RMM - Geometry Marathon 1101 - 1200

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

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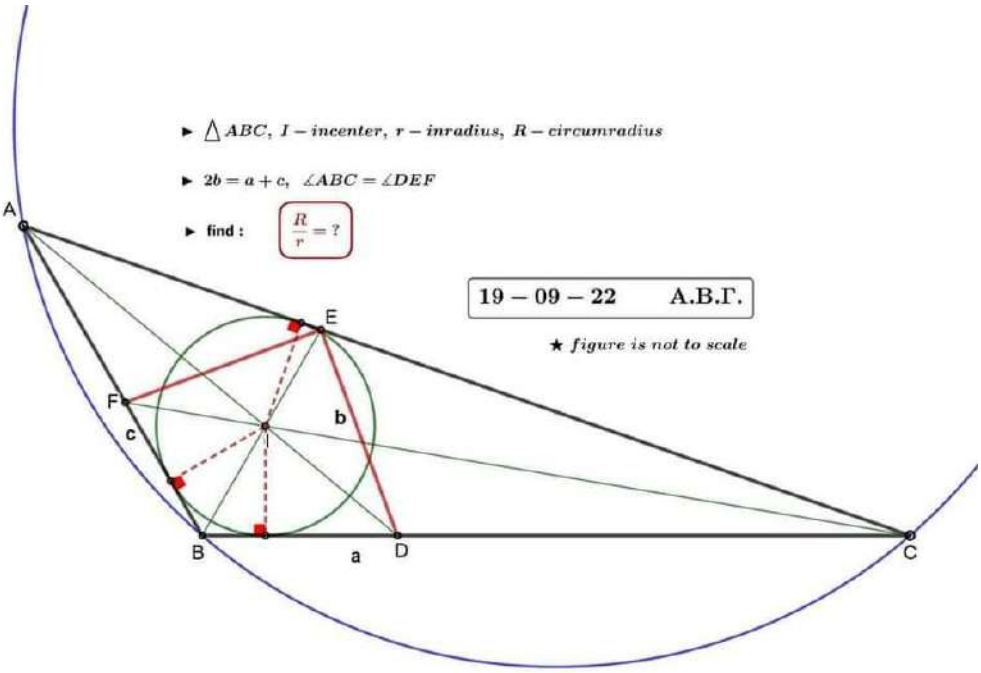
1101.



$$\mathbf{2b = a + c, and \measuredangle DEF = \measuredangle ABC \Rightarrow \frac{R}{r} = ?}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Angle} - \text{bisector theorem} \Rightarrow \frac{AF}{BF} = \frac{b}{a} \Rightarrow \frac{AF + BF}{BF} = \frac{b + a}{a}$$

$$\Rightarrow BF \overset{(i)}{=} \frac{ca}{a + b} \text{ and also, angle} - \text{bisector theorem} \Rightarrow \frac{AE}{CE} = \frac{c}{a}$$

$$\Rightarrow \frac{AE + CE}{CE} = \frac{c + a}{a} \Rightarrow CE \overset{(ii)}{=} \frac{ab}{c + a}$$

$$\text{Now, cosine law} \Rightarrow \left(BF^2 + w_b^2 - 2BF.w_b.\cos\frac{B}{2}\right) + \left(CE^2 + w_c^2 - 2CE.w_c.\cos\frac{C}{2}\right)$$

$$= 2FE^2 \overset{\text{via (i),(ii)}}{\Rightarrow} \frac{a^2b^2}{(c + a)^2} + \frac{4ca}{(c + a)^2}.s(s - b) - 2.\frac{ca}{a + b}.\frac{2ca}{c + a}.\frac{s(s - b)}{ca}$$

$$+ \frac{c^2a^2}{(a + b)^2} + \frac{4ab}{(a + b)^2}.s(s - c) - 2.\frac{ab}{c + a}.\frac{2ab}{a + b}.\frac{s(s - c)}{ab} = 2FE^2$$

$$\Rightarrow 2FE^2 = \frac{a^2b^2}{(c + a)^2} + \frac{c^2a^2}{(a + b)^2} + \frac{4ca}{c + a}.s(s - b)\left(\frac{1}{c + a} - \frac{1}{a + b}\right)$$

$$+ \frac{4ab}{a + b}.s(s - c)\left(\frac{1}{a + b} - \frac{1}{c + a}\right)$$

$$= \frac{a^2b^2}{(c + a)^2} + \frac{c^2a^2}{(a + b)^2} + \frac{4as(b - c)}{(c + a)(a + b)}.\left(\frac{c(s - b)}{c + a} - \frac{b(s - c)}{a + b}\right)$$

$$=\frac{a^2b^2}{(c+a)^2}+\frac{c^2a^2}{(a+b)^2}$$

$$+\frac{4as(b-c)}{(c+a)^2(a+b)^2}.\big(c(s-b)(s+s-c)-b(s-c)(s+s-b)\big)$$

$$=\frac{a^2b^2}{(c+a)^2}+\frac{c^2a^2}{(a+b)^2}$$

$$+\frac{4as(b-c)}{(c+a)^2(a+b)^2}.\big(cs(s-b)-bs(s-c)+c(s-b)(s-c)-c(s-b)(s-c)\big)$$

$$=\frac{a^2b^2}{(c+a)^2}+\frac{c^2a^2}{(a+b)^2}-\frac{4as(b-c)^2}{(c+a)^2(a+b)^2}.\big(s^2+(s-b)(s-c)\big)$$

$$=\frac{a^2b^2}{(c+a)^2}+\frac{c^2a^2}{(a+b)^2}-\frac{4as(b-c)^2}{(c+a)^2(a+b)^2}.\big(s^2+s^2-s(2s-a)+bc\big)$$

$$=\frac{a^2b^2}{(c+a)^2}+\frac{c^2a^2}{(a+b)^2}-\frac{4as(b-c)^2}{(c+a)^2(a+b)^2}.(as+bc)$$

$$=\frac{a^2b^2+c^2a^2-a^2(b-c)^2(a+b+c)^2-2abc(a+b+c)(b-c)^2}{(c+a)^2(a+b)^2}$$

$$=\frac{2a^3bc(a+b+c)+2a^2b^2c^2-2abc(a+b+c)(b-c)^2}{(c+a)^2(a+b)^2}$$

$$=\frac{2abc(a+b+c)\big(a^2-(b-c)^2\big)+2a^2b^2c^2}{(c+a)^2(a+b)^2}$$

$$\therefore FE^2=\frac{abc(a+b+c)\left(a^2(b+c)^2-\left(b^2-c^2\right)^2\right)+a^2b^2c^2(b+c)^2}{(c+a)^2(a+b)^2(b+c)^2}\text{ and analogs}$$

$$\Rightarrow FE^2+DE^2-DF^2\overset{(*)}{=}$$

$$\frac{abc(a+b+c)\left(a^2(b+c)^2+c^2(a+b)^2-b^2(c+a)^2-\left(b^2-c^2\right)^2-\left(a^2-b^2\right)^2+\left(c^2-a^2\right)^2\right)}{(c+a)^2(a+b)^2(b+c)^2}$$

$$+\frac{a^2b^2c^2\big((b+c)^2+(a+b)^2-(c+a)^2\big)}{(c+a)^2(a+b)^2(b+c)^2}$$

Now, $P=a^2(b+c)^2+c^2(a+b)^2-b^2(c+a)^2-\left(b^2-c^2\right)^2-\left(a^2-b^2\right)^2$

$$+\left(c^2-a^2\right)^2=2b^2c^2+2a^2b^2-2b^4+2abc(a+c-b)$$

$$=\frac{2(c+a)^2\left(c^2+a^2-\frac{(c+a)^2}{4}\right)}{4}+ca(c+a)\left(c+a-\frac{c+a}{2}\right)$$

$$=\frac{(c+a)^2}{2}.\left(c^2+a^2+ca-\frac{(c+a)^2}{4}\right)\Rightarrow P\overset{(\bullet)}{=}\frac{(c+a)^2(3c^2+3a^2+2ca)}{8}$$

Again, $Q=(b+c)^2+(a+b)^2-(c+a)^2=2b(a+b+c)-2ca$

$$=(c+a)\left(c+a+\frac{c+a}{2}\right)-2ca\Rightarrow Q\overset{(\bullet\bullet)}{=}\frac{3c^2+3a^2+2ca}{2}$$

$$\therefore(*),(\bullet),(\bullet\bullet)\Rightarrow FE^2+DE^2-DF^2$$

$$=\frac{ca\left(\frac{c+a}{2}\right)\left(c+a+\frac{c+a}{2}\right)\frac{(c+a)^2(3c^2+3a^2+2ca)}{8}+c^2a^2.\frac{(c+a)^2}{4}.\frac{3c^2+3a^2+2ca}{2}}{(a+b)^2(b+c)^2(c+a)^2}$$

$$=\frac{ca(c+a)^2(3c^2+3a^2+2ca)}{8(a+b)^2(b+c)^2(c+a)^2}\left(\frac{3(c+a)^2}{4}+ca\right)$$

$$\Rightarrow \boxed{FE^2+DE^2-DF^2 \overset{(\blacksquare)}{=} \frac{ca(c+a)^2(3c^2+3a^2+2ca)(3c+a)(c+3a)}{32(a+b)^2(b+c)^2(c+a)^2}}$$

$$\text{Now, } [DEF]+[BDF]=[BEF]+[BED] \Rightarrow \frac{1}{2}.FE.DE.\sin B+\frac{1}{2}.\frac{ca}{b+c}.\frac{ca}{a+b}.\sin B$$

$$=\frac{1}{2}.\frac{ca}{a+b}.\frac{2ca.\cos\frac{B}{2}}{c+a}.\sin\frac{B}{2}+\frac{1}{2}.\frac{ca}{b+c}.\frac{2ca.\cos\frac{B}{2}}{c+a}.\sin\frac{B}{2}$$

$$\Rightarrow FE.DE.\sin B+\frac{c^2a^2.\sin B}{(b+c)(a+b)}=\frac{c^2a^2.\sin B}{(a+b)(c+a)}+\frac{c^2a^2.\sin B}{(b+c)(c+a)}$$

$$\Rightarrow FE.DE=\frac{c^2a^2(b+c+a+b-c-a)}{(a+b)(b+c)(c+a)} \Rightarrow 2FE.DE.\cos B=\frac{4bc^2a^2\left(\frac{c^2+a^2-b^2}{2ca}\right)}{(a+b)(b+c)(c+a)}$$

$$=\frac{ca(c+a)\left(c^2+a^2-\frac{(c+a)^2}{4}\right)}{(a+b)(b+c)(c+a)}$$

$$\Rightarrow \boxed{2FE.DE.\cos(\sphericalangle DEF) \overset{(\blacksquare\blacksquare)}{=} \frac{ca(c+a)(3c^2+3a^2-2ca)}{4(a+b)(b+c)(c+a)}} \ (\because \cos B=\cos(\sphericalangle DEF))$$

$$\text{Now, } (\blacksquare),(\blacksquare\blacksquare) \Rightarrow \frac{ca(c+a)^2(3c^2+3a^2+2ca)(3c+a)(c+3a)}{32(a+b)^2(b+c)^2(c+a)^2}$$

$$=\frac{ca(c+a)(3c^2+3a^2-2ca)}{4(a+b)(b+c)(c+a)} \Rightarrow \frac{(3c^2+3a^2+2ca)(3c+a)(c+3a)}{8\left(a+\frac{c+a}{2}\right)\left(\frac{c+a}{2}+c\right)}$$

$$=3c^2+3a^2-2ca \Rightarrow \frac{(3c^2+3a^2+2ca)(3c+a)(c+3a)}{2(3c+a)(c+3a)}=3c^2+3a^2-2ca$$

$$\Rightarrow 3c^2+3a^2+2ca=6c^2+6a^2-4ca \Rightarrow 3c^2+3a^2-6ca=0 \Rightarrow 3(c-a)^2=0$$

$$\Rightarrow c=a \text{ and } \because 2b=c+a \therefore 2b=2c=2a \Rightarrow a=b=c$$

$$\Rightarrow \Delta\ ABC \text{ is equilateral} \Rightarrow \frac{R}{r}=2\ (ans)$$

1102. In any acute Δ ABC holds:

$$6max\{h_a,h_b,h_c\} \geq 3\min\{r_a,r_b,r_c\}+\sum_{cyc} n_a$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$3.\min\{r_a,r_b,r_c\}+\sum_{cyc} n_a \leq 3.\min\{r_a,r_b,r_c\}+3.max\{n_a,n_b,n_c\}$$

$$\overset{?}{\leq} 6.max\{h_a,h_b,h_c\} \Leftrightarrow max\{n_a,n_b,n_c\}+\min\{r_a,r_b,r_c\} \underset{(\bullet)}{\overset{?}{\leq}} 2.max\{h_a,h_b,h_c\}$$

$$\text{WLOG we may assume } a \leq b \leq c \text{ and then, } (\bullet) \Leftrightarrow n_a+r_a \overset{(\bullet\bullet)}{\leq} 2h_a$$

$$\text{Now}, \frac{b+c}{a} \ge \frac{R}{r} = \frac{abcs}{4F^2} = \frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

$$\Leftrightarrow (b+c)(a+b-c).(b+c-a)(c+a-b) \ge 2a^2bc$$

$$\Leftrightarrow (ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)$$

$$\ge 2a^2bc \Leftrightarrow 2a^2b^2+2a^2bc+2ab(b^2-c^2)$$

$$-(a^2+b^2-c^2)(ab+ac+b^2-c^2) \ge 2a^2bc$$

$$\Leftrightarrow 2a^2b^2-(a^2+b^2-c^2)(ab+ac)+2ab(b^2-c^2)-2ab(b^2-c^2).\cos C \ge 0$$

$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-(b^2-c^2)(ab+ac)+2ab(b^2-c^2).2\sin^2\frac{C}{2} \ge 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)-(b^2-c^2)(ab+ac)+(b^2-c^2)(c^2-(a-b)^2) \ge 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)+(b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac) \ge 0$$

$$\Leftrightarrow \left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$

$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac) \ge 0$$

$$\Leftrightarrow (c^2+a^2-b^2)(2b^2-ab-ac)+(b^2-c^2)(b^2+c^2-a^2-2ac) \ge 0$$

$$\Leftrightarrow (b^2-c^2)(b^2+c^2-2ac)-(b^2-c^2)(2b^2-ab-ac)+a^2(2b^2-ab-ac)$$

$$-a^2(b^2-c^2) \ge 0$$

$$\Leftrightarrow (b^2-c^2)\left((c^2-ca)-(b^2-ab)\right)+a^2\left((c^2-ca)+(b^2-ab)\right) \ge 0$$

$$\Leftrightarrow (c^2-ca)(b^2-c^2+a^2)+(b^2-ab)(a^2+c^2-b^2) \ge 0$$

$$\Leftrightarrow c(c-a)(a^2+b^2-c^2)+b(b-a)(c^2+a^2-b^2) \ge 0$$

$$\to \text{true} \because \Delta\,ABC \text{ being acute} \Rightarrow (a^2+b^2-c^2),(c^2+a^2-b^2) > 0 \text{ and } a \le b \le c$$

$$\Rightarrow (c-a),(b-a) \ge 0 \therefore \frac{b+c}{a} \ge \frac{R}{r}$$

$$\therefore \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}} \ge \frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \Rightarrow 2\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\cos\frac{B-C}{2} \ge 1$$

$$\Rightarrow 2\cos^2\frac{B-C}{2}-2\cos\frac{B-C}{2}\sin\frac{A}{2}-1 \overset{(*)}{\ge} 0$$

$$\text{Now, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$

$$= an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2-as^2 \Rightarrow an_a^2$$

$$= as^2+s(2bc\cos A-2bc) = as^2-4sbc\sin^2\frac{A}{2} = as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2-\frac{4\Delta^2}{s-a} = as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2-2ah_ar_a \therefore n_a^2 = s^2-2r_ah_a$$

$$\Rightarrow n_a^2+r_a^2 = s^2-2r_ah_a+r_a^2 \overset{?}{\le} 2h_a^2 \Leftrightarrow 2h_a^2+2r_ah_a \overset{?}{\ge} s^2+s^2\tan^2\frac{A}{2}$$

$$\Leftrightarrow \frac{8s^2.16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}+\frac{4.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}s.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}} \overset{?}{\ge} s^2\sec^2\frac{A}{2}$$

$$\Leftrightarrow 8\sin^2\frac{B}{2}\sin^2\frac{C}{2}+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \overset{?}{\ge} 1$$

$$\Leftrightarrow 2\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)^2+2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)\overset{?}{\geq}1$$

$$\left(\because 2\sin\frac{B}{2}\sin\frac{C}{2}=\cos\frac{B-C}{2}-\cos\frac{B+C}{2}=\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)$$

$$\Leftrightarrow 2\left(\cos^2\frac{B-C}{2}+\sin^2\frac{A}{2}-2\sin\frac{A}{2}.\cos\frac{B-C}{2}\right)+2\sin\frac{A}{2}.\cos\frac{B-C}{2}-2\sin^2\frac{A}{2}-1$$

$$\overset{?}{\geq}0\Leftrightarrow 2\cos^2\frac{B-C}{2}-2\cos\frac{B-C}{2}\sin\frac{A}{2}-1\overset{?}{\geq}0\to \text{true via } (*) \therefore n_a^2+r_a^2\leq 2h_a^2$$

$$\Rightarrow n_a+r_a\overset{CBS}{\leq}\sqrt{2(n_a^2+r_a^2)}\leq\sqrt{4h_a^2}\Rightarrow n_a+r_a\leq 2h_a\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}$$

$$\therefore \text{in any acute }\Delta ABC, 6.max\{h_a,h_b,h_c\}\geq 3.\min\{r_a,r_b,r_c\}+\sum_{cyc}n_a,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $n_a+n_b+n_c\leq 3.max\{n_a,n_b,n_c\}$ then it suffices to prove,

$$2max\{h_a,h_b,h_c\}\geq min\{r_a,r_b,r_c\}+max\{n_a,n_b,n_c\}.$$

We have :

$$\frac{{n_b}^2-{n_a}^2}{s}=\left(s-b+\frac{(c-a)^2}{b}\right)-\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$=a-b+\frac{a(c-a)^2-b(b-c)^2}{ab}$$

$$=a-b+\frac{(a-b)(a^2+ab+b^2-2ca-2bc+c^2)}{ab}=\frac{(a-b)(a+b-c)^2}{ab},$$

so if $a\leq b$ then $n_a\geq n_b$.

