ABOUT A RMM INEQUALITY

MARIN CHIRCIU - ROMANIA

If $ABCD$ - convex quadrilateral, $M \in Int(ABCD), F$ - area, p - semiperimeter, a, b, c, d - sides, then:

$$\frac{MA^4}{b} + \frac{MB^4}{c} + \frac{MC^4}{d} + \frac{MD^4}{a} \geq \frac{2F^2}{p}$$

Daniel Sitaru

Proof.

Lemma.
If $ABCD$ - convex quadrilateral, $M \in Int(ABCD), F$ - area, p - semiperimeter, a, b, c, d - sides, then:

$$MA^2 + MB^2 + MC^2 + MD^2 \geq 2F$$

Proof.
In ΔMAB we have:

$$MA^2 + MB^2 \geq 2MA \cdot MB \geq 2MA \cdot MB \cdot \sin(\angle AMB) = 4[MAB]$$

Writing the analogous relationship in the others three triangles $\Delta MBC, \Delta MCD, \Delta MDA$, we obtain:

$$MB^2 + MC^2 \geq 4[MBC], MC^2 + MD^2 \geq 4[MCD], MD^2 + MA^2 \geq 4[MDA].$$

Adding the four inequalities it follows:

$$2(MA^2 + MB^2 + MC^2 + MD^2) \geq 4F \Leftrightarrow MA^2 + MB^2 + MC^2 + MD^2 \geq 2F,$$

with equality if the angles $\angle AMB, \angle BMC, \angle CMD, \angle DMA$ have $90°$.
Let's get back to the main problem. □

Using Bergström's inequality and the Lemma, we obtain:

$$LHS = \frac{MA^4}{b} + \frac{MB^4}{c} + \frac{MC^4}{d} + \frac{MD^4}{a} \overset{\text{CS}}{\geq} \frac{MA^2 + MB^2 + MC^2 + MD^2}{a + b + c + d} \overset{\text{Lemma}}{\geq}$$

$$\overset{\text{Lemma}}{\geq} \frac{(2F)^2}{2p} = \frac{2F^2}{p} = RHS$$

Equality hold if and only if $ABCD$ - square and $M = O$. □

Remark.
The problem can be developed.
If $ABCD$ - convex quadrilateral, $M \in Int(ABCD), F$ - area, p - semiperimeter, a, b, c, d - sides and $n \in \mathbb{N}, n \geq 2$ then:

$$\frac{MA^{2n}}{b} + \frac{MB^{2n}}{c} + \frac{MC^{2n}}{d} + \frac{MD^{2n}}{a} \geq \left(\frac{F}{2}\right)^n \cdot \frac{8}{p}.$$

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$$\overset{\text{Lemma}}{\geq} \frac{(2F)^n}{4^{n-2} \cdot 2p} = \left(\frac{F}{2}\right)^n \cdot \frac{8}{p} = RHS$$

Equality holds if and only if $ABCD$ - square and $M = O$. □

Note.

For $n = 2$ we obtain the proposed problem by Daniel Sitaru in RMM 9/2022.

If $ABCD$ - convex quadrilateral, $M \in Int(ABCD), F$ - area, p - semiperimeter, a, b, c, d - sides, then:

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$$\frac{MA^{2n}}{b+c} + \frac{MB^{2n}}{c+d} + \frac{MC^{2n}}{d+a} + \frac{MD^{2n}}{a+b} \geq \left(\frac{F}{2}\right)^n \cdot \frac{4}{p}$$

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$$\overset{\text{Holder}}{\geq} \frac{(MA^2 + MB^2 + MC^2 + MD^2)^n}{4^{n-2}(2a + 2b + 2c + 2d)} \overset{\text{Lemma}}{\geq} \frac{(2F)^n}{4^{n-2} \cdot 4p} = \left(\frac{F}{2}\right)^n \cdot \frac{4}{p} = RHS.$$

Equality holds if and only if $ABCD$ - square and $M = O$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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