = 0$.

Problem 5.

Let $A \in M_2(\mathbb{R})$ such that $\mathrm{Tr}(A) > 2$. Prove that $A^n \neq I_2$, for every $n \in \mathbb{N}^*$.

Solution. Let's suppose that there is a number $n \in \mathbb{N}^*$ such that $A^n = I_2$. Then, $\det(A^n) = (\det(A))^n == 1$. Let $f_A(X) = \det(XI_2 - A) = X^2 - \mathrm{Tr}(A)X + \det(A)$ and let λ_1 and λ_2 be the eigenvalues of the matrix A. From $\lambda_1\lambda_2 = \det(A) = 1 \Rightarrow \lambda_1^n\lambda_2^n = 1$.

Because λ_1^n and λ_2^n are the eigenvalues of the matrix $A^n, A^n = I_2$ and the eigenvalues of the matrix I_2 are 1 and 1, we get that $\lambda_1^n = \lambda_2^n = 1$. Then $|\lambda_1^n| = |\lambda^2| = 1 \Rightarrow |\lambda_1| = |\lambda_2| = 1$.

Therefore: $2 = |\lambda_1^n| + |\lambda_2^n| = |\lambda_1|^n + |\lambda_2|^n = |\lambda_1| + |\lambda_2| \geq |\lambda_1 + \lambda_2| = |\mathrm{Tr}(A)| > 2$, which is false and so our assumption is false.

In conclusion: $A^n \neq I_2$, for every $n \in \mathbb{N}^*$.

Problem 6.

Let $A \in M_2(\mathbb{R})$ such that $\det(A^2 + 2A + I_2) = 0$. Prove that $\det(A) + \mathrm{Tr}(A) = -1$.

Solution. Let λ_1 and λ_2 be the eigenvalues of the matrix A. Then, the eigenvalues of the matrix $A^2 + 2A + I_2$ are $\lambda_1^2 + 2\lambda_1 + 1$ and $\lambda_2^2 + 2\lambda_2 + 1$ and so $\det(A^2 + 2A + I_2) = (\lambda_1^2 + 2\lambda_1 + 1)(\lambda_2^2 + + 2\lambda_2 + 1)$. WLOG, let $\lambda_1^2 + 2\lambda_1 + 1 = 0 \Leftrightarrow (\lambda_1 + 1)^2 = 0 \Leftrightarrow \lambda_1 = -1$.

Therefore, $\det(A) + \mathrm{Tr}(A) = \lambda_1\lambda_2 + (\lambda_1 + \lambda_2) = -\lambda_2 - 1 + \lambda_2 = -1$.

Problem 7.

Let $A \in M_n(\mathbb{C})$, $A = (a_{ij})_{i,j=\overline{1,n}}$ with its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Prove that:

$$\sum_{i,j=1}^{n} a_{ij}a_{ji} = \sum_{k=1}^{n} \lambda_k^2.$$

Solution. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A, then the eigenvalues of the matrix A^2 are $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ and $\mathrm{Tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2$.

If $B = A^2, B \in M_n(\mathbb{C}), B = (b_{ij})_{i,j=\overline{1,n}}$, then $b_{kk} = \sum_{p=1}^{n} a_{kp}a_{pk}$, $k = \overline{1,n}$.

Then, $\sum_{i,j=1}^{n} a_{ij}a_{ji} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}a_{ji}\right) = \sum_{i=1}^{n} b_{ii} = \mathrm{Tr}(A \cdot A) = \mathrm{Tr}(A^2) = \sum_{k=1}^{n} \lambda_k^2$.

Lemma 1. If $\in M_2(\mathbb{C})$, *then* $\det(A) = \frac{1}{2}\left[\left(\mathrm{Tr}(A)\right)^2 - \mathrm{Tr}(A^2)\right]$.

Proof. From Theorem 4 we get that $\mathrm{Tr}(A^2 - \mathrm{Tr}(A)A + \det(A)\, I_2) = \mathrm{Tr}(O_2) \Leftrightarrow \mathrm{Tr}(A^2) - \left(\mathrm{Tr}(A)\right)^2 + +2\det(A) = 0 \Leftrightarrow \det(A) = \frac{1}{2}\left[\left(\mathrm{Tr}(A)\right)^2 - \mathrm{Tr}(A^2)\right]$.

Lemma 2. If $A, B \in M_2(\mathbb{C})$ and $x \in \mathbb{C}$, then $\det(A + xB) = \det(A) + \left(\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)\right) \cdot x + + \det(B) \cdot x^2$.

Proof. From Lemma 1 we get that $\det(A + xB) = \frac{1}{2}\left[\left(\mathrm{Tr}(A + xB)\right)^2 - \mathrm{Tr}((A + xB)^2)\right] = \frac{1}{2}[(\mathrm{Tr}(A) + +\mathrm{Tr}(B) \cdot x)^2 - \mathrm{Tr}(A^2 + xAB + xBA + B^2x^2)] = \frac{1}{2}\left[\left(\mathrm{Tr}(A)\right)^2 + 2\mathrm{Tr}(A)\mathrm{Tr}(B) \cdot x + \left(\mathrm{Tr}(B)\right)^2 x^2 - - \mathrm{Tr}(A^2) - 2\mathrm{Tr}(AB) \cdot x - \mathrm{Tr}(B^2) \cdot x^2\right] = \frac{1}{2}\left[\left(\mathrm{Tr}(A)\right)^2 - \mathrm{Tr}(A^2)\right] + \left(\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)\right) \cdot x + + \frac{1}{2}\left[\left(\mathrm{Tr}(B)\right)^2 - \mathrm{Tr}(B^2)\right] x^2 = \det(A) + \left(\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)\right) \cdot x + \det(B) \cdot x^2$.