WLOG, we assume that $a\leq b\leq c$. Since $h_a=\frac{2F}{a},\ r_a=\frac{F}{s-a}$ (and analogs) then we have,

$$max\{h_a,h_b,h_c\}=h_a,\ \ min\{r_a,r_b,r_c\}=r_a\ \text{ and }\ max\{n_a,n_b,n_c\}=n_a.$$

Now, we have :

$${n_a}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$$

$$=-\frac{4s.sr^2}{a(s-a)}=s^2-2h_ar_a,\ \text{ and }\ r_a=s\tan\frac{A}{2}.$$

Using these results, we have :

$$(r_a+n_a)^2 \le 2({r_a}^2+{n_a}^2) = 2s^2\left(\tan^2\frac{A}{2}+1\right)-4h_ar_a = \frac{2s^2}{\cos^2\frac{A}{2}}-4h_ar_a$$

$$=\frac{2sbc}{s-a}-\frac{8s^2r^2}{a(s-a)}$$

$$=\frac{2s(abc-4sr^2)}{a(s-a)}=\frac{8s^2r(R-r)}{a(s-a)} \overset{?}{\lesssim} (2h_a)^2 \Leftrightarrow R-r \le \frac{2r(s-a)}{a} \Leftrightarrow \frac{R+r}{R} \le \frac{2sr}{Ra}$$

$$\Leftrightarrow \sum_{cyc}\cos A \le \frac{1}{a}\sum_{cyc} a\cos A \Leftrightarrow 0$$

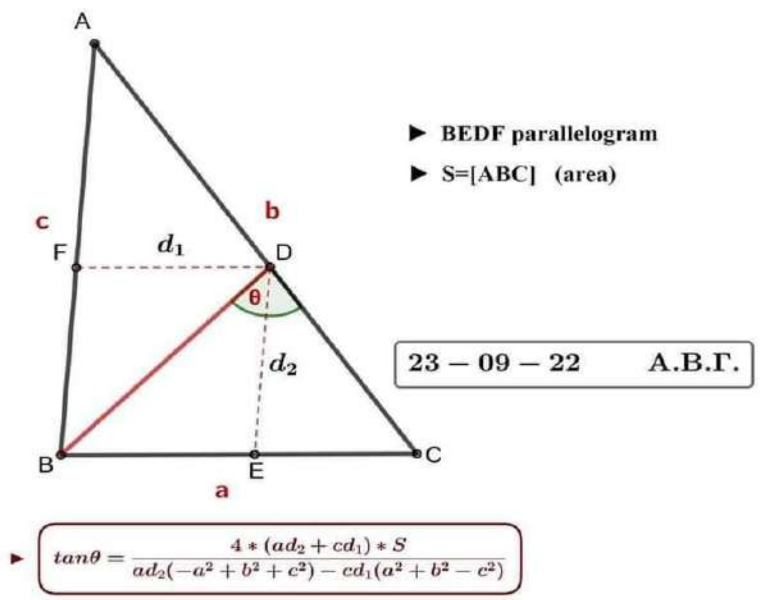
$$\le \frac{b-a}{a}.\cos B+\frac{c-a}{a}.\cos C, \text{ which is true.}$$

Then, $2h_a \ge r_a+n_a$ **or** $2max\{h_a,h_b,h_c\} \ge min\{r_a,r_b,r_c\}+max\{n_a,n_b,n_c\}$.

Therefore,

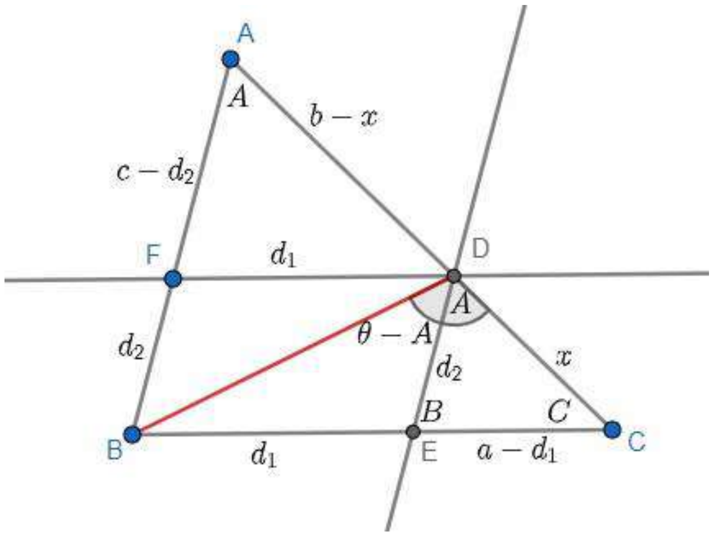
$$6.max\{h_a,h_b,h_c\} \ge 3.min\{r_a,r_b,r_c\}+n_a+n_b+n_c.$$

1103.



Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Soumava Chakraborty-Kolkata-India



$$\frac{4(ad_2+cd_1)*S}{ad_2(-a^2+b^2+c^2)-cd_1(a^2+b^2-c^2)}=\frac{4S.2R(d_2\sin A+d_1\sin C)}{ad_2.2bc.\cos A-cd_1.2ab.\cos C}$$

$$=\frac{4S.2R(d_2\sin A+d_1\sin C)}{8RS(d_2\cos A-d_1\cos C)}$$

$$\Rightarrow\frac{4(ad_2+cd_1)*S}{ad_2(-a^2+b^2+c^2)-cd_1(a^2+b^2-c^2)}\overset{(*)}{=}\frac{d_2\sin A+d_1\sin C}{d_2\cos A-d_1\cos C}$$

Now, via sine rule on Δ DEC, $\frac{a-d_1}{\sin A}=\frac{d_2}{\sin C}\Rightarrow d_2\sin A+d_1\sin C\overset{(**)}{=}a\sin C$

Again, via cot rule on Δ ABC with BD as cevian, $b\cot\theta=x\cot A-(b-x)\cot C$

$$=x\left(\frac{\cos A}{\sin A}+\frac{\cos C}{\sin C}\right)-b\cot C=x.\frac{\sin(A+C)}{\sin A.\sin C}-b\cot C=x.\frac{\frac{b}{2R}}{\frac{ac}{4R^2}}-b\cot C$$

$$\Rightarrow\cot\theta=\frac{2Rx-ac.\cot C}{ac}\Rightarrow\tan\theta=\frac{ac}{2Rx-ac.\cot C}$$

$$\overset{\text{First cosine rule on }\Delta\text{ DEC}}{=}\frac{a.2R\sin C}{2R(d_2\cos A+(a-d_1)\cos C)-ac.\cot C}$$

$$\overset{\text{via }(**)}{=}\frac{2R(d_2\sin A+d_1\sin C)}{2R(d_2\cos A+(a-d_1)\cos C)-a.2R\sin C.\frac{\cos C}{\sin C}}$$

$$=\frac{d_2\sin A+d_1\sin C}{d_2\cos A+a\cos C-d_1\cos C-a\cos C}=\frac{d_2\sin A+d_1\sin C}{d_2\cos A-d_1\cos C}$$

$$\overset{\text{via }(*)}{=}\frac{4(ad_2+cd_1)*S}{ad_2(-a^2+b^2+c^2)-cd_1(a^2+b^2-c^2)}$$

$$\therefore\tan\theta=\frac{4(ad_2+cd_1)*S}{ad_2(-a^2+b^2+c^2)-cd_1(a^2+b^2-c^2)}\text{ (QED)}$$

1104. In any acute Δ ABC,

$$2.max\{h_a^2,h_b^2,h_c^2\} \ge max\{n_a^2,n_b^2,n_c^2\} + \min\{r_a^2,r_b^2,r_c^2\}$$

$$and\ 2.max\{h_a,h_b,h_c\} \ge max\{n_a,n_b,n_c\} + \min\{r_a,r_b,r_c\}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India,

$$\text{WLOG we may assume } a \le b \le c \text{ and then : } max\{h_a^2,h_b^2,h_c^2\} = h_a^2;$$
$$max\{n_a^2,n_b^2,n_c^2\} = n_a^2; \text{ and } \min\{r_a^2,r_b^2,r_c^2\} = r_a^2$$

$$\therefore 2.max\{h_a^2,h_b^2,h_c^2\} \ge max\{n_a^2,n_b^2,n_c^2\} + \min\{r_a^2,r_b^2,r_c^2\} \Leftrightarrow 2h_a^2 \overset{(\bullet)}{\ge} n_a^2 + r_a^2$$
$$\text{and moreover, } 2.max\{h_a,h_b,h_c\} \ge max\{n_a,n_b,n_c\} + \min\{r_a,r_b,r_c\}$$
$$\Leftrightarrow 2h_a \overset{(\bullet\bullet)}{\ge} n_a + r_a$$

$$\text{Now, } \frac{b+c}{a} \ge \frac{R}{r} = \frac{abcs}{4F^2} = \frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$
$$\Leftrightarrow (b+c)(a+b-c).(b+c-a)(c+a-b) \ge 2a^2bc$$
$$\Leftrightarrow (ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)$$
$$\ge 2a^2bc \Leftrightarrow 2a^2b^2 + 2a^2bc + 2ab(b^2-c^2)$$
$$-(a^2+b^2-c^2)(ab+ac+b^2-c^2) \ge 2a^2bc$$
$$\Leftrightarrow 2a^2b^2 - (a^2+b^2-c^2)(ab+ac) + 2ab(b^2-c^2) - 2ab(b^2-c^2).\cos C \ge 0$$
$$\Leftrightarrow 2a^2b^2 - a^2(ab+ac) - (b^2-c^2)(ab+ac) + 2ab(b^2-c^2).2\sin^2\frac{C}{2} \ge 0$$
$$\Leftrightarrow a^2(2b^2-ab-ac) - (b^2-c^2)(ab+ac) + (b^2-c^2)(c^2-(a-b)^2) \ge 0$$
$$\Leftrightarrow a^2(2b^2-ab-ac) + (b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac) \ge 0$$
$$\Leftrightarrow \left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$
$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac) \ge 0$$
$$\Leftrightarrow (c^2+a^2-b^2)(2b^2-ab-ac) + (b^2-c^2)(b^2+c^2-a^2-2ac) \ge 0$$
$$\Leftrightarrow (b^2-c^2)(b^2+c^2-2ac) - (b^2-c^2)(2b^2-ab-ac) + a^2(2b^2-ab-ac)$$
$$-a^2(b^2-c^2) \ge 0$$
$$\Leftrightarrow (b^2-c^2)\left((c^2-ca)-(b^2-ab)\right) + a^2\left((c^2-ca)+(b^2-ab)\right) \ge 0$$
$$\Leftrightarrow (c^2-ca)(b^2-c^2+a^2) + (b^2-ab)(a^2+c^2-b^2) \ge 0$$
$$\Leftrightarrow c(c-a)(a^2+b^2-c^2) + b(b-a)(c^2+a^2-b^2) \ge 0$$
$$\to \text{true} \because \Delta \text{ ABC being acute} \Rightarrow (a^2+b^2-c^2),(c^2+a^2-b^2) > 0 \text{ and } a \le b \le c$$
$$\Rightarrow (c-a),(b-a) \ge 0 \therefore \frac{b+c}{a} \ge \frac{R}{r}$$
$$\therefore \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}} \ge \frac{R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \Rightarrow 2\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right)\cos\frac{B-C}{2} \ge 1$$
$$\Rightarrow 2\cos^2\frac{B-C}{2} - 2\cos\frac{B-C}{2}\sin\frac{A}{2} - 1 \overset{(*)}{\ge} 0$$
$$\text{Now, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2$$

$$= as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \therefore n_a^2 = s^2 - 2r_ah_a$$

$$\Rightarrow n_a^2 + r_a^2 = s^2 - 2r_ah_a + r_a^2 \overset{?}{\leq} 2h_a^2 \Leftrightarrow 2h_a^2 + 2r_ah_a \overset{?}{\geq} s^2 + s^2\tan^2\frac{A}{2}$$

$$\Leftrightarrow \frac{8s^2.16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}} + \frac{4.4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}s.s\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}} \overset{?}{\geq} s^2\sec^2\frac{A}{2}$$

$$\Leftrightarrow 8\sin^2\frac{B}{2}\sin^2\frac{C}{2} + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \overset{?}{\geq} 1$$

$$\Leftrightarrow 2\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right)^2 + 2\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right) \overset{?}{\geq} 1$$

$$\left(\because 2\sin\frac{B}{2}\sin\frac{C}{2} = \cos\frac{B-C}{2} - \cos\frac{B+C}{2} = \cos\frac{B-C}{2} - \sin\frac{A}{2}\right)$$

$$\Leftrightarrow 2\left(\cos^2\frac{B-C}{2} + \sin^2\frac{A}{2} - 2\sin\frac{A}{2}.\cos\frac{B-C}{2}\right) + 2\sin\frac{A}{2}.\cos\frac{B-C}{2} - 2\sin^2\frac{A}{2} - 1$$

$$\overset{?}{\geq} 0 \Leftrightarrow 2\cos^2\frac{B-C}{2} - 2\cos\frac{B-C}{2}\sin\frac{A}{2} - 1 \overset{?}{\geq} 0 \rightarrow \text{true via } (*) \therefore n_a^2 + r_a^2 \leq 2h_a^2$$

$$\Rightarrow (\bullet) \text{ is true and also}, n_a + r_a \overset{CBS}{\leq} \sqrt{2(n_a^2 + r_a^2)} \overset{?}{\leq} 2h_a \Leftrightarrow n_a^2 + r_a^2 \leq 2h_a^2$$

$$\rightarrow \text{true via } (\bullet) \Rightarrow n_a + r_a \leq 2h_a \Rightarrow (\bullet\bullet) \text{ is true} \therefore \text{in any acute } \Delta\, ABC,$$

$$2.\max\{h_a^2, h_b^2, h_c^2\} \geq \max\{n_a^2, n_b^2, n_c^2\} + \min\{r_a^2, r_b^2, r_c^2\} \text{ and}$$

$$2.\max\{h_a, h_b, h_c\} \geq \max\{n_a, n_b, n_c\} + \min\{r_a, r_b, r_c\},$$

'' ='' iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we have :

$$\frac{n_b^2 - n_a^2}{s} = \left(s - b + \frac{(c-a)^2}{b}\right) - \left(s - a + \frac{(b-c)^2}{a}\right) = a - b + \frac{a(c-a)^2 - b(b-c)^2}{ab}$$

$$= a - b + \frac{(a-b)(a^2 + ab + b^2 - 2ca - 2bc + c^2)}{ab} = \frac{(a-b)(a+b-c)^2}{ab},$$

so if $a \leq b$ then $n_a \geq n_b$. WLOG, we assume that $a \leq b \leq c$.

Since $h_a = \frac{2F}{a}$, $r_a = \frac{F}{s-a}$ (and analogs) then we have,

$$\max\{h_a, h_b, h_c\} = h_a, \quad \max\{n_a, n_b, n_c\} = n_a \quad \text{and} \quad \min\{r_a, r_b, r_c\} = r_a.$$

Now, we have :

$$n_a^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s[a^2 - (b-c)^2]}{a} = s^2 - \frac{4s(s-b)(s-c)}{a}$$

$$= -\frac{4s.sr^2}{a(s-a)} = s^2 - 2h_ar_a, \quad \text{and} \quad r_a = s\tan\frac{A}{2}.$$

Using these results, we have :

$$n_a^2 + r_a^2 = s^2\left(1 + \tan^2\frac{A}{2}\right) - 2h_ar_a = \frac{s^2}{\cos^2\frac{A}{2}} - 2h_ar_a = \frac{sbc}{s-a} - \frac{4s^2r^2}{a(s-a)} = \frac{s(abc - 4sr^2)}{a(s-a)}$$

$$=\frac{4s^2r(R-r)}{a(s-a)}\overset{?}{\lesssim}2{h_a}^2\Leftrightarrow R-r\le\frac{2r(s-a)}{a}\Leftrightarrow\frac{R+r}{R}\le\frac{2sr}{Ra}\Leftrightarrow$$

$$a\sum_{cyc}\cos A\le\sum_{cyc}a\cos A$$

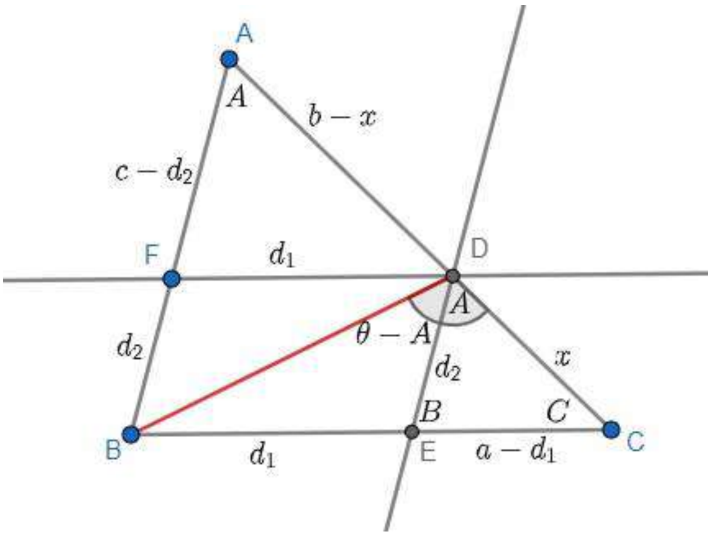
$\Leftrightarrow 0\le(b-a)\cos B+(c-a)\cos C$, which is true.

Then, $2{h_a}^2\ge{n_a}^2+{r_a}^2$ and $2h_a\ge\sqrt{2({n_a}^2+{r_a}^2)}\overset{CBS}{\gtrsim}n_a+r_a.$

$$2max\{{h_a}^2,{h_b}^2,{h_c}^2\}\ge max\{{n_a}^2,{n_b}^2,{n_c}^2\}+min\{{r_a}^2,{r_b}^2,{r_c}^2\}$$

$$2max\{h_a,h_b,h_c\}\ge max\{n_a,n_b,n_c\}+min\{r_a,r_b,r_c\}.$$

1105.



