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NEW TRIANGLE INEQUALITIES WITH BROCARD'S ANGLE

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ABSTRACT : In this paper are created new inequalities in triangle using Brocard's angle.

Panaitopol's Inequality :

In any triangle ABC, we have

$$\frac{m_a}{h_a} \le \frac{R}{2r} \text{ (and analogs)}, \qquad (1)$$

with equality if and only if the triangle ABC is equilateral.

Proof :

Considering the origin of the complex plane at the circumcenter of triangle ABC and

let z_1, z_2, z_3 be the coordinates of points A, B, C, respectively.

Using the formulas $h_a = \frac{2S}{a}$ and

$S = pr$, the desired inequality (1) can be rewritten as follows,

$am_a \le Rp$. We have :

$$2am_a = 2|z_2 - z_3|.\left|z_1 - \frac{z_2 + z_3}{2}\right| = |(z_2 - z_3)(2z_1 - z_2 - z_3)|$$

$$= |z_1(z_2 - z_3) + z_3(z_3 - z_1) + z_2(z_1 - z_2)|$$
$$\le |z_1|.|z_2 - z_3| + |z_3|.|z_3 - z_1| + |z_2|.|z_1 - z_2|$$

$= Ra + Rb + Rc = 2Rp,$

This completes the proof of (1). Equality holds if and only if the triangle ABC is equilateral.

Now, in triangle ABC, we have the following relations (see, Bogdan Fuștei – *About Nagel's and Gergonnes's cevian* – www.ssmrmh.ro),

$$2r_br_c = h_a(r_b + r_c) \quad \text{(and analogs)},$$

$$4{m_a}^2 = 4r_br_c + (b - c)^2 = 2h_a(r_b + r_c) + (b - c)^2 \quad \text{(and analogs)}$$

On the other hand, using the formulas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{p-a}$ (and analogs), we have

$$r_b + r_c = \frac{S}{p-b} + \frac{S}{p-c} = \frac{S(2p-b-c)}{(p-b)(p-c)} = \frac{4S.a}{(a-b+c)(a+b-c)} = \frac{4S.a}{a^2-(b-c)^2},$$

$$r_b + r_c - 2h_a = \frac{4S.a}{a^2-(b-c)^2} - 2h_a = \frac{2h_a.a^2}{a^2-(b-c)^2} - 2h_a = \frac{2h_a(b-c)^2}{a^2-(b-c)^2}.$$

Using these relations and identities, we have

$$(r_b+r_c)^2 - 4{m_a}^2 = (r_b+r_c)(r_b+r_c-2h_a) - (b-c)^2$$
$$= \frac{2h_a(r_b+r_c)(b-c)^2}{a^2-(b-c)^2} - (b-c)^2.$$

and since we have, $2h_a(r_b+r_c) = 4r_br_c = 4p(p-a) = (b+c)^2 - a^2$, and,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad rr_a = (p-b)(p-c), \text{ then}$$

$$(r_b+r_c)^2 - 4{m_a}^2 = \left(\frac{(b+c)^2-a^2}{a^2-(b-c)^2} - 1\right)(b-c)^2 = \frac{2(b^2+c^2-a^2)(b-c)^2}{4(p-b)(p-c)}$$

$$= \frac{2.2bc\cos A\,(b-c)^2}{4rr_a} = \frac{bc\cos A\,(b-c)^2}{rr_a}.$$

$$\Rightarrow (r_b+r_c)^2 - 4{m_a}^2 = \frac{bc\cos A\,(b-c)^2}{rr_a} \text{ (and analogs).}$$

Using the relation $\sin^2\frac{A}{2} = \frac{(p-b)(p-c)}{bc} = \frac{rr_a}{bc}$, we get

$$\boldsymbol{(r_b+r_c)^2 - 4{m_a}^2 = \frac{\cos A\,(b-c)^2}{\sin^2\frac{A}{2}}} \textbf{ (and analogs)} \qquad \textbf{(2)}$$

