Formulating The Areas of A Circle and An Ellipse, and The Length of A Continuous Function Curve by Riemann Integral and Riemann Sums

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ABSTRACT. In this article, we are applying the Riemann Integral and Riemann Sum to formulate the areas of a circle and an ellipse, and the length of a continuous function curve over a closed and finite interval.

INTRODUCTION

An ellipse is a curve formed by the collection of all points (x, y) satisfying the elliptical equation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \ldots\ldots\ldots (1.1)$$

where (h, k) is the center of the ellipse; a and b are the lengths of semi-major and semi-minor axes. Circles are special cases of ellipses, in which we obtain a circle when $a = b$ in (1.1). In this paper, we would like to explain why the area of a circle of radius r is πr^2. How can we prove that the area of an ellipse with $2a$ and $2b$ as the lengths of its major and minor axes is πab ? We will prove these formulas by the application of Riemann Integral and Riemann Sum.

Besides that, we can find the length of a continuous curve f over some interval $[a, b]$ by the application of Riemann Integral and Riemann Sum. Since the approximation method that we use to emphasize here is how to connect infinitely many points on that curve, say these points (in the respective ordering) $X_0 = \big(a, f(a)\big), X_1, X_2, X_3, \ldots\ldots, X_{n-1}, X_n = (b, f(b))$ such that n approaches to ∞ and the length of X_kX_{k+1} for each k must approach to 0, but $X_kX_{k+1} > 0$. So, we must have that

$$\sum_{k=0}^{\infty} X_kX_{k+1}$$

is the length of a continuous curve f over the closed interval $[a, b]$.

RIEMANN INTEGRAL AND RIEMANN SUM

Let $f: [a, b] \to \mathbb{R}$ be a real valued function and continuous on the closed finite interval $[a, b]$. Let us partition this closed interval $[a, b]$ to be n closed and disjoint intervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots\ldots\ldots, [x_{n-1}, x_n]$ in such a way that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

We want to make $n \to \infty$ and the values of $x_m - x_{m-1}$ for each $m = 1, 2, 3, \ldots \ldots$ approach to 0 as near as possible. Then, the Riemann sum of f under this such partition of $[a, b]$ is

$$\lim_{n\to\infty} \sum_{m=1}^{n} f(x_m^*)\,(x_m - x_{m-1}) = \int_a^b f(x)\ dx \ldots \ldots \ldots (1.2)$$

for some $x_m^* \in [x_{m-1}, x_m]$, $m = 1, 2, 3, \ldots \ldots$. Moreover, let A_f be the area of the region under the curve f on the interval $[a, b]$. Then,

$$A_f = \lim_{n\to\infty} \sum_{m=1}^{n} |f(x_m^*)|\,(x_m - x_{m-1}) = \int_a^b |f(x)|\,dx \ldots \ldots \ldots (1.3)$$

Here, the right hand sides of the equations (1.2) and (1.3) are called Riemann Integral. While, the left hand sides are called Riemann Sum.

Theorem 1:

The area of a circle with its radius length r is πr^2.

Proof :

Let us name Γ_1 to this circle. Without loss of generality, let the original point $(0, 0)$ be the center of Γ_1 such that the equation of Γ_1 is $x^2 + y^2 = r^2$. Now we consider the quarter part of Γ_1 constrained by positive x-axis and positive y-axis as shown in the diagram below.

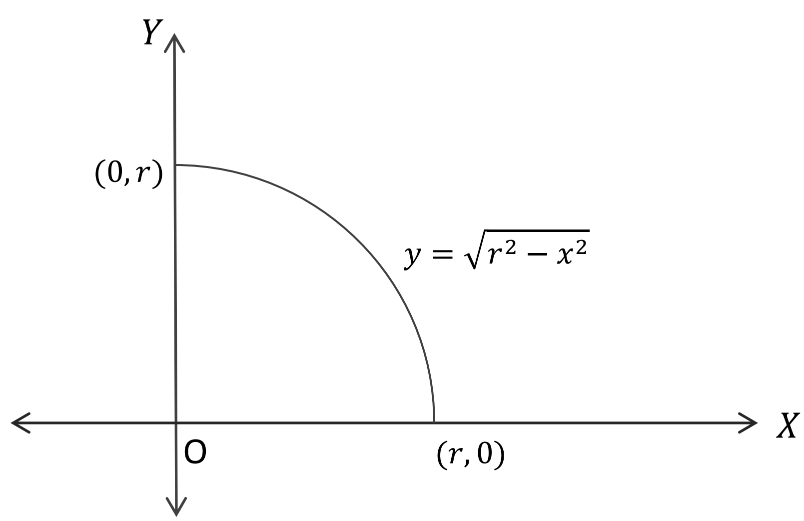

Figure $G.1$

Note that all the points on Γ_1 in the first quadrant are formed by the equation $y = \sqrt{r^2 - x^2}$ and form the quarter circle as shown in figure $G.1$. Then, the area of this quarter circle is

$$A_1 = \int_0^r \sqrt{r^2 - x^2}\, dx \dots\dots\dots (1.4)$$

Substituting $x = r\sin\theta$ to (1.4), we have $dx = r\cos\theta\, d\theta$ and

$$A_1 = \int_0^{\frac{\pi}{2}} (r\cos\theta)^2\ d\theta = \frac{r^2}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta + 1\ d\theta = \frac{r^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{\pi r^2}{4} \dots\dots\dots (1.5)$$

Consequently, the area of a full circle Γ_1 is $4A_1 = \pi r^2$. □

Theorem 2:

The area of an ellipse where $2a$ and $2b$ are the lengths of its major and minor axes is πab.