Prove that :

$$\frac{d_1}{d_2}=\frac{a(-a^2+b^2+c^2)}{c(a^2+b^2-c^2)}\Rightarrow\Delta\ \text{BDC is right}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4(ad_2+cd_1)*S}{ad_2(-a^2+b^2+c^2)-cd_1(a^2+b^2-c^2)}=\frac{4S.2R(d_2\sin A+d_1\sin C)}{ad_2.2bc.\cos A-cd_1.2ab.\cos C}$$

$$=\frac{4S.2R(d_2\sin A+d_1\sin C)}{8RS(d_2\cos A-d_1\cos C)}$$

$$\Rightarrow\frac{4(ad_2+cd_1)*S}{ad_2(-a^2+b^2+c^2)-cd_1(a^2+b^2-c^2)}\overset{(*)}{=}\frac{d_2\sin A+d_1\sin C}{d_2\cos A-d_1\cos C}$$

Now, via sine rule on Δ DEC, $\frac{a-d_1}{\sin A}=\frac{d_2}{\sin C}\Rightarrow d_2\sin A+d_1\sin C\overset{(**)}{=}a\sin C$

Again, via cot rule on Δ ABC with BD as cevian, $b\cot\theta=x\cot A-(b-x)\cot C$

$$= x\left(\frac{\cos A}{\sin A} + \frac{\cos C}{\sin C}\right) - b\cot C = x.\frac{\sin(A+C)}{\sin A.\sin C} - b\cot C = x.\frac{\frac{b}{2R}}{\frac{ac}{4R^2}} - b\cot C$$

$$\Rightarrow \cot\theta = \frac{2Rx - ac.\cot C}{ac} \Rightarrow \tan\theta = \frac{ac}{2Rx - ac.\cot C}$$

$$\overset{\text{First cosine rule on } \Delta\, DEC}{=} \frac{a.2R\sin C}{2R(d_2\cos A + (a-d_1)\cos C) - ac.\cot C}$$

$$\overset{\text{via } (**)}{=} \frac{2R(d_2\sin A + d_1\sin C)}{2R(d_2\cos A + (a-d_1)\cos C) - a.2R\sin C.\frac{\cos C}{\sin C}}$$

$$= \frac{d_2\sin A + d_1\sin C}{d_2\cos A + a\cos C - d_1\cos C - a\cos C} = \frac{d_2\sin A + d_1\sin C}{d_2\cos A - d_1\cos C}$$

$$\overset{\text{via } (*)}{=} \frac{4(ad_2 + cd_1) * S}{ad_2(-a^2+b^2+c^2) - cd_1(a^2+b^2-c^2)}$$

$$\therefore \tan\theta = \frac{4(ad_2 + cd_1) * S}{ad_2(-a^2+b^2+c^2) - cd_1(a^2+b^2-c^2)} \text{ and } \because \frac{d_1}{d_2} = \frac{a(-a^2+b^2+c^2)}{c(a^2+b^2-c^2)}$$

$$\therefore ad_2(-a^2+b^2+c^2) - cd_1(a^2+b^2-c^2) = 0 \Rightarrow \tan\theta = \frac{4(ad_2+cd_1) * S}{0}$$

$$\Rightarrow \theta = 90° \Rightarrow \Delta\, BDC \text{ is right (QED)}$$

1106. In ΔABC holds :

$$m(\angle A) \le 90 \Leftrightarrow {n_a}^2 \ge r_a(r_a - 2h_a).$$

$$m(\angle A) \ge 90 \Leftrightarrow {n_a}^2 \le r_a(r_a - 2h_a).$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

$${n_a}^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s[a^2-(b-c)^2]}{a} = s^2 - \frac{4s(s-b)(s-c)}{a}$$

$$= s^2 - \frac{4s.sr^2}{a(s-a)} = \frac{{r_a}^2}{\tan^2\frac{A}{2}} - 2r_ah_a = r_a\left(\frac{r_a}{\tan^2\frac{A}{2}} - 2h_a\right).$$

Then $m(\angle A) \le 90 \Leftrightarrow \tan\frac{A}{2} \le 1 \Leftrightarrow {n_a}^2 = r_a\left(\frac{r_a}{\tan^2\frac{A}{2}} - 2h_a\right) \ge r_a(r_a - 2h_a),$

$$m(\angle A) \ge 90 \Leftrightarrow \tan\frac{A}{2} \ge 1 \Leftrightarrow {n_a}^2 = r_a\left(\frac{r_a}{\tan^2\frac{A}{2}} - 2h_a\right) \le r_a(r_a - 2h_a).$$

1107. In any acute Δ ABC with $a = \min\{a, b, c\}$, the following relationship holds :

$$4r\left(\sum_{cyc}\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}} - 1\right) \geq R + h_b + h_c$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since, $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$, then, $\frac{m_a}{w_a} \geq \frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{b+c}{2bc\cos\frac{A}{2}} = \frac{(b+c)^2}{4bc}$.

By Tereshin and CBS inequalities, we have,

$$\frac{m_b m_c}{h_b h_c} \geq \frac{(c^2+a^2)(a^2+b^2)}{(4R)^2}\cdot\frac{(2R)^2}{ca.ab} \geq \frac{(ca+ab)^2}{4a^2bc} = \frac{(b+c)^2}{4bc}.$$

Using these results, we obtain,

$$\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}} \geq \frac{(b+c)^2}{4bc} \quad \text{(and analogs)}$$

Then,

$$4r\left(\sum_{cyc}\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}} - 1\right) \geq r\left(\sum_{cyc}\frac{(b+c)^2}{bc} - 4\right) = r\left(\frac{(a+b+c)(ab+bc+ca)}{abc} - 1\right)$$

$$= \frac{r.2s.2R(h_a+h_b+h_c)}{4Rsr} - r = h_a + h_b + h_c - r \overset{?}{\geq} R + h_b + h_c \Leftrightarrow h_a \geq R + r$$

$$\Leftrightarrow \frac{2F}{R} \geq a.\frac{R+r}{R} \Leftrightarrow \sum_{cyc} a\cos A \geq a.\sum_{cyc}\cos A \Leftrightarrow (b-a)\cos B + (c-a)\cos C \geq 0,$$

which is true because ΔABC is acute and $a = min(a,b,c)$.

So the proof is completed. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$4r\left(\sum_{cyc}\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}} - 1\right) - R - h_b - h_c \overset{\text{Lascu + Tereshin}}{\geq}$$

$$4r\left(\sum_{cyc}\sqrt{\frac{\frac{(b+c)^2}{2}\cos\frac{A}{2}\cdot\frac{(c^2+a^2)(a^2+b^2)}{16R^2}}{2bc\cos\frac{A}{2}\cdot\frac{ca.ab}{4R^2}}}\right)-4r-R-h_b-h_c$$

$$\overset{CBS}{\geq} 4r\sum_{cyc}\sqrt{\frac{(b+c)^2}{4bc}\cdot\frac{a^2(b+c)^2}{4a^2bc}}-4r-R-h_b-h_c$$

$$=4r\sum_{cyc}\frac{a(b^2+c^2+2bc)}{4abc}-4r-R-h_b-h_c$$

$$=\frac{\sum_{cyc}(ab(2s-c))+6abc}{4Rs}-4r-R-h_b-h_c=\frac{ab+bc+ca}{2R}-r-R-\frac{ca+ab}{2R}$$

$$=\frac{4Rrs}{a.2R}-r-R=r.\frac{a+b+c}{a}-r-R$$

$$\therefore 4r\left(\sum_{cyc}\sqrt{\frac{m_am_bm_c}{w_ah_bh_c}}-1\right)-R-h_b-h_c\overset{(\bullet)}{\geq} r.\frac{b+c}{a}-R$$

$$\text{Now, }\frac{b+c}{a}\geq\frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

$$\Leftrightarrow (b+c)(a+b-c).(b+c-a)(c+a-b)\geq 2a^2bc$$

$$\Leftrightarrow (ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)$$

$$\geq 2a^2bc\Leftrightarrow 2a^2b^2+2a^2bc+2ab(b^2-c^2)$$

$$-(a^2+b^2-c^2)(ab+ac+b^2-c^2)\geq 2a^2bc$$

$$\Leftrightarrow 2a^2b^2-(a^2+b^2-c^2)(ab+ac)+2ab(b^2-c^2)-2ab(b^2-c^2).\cos C\geq 0$$

$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-(b^2-c^2)(ab+ac)+2ab(b^2-c^2).2\sin^2\frac{C}{2}\geq 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)-(b^2-c^2)(ab+ac)+(b^2-c^2)(c^2-(a-b)^2)\geq 0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)+(b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac)\geq 0$$

$$\Leftrightarrow \left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$

$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac)\geq 0$$

$$\Leftrightarrow (c^2+a^2-b^2)(2b^2-ab-ac)+(b^2-c^2)(b^2+c^2-a^2-2ac)\geq 0$$

$$\Leftrightarrow (b^2-c^2)(b^2+c^2-2ac)-(b^2-c^2)(2b^2-ab-ac)+a^2(2b^2-ab-ac)$$

$$-a^2(b^2-c^2)\geq 0$$

$$\Leftrightarrow (b^2-c^2)\left((c^2-ca)-(b^2-ab)\right)+a^2\left((c^2-ca)+(b^2-ab)\right)\geq 0$$

$$\Leftrightarrow (c^2-ca)(b^2-c^2+a^2)+(b^2-ab)(a^2+c^2-b^2)\geq 0$$

$$\Leftrightarrow c(c-a)(a^2+b^2-c^2)+b(b-a)(c^2+a^2-b^2)\geq 0$$

$$\rightarrow \text{true} \because \Delta ABC \text{ being acute} \Rightarrow (a^2+b^2-c^2),(c^2+a^2-b^2)>0 \text{ and}$$

$$a=\min\{a,b,c\}\Rightarrow a\leq b,c\Rightarrow (c-a),(b-a)\geq 0\Rightarrow (c-a),(b-a)\geq 0$$

$$\therefore \frac{b+c}{a}\geq\frac{R}{r}\Rightarrow r.\frac{b+c}{a}-R\overset{(\bullet\bullet)}{\geq}0$$

$$\therefore (\bullet)(\bullet\bullet)\Rightarrow 4r\left(\sum_{cyc}\sqrt{\frac{m_am_bm_c}{w_ah_bh_c}}-1\right)-R-h_b-h_c\geq 0$$

$$\Rightarrow \text{in any acute } \Delta ABC \text{ with } a=\min\{a,b,c\},$$

$$4r\left(\sum_{cyc}\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}}-1\right)\geq R+h_b+h_c \text{ ''='' iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1108. In any acute Δ ABC, the following relationship holds :

$$\frac{3r}{2R}.\sqrt{6}\leq\sum_{cyc}\frac{m_a}{\sqrt{b^2+c^2}}\leq\sqrt{3\left(1+\frac{r}{4R}\right)}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

$$\sum_{cyc}\frac{m_a}{\sqrt{b^2+c^2}}\overset{Tereshin}{\geq}\sum_{cyc}\frac{\sqrt{b^2+c^2}}{4R}\overset{CBS}{\geq}\sum_{cyc}\frac{b+c}{4\sqrt{2}R}=\frac{s}{\sqrt{2}R}\overset{Mitrinovic}{\geq}\frac{3\sqrt{3}r}{\sqrt{2}R}=\frac{3r}{2R}\sqrt{6}.$$

Now, since ΔABC is acute, then we have,

$$\sum_{cyc}\frac{m_a}{\sqrt{b^2+c^2}}\overset{CBS}{\leq}\sqrt{3\sum_{cyc}\frac{m_a{}^2}{b^2+c^2}}$$

$$=\sqrt{\frac{3}{4}\sum_{cyc}\left(1+\frac{b^2+c^2-a^2}{b^2+c^2}\right)}\overset{AM-GM}{\leq}\sqrt{\frac{3}{4}\sum_{cyc}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)}$$

$$=\sqrt{\frac{3}{4}\sum_{cyc}(1+\cos A)}=\sqrt{\frac{3}{4}\left(4+\frac{r}{R}\right)}=\sqrt{3\left(1+\frac{r}{4R}\right)}.$$

So the proof is completed. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2}{s(s-a)}-1\overset{?}{\leq}\frac{b^2+c^2}{2bc}-1\Leftrightarrow\frac{(b-c)^2+4s(s-a)-4s(s-a)}{4s(s-a)}\overset{?}{\leq}\frac{(b-c)^2}{2bc}$$

$$\Leftrightarrow(b-c)^2.\left(\frac{1}{2bc}-\frac{1}{4s(s-a)}\right)\overset{?}{\geq}0\Leftrightarrow(b-c)^2.\left(\frac{4s(s-a)-2bc}{8sbc(s-a)}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow(b-c)^2.\frac{b^2+c^2-a^2}{8sbc(s-a)}\overset{?}{\geq}0\Leftrightarrow(b-c)^2.\frac{\cos A}{4s(s-a)}\overset{?}{\geq}0\rightarrow\text{true}\because\cos A>0$$

$$\text{in acute triangles}\Rightarrow\frac{m_a^2}{s(s-a)}\leq\frac{b^2+c^2}{2bc}\Rightarrow\frac{m_a}{\sqrt{b^2+c^2}}\leq\sqrt{\frac{s(s-a)}{2bc}}$$

$$\Rightarrow \frac{m_a}{\sqrt{b^2+c^2}} \le \frac{1}{\sqrt{2}}\cos\frac{A}{2} \text{ and analogs} \Rightarrow \sum_{cyc}\frac{m_a}{\sqrt{b^2+c^2}} \le \frac{1}{\sqrt{2}}.\sum_{cyc}\cos\frac{A}{2}$$

$$\overset{CBS}{\le} \sqrt{\frac{3}{4}}.\sqrt{\sum_{cyc}2\cos^2\frac{A}{2}} = \sqrt{\frac{3}{4}}.\sqrt{\sum_{cyc}(1+\cos A)} = \sqrt{\frac{3}{4}}.\sqrt{3+1+\frac{r}{R}} = \sqrt{3\left(1+\frac{r}{4R}\right)}$$

$$\Rightarrow \sum_{cyc}\frac{m_a}{\sqrt{b^2+c^2}} \le \sqrt{3\left(1+\frac{r}{4R}\right)}\ \forall \text{ acute } \Delta\, ABC$$

$$\text{Again}, \sum_{cyc}\frac{m_a}{\sqrt{b^2+c^2}} \overset{A-G}{\ge} 3.\sqrt[3]{\frac{m_a m_b m_c}{\sqrt{(b^2+c^2)(c^2+a^2)(a^2+b^2)}}}$$

$$\overset{\substack{Lascu+A-G\\ and\\ Bandila}}{\ge} 3.\sqrt[3]{\frac{\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}}{\sqrt{\frac{R}{r}.bc}.\sqrt{\frac{R}{r}.ca}\sqrt{\frac{R}{r}.ab}}} = 3.\sqrt[3]{\frac{rs^2}{\frac{R}{r}.\sqrt{\frac{R}{r}}.4Rrs}} \overset{?}{\ge} \frac{3r}{2R}.\sqrt{6}$$

$$\Leftrightarrow \frac{rs^2}{\frac{R}{r}.\sqrt{\frac{R}{r}}.4Rrs} \overset{?}{\ge} \frac{r^3}{8R^3}.6\sqrt{6} \Leftrightarrow \frac{r^2s^4}{\frac{R^3}{r^3}.16R^2r^2s^2} \overset{?}{\ge} \frac{r^6}{64R^6}.216 \Leftrightarrow Rs^2 \overset{?}{\ge} 54r^3 \to \text{true}$$

$$\because R \overset{Euler}{\ge} 2r \text{ and } s^2 \overset{Mitrinovic}{\ge} 27r^2 \Rightarrow Rs^2 \ge 54r^3$$

$$\therefore \sum_{cyc}\frac{m_a}{\sqrt{b^2+c^2}} \ge \frac{3r}{2R}.\sqrt{6}\ \forall\ \Delta\, ABC \text{ (QED)}$$

1109. In any Δ ABC, the following relationship holds :

$$\frac{9r}{4rs} \le \sum_{cyc}\frac{\cos A}{b+c} \le \frac{9R}{16rs}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

$$abc\sum_{cyc}\frac{\cos A}{b+c} = \sum_{cyc}\frac{a(b^2+c^2-a^2)}{2(b+c)} = \sum_{cyc}\frac{a[2(b+c)^2-(a^2+b^2+c^2)-4bc]}{2(b+c)}$$

$$= \sum_{cyc}\left(a(b+c) - \frac{(a+b+c)(a^2+b^2+c^2)+4abc}{2(b+c)} + \frac{a^2+b^2+c^2}{2}\right)$$

$$= 2\sum_{cyc}bc - \frac{(a+b+c)(a^2+b^2+c^2)+4abc}{2}.\sum_{cyc}\frac{1}{b+c} + \frac{3}{2}\sum_{cyc}a^2$$

$$= 2(s^2+r^2+4Rr) - \frac{2s.2(s^2-r^2-4Rr)+16Rsr}{2} . \frac{5s^2+r^2+4Rr}{2s(s^2+r^2+2Rr)} + 3(s^2-r^2-4Rr)$$

$$= 2Rr + 8r^2 - \frac{4r^2(3R^2+5Rr+2r^2)}{s^2+r^2+2Rr}$$

Now, by Gerretsen's inequality, we have,

$$abc\sum_{cyc}\frac{\cos A}{b+c} \ge 2Rr + 8r^2 - \frac{4r^2(3R^2+5Rr+2r^2)}{(16Rr-5r^2)+r^2+2Rr}$$

$$= 9r^2 + \frac{r(R-2r)(12R+r)}{9R-2r} \overset{Euler}{\ge} 9r^2,$$

$$\text{then,} \quad \sum_{cyc}\frac{\cos A}{b+c} \ge \frac{9r^2}{abc} = \frac{9r}{4Rs}.$$

$$\text{and,} \quad abc\sum_{cyc}\frac{\cos A}{b+c} \le 2Rr + 8r^2 - \frac{4r^2(3R^2+5Rr+2r^2)}{(4R^2+4Rr+3r^2)+r^2+2Rr}$$

$$= \frac{9R^2}{4} - \frac{(R-2r)(18R^3+47R^2r+48Rr^2+24r^4)}{9R-2r} \overset{Euler}{\le} \frac{9R^2}{4},$$

$$\text{then,} \quad \sum_{cyc}\frac{\cos A}{b+c} \le \frac{9R^2}{4abc} = \frac{9R}{16rs}.$$

which completes the proof. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\cos B + \cos C = \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{bc^2+a^2b-b^3+ca^2+b^2c-c^3}{2abc}$$

$$= \frac{bc(b+c)-(b+c)(b^2-bc+c^2)+a^2(b+c)}{2abc} \therefore \sum_{cyc}\frac{\cos A}{b+c}$$

$$= \left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\frac{1}{b+c}\right) - \sum_{cyc}\frac{bc(b+c)-(b+c)(b^2-bc+c^2)+a^2(b+c)}{2abc(b+c)}$$

$$= \frac{R+r}{R} . \frac{(2\sum_{cyc}ab+\sum_{cyc}a^2)+\sum_{cyc}ab}{2s(s^2+2Rr+r^2)} - \frac{1}{8Rrs}\left(2\sum_{cyc}ab - \sum_{cyc}a^2\right)$$

$$= \frac{R+r}{R} . \frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)} - \frac{2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)}{8Rrs}$$

$$\Rightarrow \sum_{cyc} \frac{\cos A}{b+c} \stackrel{(\bullet)}{=} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2+2Rr+r^2)} \therefore \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9R}{16rs}$$

$$\stackrel{\text{via } (\bullet)}{\Leftrightarrow} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2+2Rr+r^2)} \le \frac{9R}{16rs}$$

$$\Leftrightarrow (9R^2 - 8Rr - 32r^2)s^2 + Rr(18R^2 + 41Rr + 8r^2) \stackrel{(*)}{\ge} 0$$

$\boxed{Case\ 1}$ $9R^2 - 8Rr - 32r^2 \ge 0$ and then : LHS of $(*) \ge Rr(18R^2 + 41Rr + 8r^2)$ $> 0 \Rightarrow (*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $9R^2 - 8Rr - 32r^2 < 0$ and then : LHS of $(*)$

$$= -\left(-(9R^2 - 8Rr - 32r^2)\right)s^2 + Rr(18R^2 + 41Rr + 8r^2) \stackrel{\text{Gerretsen}}{\ge}$$

$$-\left(-(9R^2 - 8Rr - 32r^2)\right)(4R^2 + 4Rr + 3r^2) + Rr(18R^2 + 41Rr + 8r^2) \stackrel{?}{\ge} 0$$

$$\Leftrightarrow 18t^4 + 11t^3 - 46t^2 - 72t - 48 \stackrel{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(18t^3 + 47t^2 + 48t + 24) \stackrel{?}{\ge} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$\therefore$ combining cases 1 and 2, $(*)$ is true $\forall \Delta ABC \therefore$ in any ΔABC, $\sum_{cyc} \frac{\cos A}{b+c} \le \frac{9R}{16rs}$

$$\text{Again}, \sum_{cyc} \frac{\cos A}{b+c} \ge \frac{9r}{4rs} \stackrel{\text{via } (\bullet)}{\Leftrightarrow} \frac{(R+4r)s^2 - Rr(4R+r)}{2Rs(s^2+2Rr+r^2)} \ge \frac{9r}{4rs}$$

$$\Leftrightarrow (2R-r)s^2 \stackrel{(**)}{\ge} r(8R^2 + 20Rr + 9r^2)$$

$$\text{Now}, (2R-r)s^2 \stackrel{\text{Gerretsen}}{\ge} (2R-r)(16Rr - 5r^2) \stackrel{?}{\ge} r(8R^2 + 20Rr + 9r^2)$$

$$\Leftrightarrow 12R^2 - 23Rr - 2r^2 \stackrel{?}{\ge} 0 \Leftrightarrow (12R+r)(R-2r) \stackrel{?}{\ge} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\ge} 2r$$