Using this identity and the formulas $a = 4R\,sin\frac{A}{2}cos\frac{A}{2}$ and $r_b + r_c = 4R\cos^2\frac{A}{2}$, we get

$$r_b + r_c - \frac{4{m_a}^2}{r_b+r_c} = \frac{\cos A\,(b-c)^2}{4R\cos^2\frac{A}{2}.\sin^2\frac{A}{2}} = \frac{4R\cos A\,(b-c)^2}{a^2},$$

then we have the following new identity

$$\boldsymbol{r_b + r_c - \frac{4m_a{}^2}{r_b + r_c} = 4R\cos A\left(\frac{b-c}{a}\right)^2 \quad (\text{and analogs})} \tag{3}$$

Now, by the arithmetic – geometric mean inequality, we have

$$\boldsymbol{r_b + r_c + \frac{4m_a{}^2}{r_b + r_c} \geq 2\sqrt{(r_b + r_c).\frac{4m_a{}^2}{r_b + r_c}} = 4m_a} \tag{4}$$

From the results (3) and (4), we have

$$\boldsymbol{2(r_b + r_c) \geq 4m_a + 4R\cos A\left(\frac{b-c}{a}\right)^2.}$$

Which gives us the following inequality, in any triangle ABC, we have

$$\boldsymbol{\frac{r_b + r_c}{2} \geq m_a + R\cos A\left(\frac{b-c}{a}\right)^2 \quad (\text{and analogs}).}$$

Again, from the results (2) and (3), we have

$$\boldsymbol{4R\cos A\left(\frac{b-c}{a}\right)^2 + \frac{8m_a{}^2}{r_b + r_c} \geq 4m_a,}$$

or,

$$\boldsymbol{R\cos A\left(\frac{b-c}{a}\right)^2 + \frac{2m_a{}^2}{r_b + r_c} \geq m_a} \tag{5}$$

On the other hand, using the relations $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ and

$\cos^2\frac{A}{2} = \frac{1 + \cos A}{2}$, we have

$$4m_a{}^2 = 2(b^2 + c^2) - a^2 = (b-c)^2 + 2bc + (b^2 + c^2 - a^2)$$
$$= (b-c)^2 + 2bc + 2bc\cos A$$

$$= (b-c)^2 + 2bc(1 + \cos A) = (b-c)^2 + 4bc\cos^2\frac{A}{2}.$$

and by the formulas $r_b + r_c = 4R\cos^2\frac{A}{2}$ and $h_a = \frac{bc}{2R}$, we obtain

$$\frac{2{m_a}^2}{r_b+r_c}=\frac{4bc\cos^2\frac{A}{2}+(b-c)^2}{2.4R\cos^2\frac{A}{2}}=h_a+\frac{(b-c)^2}{8R\cos^2\frac{A}{2}}.$$

Note that $a=4R\,sin\frac{A}{2}cos\frac{A}{2}$, we obtain the following identity

$$\boldsymbol{\frac{2{m_a}^2}{r_b+r_c}=h_a+2R\sin^2\frac{A}{2}\left(\frac{b-c}{a}\right)^2}\quad \textbf{(and analogs)}$$

Replacing this identity in (5), we get

$$\boldsymbol{h_a+R\left(\cos A+2\sin^2\frac{A}{2}\right)\left(\frac{b-c}{a}\right)^2\geq m_a}.$$

Using the relation

$2\sin^2\frac{A}{2}=1-\cos A$, we obtain the following inequality, in any triangle ABC

$$\boldsymbol{m_a\leq h_a+R\left(\frac{b-c}{a}\right)^2}\quad \textbf{(and analogs)}\qquad (6)$$