Proof :

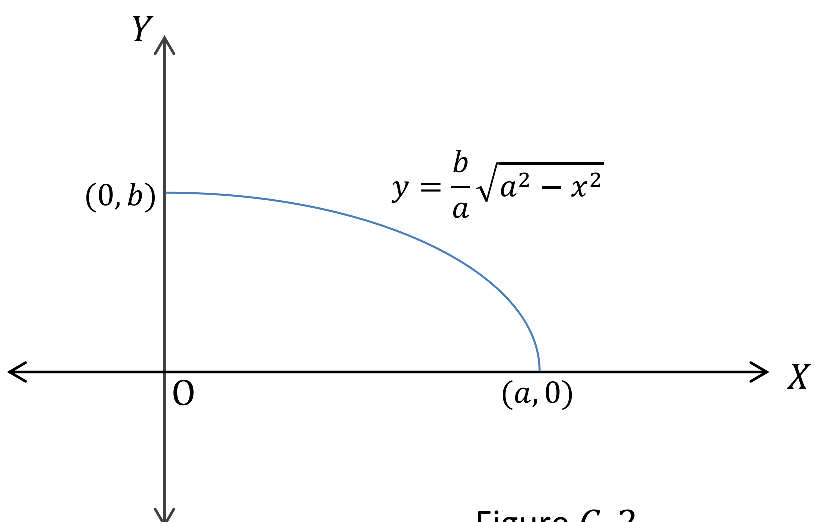

Figure $G.2$

Let Γ_2 be its name of the full ellipse. WLOG, this ellipse has the center $(0,0)$. Then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of Γ_2. Note that all points on Γ_2 that are positioned in the first quadrant are formed by the equation $y = \frac{b}{a}\sqrt{a^2 - x^2}$. As we can see in figure $G.2$, the area of a quarter ellipse of Γ_2 is

$$A_2 = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\ dx \ldots\ldots\ldots (1.6)$$

We substitute $x = a\sin\theta$ to (1.6) such that $dx = a\cos\theta\, d\theta$,and we obtain

$$A_2 = \frac{b}{a}\int_0^{\frac{\pi}{2}} a^2(\cos\theta)^2\ d\theta = \frac{ab}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta + 1\ d\theta = \frac{ab}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{\pi ab}{4} \ldots\ldots\ldots (1.7)$$

Hence, the area of ellipse Γ_2 is $4A_2 = \pi ab$. □

Theorem 3:

Let $y = f(x)$ be a real-valued function and continuous on the closed interval $[a, b]$ where a and b are finite. Then the length of the curve of f over the interval $[a, b]$ is formulated by

$$\int_a^b \sqrt{1 + \left(\frac{df(x)}{dx}\right)^2}\ dx$$

Proof :

Let $X_0(a_0, f(a_0)), X_1(a_1, f(a_1)), X_2(a_2, f(a_2)), \ldots\ldots, X_n(a_n, f(a_n))$ be the points on the curve f so that $a = a_0 < a_1 < a_2 < \cdots < a_n = b$. By taking $n \to \infty$ so that $\forall k \in \mathbb{N}, a_k - a_{k-1} \to 0$, i.e. the length of $X_k X_{k-1}$ approaches to 0 for all k. Then we obtain that the length of the curve of f over the interval $[a, b]$ is

$$\lim_{n\to\infty} \sum_{k=1}^{n} X_k X_{k-1}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(a_k - a_{k-1})^2 + \left(f(a_k) - f(a_{k-1})\right)^2}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (a_k - a_{k-1}) \sqrt{1 + \left(\frac{f(a_k) - f(a_{k-1})}{a_k - a_{k-1}}\right)^2}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (a_k - a_{k-1})\sqrt{1 + (f'(a_{k-1}))^2}$$

$$= \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx \dots \dots \dots (1.8)$$

where $f'(x)$ is the first derivation of f under x. □

REFERENCES

1. https://simple.wikipedia.org/wiki/Ellipse
2. https://mathworld.wolfram.com/Circle.html
3. https://www.mit.edu/~ehjin/files/Integration_And_Summation.pdf (Chapter 2 : Riemann Sums, page 4)
4. https://mathinsight.org/length_curves_refresher