$\Rightarrow (**)$ is true $\forall \Delta ABC \therefore$ in any ΔABC, $\sum_{cyc} \frac{\cos A}{b+c} \ge \frac{9r}{4rs} \therefore$ in any ΔABC,

$\frac{9r}{4rs} \le \sum_{cyc} \frac{\cos A}{b+c} \le \frac{9R}{16rs}$, equalities iff ΔABC is equilateral (QED)

1110. In any ΔABC, the following relationship holds :

$$\frac{24r^2}{F} \le \sum_{cyc} \frac{\sec^2 \frac{A}{2}}{\sin A} \le \frac{3(7R^2 + 4r^2)}{4F}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a \ge b \ge c$ and then, $\sec^2 \frac{A}{2} \ge \sec^2 \frac{B}{2} \ge \sec^2 \frac{C}{2}$

$$\text{and } \frac{1}{\sin A} \le \frac{1}{\sin B} \le \frac{1}{\sin C} \therefore \sum_{cyc} \frac{\sec^2 \frac{A}{2}}{\sin A} \stackrel{\text{Chebyshev}}{\le} \frac{1}{3}\left(\sum_{cyc} \sec^2 \frac{A}{2}\right)\left(\sum_{cyc} \frac{1}{\sin A}\right)$$

$$=\frac{s^2+(4R+r)^2}{3s^2}\cdot\frac{2R(s^2+4Rr+r^2)}{4Rrs}\overset{?}{\le}\frac{3(7R^2+4r^2)}{4F}=\frac{3(7R^2+4r^2)}{4rs}$$

$$\Leftrightarrow 2s^4-(31R^2-24Rr+32r^2)s^2+2r(4R+r)^3\underset{(*)}{\overset{?}{\le}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\le}(8R^2+8Rr+6r^2)s^2-(31R^2-24Rr+32r^2)s^2$$

$$+2r(4R+r)^3\overset{?}{\le}0\Leftrightarrow(23R^2-32Rr+26r^2)s^2\underset{(**)}{\overset{?}{\ge}}2r(4R+r)^3$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\ge}(23R^2-32Rr+26r^2)(16Rr-5r^2)\overset{?}{\ge}2r(4R+r)^3$$

$$\Leftrightarrow 240t^3-723t^2+552t-132\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(122t(t-2)+118t^2+t+66)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{\sec^2\frac{A}{2}}{\sin A}\le\frac{3(7R^2+4r^2)}{4F}$$

$$\text{Now, }f(x)=\sec\frac{x}{2}\ \forall x\in(0,\pi)\text{ is convex}\because f''(x)=\frac{\sec\frac{x}{2}\tan^2\frac{x}{2}+\sec^3\frac{x}{2}}{4}>0$$

$$\therefore\sum_{cyc}\frac{\sec^2\frac{A}{2}}{\sin A}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}\sec\frac{A}{2}\right)^2}{\frac{s}{R}}\overset{\text{Jensen}}{\ge}\frac{R\left(3\sec\frac{\pi}{6}\right)^2}{s}=\frac{12R}{s}\overset{\text{Euler}}{\ge}\frac{24r}{s}=\frac{24r^2}{F}$$

$$\therefore\sum_{cyc}\frac{\sec^2\frac{A}{2}}{\sin A}\ge\frac{24r^2}{F}\text{ and hence, in any }\Delta\,ABC,\frac{24r^2}{F}\le\sum_{cyc}\frac{\sec^2\frac{A}{2}}{\sin A}\le\frac{3(7R^2+4r^2)}{4F},$$

$$''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1111. In ΔABC the following relationship holds,

$$2\sum_{cyc}\sqrt{\frac{m_am_b}{h_ah_b}}\ge\sum_{cyc}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By *Tereshin* and *CBS* inequalities, we have,

$$2\sqrt{\frac{m_am_b}{h_ah_b}}\ge2\sqrt{\frac{(b^2+c^2)(c^2+a^2)}{(4R)^2}\cdot\frac{(2R)^2}{bc.ca}}\ge\sqrt{\frac{(bc+ca)^2}{abc^2}}=\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}.$$

Similarly, we have,

$$2\sqrt{\frac{m_bm_c}{h_bh_c}}\ge\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}},\quad 2\sqrt{\frac{m_cm_a}{h_ch_a}}\ge\sqrt{\frac{c}{a}}+\sqrt{\frac{a}{c}}.$$

Adding these inequalities yields the desired result.

Equality holds if and only if ΔABC is equilateral.

1112. In ΔABC the following relationship holds :

$$6r\sum_{cyc}\frac{r_a}{s+n_a}+\sum_{cyc}n_a\geq 3s$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine-Ben Ajiba-Tanger-Morocco

We have :

$$n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$$

$$=-\frac{4s.sr^2}{a(s-a)}=s^2-2h_ar_a\ \Rightarrow\ \frac{r_a}{s+n_a}=\frac{s-n_a}{2h_a}\ \text{(and analogs)}$$

Also, we have,

$$\frac{n_b{}^2-n_a{}^2}{s}=\left(s-b+\frac{(c-a)^2}{b}\right)-\left(s-a+\frac{(b-c)^2}{a}\right)=a-b+\frac{a(c-a)^2-b(b-c)^2}{ab}$$

$$=a-b+\frac{(a-b)(a^2+ab+b^2-2ca-2bc+c^2)}{ab}=\frac{(a-b)(a+b-c)^2}{ab},$$

so if $a\geq b$ then $n_a\leq n_b$ or $s-n_a\geq s-n_b$.

WLOG, we assume that $a\geq b\geq c$. Since $h_a=\frac{2F}{a}$ (and analogs) then we have,

$$s-n_a\geq s-n_b\geq s-n_c\ \text{and}\ \frac{1}{h_a}\geq\frac{1}{h_b}\geq\frac{1}{h_c}.$$

Using Chebyshev's inequality, we have,

$$6r\sum_{cyc}\frac{r_a}{s+n_a}=3r\sum_{cyc}\frac{s-n_a}{h_a}\geq r\sum_{cyc}\frac{1}{h_a}.\sum_{cyc}(s-n_a)=3s-\sum_{cyc}n_a.$$

Therefore,

$$6r\sum_{cyc}\frac{r_a}{s+n_a}+\sum_{cyc}n_a\geq 3s.$$

1113. In ΔABC the following relationship holds:

$$\frac{a}{9R^2-a^2}+\frac{b}{9R^2-b^2}+\frac{c}{9R^2-c^2}<\frac{4}{s}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by George Florin Șerban-Romania

Show that $\frac{a}{9R^2-a^2}\le\frac{2}{5R}$, $9R^2-a^2=9R^2-4R^2\sin^2 A=R^2(3-2\sin A)(3+2\sin A)>0$,

$\frac{2R\sin A}{R^2(9-4\sin^2 A)}\le\frac{2}{5R}$, $5\sin A\le 9-4\sin^2 A$, $4\sin^2 A+5\sin A-9\le 0$,

$(\sin A-1)(4\sin A+9)\le 0$, true, because $\sin A\le 1$, $\sin A-1\le 0$ and $4\sin A+9\ge -4+9=5>0$.

Equality is if $\mu(A)=\frac{\pi}{2}$. Then $\sum_{cyc}\frac{a}{9R^2-a^2}<\frac{6}{5R}<\frac{4}{s}$, $s<\frac{10R}{3}$, Mitrinovic inequality,

$s\le\frac{3\sqrt{3}R}{2}<\frac{10R}{3}$, then $9\sqrt{3}<20$, $243<400$, true, then $\sum_{cyc}\frac{a}{9R^2-a^2}<\frac{4}{s}$.

1114. $ABCD$ **– convex quadrilateral,** $M\in Int(ABCD)$, F **– area,**

s **– semiperimeter,**

a,b,c,d **– sides. Prove that:**

$$\frac{MA^4}{b}+\frac{MB^4}{c}+\frac{MC^4}{d}+\frac{MD^4}{a}\ge\frac{2F^2}{s}.$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$MA^2+MB^2\ge 2MA\cdot MB\ge 2MA\cdot MB\cdot\sin M=4F_{MAB}$$
$$Analogously, MB^2+MC^2\ge 4F_{MBC}, MC^2+MD^2\ge 4F_{MDC} \text{ and}$$
$$MD^2+MA^2\ge 4F_{MAD}. \text{By adding, we get:}$$
$$2(MA^2+MB^2+MC^2+MD^2)\ge 4(F_{MAB}+F_{MBC}+F_{MDC}+F_{MAD})\Leftrightarrow$$
$$MA^2+MB^2+MC^2+MD^2\ge 2F_{ABCD}$$
$$\frac{MA^4}{b}+\frac{MB^4}{c}+\frac{MC^4}{d}+\frac{MD^4}{a}=\frac{(MA^2)^2}{b}+\frac{(MB^2)^2}{c}+\frac{(MC^2)^2}{d}+\frac{(MD^2)^2}{a}\ge$$
$$\overset{Bergstrom}{\ge}\frac{(MA^2+MB^2+MC^2+MD^2)^2}{a+b+c+d}\ge\frac{4F_{ABCD}^2}{2s}=\frac{2F^2}{s}.$$

Solution 2 by George Florin Șerban-Romania

$$F=[AMB]+[MBC]+[MCD]+[MAD]=$$
$$=\frac{AM\cdot MB\cdot\sin(AMB)}{2}+\frac{MC\cdot MB\cdot\sin(CMB)}{2}+\frac{MC\cdot MD\cdot\sin(CMD)}{2}$$

$$+\frac{MA\cdot MD\cdot\sin(AMD)}{2}\le\frac{AM\cdot MB}{2}+\frac{MB\cdot MC}{2}+\frac{MC\cdot MD}{2}+\frac{MD\cdot MA}{2}\le$$

$$\le\frac{MA^2+MB^2}{4}+\frac{MC^2+MB^2}{4}+\frac{MC^2+MD^2}{4}+\frac{MD^2+MA^2}{4}=$$

$$=\frac{1}{2}(MA^2+MB^2+MC^2+MD^2),\ because\ 2xy\le x^2+y^2\Leftrightarrow(x-y)^2\ge 0,$$

$$(\forall)x,y\in\mathbb{R}$$

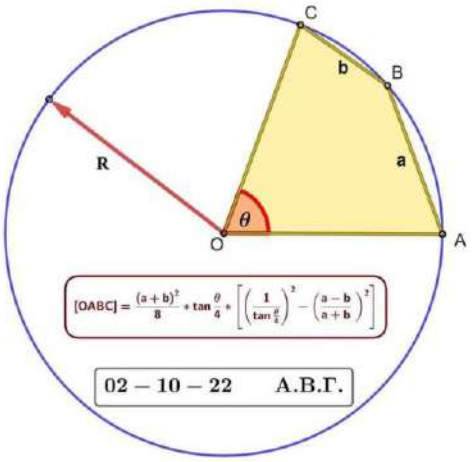
$$Then,\ MA^2+MB^2+MC^2+MD^2\ge 2F\ and\ hence,$$

$$\frac{MA^4}{b}+\frac{MB^4}{c}+\frac{MC^4}{d}+\frac{MD^4}{a}=\frac{(MA^2)^2}{b}+\frac{(MB^2)^2}{c}+\frac{(MC^2)^2}{d}+\frac{(MD^2)^2}{a}\ge$$

$$\overset{Bergstrom}{\ge}\frac{(MA^2+MB^2+MC^2+MD^2)^2}{a+b+c+d}\ge\frac{4F_{ABCD}^2}{2s}=\frac{2F^2}{s}.$$

Equality holds if and only if ABCD is a square.

1115.



Prove that :

$$[OABC]=\frac{(a+b)^2}{8}.\tan\frac{\theta}{4}.\left(\left(\frac{1}{\tan\frac{\theta}{4}}\right)^2-\left(\frac{a-b}{a+b}\right)^2\right)$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via cosine law, } R^2+R^2-2R^2\cos\theta=a^2+b^2-2ab\cos\left(180^\circ-\frac{\theta}{2}\right)$$

$$\Rightarrow 4R^2\sin^2\frac{\theta}{2} = a^2+b^2+2ab\cos\frac{\theta}{2} \therefore [OABC] = [AOC]+[ABC]$$

$$= \frac{1}{2}.R^2.\sin\theta + \frac{1}{2}.ab.\sin\left(180^\circ - \frac{\theta}{2}\right) = \frac{a^2+b^2+2ab\cos\frac{\theta}{2}}{8\sin^2\frac{\theta}{2}}.\sin\theta + \frac{ab.\sin\frac{\theta}{2}}{2}$$

$$= \frac{(a^2+b^2)\sin\theta}{8\sin^2\frac{\theta}{2}} + ab.\left(\frac{\sin\theta\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} + \frac{\sin\frac{\theta}{2}}{2}\right)$$

$$= \frac{(a^2+b^2)\sin\theta}{8\sin^2\frac{\theta}{2}} + ab.\frac{\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}+\sin^3\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= \frac{(a^2+b^2)\sin\theta}{8\sin^2\frac{\theta}{2}} + \frac{ab.\sin\frac{\theta}{2}\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)}{2\sin^2\frac{\theta}{2}} \Rightarrow [OABC] \overset{(*)}{=} \frac{(a^2+b^2)\sin\theta}{8\sin^2\frac{\theta}{2}} + \frac{ab}{2\sin\frac{\theta}{2}}$$

$$\text{Now, } \frac{(a+b)^2}{8}.\tan\frac{\theta}{4}.\left(\left(\frac{1}{\tan\frac{\theta}{4}}\right)^2 - \left(\frac{a-b}{a+b}\right)^2\right)$$

$$= \frac{(a+b)^2}{8}.\tan\frac{\theta}{4}.\frac{(a+b)^2-(a-b)^2.\tan^2\frac{\theta}{4}}{\tan^2\frac{\theta}{4}.(a+b)^2}$$

$$= \frac{(a^2+b^2)\left(1-\frac{\sin^2\frac{\theta}{4}}{\cos^2\frac{\theta}{4}}\right)}{8\tan\frac{\theta}{4}} + \frac{2ab\left(1+\tan^2\frac{\theta}{4}\right)}{8\tan\frac{\theta}{4}}$$

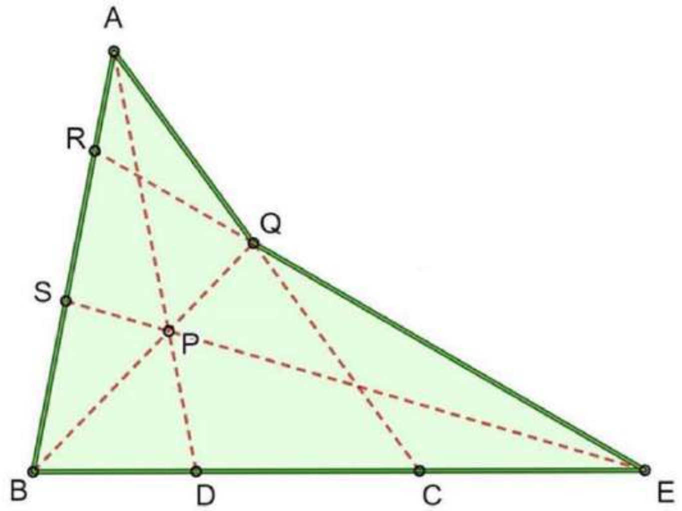
$$= \frac{(a^2+b^2).\cos\frac{\theta}{2}}{8\tan\frac{\theta}{4}.\cos^2\frac{\theta}{4}} + \frac{ab}{4\tan\frac{\theta}{4}.\cos^2\frac{\theta}{4}} = \frac{(a^2+b^2).\cos\frac{\theta}{2}}{8\cos\frac{\theta}{4}.\sin\frac{\theta}{4}} + \frac{ab}{4\cos\frac{\theta}{4}.\sin\frac{\theta}{4}}$$

$$= \frac{(a^2+b^2).\left(2\cos\frac{\theta}{2}.\sin\frac{\theta}{2}\right)}{8\sin\frac{\theta}{2}.\sin\frac{\theta}{2}} + \frac{ab}{2\sin\frac{\theta}{2}} \Rightarrow \frac{(a+b)^2}{8}.\tan\frac{\theta}{4}.\left(\left(\frac{1}{\tan\frac{\theta}{4}}\right)^2 - \left(\frac{a-b}{a+b}\right)^2\right)$$

$$\overset{(**)}{=} \frac{(a^2+b^2)\sin\theta}{8\sin^2\frac{\theta}{2}} + \frac{ab}{2\sin\frac{\theta}{2}} \therefore (*),(**) \Rightarrow$$

$$[OABC] = \frac{(a+b)^2}{8}.\tan\frac{\theta}{4}.\left(\left(\frac{1}{\tan\frac{\theta}{4}}\right)^2 - \left(\frac{a-b}{a+b}\right)^2\right) \text{ (QED)}$$

1116.



$ARBCEQ$ – complete quadrilateral, AD, ES – cevians, $AD \cap ES = \{P\}$

$\frac{AP}{PD} + \frac{AR}{RB} = \frac{AQ}{QC} + \frac{AS}{SB}$. Prove that B, P, Q – are collinear.

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution by Adrian Popa-Romania

$$We\ must\ show: \frac{QA}{QC} \cdot \frac{BC}{BD} \cdot \frac{PD}{DA} = 1.$$

$$In\ \Delta ABC: R \to Q \to C: \frac{QA}{QC} \cdot \frac{EC}{EB} \cdot \frac{RB}{RA} = 1 \Rightarrow \frac{QA}{QC} = \frac{EB}{EC} \cdot \frac{RA}{RB}; \quad (1)$$

$$In\ \Delta ABD: S \to P \to E: \frac{SA}{SB} \cdot \frac{EB}{ED} \cdot \frac{PD}{PA} = 1 \Rightarrow \frac{SA}{SB} = \frac{ED}{EB} \cdot \frac{PA}{PD}$$

$$\frac{QA}{QC} + \frac{SA}{SB} = \frac{EB}{EC} \cdot \frac{RA}{RB} + \frac{ED}{EB} \cdot \frac{PA}{PD} = \frac{AR}{RB} + \frac{AP}{PD}$$

$$\frac{AR}{RB}\left(\frac{EB}{EC} - 1\right) = \frac{AP}{PD}\left(1 - \frac{ED}{EB}\right)$$

$$\frac{AR}{RB} \cdot \frac{BC}{EC} = \frac{AP}{PD} \cdot \frac{BD}{EB} \Rightarrow \frac{PD}{PA} \cdot \frac{EB}{BD} = \frac{RB}{AR} \cdot \frac{EC}{BC} \Rightarrow$$

$$\frac{PD}{PA} \cdot \frac{EB}{EC} = \frac{RB}{AR} \cdot \frac{BD}{BC} \Rightarrow \frac{PD}{PA} \cdot \frac{BC}{BD} = \frac{RB}{AR} \cdot \frac{EC}{EB}$$

$$\frac{QA}{QC} \cdot \frac{BC}{BD} \cdot \frac{PD}{PA} = \frac{QA}{QC} \cdot \frac{RB}{RA} \cdot \frac{EC}{EB} \overset{(1)}{=} \frac{EB}{EC} \cdot \frac{RA}{RB} \cdot \frac{RB}{RA} \cdot \frac{EC}{EB} = 1$$

From Menelaus reverse theorem $\Rightarrow B, P, Q -$ *collinear*.