By the results (3) and (4), the equality in (6) holds if

$$r_b+r_c=\frac{4{m_a}^2}{r_b+r_c}\Leftrightarrow\ \cos A\left(\frac{b-c}{a}\right)^2=0,\text{i.e. } b=c \text{ or } A=\frac{\pi}{2}.$$

From this result, we have

$$\boldsymbol{\sqrt{\frac{m_a-h_a}{R}}\leq\frac{|b-c|}{a}},$$

Then we have, in any triangle ABC, the follwing inequality

$$\boldsymbol{a\sqrt{\frac{m_a-h_a}{R}}\leq|b-c|}\quad \textbf{(and analogs)}\qquad (7)$$

Adding this inequality with similar ones and using the identity

$$|a-b|+|b-c|+|c-a|=2\big(max(a,b,c)-min(a,b,c)\big),$$

we obtain the following inequality, in any triangle ABC

$$\boldsymbol{\frac{1}{2}\left(a\sqrt{\frac{m_a-h_a}{R}}+b\sqrt{\frac{m_b-h_b}{R}}+c\sqrt{\frac{m_c-h_c}{R}}\right)\le max(a,b,c)-min(a,b,c).} \qquad (8)$$

From the inequality (6), we have

$$\frac{m_a}{h_a}\le 1+\frac{R(b-c)^2}{a^2h_a},$$

and by the formulas $R=\frac{abc}{4S}$ and $ah_a=2S$, we obtain

$$\boldsymbol{\frac{m_a}{h_a}\le 1+\frac{bc(b-c)^2}{8S^2}\ (\text{and analogs})} \qquad (9)$$

Since m_a, m_b, m_c can be the sides of triangle with area

$$S_m=\frac{3S}{4}, \text{median } \overline{m_a}=\frac{3a}{4}\ (\text{and analogs}) \text{ altitude}$$

$$\overline{h_a}=\frac{2S_m}{m_a}=\frac{3S}{2m_a}\ (\text{and analogs}), \text{then by using the inequality (9) in } \Delta m_am_bm_c,$$

we obtain

$$\boldsymbol{\frac{m_a}{h_a}\le 1+\frac{2m_bm_c(m_b-m_c)^2}{9S^2}\ (\text{and analogs})} \qquad (10)$$

Now, in any triangle ABC, we have the following relation

$$4{m_a}^2={n_a}^2+{g_a}^2+2r_br_c \quad (\text{and analogs})$$

Using this relation and the identity (2), we have

$${r_b}^2+{r_c}^2=(4{m_a}^2-2r_br_c)+[(r_b+r_c)^2-4{m_a}^2]={n_a}^2+{g_a}^2+\frac{\cos A\,(b-c)^2}{\sin^2\frac{A}{2}}.$$

Then we obtain the following identity

$${r_b}^2+{r_c}^2={n_a}^2+{g_a}^2+\frac{\cos A\,(b-c)^2}{\sin^2\frac{A}{2}} \quad (\text{and analogs})$$

Which gives us the following inequality, in any non– obtuse triangle ABC, we have

$$\boldsymbol{r_b}^2 + \boldsymbol{r_c}^2 \geq \boldsymbol{n_a}^2 + \boldsymbol{g_a}^2 \quad \textbf{(and analogs)} \qquad \textbf{(11)}$$

with equality if $b = c$ or $A = \frac{\pi}{2}$.

In this part, we will prove the following inequality, in any triangle ABC, we have

$$\frac{\boldsymbol{m_b}}{\boldsymbol{h_c}} + \frac{\boldsymbol{m_c}}{\boldsymbol{h_b}} \leq \frac{\boldsymbol{R}}{\boldsymbol{r}} \quad \textbf{(and analogs)} \qquad \textbf{(12)}$$

Using the result (6) and the formulas $\mathrm{R} = \frac{\mathrm{abc}}{4\mathrm{S}}, h_a = \frac{2S}{a}$ (and analogs), we have