Lemma 3. If $A, B \in M_2(\mathbb{C})$, then $\det(A + B) + \det(A - B) = 2\left(\det(A) + \det(B)\right)$.

Proof. From Lemma 2 we have:

$$\begin{cases} x = 1 \Rightarrow \det(A+B) = \det(A) + \big(\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)\big) + \det(B) \\ x = -1 \Rightarrow \det(A-B) = \det(A) - \big(\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)\big) + \det(B) \end{cases} \overset{(+)}{\Longrightarrow} \text{Lemma 3.}$$

Problem 8.

Let $A, B \in M_2(\mathbb{R})$ such that $AB = BA$. Prove that $\det(A^2 + B^2) \geq 0$.

Solution. From $AB = BA$, it is obvious to see that $A^2 + B^2 = (A + iB)(A - iB)$, where $i^2 = -1$. Then, $\det(A^2 + B^2) = \det(A + iB) \cdot \det(A - iB)$.

Let $f(x) = \det(A + xB)$. From Lemma 2, we get that:

$$\begin{cases} x = i \Rightarrow \det(A+iB) = \det(A) + \big(\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)\big) \cdot i - \det(B) \ (1) \\ x = -i \Rightarrow \det(A-iB) = \det(A) - (\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB) \cdot i - \det(B) \ (2) \end{cases}.$$

$(1) \cdot (2) \Rightarrow \det(A + iB) \cdot \det(A - iB) =$
$[\det(A) - \det(B)]^2 + [\mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)]^2 \underbrace{\geq}_{A,B \in M_2(\mathbb{R})} 0.$

In conclusion, $\det(A^2 + B^2) \geq 0$.

Problem 9.

Let $A, B \in M_2(\mathbb{R})$ such that $\det(AB + BA) \leq 0$. Prove that $\det(A^2 + B^2) \geq 0$.

Solution. Let $X = A^2 + B^2$ and $Y = AB + BA$. Then $\begin{cases} X + Y = A^2 + B^2 + AB + BA = (A+B)^2 \\ X - Y = A^2 + B^2 - AB - BA = (A-B)^2 \end{cases}$.
From Lemma 3 we get that $\det(X + Y) + \det(X - Y) = 2(\det(X) + \det(Y)) \Leftrightarrow (\det(A + B))^2 +$ $+ (\det(A - B))^2 = 2(\det(A^2 + B^2) + \det(AB + BA))$.

If $\det(A^2 + B^2) < 0 \Rightarrow 2(\det(A^2 + B^2) + \det(AB + BA)) < 0$, which is impossible because $(\det(A + B))^2 + (\det(A - B))^2 \geq 0$.

In conclusion, $\det(A^2 + B^2) \geq 0$.

Problem 10.

Let $A, B, C \in M_n(\mathbb{R})$ such that $AB = BA, BC = CB, CA = AC$. Prove that:

$$\det(A^2 + B^2 + C^2 - AB - BC - CA) \geq 0.$$

Solution 1.

$A^2+B^2+C^2-AB-BC-CA=\left(\frac{1}{4}A^2+\frac{1}{4}B^2+C^2\right)+\left(\frac{1}{2}AB-BC-CA\right)+\left(\frac{3}{4}A^2+\frac{3}{4}B^2-\frac{3}{2}AB\right)==\left(\frac{1}{2}A+\frac{1}{2}B-C\right)^2+\left[\frac{\sqrt{3}}{2}(A-B)\right]^2.$

Let $X=\frac{1}{2}A+\frac{1}{2}B-C$ and $Y=\frac{\sqrt{3}}{2}(A-B)$. From $AB=BA, BC=CB, CA=AC \Rightarrow XY=YX$ and from Problem 8 we know that $\det(X^2+Y^2)\geq 0 \Leftrightarrow \det(A^2+B^2+C^2-AB-BC-CA)\geq 0.$

Observation. It is obvious that $X, Y \in M_n(\mathbb{R})$.

Solution 2. Let $\mathcal{E}\notin\mathbb{R}$, $\mathcal{E}^3=1$. Because $A, B, C \in M_n(\mathbb{R})$, it is easy to see that:

$$A^2+B^2+C^2-AB-BC-CA=(A+\mathcal{E}B+\mathcal{E}^2C)\cdot\left(\overline{A+\mathcal{E}B+\mathcal{E}^2C}\right)\ (1).$$

From $(1)\Rightarrow \det(A^2+B^2+C^2-AB-BC-CA)=|\det(A+\mathcal{E}B+\mathcal{E}^2C)|^2\geq 0.$

References:

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