1117. In $\Delta ABC, AB = AC, M \in (AB), N \in (BC), AN = 4NB, BM = 2MC$
$\{P\} = MN \cap AC$. Prove that:

$$\frac{15BC^2 + 32F}{28AB} < BP\cos B + AP\sin B < \frac{17}{7} \cdot \max(AB, BC)$$

Proposed by Radu Diaconu-Romania

Solution by George Florin Șerban-Romania

ΔABC, **T.Menelaos,** $\frac{AN}{BN}\frac{BM}{CM}\frac{CP}{AP} = 1,\ AB = AC = b, BC = a,\ AP = 8PC,\ PC = \frac{b}{7}, AP = \frac{8b}{7}.$

ΔABP, **T.Stewart,** $BC^2 AP = AB^2 PC + BP^2 AC - AC \cdot PC \cdot AP,\ \frac{8a^2 b}{7} = \frac{b^3}{7} + BP^2 \cdot b - \frac{8b^3}{49},$

$BP^2 = \frac{56a^2 + b^2}{49},\ BP\cos B + AP\sin B \le \sqrt{(BP^2 + AP^2)(\cos^2 B + \sin^2 B)} = \sqrt{\frac{56a^2 + 65b^2}{49}},$

C.B.S. inequality, $\max(AB, BC) \ge AB, \max(AB, BC) \ge BC,$ **then** $\max(AB, BC) \ge \frac{AB + BC}{2}.$

Show that $\sqrt{\frac{56a^2 + 65b^2}{49}} < \frac{17}{7} \cdot \frac{AB + BC}{2},\ 2\sqrt{56a^2 + 65b^2} < 17(a + b),$

$224a^2 + 260b^2 < 289a^2 + 289b^2 + 578ab,\ 65a^2 + 29b^2 + 578ab > 0,$ **true,then**

$BP\cos B + AP\sin B \le \sqrt{\frac{56a^2 + 65b^2}{49}} < \frac{17}{7} \cdot \frac{AB + BC}{2} \le \frac{17}{7}\max(AB, BC),$ **true,then**

$BP\cos B + AP\sin B < \frac{17}{7}\max(AB, BC).\ \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{a^2}{2ac} = \frac{a}{2b} < 1,$ **then** $a < 2b.$

Show that $BP\cos B > \frac{15BC^2}{28AB},\ \frac{a\sqrt{56a^2 + b^2}}{14b} > \frac{15a^2}{28b},\ 2\sqrt{56a^2 + b^2} > 15a,$

$224a^2 + 4b^2 > 225a^2,\ 4b^2 > a^2,\ a < 2b,$ **true,then** $BP\cos B > \frac{15BC^2}{28AB}.$ **Show that**

$AP\sin B > \frac{32F}{28AB},\ \frac{8b}{7}\sin B > \frac{8F}{7b},\ \frac{8b}{7}\sin B > \frac{8 \cdot \frac{ab\sin B}{2}}{7b} = \frac{4a\sin B}{7},$ **then** $4a < 8b, a < 2b,$

true,then $BP\cos B > \frac{15BC^2}{28AB}.$ **Then** $BP\cos B + AP\sin B > \frac{15BC^2 + 32F}{28AB},$ **true.Then**

$$\frac{15BC^2 + 32F}{28AB} < BP\cos B + AP\sin B < \frac{17}{7}\max(AB, BC).$$

1118. In ΔABC, $x^2\sin A - 4x + 5 - \sin A > 0, \forall x > 0$ and $F \geq \min(m_a^2, m_b^2, m_c^2)$

Prove that ΔABC is right isoscele.

Proposed by Radu Diaconu-Romania

Solution by George Florin Șerban-Romania

$\sin A > 0, \Delta = 16 - 4(\sin A)(5 - \sin A) = 4\sin^2 A - 20\sin A + 16 = 4(\sin^2 A - 5\sin A + 4) = 4(\sin A - 1)(\sin A - 4) \leq 0$

and $\Delta = 4(\sin A - 1)(\sin A - 4) \geq 0$, because $\sin A \leq 1 < 4$ then $\Delta = 0$. If

$\sin A - 4 = 0, \sin A = 4$, false.If $\sin A - 1 = 0, \sin A = 1, \mu(A) = \frac{\pi}{2}$. If $b \leq c < a$ or $c \leq b < a$ then

$m_b \geq m_c > m_a$ or $m_c \geq m_b > m_a$ then $m_b^2 \geq m_c^2 > m_a^2$ or $m_c^2 \geq m_b^2 > m_a^2$ then

$\min\{m_a^2, m_b^2, m_c^2\} = m_a^2 = \frac{a^2}{4} = \frac{b^2 + c^2}{4} \leq S = \frac{bc}{2}$, then $b^2 + c^2 \leq 2bc$, $0 \leq (b - c)^2 \leq 0$, then

$b - c = 0$, $b = c$, $\mu(B) = \mu(C) = \frac{\pi}{4}, \mu(A) = \frac{\pi}{2}$. ΔABC right isoscele triangle.

1119. In any Δ ABC, the following relationship holds:

$$\frac{ab(a+c)}{b+c} + \frac{bc(b+a)}{c+a} + \frac{ca(c+b)}{a+b} < \frac{6R^3}{r}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{ab(a+c)}{b+c} + \frac{bc(b+a)}{c+a} + \frac{ca(c+b)}{a+b} = \sum_{cyc} \frac{ab(2s-b)}{b+c}$$

$$= 2s\sum_{cyc}\frac{a(b+c-c)}{b+c} - \sum_{cyc}\frac{a^2b^2}{ab+ac} \overset{\text{Bergstrom}}{\leq} 2s.\sum_{cyc} a - 2s.4Rrs.\sum_{cyc}\frac{1}{b^2+bc}$$

$$-\frac{\left(\sum_{cyc} ab\right)^2}{2\sum_{cyc} ab} \overset{\text{Bergstrom}}{\leq} 4s^2 - 8Rrs^2.\frac{9}{\sum_{cyc} a^2 + \sum_{cyc} ab} - \frac{s^2+4Rr+r^2}{2}$$

$$\leq 4s^2 - 8Rrs^2.\frac{9}{2\sum_{cyc} a^2} - \frac{s^2+4Rr+r^2}{2} \overset{\text{Leibnitz}}{\leq} 4s^2 - 8Rrs^2.\frac{9}{18R^2}$$

$$-\frac{s^2+4Rr+r^2}{2} = 4s^2 - \frac{4rs^2}{R} - \frac{s^2+4Rr+r^2}{2} \overset{?}{<} \frac{6R^3}{r}$$

$$\Leftrightarrow \frac{6R^3}{r} + \frac{s^2+4Rr+r^2}{2} + \frac{4rs^2}{R} \overset{?}{>} 4s^2 \Leftrightarrow \frac{12R^4 + Rr(s^2+4Rr+r^2) + 8r^2s^2}{2Rr} \overset{?}{>} 4s^2$$

$$\Leftrightarrow (7Rr-8r^2)s^2 \underset{(*)}{\overset{?}{<}} 12R^4+4R^2r^2+Rr^3$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\leq} (7Rr-8r^2)(4R^2+4Rr+3r^2) \overset{?}{<} 12R^4+4R^2r^2+Rr^3$$

$$\Leftrightarrow 3t^4-7t^3+2t^2+3t+6 \overset{?}{>} 0 \ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(3t^2+5t+10)+23\right)+12 \overset{?}{>} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \text{in any } \Delta\, ABC, \frac{ab(a+c)}{b+c}+\frac{bc(b+a)}{c+a}+\frac{ca(c+b)}{a+b}<\frac{6R^3}{r} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

$$\sum_{cyc}\frac{ab(c+a)}{b+c}=\sum_{cyc}a(c+a)\left(1-\frac{c}{b+c}\right)=\sum_{cyc}(ca+a^2)-\sum_{cyc}\frac{ca(c+a)}{b+c}$$

$$\overset{AM-GM}{\leq} 2\sum_{cyc}a^2-3\sqrt[3]{(abc)^2} \overset{Leibniz}{\leq} 2.9R^2-3\sqrt[3]{(4Rsr)^2} \overset{Cosnita-Turtoiu}{\leq} 18R^2-3\sqrt[3]{8R^2r^2.27Rr}$$

$$=18R^2-18Rr=\frac{9R^3}{2r}-\frac{9R(R-2r)^2}{2r}\leq\frac{9R^3}{2r}<\frac{6R^3}{r}, \text{as desired.}$$

1120. In any Δ ABC, the following relationship holds :

$$\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}<\frac{4\sqrt{3}}{w_a+w_b+m_c}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{4\sqrt{3}}{w_a+w_b+m_c} \overset{\text{Lessel-Pelling}}{\geq} \frac{4\sqrt{3}}{\sqrt{3}s} \therefore \text{in order to prove :}$$

$$\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}<\frac{4\sqrt{3}}{w_a+w_b+m_c}, \text{it suffices to prove :}$$

$$\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2} \overset{(\bullet)}{<} \frac{4}{s}$$

$$\text{Now, }\sum_{cyc}\frac{a}{b^2+c^2}=\frac{1}{(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)-a^2b^2c^2}.\sum_{cyc}\left(a\left(\sum_{cyc}a^2b^2+a^4\right)\right)$$

$$=\frac{\sum_{cyc}a^5+(\sum_{cyc}a)(\sum_{cyc}a^2b^2)}{(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)-a^2b^2c^2} \overset{(*)}{=} \sum_{cyc}\frac{a}{b^2+c^2}$$

$$\text{Now}, \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^4\right) = \sum_{cyc} a^5 + \sum_{cyc}\left(ab\left(\sum_{cyc} a^3 - c^3\right)\right)$$

$$\Rightarrow \left(\sum_{cyc} a\right)\left(2\sum_{cyc} a^2b^2 - 16r^2s^2\right) = \sum_{cyc} a^5 + \left(\sum_{cyc} a^3\right)\left(\sum_{cyc} ab\right) - abc\left(\sum_{cyc} a^2\right)$$

$$\Rightarrow \sum_{cyc} a^5 + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right)$$

$$= \left(\sum_{cyc} a\right)\left(3\sum_{cyc} a^2b^2 - 16r^2s^2\right) - \left(\sum_{cyc} a^3\right)\left(\sum_{cyc} ab\right) + abc\left(\sum_{cyc} a^2\right)$$

$$= 2s\left(3(s^2+4Rr+r^2)^2 - 48Rrs^2 - 16r^2s^2\right) - 2s(s^2+4Rr+r^2)(s^2-6Rr-3r^2)$$

$$+8Rrs(s^2-4Rr-r^2) \Rightarrow \frac{\sum_{cyc} a^5 + (\sum_{cyc} a)(\sum_{cyc} a^2b^2)}{(\sum_{cyc} a^2)(\sum_{cyc} a^2b^2) - a^2b^2c^2}$$

$$= \frac{2s\left(3(s^2+4Rr+r^2)^2 - 48Rrs^2 - 16r^2s^2\right) - 2s(s^2+4Rr+r^2)(s^2-6Rr-3r^2)}{2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2 - 16Rrs^2) - 16R^2r^2s^2}$$

$$+\frac{8Rrs(s^2-4Rr-r^2)}{2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2) - 16R^2r^2s^2} \overset{(**)}{=} \sum_{cyc} \frac{a}{b^2+c^2}$$

$$\therefore (*), (**) \Rightarrow (\bullet) \Leftrightarrow$$

$$\frac{2s\left(3(s^2+4Rr+r^2)^2 - 48Rrs^2 - 16r^2s^2\right) - 2s(s^2+4Rr+r^2)(s^2-6Rr-3r^2)}{2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2) - 16R^2r^2s^2}$$

$$+\frac{8Rrs(s^2-4Rr-r^2)}{2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2) - 16R^2r^2s^2} < \frac{4}{s}$$

$$\Leftrightarrow s^6 - (15Rr - 6r^2)s^4 + r^2s^2(52R^2 - 3Rr - 5r^2) - 2r^3(4R+r)^3 \overset{(\bullet\bullet)}{>} 0$$

$$\text{Now, LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq} (Rr + r^2)s^4 + r^2s^2(52R^2 - 3Rr - 5r^2) - 2r^3(4R+r)^3$$

$$\overset{?}{>} 0 \Leftrightarrow (R+r)s^4 + r(52R^2 - 3Rr - 5r^2)s^2 - 2r^2(4R+r)^3 \underset{(\bullet\bullet\bullet)}{\overset{?}{>}} 0$$

$$\text{Again, LHS of } (\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\geq} \left((R+r)(16Rr - 5r^2) + r(52R^2 - 3Rr - 5r^2)\right)s^2$$

$$-2r^2(4R+r)^3 \Leftrightarrow (34R^2 + 4Rr - 5r^2)s^2 \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{>}} r(4R+r)^3$$

$$\text{Once again, LHS of } (\bullet\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\geq} (34R^2 + 4Rr - 5r^2)(16Rr - 5r^2) \overset{?}{>} r(4R+r)^3$$

$$\Leftrightarrow 240t^3 - 77t^2 - 56t + 12 \overset{?}{>} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(240t^2 + 403t + 750) + 1512 \overset{?}{>} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \therefore \text{in any } \Delta\, ABC,$$

$$\frac{a}{b^2+c^2} + \frac{b}{c^2+a^2} + \frac{c}{a^2+b^2} < \frac{4\sqrt{3}}{w_a + w_b + m_c} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Lessel – Pelling's inequality, we have,

$$w_a + w_b + m_c \le s\sqrt{3},$$

so it suffices to prove that,

$$\frac{a}{b^2+c^2} + \frac{b}{c^2+a^2} + \frac{c}{a^2+b^2} < \frac{8}{a+b+c}.$$

We have,

$$(a+b+c).\frac{a}{b^2+c^2} = \frac{a^2}{b^2+c^2} + \frac{a(b+c)}{b^2+c^2} \overset{CBS}{\le} \frac{3a^2}{2(b^2+c^2)+(b^2+c^2)} + \frac{2a}{b+c}$$

$$\overset{CBS}{\le} \frac{3a^2}{(b+c)^2+b^2+c^2} + \frac{4a}{2(b+c)} \overset{b+c>a}{<} \frac{3a^2}{a^2+b^2+c^2} + \frac{4a}{a+b+c}$$

Then,

$$(a+b+c)\left(\frac{a}{b^2+c^2} + \frac{b}{c^2+a^2} + \frac{c}{a^2+b^2}\right) < \sum_{cyc}\left(\frac{3a^2}{a^2+b^2+c^2} + \frac{4a}{a+b+c}\right) = 3+4 < 8,$$

which completes the proof.

1121. In ΔABC the following relationship holds:

$$\frac{32s^5 - a^5 - b^5 - c^5}{8s^3 - a^3 - b^3 - c^3} \ge 120r^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Şerban-Romania

$$We\ use\ the\ formulas:$$

$$(a+b+c)^5 = 5(a+b)(b+c)(c+a)\left(\sum_{cyc} a^2 + \sum_{cyc} ab\right)$$

$$(a+b+c)^3 - a^3 - b^3 - c^3 = 3(a+b)(b+c)(c+a)$$

$$\frac{32s^5 - a^5 - b^5 - c^5}{8s^3 - a^3 - b^3 - c^3} = \frac{(a+b+c)^5 - a^5 - b^5 - c^5}{(a+b+c)^3 - a^3 - b^3 - c^3} =$$

$$= \frac{5(a+b)(b+c)(c+a)\left(\sum_{cyc} a^2 + \sum_{cyc} ab\right)}{3(a+b)(b+c)(c+a)} =$$

$$=\frac{5(\sum a^2+\sum ab)}{3}=\frac{5(48Rr-2r^2-8Rr+s^2+r^2+4Rr)}{3}=$$

$$=\frac{5(3s^2-4Rr-r^2)}{3}$$

By Gerretsen's and Euler's inequalities, we have:

$$\frac{32s^5-a^5-b^5-c^5}{8s^3-a^3-b^3-c^3}=\frac{5(3s^2-4Rr-r^2)}{3}\ge\frac{5(48Rr-15r^2-4Rr-r^2)}{3}=$$

$$=\frac{5(44Rr-16r^2)}{3}=\frac{220Rr-80r^2}{3}\ge\frac{440r^2-80r^2}{3}=\frac{360r^2}{2}=120r^2.$$

Therefore, $\frac{32s^5-a^5-b^5-c^5}{8s^3-a^3-b^3-c^3}\ge 120r^2$. *Equality holds for* $a=b=c$.

Solution 2 by Tapas Das-India

Let $x=a+b+c,\ y=ab+bc+ca,\ z=abc.$

$$\sum_{cyc}a^5=x^5-5x^3y+5xy^2+5x^2z-5yz$$

$$(a+b+c)^5-a^5-b^5-c^5=x^5-\left(x^5-5x^3y+5xy^2+5x^2z-5yz\right)=$$

$$=5(x^3y-xy^2-x^2z+yz)$$

$$(a+b+c)^3-a^3-b^3-c^3=x^3-(x^3-3xy+3z)=3(xy-z)$$

$$\frac{32s^5-a^5-b^5-c^5}{8s^3-a^3-b^3-c^3}=\frac{5}{3}\cdot\frac{x^3y-xy^2-x^2z+yz}{xy-z}=$$

$$=\frac{5}{3}\cdot\frac{(xy-z)(x^2-y)}{xy-z}=\frac{5}{3}(x^2-y)=\frac{5}{3}[(a+b+c)^2-(ab+bc+ca)]=$$

$$=\frac{5}{3}(a^2+b^2+c^2+ab+bc+ca)\overset{AM-GM}{\ge}\frac{5}{3}\left[3(abc)^{\frac{2}{3}}+3(abc)^{\frac{2}{3}}\right]=\frac{5}{3}\cdot 6(abc)^{\frac{2}{3}}=$$

$$=10(4Rrs)^{\frac{2}{3}}=10\left(4\cdot 2r\cdot r\cdot 3\sqrt{3}r\right)^{\frac{2}{3}}=10\left(8r^3\cdot 3^{\frac{3}{2}}\right)^{\frac{2}{3}}=120r^2.$$

1122. **Tom said : "In Δ ABC, if $a+b>3c$, then : $R<4r$"**

and Fred said : "In Δ ABC, if $a+b<3c$, then : $R>4r$".

Who is true and who is false and why ?