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \left(\frac{h_b}{h_c} + \frac{R(c-a)^2}{h_c b^2}\right) + \left(\frac{h_c}{h_b} + \frac{R(a-b)^2}{h_b c^2}\right)$$
$$= \left(\frac{c}{b} + \frac{c^2 a(c-a)^2}{8bS^2}\right) + \left(\frac{b}{c} + \frac{b^2 a(a-b)^2}{8cS^2}\right).$$

Then we have,

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \leq \frac{8S^2(b^2+c^2) + c^3 a(c-a)^2 + b^3 a(a-b)}{8bcS^2}$$

So to prove (12) it suffices to prove that

$$8S^2(b^2+c^2) + c^3 a(c-a)^2 + ab^3(a-b) \leq 8bcS^2 . \frac{R}{r}$$

and by the formulas $4RS = abc, S = pr$ and the following identity

$$16S^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4),$$

the last inequality is equivalent to

$$[2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)](b^2+c^2) + 2c^3 a(c-a)^2 + 2ab^3(a-b)$$
$$\leq 2ab^2c^2(a+b+c).$$

which, after expanding and simplifying, equivalent to

$$(b^2+c^2)a^4 - 2(b^3+c^3)a^3 + 2(b^4 - b^2c^2 + c^4)a^2 - 2(b+c)(b^2+bc+c^2)(b-c)^2 a$$
$$+(b^2+c^2)(b^2-c^2)^2 \geq 0$$

or,

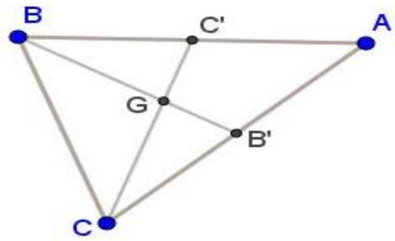
$$(b^2+c^2)\left(a^2-\frac{(b^3+c^3)a}{b^2+c^2}\right)^2+\frac{(b^2+bc+c^2)^2}{b^2+c^2}.\left(a-\frac{(b^2+c^2)(b+c)}{b^2+bc+c^2}\right)^2(b-c)^2\geq 0,$$

which is true and the proof of (12) is complete. Equality holds if b = c.

Now, we will prove the following inequality, in any triangle ABC, we have

$$\mathbf{2\frac{m_a}{h_a}\leq\frac{m_b}{h_c}+\frac{m_c}{h_b}}\quad\textbf{(and analogs)}\qquad\textbf{(13)}$$

Let B', C' be the midpoints of AC, AB, and let G be the centroid of triangle ABC.



By Ptolemy's inequality in the quadrilateral $AB'GC'$, we have

$$B'C'.GA\leq AC'.GB'+AB'.GC',$$

or more explicitly,

$$\frac{a}{2}.\frac{2m_a}{3}\leq\frac{c}{2}.\frac{m_b}{3}+\frac{b}{2}.\frac{m_c}{3}$$

which is equialent to

$$2\frac{m_a}{h_a}\leq\frac{m_b}{h_c}+\frac{m_c}{h_b}.$$

From the results (12) and (13), we obtain the following refinement

of Panaitopol's inequality :

$$\mathbf{2\frac{m_a}{h_a}\leq\frac{m_b}{h_c}+\frac{m_c}{h_b}\leq\frac{R}{r}}\quad\textbf{(and analogs)}\qquad\textbf{(14)}$$

Now, using the formula

$h_a = \frac{bc}{2R}$ (and analogs), we obtain the equivalent expression of (9) :

$$\frac{m_b}{b} + \frac{m_c}{c} \le \frac{a}{2r} \quad \textbf{(and analogs)} \tag{15}$$

We have, in any triangle ABC the following relation

$$p^2 = {n_a}^2 + 2r_a h_a \quad \text{(and analogs)}$$

This relation can be rewritten as follows

$$\frac{p + n_a}{h_a} = \frac{2r_a}{p - n_a} \quad \text{(and analogs)}$$

By this relation and the formulas $h_a = \frac{2S}{a}$, $S = pr$, we have

$$\frac{a}{2r} = \frac{p}{h_a} = \frac{p + n_a}{h_a} - \frac{n_a}{h_a} = \frac{2r_a}{p - n_a} - \frac{n_a}{h_a} \quad \text{(and analogs)}$$

From the result (15), we obtain

$$\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a} \le \frac{2r_a}{p - n_a} \quad \textbf{(and analogs)} \tag{16}$$

or,

$$\left(\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a}\right)^{-1} \ge \frac{p - n_a}{2r_a} \quad \textbf{(and analogs)}$$

Adding this inequality with similar ones and using the identity

$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$, we obtain the following inequality

$$\sum_{cyc} \left(\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a}\right)^{-1} \ge \frac{p}{2r} - \frac{1}{2}\left(\frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c}\right) \tag{17}$$

From the inequality (16), we have

$$\frac{1}{2}(p - n_a)\left(\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a}\right) \le r_a \quad \textbf{(and analogs)}$$

Adding this inequality with similar ones and using the identity

$r_a + r_b + r_c = 4R + r$, we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p - n_a)\left(\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a}\right) \le 4R + r \qquad (18)$$

We have the following identity

$$\frac{{g_a}^2}{h_a} + \frac{{g_b}^2}{h_b} + \frac{{g_c}^2}{h_c} = 2R + 5r.$$

Then we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p - n_a)\left(\frac{m_b}{b} + \frac{m_c}{c} + \frac{n_a}{h_a}\right) \le \sum_{cyc}\frac{{g_a}^2}{h_a} + 2(R - 2r) \qquad (19)$$

Again, by the relation $p^2 = {n_a}^2 + 2r_a h_a$ (and analogs), we have

$$\frac{p - n_a}{h_a} = \frac{2r_a}{p + n_a} \quad \text{(and analogs)}$$

By this relation and the formulas $h_a = \frac{2S}{a}, S = pr$, we have

$$\frac{a}{2r} = \frac{p}{h_a} = \frac{p - n_a}{h_a} + \frac{n_a}{h_a} = \frac{2r_a}{p + n_a} + \frac{n_a}{h_a} \quad \text{(and analogs)}$$

From the result (15), we obtain

$$\frac{m_b}{b} + \frac{m_c}{c} - \frac{n_a}{h_a} \le \frac{2r_a}{p + n_a} \quad \text{(and analogs)} \qquad (20)$$

or,

$$\frac{1}{2}(p + n_a)\left(\frac{m_b}{b} + \frac{m_c}{c} - \frac{n_a}{h_a}\right) \le r_a \quad \text{(and analogs)}$$

Adding this inequality with similar ones and using the identity

$r_a + r_b + r_c = 4R + r$, we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p + n_a)\left(\frac{m_b}{b} + \frac{m_c}{c} - \frac{n_a}{h_a}\right) \le 4R + r \qquad (21)$$

Using the following identity

$$\frac{g_a{}^2}{h_a}+\frac{g_b{}^2}{h_b}+\frac{g_c{}^2}{h_c}=2R+5r.$$

we obtain the following inequality

$$\frac{1}{2}\sum_{cyc}(p+n_a)\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\le\sum_{cyc}\frac{g_a{}^2}{h_a}+2(R-2r) \tag{22}$$

In this part, we will prove the following inequality, in any non– obtuse triangle ABC, we have

$$\frac{2m_a{}^2}{r_b+r_c}\le\frac{b^2+c^2}{4R}\quad\text{(and analogs)} \tag{23}$$

By the formulas $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ and $\cos^2\dfrac{A}{2}=\dfrac{1+\cos A}{2}$, we have the two relations

$$4m_a{}^2=2(b^2+c^2)-a^2=b^2+c^2+(b^2+c^2-a^2)=b^2+c^2+2bc\cos A$$

$$r_b+r_c=4R\cos^2\frac{A}{2}=2R(1+\cos A).$$

Using these relations, the inequality (23) is succesively equivalent to

$$4m_a{}^2\le\frac{r_b+r_c}{2R}(b^2+c^2)\ \Leftrightarrow\ b^2+c^2+2bc\cos A\le(1+\cos A)(b^2+c^2)\ \Leftrightarrow 0\le(b-c)^2\cos A,$$

which is true because $\cos A\ge 0$. Equality in (23) holds if $b=c$ or $A=\dfrac{\pi}{2}$.