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2\cos^2 15° - 1 = \frac{\sqrt{3}}{2} \Rightarrow \cos^2 15° = \frac{4+2\sqrt{3}}{8} = \frac{\left(\sqrt{3}+1\right)^2}{8}$$

$$\Rightarrow \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}} \Rightarrow 1 - 2\sin^2 7.5° = \frac{\sqrt{3}+1}{2\sqrt{2}} \Rightarrow 2\sin^2 7.5° = \frac{4-\sqrt{6}-\sqrt{2}}{4}$$

$$\Rightarrow \sin 7.5° \overset{(*)}{=} \frac{1}{2}.\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}\ and\ 1 - 2\sin^2 15° = \frac{\sqrt{3}}{2} \Rightarrow \sin^2 15° = \frac{4-2\sqrt{3}}{8}$$

$$= \frac{\left(\sqrt{3}-1\right)^2}{8} \Rightarrow \sin 15° \overset{(**)}{=} \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\text{Now}, a + b > 3c \Leftrightarrow 2s > 4c \Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} > 8R\cos\frac{C}{2}\sin\frac{C}{2}$$

$$\Leftrightarrow \cos\frac{A+B}{2} + \cos\frac{A-B}{2} > 4\sin\frac{C}{2} \therefore a + b > 3c \Leftrightarrow \cos\frac{A-B}{2} > 3\sin\frac{C}{2} \to (1)$$

$$\text{Also}, R < 4r \Leftrightarrow R < 16R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow \frac{1}{8} < \sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)$$

$$\therefore R < 4r \Leftrightarrow \sin^2\frac{C}{2} - \sin\frac{C}{2}.\cos\frac{A-B}{2} + \frac{1}{8} < 0 \to (2)$$

$$\text{Let } A = B \text{ and } C = 15° \text{ and then}, \cos\frac{A-B}{2} = 1 > 3 * \frac{1}{2}.\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}$$

$$\overset{\text{via }(*)}{=} 3 * \sin 7.5° = 3\sin\frac{C}{2} \overset{\text{via }(1)}{\Rightarrow} a + b > 3c \text{ and in order to prove } t^2 - t + \frac{1}{8} > 0,$$

$$\text{it suffices to prove}: t < \frac{1 - \sqrt{1 - 4.\frac{1}{8}}}{2} = \frac{1 - \frac{1}{\sqrt{2}}}{2} \therefore \text{in order to prove}:$$

$$\sin^2 7.5° - \sin 7.5° + \frac{1}{8} > 0, \text{it suffices to prove}: \sin 7.5° < \frac{1 - \frac{1}{\sqrt{2}}}{2}$$

$$\overset{\text{via }(*)}{\Leftrightarrow} \frac{1}{2}.\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}} < \frac{1 - \frac{1}{\sqrt{2}}}{2} \Leftrightarrow \frac{4-\sqrt{6}-\sqrt{2}}{2} < \frac{3-2\sqrt{2}}{2}$$

$$\Leftrightarrow 1 + \sqrt{2} < \sqrt{6} \Leftrightarrow 3 + 2\sqrt{2} < 6 \Leftrightarrow 2\sqrt{2} < 3 \Leftrightarrow 8 < 9 \to true$$

$$\therefore \sin^2 7.5° - \sin 7.5° + \frac{1}{8} > 0 \Rightarrow \sin^2\frac{C}{2} - \sin\frac{C}{2}.\cos\frac{A-B}{2} + \frac{1}{8} > 0 \overset{\text{via }(2)}{\Rightarrow} R > 4r$$

$$\therefore \text{in } \Delta\, ABC, \text{if } a + b > 3c, then\ for\ A = B \text{ and } C = 15°$$

$$(\text{for such values}, a + b > 3c\ is\ preserved), \text{we have } R > 4r$$

$$\therefore \boxed{\text{in } \Delta\, ABC, \text{if } a + b > 3c, then: R < 4r \text{ isn't always true}}$$

$$\therefore \text{Tom's assertion isn't always true}$$

$$\text{Again, let } A = B \text{ and } C = 105° \text{ and then}, 1 - 2\sin^2 52.5° = \cos 105° = -\cos 75°$$

$$= -\sin 15° \overset{\text{via }(**)}{=} -\frac{\sqrt{3}-1}{2\sqrt{2}} \Rightarrow \sin^2 52.5° = \frac{1 + \frac{\sqrt{3}-1}{2\sqrt{2}}}{2} = \frac{4+\sqrt{6}-\sqrt{2}}{8}$$

$$\Rightarrow \sin 52.5° = \frac{1}{2}.\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}} \therefore \cos\frac{A-B}{2} = 1 < 3 * \frac{1}{2}.\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}$$

$$= 3 * \sin 52.5^\circ = 3\sin\frac{C}{2} \overset{via\,(1)}{\Rightarrow} a + b < 3c \text{ and } \sin^2 52.5^\circ - \sin 52.5^\circ + \frac{1}{8} =$$

$$\frac{4+\sqrt{6}-\sqrt{2}}{8} - \frac{1}{2}\cdot\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}} + \frac{1}{8} \approx -0.0389438 < 0$$

$$\Rightarrow \sin^2\frac{C}{2} - \sin\frac{C}{2}\cdot\cos\frac{A-B}{2} + \frac{1}{8} < 0 \overset{via\,(2)}{\Rightarrow} R < 4r$$

$\therefore$ in Δ ABC, if $a + b < 3c$, then for $A = B$ and $C = 105^\circ$

(for such values, $a + b < 3c$ is preserved), we have $R < 4r$

$\therefore$ $\boxed{\text{in } \Delta \text{ ABC, if } a+b<3c, \text{ then : } R > 4r \text{ isn't always true}}$

$\therefore$ Fred's assertion isn't always true

$\therefore$ $\boxed{\text{Tom's as well as Fred's respective assertions aren't always true}}$ (ans)

1123. In Δ ABC, prove or disprove that :

$$\frac{(a^2+b^2+c^2)^4}{(ab+bc+ca)^3} + \frac{R^3}{8r} \geq r^2 + \sum_{cyc}\frac{a^8}{b^3c^3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

We shall prove that for right triangles with $\frac{R}{r} \geq 3$, holds :

$$\frac{(a^2+b^2+c^2)^4}{(ab+bc+ca)^3} + \frac{R^3}{8r} < r^2 + \sum_{cyc}\frac{a^8}{b^3c^3}$$

$$\text{Now, } \sum_{cyc}\frac{a^8}{b^3c^3} = \frac{1}{128R^3r^3s^4}\cdot\left(\sum_{cyc}a^{11}\right)\left(\sum_{cyc}a\right) \overset{CBS}{\geq} \frac{1}{128R^3r^3s^4}\cdot\left(\sum_{cyc}a^6\right)^2$$

$$\geq \frac{1}{128R^3r^3s^4}\left(\frac{1}{3}\left(\sum_{cyc}a^3\right)^2\right)^2 = \frac{16s^4(s^2-6Rr-3r^2)^4}{9*128R^3r^3s^4} = \frac{(s^2-6Rr-3r^2)^4}{72R^3r^3}$$

$$\overset{?}{>} \frac{(a^2+b^2+c^2)^4}{(ab+bc+ca)^3} + \frac{R^3-8r^3}{8r}$$

$$\Leftrightarrow \frac{((2R+r)^2-6Rr-3r^2)^4}{72R^3r^3} \overset{?}{>} \frac{16((2R+r)^2-4Rr-r^2)^4}{((2R+r)^2+4Rr+r^2)^3} + \frac{R^3-8r^3}{8r}$$

$$\left(\because \Delta \text{ ABC is right} \Rightarrow \cos A\cos B\cos C = 0 \Rightarrow \frac{s^2-(2R+r)^2}{4R^2} = 0 \Rightarrow s = 2R+r\right)$$

$$\Leftrightarrow \frac{((2R+r)^2-6Rr-3r^2)^4}{72R^3r^3}$$

$$\overset{?}{>}\frac{128r\left((2R+r)^2-4Rr-r^2\right)^4+\left(R^3-8r^3\right)\left((2R+r)^2+4Rr+r^2\right)^3}{8r((2R+r)^2+4Rr+r^2)^3}$$

$$\Leftrightarrow 16384t^{14}+65536t^{13}+15808t^{12}-519552t^{11}-219744t^{10}+223872t^9$$
$$+383696t^8+51104t^7-157000t^6-67712t^5+33664t^4+36928t^3+12800t^2$$
$$+2048t+128\overset{?}{>}0\left(t=\frac{R}{r}\right)$$
$$(t-3)\left(16384t^{13}+114688t^{12}+359872t^{11}+560064t^{10}+1460448t^9\right)$$
$$+(t-3)\left(4605216t^8 14199344t^7+42649136t^6+127790408t^5+383303512t^4\right)$$
$$+(t-3)\left(1149944200t^3+3449869528t^2+10349621384t+31048866200\right)$$
$$+93146598728\to\text{true}\because t=\frac{R}{r}\geq 3\therefore\text{for right triangles with }\frac{R}{r}\geq 3,\text{holds}:$$
$$\sum_{cyc}\frac{a^8}{b^3c^3}>\frac{\left(a^2+b^2+c^2\right)^4}{(ab+bc+ca)^3}+\frac{R^3-8r^3}{8r}\Rightarrow\frac{\left(a^2+b^2+c^2\right)^4}{(ab+bc+ca)^3}+\frac{R^3}{8r}<r^2+\sum_{cyc}\frac{a^8}{b^3c^3}$$
$$\Rightarrow\boxed{\frac{\left(a^2+b^2+c^2\right)^4}{(ab+bc+ca)^3}+\frac{R^3}{8r}\geq r^2+\sum_{cyc}\frac{a^8}{b^3c^3}\text{ isn't true }\forall\,\Delta\,ABC}$$

1124. In ΔABC *the following relationship holds*:

$$1008+\frac{1}{\sin^4\frac{A}{2}}+\frac{1}{\sin^4\frac{B}{2}}+\frac{1}{\sin^4\frac{C}{2}}\leq 1056\left(\frac{R}{2r}\right)^4$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x=\sin\frac{A}{2},\ \ y=\sin\frac{B}{2},\ \ z=\sin\frac{C}{2}$ and $t=xyz=\frac{r}{4R}\leq\frac{1}{8}$. The problem becomes,

$$1008+\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\leq\frac{33}{128t^4}.$$

Since $x^2+y^2+z^2+2xyz=1$ then we have,

$$1008+\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}=1008+\frac{\left(x^2y^2+y^2z^2+z^2x^2\right)^2-2x^2y^2z^2\left(x^2+y^2+z^2\right)}{t^4}$$

$$\leq 1008+\frac{\left(\frac{\left(x^2+y^2+z^2\right)^2}{3}\right)^2-2t^2(1-2t)}{t^4}=1008+\frac{(1-2t)^4-18t^2(1-2t)}{9t^4}$$

$$=\frac{33}{128t^4}-\frac{(1-8t)\left(169+2376t+18240t^2+145408t^3\right)}{9t^4}\leq\frac{33}{128t^4},$$

because $t\leq\frac{1}{8}$. So the proof is completed. Equality holds iff ΔABC is equilateral.

1125. In ΔABC the following relationship holds:

$$\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\geq\frac{36r^2}{\sqrt[4]{3R^2}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\overset{AM-GM}{\geq}3\left[(abc)^{\frac{3}{2}}\right]^{\frac{1}{3}}=3\sqrt{abc}$$

$$We\ need\ to\ show: 3\sqrt{abc}\geq\frac{36r^2}{\sqrt[4]{3R^2}}\Leftrightarrow$$

$$3\sqrt{4RF}\cdot\sqrt{R}\cdot 3^{\frac{1}{4}}\geq 36r^2\Leftrightarrow 2R\sqrt{rs}\cdot 3^{\frac{1}{4}}\geq 12r^2\Leftrightarrow$$

$$2\cdot 2r\sqrt{r3\sqrt{3}r}\cdot 3^{\frac{1}{4}}\geq 12r^2\Leftrightarrow 4r^2\cdot 3^{\frac{3}{4}}\cdot 3^{\frac{1}{4}}\geq 12r^2\Leftrightarrow$$

$$12r^2\geq 12r^2\ (true!)$$

Solution 2 by George Florin Șerban-Romania

$$Let\ be\ f:(0,\infty)\to\mathbb{R}, f(x)=\sqrt{x}-concave\ function, then\ by\ Jensen's\ inequality:$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\leq 3\sqrt{\frac{a+b+c}{3}}=\sqrt{6s}$$

$$By\ Bergstrom's\ inequality:$$

$$\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\geq\frac{(\sum a)^2}{\sum\sqrt{a}}\geq\frac{4s^2}{\sqrt{6s}}=\frac{4s^2\sqrt{6s}}{6s}=\frac{2s\sqrt{6s}}{3}\geq\frac{36r^2}{\sqrt[4]{3R^2}}\Leftrightarrow$$

$$\frac{s\sqrt{6s}}{3}\geq\frac{18r^2}{\sqrt[4]{3R^2}}\Leftrightarrow\frac{s^4\cdot 36r^2}{81}\geq\frac{18^4r^8}{3R^2}\Leftrightarrow\frac{s^6}{27}\geq\frac{18^3r^8}{2R^2}$$

$$By\ Mitrinovic's\ inequality:$$

$$\frac{s^6}{27}\geq\frac{\left(3\sqrt{3}r\right)^6}{27}\geq\frac{18^3r^8}{2R^2}\Leftrightarrow\frac{3^9r^6}{3^3}\geq\frac{3^6\cdot 4r^8}{R^2}\Leftrightarrow R^2\geq 4r^2\Leftrightarrow R\geq 2r\ (Euler).$$

$$Equality\ holds\ for\ a=b=c.$$

$$Therefore,\ \frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\geq\frac{36r^2}{\sqrt[4]{3R^2}}$$

1126. In ΔABC the following relationship holds,

$$\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin\frac{A}{4}+\cos\frac{A}{4}}\ge\frac{3\sqrt{2}r}{R}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have, $\frac{\cos\frac{A}{2}}{\sin\frac{A}{4}+\cos\frac{A}{4}}=\frac{\cos\frac{A}{2}}{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{A}{4}\right)}=\frac{\sin\frac{B+C}{2}}{\sqrt{2}\cos\frac{B+C}{4}}=\sqrt{2}\sin\frac{B+C}{4}$.

Since $f(x)=\sin\frac{x}{2}$ is concave on $(0,\pi)$, then by Popoviciu's inequality, we have,

$$\sum_{cyc}\sin\frac{B+C}{4}=\sum_{cyc}f\left(\frac{B+C}{2}\right)\ge\frac{1}{2}\left(\sum_{cyc}f(A)+3f\left(\frac{A+B+C}{3}\right)\right)=\frac{1}{2}\left(\sum_{cyc}\sin\frac{A}{2}+\frac{3}{2}\right)$$

$$\overset{AM-GM}{\ge}\frac{1}{2}\left(3\sqrt[3]{\prod_{cyc}\sin\frac{A}{2}}+\frac{3}{2}\right)=\frac{3}{2}\left(\sqrt[3]{\frac{r}{4R}}+\frac{1}{2}\right)\overset{Euler}{\ge}\frac{3}{2}\left(\frac{r}{R}+\frac{r}{R}\right)=\frac{3r}{R}.$$

Which completes the proof. Equality holds iff ΔABC is equilateral.

1127. In ΔABC the following relationship holds:

$$\frac{1}{2}-\frac{r}{2R}-\frac{r^2}{4R^2}\le\sum_{cyc}sin^4\frac{A}{2}\le 1-\frac{2r}{R}+\frac{3r^2}{4R^2}$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Șerban-Romania

By Gerretsen inequality:

$$4R^2+4Rr+3r^2\ge s^2\ge 16Rr-5r^2$$

$$-16Rr+5r^2\ge -s^2\ge -4R^2-4Rr-3r^2$$

$$8R^2+r^2-16Rr+5r^2\ge 8R^2+r^2-s^2\ge 8R^2+r^2-4R^2-4Rr-3r^2$$

$$4R^2-4Rr-2r^2\le 8R^2+r^2-s^2\le 8R^2-16Rr+6r^2$$

$$\frac{4R^2-4Rr-2r^2}{8R^2}\le\frac{8R^2+r^2-s^2}{8R^2}\le\frac{8R^2-16Rr+6r^2}{8R^2}$$

$$\frac{1}{2}-\frac{r}{2R}-\frac{r^2}{4R^2}\le\frac{8R^2+r^2-s^2}{8R^2}\le 1-\frac{2r}{R}+\frac{3r^2}{4R^2}$$

$$\frac{1}{2}-\frac{r}{2R}-\frac{r^2}{4R^2}\le\sum_{cyc}sin^4\frac{A}{2}\le 1-\frac{2r}{R}+\frac{3r^2}{4R^2}$$

Equality for $a=b=c$.

1128. If in ΔABC, $\sqrt[3]{sin^2A+sinA+6}+4sin^2A+4sinA=10$ and

$m(\sphericalangle B)=2m(\sphericalangle C)$ then:

$$\frac{OI^2}{OI_a^2}=\frac{2-\sqrt{3}}{4+\sqrt{3}}\ and\ \frac{r}{R}+\frac{R}{r}>3$$

Proposed by Radu Diaconu-Romania

Solution by George Florin Șerban-Romania

$\sin^2 A+\sin A=x,\ \sqrt[3]{x+6}=10-4x,\ (10-4x)^3=x+6,\ 1000-1200x+480x^2-64x^3=x+6,$

$64x^3-480x^2+1201x-994=0,\ 64x^3-128x^2-352x^2+704x+497x-994=0,$

$64x^2(x-2)-352x(x-2)+497(x-2)=0,\ (x-2)(64x^2-352x+497)=0.$ If

$64x^2-352x+497=0,\ \Delta=123904-127232=-3328<0,$ false. If $x-2=0, x=2,$

$\sin^2 A+\sin A-2=0,\ (\sin A-1)(\sin A+2)=0.$ If $\sin A+2=0, \sin A=-2,$ false. If

$\sin A-1=0, \sin A=1, m(\sphericalangle A)=90^0,\ m(\sphericalangle B)+m(\sphericalangle C)=90^0=3m(\sphericalangle C),$

$m(\sphericalangle C)=30^0, m(\sphericalangle B)=60^0,\ BC=a,\ T.30^0\Rightarrow AB=c=\frac{a}{2}, \cos 30^0=\frac{\sqrt{3}}{2}=\frac{b}{a}, AC=b=\frac{a\sqrt{3}}{2}.$

$p=2R+r,\ \frac{a+\frac{a\sqrt{3}}{2}+\frac{a}{2}}{2}=a+r,\ a+\frac{a\sqrt{3}}{2}+\frac{a}{2}=2a+2r,\ 3a+a\sqrt{3}=4a+4r,\ r=\frac{a\sqrt{3}-a}{4},$

$\frac{r}{R}+\frac{R}{r}=\frac{a(\sqrt{3}-1)}{2a}+\frac{2a}{a(\sqrt{3}-1)}=\frac{\sqrt{3}-1}{2}+\frac{2}{\sqrt{3}-1}=\frac{\sqrt{3}-1}{2}+\sqrt{3}+1=\frac{3\sqrt{3}+1}{2}>3,$