Using the identity (3), we obtain the following inequality, in any non– obtuse triangle ABC, we have

$$r_b+r_c\le\frac{b^2+c^2}{2R}+4R\cos A\left(\frac{b-c}{a}\right)^2\quad\text{(and analogs)} \tag{24}$$

By the inequality (23) and the formulas $h_a=\dfrac{bc}{2R}, s_a=\dfrac{2bcm_a}{b^2+c^2}$, we obtain the following inequality, in any non– obtuse triangle ABC holds

$$\frac{r_b + r_c}{2} \geq \frac{m_a s_a}{h_a} \geq m_a \text{ (and analogs)} \qquad (25)$$

In this part, we will prove the following inequality chains, in any triangle ABC,

if ω − Brocard's angle, we have

$$\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \geq \frac{1}{\sin\omega} \geq max\left\{2\frac{m_b + m_c}{h_b + h_c}, \frac{m_b}{h_b} + \frac{m_c}{h_c}\right\} \text{ (and analogs)} \qquad (26)$$

$$\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \geq max\left\{\frac{1}{\sin\omega}, 2\frac{m_a}{h_a}\right\} \geq \frac{b}{c} + \frac{c}{b} \geq \frac{c+a}{a+b} + \frac{a+b}{c+a} \text{ (and analogs)} \qquad (27)$$

$$\frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b} \geq max\left\{\frac{1}{\sin\omega}, 2\frac{m_a}{h_a}\right\} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \geq \frac{m_a + m_b}{m_b + m_c} + \frac{m_b + m_c}{m_a + m_b} \text{ (and analogs)} \qquad (28)$$

Lemma 1. In triangle ABC, ω − Brocard's angle, we have

$$\frac{m_b}{h_c} + \frac{m_c}{h_b} \geq \frac{1}{\sin\omega} \text{ (and analogs)} \qquad (29)$$

Proof. Using the known median formulae we have

$$4cm_b = \sqrt{4c^2(2c^2 + 2a^2 - b^2)}$$

$$= \sqrt{(3c^2 + a^2 - b^2)^2 + 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)},$$

by the identity $16S^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$, we get

$$\frac{m_b}{h_c} = \frac{cm_b}{2S} = \frac{\sqrt{(3c^2 + a^2 - b^2)^2 + (4S)^2}}{8S}.$$

Similarly, we get

$$\frac{m_c}{h_b} = \frac{\sqrt{(3b^2 + a^2 - c^2)^2 + (4S)^2}}{8S}.$$

By the triangle inequality, we have

$$\sqrt{x^2 + y^2} + \sqrt{z^2 + t^2} \geq \sqrt{(x+z)^2 + (y+t)^2},$$

for all real numbers x, y, z, t, with equality if $xt = yz$.

Using this inequality, the formula $\sin\omega = \dfrac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ and the identity

$16S^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$, we obtain

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}=\frac{\sqrt{(3c^2+a^2-b^2)^2+(4S)^2}+\sqrt{(3b^2+a^2-c^2)^2+(4S)^2}}{8S}$$

$$\geq \frac{\sqrt{(2a^2+2b^2+2c^2)^2+(8S)^2}}{8S}=\frac{\sqrt{16(a^2b^2+b^2c^2+c^2a^2)}}{8S}=\frac{1}{\sin\omega},$$

which completes the proof of (29).