$3\sqrt{3}>5\Rightarrow 27>25,$ true, then $\frac{r}{R}+\frac{R}{r}>3.$ $OI^2=R^2-2Rr,\ OI_a^2=R^2+2R\cdot r_a,$ Euler,

$$r_a=\frac{S}{p-a}=\frac{bc}{-a+b+c}=\frac{\frac{a^2\sqrt{3}}{4}}{-a+\frac{a}{2}+\frac{a\sqrt{3}}{2}}=\frac{a^2\sqrt{3}}{4}\cdot\frac{2}{a\sqrt{3}-a}=\frac{a\sqrt{3}}{2(\sqrt{3}-1)}=\frac{a\sqrt{3}(\sqrt{3}+1)}{4}=\frac{a(3+\sqrt{3})}{4},$$

$$\frac{OI^2}{OI_a^2}=\frac{R(R-2r)}{R(R+2r_a)}=\frac{R-2r}{R+2r_a}=\frac{\frac{a}{2}-\frac{a\sqrt{3}-a}{2}}{\frac{a}{2}+\frac{a(3+\sqrt{3})}{2}}=\frac{a(2-\sqrt{3})}{a(4+\sqrt{3})}=\frac{2-\sqrt{3}}{4+\sqrt{3}},\ \text{true, then}\ \frac{OI^2}{OI_a^2}=\frac{2-\sqrt{3}}{4+\sqrt{3}}.$$

1129. If H –orthocenter in $\Delta ABC, AH=a', BH=b', CH=c'$ then:

$$[ABC]=\frac{1}{4}(aa'+bb'+cc')$$

Proposed by Alpaslan Ceran-Turkiye

Solution by Daniel Sitaru-Romania

Let be AA', BB' –altitudes.

$$\sphericalangle B'BA=\frac{\pi}{2}-\sphericalangle C, \quad cos\left(\frac{\pi}{2}-\sphericalangle C\right)=\frac{BA'}{BH}\Rightarrow BA'=BHsinC$$

$$\Delta BHA'\sim\Delta AHB'\Rightarrow\frac{AH}{BH}=\frac{B'A}{A'B}$$

$$AH=\frac{BH\cdot B'A}{A'B}=\frac{BH\cdot ccosA}{BH\cdot sinC}=\frac{2RsinC\cdot cosA}{sinC}=2RcosA$$

$$aa'+bb'+cc'=a\cdot 2RcosA+b\cdot 2RcosB+c\cdot 2RcosC=$$
$$=2R^2(2sinAcosA+2sinBcosB+2sinCcosC)=$$
$$=2R^2(sin2A+sin2B+sin2C)=$$

$$=2R^2\cdot 4sinAsinBsinC=2R^2\cdot 4\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot sinC=2\cdot absinC=4F$$

$$[ABC]=\frac{1}{4}(aa'+bb'+cc')$$

1130. In ΔABC, G –centroid, the following relationship holds:

$$\frac{2F}{R^2}\le\sum_{cyc}sin(\sphericalangle BGC)\le\frac{27R^2}{8F}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Daniel Sitaru-Romania

Lemma 1

In ΔABC the following relationship holds:

$$m_bm_c\le\frac{2a^2+bc}{4}\quad(1)$$

Proof:

$$m_bm_c\le\frac{2a^2+bc}{4}\Leftrightarrow 16m_b^2m_c^2\le(2a^2+bc)^2\Leftrightarrow$$

$$(2a^2+2c^2-b^2)(2a^2+2b^2-c^2)\le(2a^2+bc)^2$$

$$a^2b^2+a^2c^2+2b^2c^2-c^4-b^4-2a^2bc\le 0$$

$$a^2(b-c)^2-(b-c)^2(b+c)^2\le 0$$

$$(b-c)^2(a+b+c)(b+c-a)\ge 0\ (True)$$

Equality holds for $b=c$.

Lemma 2

In ΔABC the following relationship holds:

$$\sum_{cyc} m_b m_c \le \frac{27R^2}{4}\quad (2)$$

Proof:

$$\sum_{cyc} m_b m_c \overset{(1)}{\le} \sum_{cyc}\frac{2a^2+bc}{4}=\frac{1}{4}\left(2\sum_{cyc}a^2+\sum_{cyc}bc\right)=$$

$$=\frac{1}{4}(2\cdot 2(s^2-r^2-4Rr)+s^2+r^2+4Rr)=\frac{1}{4}(5s^2-3r^2-12Rr)\overset{GERRETSEN}{\le}$$

$$\le\frac{1}{4}(5(4R^2+4r^2+3Rr)-3r^2-12Rr)=5R^2+2Rr+3r^2\overset{EULER}{\le}$$

$$\le 5R^2++2R\cdot\frac{R}{2}+3\cdot\frac{R^2}{4}=\frac{27R^2}{4}$$

Equality holds for $a=b=c$.

Back to the problem:

$$[BGC]=\frac{F}{3}=\frac{1}{2}\cdot\frac{2}{3}m_b\cdot\frac{2}{3}m_c\cdot sin(\sphericalangle BGC)\Rightarrow sin(\sphericalangle BGC)=\frac{3F}{2m_bm_c}$$

$$\sum_{cyc} sin(\sphericalangle BGC)=\sum_{cyc}\frac{3F}{2m_bm_c}=\frac{3F}{2}\sum_{cyc}\frac{1}{m_bm_c}\le$$

$$\le\frac{3F}{2}\sum_{cyc}\frac{1}{\sqrt{s(s-b)\cdot s(s-c)}}=\frac{3F}{2}\cdot\frac{1}{F\sqrt{s}}\sum_{cyc}\sqrt{s-a}\overset{CBS}{\le}$$

$$\le\frac{3}{2\sqrt{s}}\cdot\sqrt{(1+1+1)(s-a+s-b+s-c)}=\frac{3\sqrt{3}}{2}\le\frac{27R^2}{8F}\Leftrightarrow$$

$$\Leftrightarrow 12\sqrt{3}F\le 27R^2$$

$$12\sqrt{3}F=4\sqrt{3}rs\overset{EULER}{\le}12\sqrt{3}\cdot\frac{R}{2}\cdot s\overset{MITRINOVIC}{\le}6\sqrt{3}\cdot R\cdot\frac{3R\sqrt{3}}{2}=27R^2$$

$$\sum_{cyc} sin(\measuredangle BGC)=\sum_{cyc}\frac{3F}{2m_bm_c}=\frac{3F}{2}\sum_{cyc}\frac{1}{m_bm_c}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{3F}{2}\cdot\frac{(1+1+1)^2}{m_am_b+m_bm_c+m_cm_a}\overset{(2)}{\geq}\frac{3F}{2}\cdot\frac{9}{\frac{27R^2}{4}}=\frac{2F}{R^2}$$

Equality holds for $a=b=c$.

1131. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}\sum_{cyc}\frac{b+c}{a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c=4R\cos^2\frac{A}{2}\therefore r_b+r_c-h_a\overset{?}{\geq}\frac{b^2+c^2}{4R}\Leftrightarrow 4R\cos^2\frac{A}{2}\overset{?}{\geq}\frac{b^2+c^2+2bc}{4R}$$

$$\Leftrightarrow 16R^2\cos^2\frac{A}{2}\overset{?}{\geq}(b+c)^2=16R^2\cos^2\frac{A}{2}.\cos^2\frac{B-C}{2}\Leftrightarrow\cos^2\frac{B-C}{2}\overset{?}{\leq}1\to\text{true}$$

$$\therefore\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{\sqrt{m_a.\frac{b^2+c^2}{4R}}}{h_a}\overset{\text{Tereshin}}{\geq}\frac{\sqrt{\frac{b^2+c^2}{4R}.\frac{b^2+c^2}{4R}}}{\frac{bc}{2R}}$$

$$\Rightarrow\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\text{ and analogs}$$

$$\therefore\sum_{cyc}\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\sum_{cyc}\frac{b+c}{a}$$

$$\therefore\text{ in any }\Delta\text{ ABC},\sum_{cyc}\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}\sum_{cyc}\frac{b+c}{a},$$

$''=''$ iff Δ ABC is equilateral (QED)

1132. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{r_br_c}{a^2}\geq\frac{r}{2R}+2\prod_{cyc}\frac{r_a}{w_a}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_br_c}{a^2}\ge\frac{r}{2R}+2\prod_{cyc}\frac{r_a}{w_a}\Leftrightarrow\frac{s(s-a)b^2c^2}{16R^2r^2s^2}\ge\frac{r}{2R}+2rs^2.\prod_{cyc}\frac{b+c}{2bc\cos\frac{A}{2}}$$

$$\Leftrightarrow\frac{s^2\left(\left(\sum_{cyc}ab\right)^2-2.4Rrs.2s\right)-s.4Rrs.\sum_{cyc}ab}{16R^2r^2s^2}\ge\frac{r}{2R}+2rs^2.\frac{2s(s^2+2Rr+r^2)}{8.16R^2r^2s^2.\frac{s}{4R}}$$

$$\Leftrightarrow\frac{(s^2+4Rr+r^2)(s^2+4Rr+r^2-4Rr)-16Rrs^2}{16R^2r^2}\ge\frac{4r^2+s^2+2Rr+r^2}{8Rr}$$

$$\Leftrightarrow s^4-(14Rr-2r^2)s^2-r^2(4R^2+6Rr-r^2)\overset{(*)}{\ge}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(2Rr-3r^2)s^2-r^2(4R^2+6Rr-r^2)\overset{?}{\ge}0$$

$$\Leftrightarrow(2R-3r)s^2-r(4R^2+6Rr-r^2)\underbrace{\overset{?}{\ge}}_{(**)}0$$

$$\text{Again, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(2R-3r)(16Rr-5r^2)-r(4R^2+6Rr-r^2)$$

$$=4(7R-2r)(R-2r)\overset{\text{Euler}}{\ge}0\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\text{ in any }\Delta\,ABC,\sum_{cyc}\frac{r_br_c}{a^2}\ge\frac{r}{2R}+2\prod_{cyc}\frac{r_a}{w_a},''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1133. In any acute $\Delta\,ABC$ with $a=\min\{a,b,c\}$, the following relationship holds :

$$\frac{w_a}{h_a}.\sqrt{\frac{b+c}{b+c-a}}\le\sqrt{2}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

$$\frac{w_a}{h_a}\sqrt{\frac{b+c}{b+c-a}}=\frac{2\sqrt{bc.s(s-a)}}{b+c}.\frac{2R}{bc}.\sqrt{\frac{b+c}{2(s-a)}}=\sqrt{\frac{8R^2s}{bc(b+c)}}=\sqrt{\frac{2Ra}{r(b+c)}}\overset{?}{\le}\sqrt{2}$$

$$\Leftrightarrow\frac{R}{r}\le\frac{b+c}{a}\Leftrightarrow\frac{R+r}{r}\le\frac{2s}{a}\Leftrightarrow a.\frac{R+r}{R}\le\frac{2F}{R}\Leftrightarrow a\sum_{cyc}\cos A\le\sum_{cyc}a\cos A$$

$$\Leftrightarrow 0\le(b-a)\cos B+(c-a)\cos C,$$

which is true. So the proof is completed.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{w_a}{h_a}\cdot\sqrt{\frac{b+c}{b+c-a}}=\frac{2abc.\cos\frac{A}{2}}{2rs(b+c)}\cdot\sqrt{\frac{b+c}{b+c-a}}$$

$$=\frac{4Rrs}{rs(b+c)}\cdot\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{b+c}{2(s-a)}}\overset{?}{\le}\sqrt{2}\Leftrightarrow\frac{16R^2}{(b+c)^2}\cdot\frac{sa}{4Rrs}\cdot\frac{b+c}{2}\overset{?}{\le}2$$

$$\Leftrightarrow\frac{a}{b+c}\overset{?}{\le}\frac{r}{R}\Leftrightarrow\frac{b+c}{a}\overset{?}{\ge}\frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

$$\Leftrightarrow(b+c)(a+b-c).(b+c-a)(c+a-b)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow(ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow 2a^2b^2+2a^2bc+2ab(b^2-c^2)-(a^2+b^2-c^2)(ab+ac+b^2-c^2)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow 2a^2b^2-(a^2+b^2-c^2)(ab+ac)+2ab(b^2-c^2)-2ab(b^2-c^2).\cos C\overset{?}{\ge}0$$

$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-(b^2-c^2)(ab+ac)+2ab(b^2-c^2).2\sin^2\frac{C}{2}\overset{?}{\ge}0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)-(b^2-c^2)(ab+ac)+(b^2-c^2)(c^2-(a-b)^2)\overset{?}{\ge}0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)+(b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac)\overset{?}{\ge}0$$

$$\Leftrightarrow\left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$

$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac)\overset{?}{\ge}0$$

$$\Leftrightarrow(c^2+a^2-b^2)(2b^2-ab-ac)+(b^2-c^2)(b^2+c^2-a^2-2ac)\overset{?}{\ge}0$$

$$\Leftrightarrow(b^2-c^2)(b^2+c^2-2ac)-(b^2-c^2)(2b^2-ab-ac)$$

$$+a^2(2b^2-ab-ac)-a^2(b^2-c^2)\overset{?}{\ge}0$$

$$\Leftrightarrow(b^2-c^2)\left((c^2-ca)-(b^2-ab)\right)+a^2\left((c^2-ca)+(b^2-ab)\right)\overset{?}{\ge}0$$

$$\Leftrightarrow(c^2-ca)(b^2-c^2+a^2)+(b^2-ab)(a^2+c^2-b^2)\overset{?}{\ge}0$$

$$\Leftrightarrow c(c-a)(a^2+b^2-c^2)+b(b-a)(c^2+a^2-b^2)\overset{?}{\ge}0$$

$\rightarrow$ **true** $\because\Delta ABC$ **being acute** $\Rightarrow(a^2+b^2-c^2),(c^2+a^2-b^2)>0$ and
$a=\min\{a,b,c\}\Rightarrow a\le b,c\Rightarrow(c-a),(b-a)\ge 0\therefore$ **in any acute** ΔABC

with $a=\min\{a,b,c\},\frac{w_a}{h_a}\cdot\sqrt{\frac{b+c}{b+c-a}}\le\sqrt{2},''=''$ **iff** ΔABC **is equilateral (QED)**

1134. In any ΔABC, the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_am_bm_c}{w_ah_bh_c}}-1\right)}\ge\frac{1}{2}\sum_{cyc}\frac{b+c}{a}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since, $m_a \geq \frac{b+c}{2}\cos\frac{A}{2}$, then, $\frac{m_a}{w_a} \geq \frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{b+c}{2bc\cos\frac{A}{2}} = \frac{(b+c)^2}{4bc}$.

Also, by Tereshin and CBS inequalities, we have,

$$\frac{m_b m_c}{h_b h_c} \geq \frac{(c^2+a^2)(a^2+b^2)}{(4R)^2}\cdot\frac{(2R)^2}{ca.ab} \geq \frac{(ca+ab)^2}{4a^2bc} = \frac{(b+c)^2}{4bc}.$$

Using these results, we obtain,

$$\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}} \geq \frac{(b+c)^2}{4bc} \quad \text{(and analogs)}$$

Now, let us prove that, in any ΔABC, $\frac{1}{\sin\omega} \geq \frac{b}{c}+\frac{c}{b}$ (1)

$$(1) \Leftrightarrow 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2} \geq (b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow} 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$$

$$\geq (2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow 0 \geq -a^4(b^2+c^2)^2 + 2(b^4+c^4)(a^2b^2+c^2a^2) - (b^4+c^4)^2$$

$$= -[a^2(b^2+c^2)-(b^4+c^4)]^2,$$

which is true.

Then,

$$\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}}-1\right) \geq 2\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{(b+c)^2}{2bc}-1\right) = \left(\frac{b}{c}+\frac{c}{b}\right)^2 \quad \text{(and analogs)}$$

Therefore,

$$\sum_{cyc}\sqrt{\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}}-1\right)} \geq \sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right) = \sum_{cyc}\frac{b+c}{a}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_a m_b m_c}{w_a h_b h_c}}-1\right)} \overset{Lascu + Tereshin}{\geq}$$

$$\sqrt{\frac{\frac{b^2+c^2}{4R}}{\frac{bc}{2R}}\left(2\sqrt{\frac{\frac{(b+c)^2}{2}\cos\frac{A}{2}\cdot\frac{(c^2+a^2)(a^2+b^2)}{16R^2}}{2bc\cos\frac{A}{2}\cdot\frac{ca.ab}{4R^2}}}-1\right)}\overset{CBS}{\geq}\sqrt{\frac{b^2+c^2}{2bc}\left(2\sqrt{\frac{(b+c)^2.(ca+ab)^2}{16a^2b^2c^2}}-1\right)}$$

$$=\sqrt{\left(\frac{b^2+c^2}{2bc}\right)\left(\frac{(b+c)^2}{2bc}-1\right)}=\sqrt{\left(\frac{b^2+c^2}{2bc}\right)\left(\frac{b^2+c^2}{2bc}\right)}$$

$$\therefore\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_am_bm_c}{w_ah_bh_c}}-1\right)}\geq\frac{b^2+c^2}{2bc}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_am_bm_c}{w_ah_bh_c}}-1\right)}\geq\frac{1}{2}\sum_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2}\sum_{cyc}\frac{b+c}{a}$$

$$\therefore\text{ in any }\Delta\text{ ABC},\sum_{cyc}\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_am_bm_c}{w_ah_bh_c}}-1\right)}\geq\frac{1}{2}\sum_{cyc}\frac{b+c}{a},$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

1135. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_br_c}{a^2}\geq\frac{r}{2R}+\frac{2r_ar_br_c}{w_aw_bw_c}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x=\frac{b+c}{a},\ y=\frac{c+a}{b},\ z=\frac{a+b}{c}$. We have, $xyz=x+y+z+2$, and,

$$\bullet\sum_{cyc}\frac{r_br_c}{a^2}=\sum_{cyc}\frac{s(s-a)}{a^2}=\sum_{cyc}\frac{(b+c)^2-a^2}{4a^2}=\frac{x^2+y^2+z^2-3}{4}.$$

$$\bullet\frac{2r}{R}=\frac{(b+c-a)(c+a-b)(a+b-c)}{abc}=(x-1)(y-1)(z-1)$$

$$=xyz-\sum_{cyc}xy+\sum_{cyc}x-1$$

$$=2\sum_{cyc}x+1-\sum_{cyc}xy.$$

$$\bullet\ \frac{r_a r_b r_c}{w_a w_b w_c}=\frac{s^2 r.\prod_{cyc}(b+c)}{8(abc)^2.\prod_{cyc}\cos\frac{A}{2}}=\frac{s^2 r}{8.4Rsr.\frac{s}{4R}}.\prod_{cyc}\frac{b+c}{a}=\frac{xyz}{8}=\frac{x+y+z+2}{8}.$$

So it suffices to prove that,

$$\sum_{cyc}x^2-3\ge\left(2\sum_{cyc}x+1-\sum_{cyc}xy\right)+\left(\sum_{cyc}x+2\right)\Leftrightarrow\frac{1}{2}\sum_{cyc}(x+y)^2\ge 3\sum_{cyc}x+6,$$

Let $p=x+y+z=\sum_{cyc}\frac{b+c}{a}\overset{AM-GM}{\ge}6.$ We have,

$$\frac{1}{2}\sum_{cyc}(x+y)^2\overset{CBS}{\ge}\frac{1}{2}.\frac{\left(\sum_{cyc}(x+y)\right)^2}{3}=\frac{2p^2}{3}=3p+6+(p-6)\left(\frac{2p}{3}+1\right)\ge 3p+6,$$

which completes the proof. Equality holds iff ΔABC is equilateral.