The equality in (29) holds if $(3c^2+a^2-b^2).4S=(3b^2+a^2-c^2).4S$, i.e. $b=c$.

Lemma 2. In triangle ABC, we have

$$\mathbf{4m_bm_c \leq 2a^2+bc} \tag{30}$$

Proof. Using the known median formulae we have

$$(4m_bm_c)^2=(2c^2+2a^2-b^2)(2b^2+2a^2-c^2)$$
$$=4a^4+2a^2(b^2+c^2)-(2b^4-5b^2c^2+2c^4)$$

$$=(2a^2+bc)^2+2a^2(b^2+c^2-2bc)-2(b^4-2b^2c^2+c^4)$$

$$=(2a^2+bc)^2-2[(b+c)^2-a^2](b-c)^2 \leq (2a^2+bc)^2,$$

the last inequality is true by $b+c>a$. Equality holds if b = c.

Lemma 3. In triangle ABC, $\boldsymbol{\omega}$ − Brocard's angle, we have

$$\frac{\mathbf{1}}{\sin\boldsymbol{\omega}} \geq 2\frac{\boldsymbol{m_b+m_c}}{\boldsymbol{h_b+h_c}} \quad \textbf{(and analogs)} \tag{31}$$

Proof. By the formulas $h_a=\dfrac{2S}{a}$ (and analogs) and $\sin\omega=\dfrac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$, the

inequality (31) can be rewritten as follows

$$\sqrt{a^2b^2+b^2c^2+c^2a^2} \geq \frac{2bc(m_b+m_c)}{b+c}.$$

Using the known median formulae and the inequality (30), we have

$$\left(2bc(m_b+m_c)\right)^2 = (bc)^2(4{m_b}^2+4{m_c}^2+2.4m_bm_c)$$
$$\le (bc)^2[(2c^2+2a^2-b^2)+(2b^2+2a^2-c^2)+2(2a^2+bc)] = (bc)^2[8a^2+(b+c)^2]$$

By the AM − GM ineuality, we have

$$8(bc)^2 = 4bc.2bc \le (b+c)^2(b^2+c^2),$$

Then

$$\left(\frac{2bc(m_b+m_c)}{b+c}\right)^2 \le \frac{(b+c)^2(b^2+c^2)a^2+(bc)^2(b+c)^2}{(b+c)^2} = a^2b^2+b^2c^2+c^2a^2,$$

which completes the proof of (31). Equality holds if b = c.

Lemma 4. In triangle ABC, ω − Brocard's angle, we have

$$\frac{1}{\sin\omega} \ge \frac{m_b}{h_b}+\frac{m_c}{h_c} \quad \text{(and analogs)} \tag{32}$$

Proof. By the formulas $h_a = \frac{2S}{a}$ (and analogs) and $\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$, the

inequality (32) can be rewritten as follows

$$\sqrt{a^2b^2+b^2c^2+c^2a^2} \ge bm_b+cm_c.$$

By the following inequality $(x+y)^2$
$\le 2(x^2+y^2)$, for all real numbers x, y, and using the known

median formulae, we have

$$(bm_b+cm_c)^2 \le 2(b^2{m_b}^2+c^2{m_c}^2) = \frac{b^2(2c^2+2a^2-b^2)+c^2(2b^2+2a^2-c^2)}{2}$$
$$= (a^2b^2+b^2c^2+c^2a^2) - \frac{(b^2-c^2)^2}{2} \le a^2b^2+b^2c^2+c^2a^2,$$

which completes the proof of (32). Equality holds if b = c.

From the inequalities (12), (29), (31) and (32) yields the desired inequality chain (26).

Lemma 5. In triangle ABC, ω − Brocard's angle, we have

$$\frac{1}{\sin\omega} \geq \frac{b}{c} + \frac{c}{b} \tag{33}$$

Proof. By the formula $\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$, the inequality (33) can be rewritten as

$$bc\sqrt{a^2b^2+b^2c^2+c^2a^2} \geq 2S(b^2+c^2).$$

Using the identity $16S^2 = 2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)$, we have

$$4\left(2S(b^2+c^2)\right)^2 = [2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)](2b^2c^2+b^4+c^4)$$

$$= 4b^2c^2(a^2b^2+b^2c^2+c^2a^2) - a^4(b^2+c^2)^2 + 2(b^4+c^4)(a^2b^2+c^2a^2) - (b^4+c^4)^2$$

$$= 4b^2c^2(a^2b^2+b^2c^2+c^2a^2) - [a^2(b^2+c^2) - (b^4+c^4)]^2$$

$$\leq 4b^2c^2(a^2b^2+b^2c^2+c^2a^2),$$

which completes the proof of (33).

By Tereshin's inequality, we have

$$m_a \geq \frac{b^2+c^2}{4R} \quad \text{(and analogs)}$$

and by the formula $h_a = \frac{bc}{2R}$, we obtain

$$2\frac{m_a}{h_a} \geq \frac{b^2+c^2}{bc} = \frac{b}{c} + \frac{c}{b} \quad \text{(and analogs)} \tag{34}$$

Lemma 6. If a, b, c be positive real numbers, then we have

$$\frac{b}{c} + \frac{c}{b} \geq \frac{c+a}{a+b} + \frac{a+b}{c+a}. \tag{35}$$

Proof. The desired inequality is successively equivalent to

$$\frac{b}{c} - \frac{a+b}{c+a} \geq \frac{c+a}{a+b} - \frac{c}{b} \Leftrightarrow \frac{a(b-c)}{c(c+a)} \geq \frac{a(b-c)}{b(a+b)} \Leftrightarrow \frac{a(b-c)[b(a+b)-c(c+a)]}{bc(c+a)(a+b)} \geq 0$$

$$\Leftrightarrow \frac{a(a+b+c)(b-c)^2}{bc(c+a)(a+b)} \geq 0,$$

which is true and the proof of (35) is complete. Equality holds if $b = c$.

From the inequalities (12), (13), (29), (33), (34) and (35) yields the desired inequality chain (27).

Since m_a, m_b, m_c can be the sides of triangle with area $S_m = \frac{3S}{4}$,

median $\overline{m_a} = \frac{3a}{4}$ (and analogs) altitude

$\overline{h_a} = \frac{2S_m}{m_a} = \frac{3S}{2m_a}$ (and analogs), and by the formula $\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$,

and the identity ${m_a}^2{m_b}^2 + {m_b}^2{m_c}^2 + {m_c}^2{m_a}^2 = \frac{9}{16}(a^2b^2+b^2c^2+c^2a^2)$, then we have

$$\frac{\overline{m_a}}{\overline{h_a}} = \frac{m_a}{h_a} \text{ and } \sin\omega_m = \frac{2S_m}{\sqrt{{m_a}^2{m_b}^2 + {m_b}^2{m_c}^2 + {m_c}^2{m_a}^2}} = \frac{2S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}} = \sin\omega.$$

Applying the inequalities (33), (34) and (35) in $\Delta m_a m_b m_c$ and using the previous results, we obtain the following inequalities

$$\boldsymbol{\frac{1}{\sin\omega} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \quad \text{(and analogs)}} \tag{36}$$

$$\boldsymbol{2\frac{m_a}{h_a} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \quad \text{(and analogs)}} \tag{37}$$

$$\boldsymbol{\frac{m_b}{m_c} + \frac{m_c}{m_b} \geq \frac{m_a+m_b}{m_b+m_c} + \frac{m_b+m_c}{m_a+m_b} \quad \text{(and analogs)}} \tag{38}$$

From the inequalities (9), (10), (26), (33), (34) and (35)

yields the desired inequality chain (28).

Reference :

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