1136. In any Δ ABC, the following relationship holds :

$$\frac{a^3}{a^4+b^4}+\frac{b^3}{b^4+c^4}+\frac{c^3}{c^4+a^4}<\frac{144\sqrt{3}(R^3-2Rr^2)}{s^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{a^4+b^4}+\frac{b^3}{b^4+c^4}+\frac{c^3}{c^4+a^4}\overset{\text{Chebyshev}}{\le}\sum_{cyc}\frac{2\left((a^3+b^3)-b^3\right)}{(a+b)(a^3+b^3)}$$

$$=2\sum_{cyc}\frac{1}{a+b}-2\sum_{cyc}\frac{b^3}{(a+b)(a^3+b^3)}<2\sum_{cyc}\frac{(b+c)(c+a)}{(a+b)(b+c)(c+a)}$$

$$=\frac{\sum_{cyc}ab+\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)}{s(s^2+2Rr+r^2)}=\frac{s^2+4Rr+r^2+4s^2}{s(s^2+2Rr+r^2)}\overset{?}{<}\frac{144\sqrt{3}(R^3-2Rr^2)}{s^4}$$

$$\Leftrightarrow s.s^2(5s^2+4Rr+r^2)\underbrace{\overset{?}{<}}_{(*)}144\sqrt{3}(R^3-2Rr^2)(s^2+2Rr+r^2)$$

$$\text{Now, LHS of }(*)\overset{\text{Mitrinovic + Gerretsen}}{\le}\frac{3\sqrt{3}R}{2}(4R^2+4Rr+3r^2)(5s^2+4Rr+r^2)$$

$$\overset{?}{<}144\sqrt{3}(R^3-2Rr^2)(s^2+2Rr+r^2)$$

$$\Leftrightarrow(76R^2-20Rr-207r^2)s^2+r(176R^3+76R^2r-400Rr^2-195r^3)\underbrace{\overset{?}{>}}_{(**)}0$$

$$\because 76R^2-20Rr-207r^2=(R-2r)(76R+132r)+57r^2\overset{\text{Euler}}{\ge}57r^2>0$$

$$\therefore \text{LHS of }(**)\overset{\text{Gerretsen}}{\ge}(76R^2-20Rr-207r^2)(16Rr-5r^2)$$

$$+r(176R^3+76R^2r-400Rr^2-195r^3)\overset{?}{>}0 \Leftrightarrow 116t^3-52t^2-301t+70\overset{?}{>}0$$

$$\left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(116t^2+180t+59)+188\overset{?}{>}0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \text{in any } \Delta\, ABC,$$

$$\frac{a^3}{a^4+b^4}+\frac{b^3}{b^4+c^4}+\frac{c^3}{c^4+a^4}<\frac{144\sqrt{3}(R^3-2Rr^2)}{s^4} \text{ (QED)}$$

1137. In $\Delta\, ABC$, $I \to$ incenter, $r_A, r_B, r_C \to$ inradii of $\Delta\, BIC$, $\Delta\, CIA$, $\Delta\, AIB$ respectively.

$$\text{Prove that}: \frac{r_A+r_B+r_C}{r} \geq 3(2\sqrt{3}-3)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

Let R_A, R_B, R_C be circumradii of $\Delta\, BIC$, $\Delta\, CIA$, $\Delta\, AIB$ respectively. Then :

$$2R_A.\sin\left(180°-\frac{B+C}{2}\right)=a \Rightarrow 2R_A\cos\frac{A}{2}=4R\cos\frac{A}{2}\sin\frac{A}{2} \Rightarrow R_A=2R\sin\frac{A}{2}$$

$$\Rightarrow r_A=4*2R\sin\frac{A}{2}*\sin\frac{B}{4}.\sin\frac{C}{4}.\sin\frac{\pi+A}{4} \Rightarrow \frac{r_A}{r}$$

$$=\frac{4*2R\sin\frac{A}{2}*\sin\frac{B}{4}.\sin\frac{C}{4}.\sin\frac{\pi+A}{4}}{4R\sin\frac{A}{2}*\left(2\sin\frac{B}{4}\cos\frac{B}{4}\right)*\left(2\sin\frac{C}{4}\cos\frac{C}{4}\right)} \Rightarrow \frac{r_A}{r}=\frac{\sin\frac{\pi+A}{4}}{2\cos\frac{B}{4}\cos\frac{C}{4}} \text{ and analogs}$$

$$\Rightarrow \frac{r_A+r_B+r_C}{r}\overset{A-G}{\geq}\frac{3}{2}\sqrt[3]{\prod_{cyc}\frac{\sin\frac{\pi+A}{4}}{\cos^2\frac{A}{4}}}\overset{?}{\geq}3(2\sqrt{3}-3)$$

$$\Leftrightarrow \ln\sqrt[3]{\prod_{cyc}\frac{\sin\frac{\pi+A}{4}}{\cos^2\frac{A}{4}}}\overset{?}{\geq}\ln(4\sqrt{3}-6) \Leftrightarrow \sum_{cyc}\left(\ln\frac{\sin\frac{\pi+A}{4}}{\cos^2\frac{A}{4}}\right)\underset{(*)}{\overset{?}{\geq}}3.\ln(4\sqrt{3}-6)$$

$$\text{Let } f(x)=\ln\frac{\sin\frac{\pi+x}{4}}{\cos^2\frac{x}{4}} \ \forall x\in(0,\pi) \text{ and then}: f''(x)=\frac{\sec^2\frac{x}{4}}{8}-\frac{\operatorname{cosec}^2\frac{x+\pi}{4}}{16}$$

$$=\frac{2\left(\frac{1}{\sqrt{2}}.\sin\frac{x}{4}+\frac{1}{\sqrt{2}}.\cos\frac{x}{4}\right)^2-\cos^2\frac{x}{4}}{16\cos^2\frac{x}{4}*\sin^2\frac{x+\pi}{4}}=\frac{\sin^2\frac{x}{4}+\cos^2\frac{x}{4}+2\sin\frac{x}{4}.\cos\frac{x}{4}-\cos^2\frac{x}{4}}{16\cos^2\frac{x}{4}*\sin^2\frac{x+\pi}{4}}$$

$$=\frac{\sin^2\frac{x}{4}+\sin\frac{x}{2}}{16\cos^2\frac{x}{4}*\sin^2\frac{x+\pi}{4}}>0 \Rightarrow f(x) \text{ is convex on } (0,\pi)$$

$$\therefore \sum_{cyc}\left(\ln\frac{\sin\frac{\pi+A}{4}}{\cos^2\frac{A}{4}}\right)\overset{\text{Jensen}}{\geq}3.\ln\frac{\sin\frac{\pi+\frac{\pi}{3}}{4}}{\cos^2\frac{\pi}{12}}=3.\ln\frac{2.\frac{\sqrt{3}}{2}}{1+\cos\frac{\pi}{6}}=3.\ln\frac{\sqrt{3}}{1+\frac{\sqrt{3}}{2}}$$

$$= 3.\ln\frac{2\sqrt{3}(2-\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = 3.\ln(4\sqrt{3}-6) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{r_A + r_B + r_C}{r} \geq 3(2\sqrt{3}-3), ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1138. **Prove that in any acute triangle ABC the following relationships holds :**

$$\frac{9F^2(s^2+r^2+4Rr)^{n-1}}{4^{n-1}} \leq m_a{}^{2n}h_bh_c + m_b{}^{2n}h_ch_a + m_c{}^{2n}h_ah_b \leq 2^{2n-3}.R^{2n+3}.\frac{9}{r}.\sum_{cyc}\cos^{4n}\frac{\hat{A}}{2},$$

where $n \in \mathbb{N}^*$.

Proposed by Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have,

$$\sum_{cyc} m_a{}^{2n}h_bh_c = (2F)^2.\sum_{cyc}\frac{m_a{}^{2n}}{bc} \geq 4F^2.\frac{(m_a{}^2+m_b{}^2+m_c{}^2)^n}{3^{n-2}(bc+ca+ab)} = \frac{4F^2}{3^{n-2}}.\left(\frac{3}{4}\right)^n.\frac{(a^2+b^2+c^2)^n}{ab+bc+ca}$$

$$\geq \frac{9F^2}{4^{n-1}}.\frac{(ab+bc+ca)^n}{ab+bc+ca} = \frac{9F^2(s^2+r^2+4Rr)^{n-1}}{4^{n-1}}.$$

Now, since ΔABC is acute then we have, $m_a \leq 2R\cos^2\frac{\hat{A}}{2}$ (and analogs)

Also, if $a \geq b \geq c$ then we have, $m_a \leq m_b \leq m_c$ and

$h_bh_c \geq h_ch_a \geq h_ah_b$, so by Chebyshev's

inequality, we have,

$$\sum_{cyc} m_a{}^{2n}h_bh_c \leq \frac{1}{3}.\sum_{cyc}h_bh_c.\sum_{cyc}m_a{}^{2n} \leq \frac{1}{3}.\frac{2s^2r}{R}.\sum_{cyc}\left(2R\cos^2\frac{\hat{A}}{2}\right)^{2n}$$

$$= 2^{2n-3}.R^{2n+3}.\frac{9}{r}.\left(\frac{2s.2r}{3\sqrt{3}R^2}\right)^2.\sum_{cyc}\cos^{4n}\frac{\hat{A}}{2} \overset{Mitrinovic\,\&\,Euler}{\leq} 2^{2n-3}.R^{2n+3}.\frac{9}{r}.\sum_{cyc}\cos^{4n}\frac{\hat{A}}{2},$$

which complete the proof. Equality holds iff ΔABC is equilateral.

1139. In any ΔABC, the following relationship holds :

$$\sum_{cyc}\frac{a^2}{a^2+b^2}+\frac{R^4}{r^4}\ge 16+\sum_{cyc}\frac{a^3}{a^3+b^3}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{a^2+b^2}+\frac{R^4}{r^4}\ge 16+\sum_{cyc}\frac{a^3}{a^3+b^3}\Leftrightarrow\sum_{cyc}\frac{a^2}{a^2+b^2}+\frac{R^4}{r^4}\ge$$

$$16+\sum_{cyc}\frac{a^3+b^3-b^3}{a^3+b^3}\Leftrightarrow\sum_{cyc}\frac{a^2}{a^2+b^2}+\sum_{cyc}\frac{b^3}{a^3+b^3}+\frac{R^4-19r^4}{r^4}\overset{(*)}{\ge}0$$

$$\text{Now, }\sum_{cyc}\frac{a^2}{a^2+b^2}+\sum_{cyc}\frac{b^3}{a^3+b^3}=\sum_{cyc}\frac{a^3}{a^3+ab^2}+\sum_{cyc}\frac{a^3}{a^3+b^3}$$

$$\overset{\text{Holder}}{\ge}\frac{8s^3}{3\left(\sum_{cyc}a^3+\sum_{cyc}ab^2\right)}+\frac{8s^3}{3\left(\sum_{cyc}a^3+\sum_{cyc}a^3\right)}\overset{A-G}{\ge}\frac{16s^3}{3\left(\sum_{cyc}a^3+\sum_{cyc}a^3\right)}$$

$$=\frac{8s^3}{3.2s(s^2-6Rr-3r^2)}\therefore\text{ LHS of }(*)\ge\frac{4s^2}{3(s^2-6Rr-3r^2)}+\frac{R^4-19r^4}{r^4}\overset{?}{\ge}0$$

$$\Leftrightarrow(3R^4-53r^4)s^2\overset{?}{\ge}3(6Rr+3r^2)(R^4-19r^4)$$

$$\Leftrightarrow(3R^4-48r^4)s^2-5r^4s^2\underset{(**)}{\overset{?}{\ge}}3(6Rr+3r^2)(R^4-19r^4)$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\ge}(3R^4-48r^4)(16Rr-5r^2)-5r^4(4R^2+4Rr+3r^2)$$

$$\overset{?}{\ge}3(6Rr+3r^2)(R^4-19r^4)\Leftrightarrow 15t^5-12t^4-10t^2-223t+198\overset{?}{\ge}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(15t^4+18t^3+36t^2+62t(t-2)+25)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)$$

$$\Rightarrow(*)\text{ is true}\therefore\text{ in }\Delta ABC,$$

$$\sum_{cyc}\frac{a^2}{a^2+b^2}+\frac{R^4}{r^4}\ge 16+\sum_{cyc}\frac{a^3}{a^3+b^3},''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows :

$$\sum_{cyc}\frac{a^2}{a^2+b^2}+\sum_{cyc}\frac{b^3}{a^3+b^3}+\frac{R^4}{r^4}\ge 19\qquad(1)$$

By Hölder and Gerretsen inequalities, we have,

$$LHS_{(1)}\ge\frac{(a+b+c)^2}{2(a^2+b^2+c^2)}+\frac{(a+b+c)^3}{\sum_{cyc}(a+b).\sum_{cyc}(a^2-ab+b^2)}+\frac{R^4}{r^4}$$

$$=\frac{s^2}{s^2-r^2-4Rr}+\frac{2s^2}{3s^2-5(r^2+4Rr)}+\frac{R^4}{r^4}\ge\frac{16Rr-5r^2}{4R^2+2r^2}+\frac{2(16Rr-5r^2)}{12R^2-8Rr+4r^2}+\frac{R^4}{r^4}$$

$$\overset{CBS\ \&\ Euler}{\geq}\frac{4(16Rr-5r^2)}{(4R^2+2r^2)+(6R^2-4Rr+2r^2)}+\frac{8R}{r}=\frac{2(16Rr-5r^2)}{5R^2-2Rr+2r^2}+\frac{8R}{r}$$

$$=19+\frac{(R-2r)(40R^2-31Rr+24r^2)}{(5R^2-2Rr+2r^2)r}\overset{Euler}{\geq}19=RHS_{(2)},$$

which complete the proof. Equality holds iff ΔABC is equilateral.

1140. In *any* Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{1}{4\tan^2\frac{A}{2}-\tan\frac{B}{2}\tan\frac{C}{2}+2}\geq 1$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}\ \text{and}\ \sum_{cyc}\frac{1}{4\tan^2\frac{A}{2}-\tan\frac{B}{2}\tan\frac{C}{2}+2}\geq 1$$

$$\Leftrightarrow\sum_{cyc}\frac{s^2}{4r_a^2-r_br_c+2s^2}\overset{(*)}{\geq}1$$

$$\text{Now, }\prod_{cyc}(4r_a^2-r_br_c+2s^2)=63r_a^2r_b^2r_c^2-16\sum_{cyc}r_a^3r_b^3+32s^2\sum_{cyc}r_a^2r_b^2$$

$$+4r_ar_br_c\sum_{cyc}r_a^3-8s^2\sum_{cyc}\left(r_br_c\left(\sum_{cyc}r_a^2-r_a^2\right)\right)-4s^4\sum_{cyc}r_br_c+2rs^4\sum_{cyc}r_a$$

$$+16s^4\sum_{cyc}r_a^2+8s^6=63r^2s^4-16\left(\left(\sum_{cyc}r_br_c\right)^3-3\prod_{cyc}(r_ar_b+r_br_c)\right)$$

$$+32s^2\left(\left(\sum_{cyc}r_br_c\right)^2-2rs^2(4R+r)\right)+4rs^2\left(\left(\sum_{cyc}r_a\right)^3-3\prod_{cyc}(r_b+r_c)\right)$$

$$-8s^2\left(\sum_{cyc}r_br_c\right)\left(\left(\sum_{cyc}r_br_c\right)^2-2rs^2(4R+r)\right)+8s^2.rs^2\left(\sum_{cyc}r_a\right)-4s^6$$

$$+2rs^4(4R+r)+16s^4\left(\left(\sum_{cyc} r_b r_c\right)^2-2rs^2(4R+r)\right)+8s^6$$

$$\overset{via\ (i)\ and\ analogs}{=} 63r^2s^4-16\left(s^6-3rs^2.64R^3.\frac{s^2}{16R^2}\right)+32s^2\left(s^4-2rs^2(4R+r)\right)$$

$$+4rs^2\left((4R+r)^3-3.64R^3.\frac{s^2}{16R^2}\right)+8s^4\left(s^4-2rs^2(4R+r)\right)$$

$$+8r(4R+r)s^4+2rs^4(4R+r)+2s^6$$

$$\Rightarrow \prod_{cyc}(4r_a^2-r_br_c+2s^2)\overset{(\bullet)}{=} s^2\left(4s^4+(128R^2-8Rr+17r^2)s^2+4r(4R+r)^3\right)$$

$$\text{Again}, \sum_{cyc}(4r_b^2-r_cr_a+2s^2)(4r_c^2-r_ar_b+2s^2)$$

$$=16\sum_{cyc} r_a^2r_b^2-4\sum_{cyc}\left(r_br_c\left(\sum_{cyc} r_a^2-r_a^2\right)\right)+16s^2\sum_{cyc} r_a^2$$

$$+rs^2(4R+r)-4s^2\left(\sum_{cyc} r_br_c\right)+12s^4$$

$$=16\left(\left(\sum_{cyc} r_br_c\right)^2-2rs^2(4R+r)\right)-4s^2\left((4R+r)^2-2s^2\right)+4rs^2(4R+r)$$

$$+16s^2\left((4R+r)^2-2s^2\right)+rs^2(4R+r)-4s^4+12s^4$$

$$\Rightarrow \sum_{cyc}(4r_b^2-r_cr_a+2s^2)(4r_c^2-r_ar_b+2s^2)\overset{(\bullet\bullet)}{=} s^2(192R^2-12Rr-15r^2)$$

$$\text{Now}, (*) \Leftrightarrow \frac{s^2.\sum_{cyc}(4r_b^2-r_cr_a+2s^2)(4r_c^2-r_ar_b+2s^2)}{\prod_{cyc}(4r_a^2-r_br_c+2s^2)}$$

$$\overset{via\ (\bullet\bullet)}{\Leftrightarrow} \frac{s^4(192R^2-12Rr-15r^2)}{s^2\left(4s^4+(128R^2-8Rr+17r^2)s^2+4r(4R+r)^3\right)}\overset{?}{\geq} 1$$

$$\Leftrightarrow s^4-s^2(16R^2-Rr-8r^2)+r(4R+r)^3\ \underset{(**)}{\overset{?}{\leq}}\ 0$$

$$\text{Now, LHS of } (*) \overset{Gerretsen}{\leq} s^2(4R^2+4Rr+3r^2)-s^2(16R^2-Rr-8r^2)$$

$$+r(4R+r)^3\overset{?}{\leq}0 \Leftrightarrow (12R^2-5Rr-11r^2)s^2\ \underset{(***)}{\overset{?}{\geq}}\ r(4R+r)^3$$

$$\text{Again, LHS of } (*) \overset{Gerretsen}{\geq} (12R^2-5Rr-11r^2)(16Rr-5r^2)\overset{?}{\geq} r(4R+r)^3$$

$$\Leftrightarrow 128t^3-188t^2-163t+54\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(128t^2+14(t-2)+54t+1\right)\overset{?}{\geq}0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (***) \Rightarrow (**)$$

$$\Rightarrow (*) \text{ is true} \Rightarrow \text{in any } \Delta\, ABC,$$

$$\sum_{cyc}\frac{1}{4\tan^2\frac{A}{2}-\tan\frac{B}{2}\tan\frac{C}{2}+2}\ge 1,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=\tan\frac{B}{2}\tan\frac{C}{2},\ \ y:=\tan\frac{C}{2}\tan\frac{A}{2},\ \ z:=\tan\frac{A}{2}\tan\frac{B}{2}$. We have, $x+y+z=1$ and,

$$\sum_{cyc}\frac{1}{4\tan^2\frac{A}{2}-\tan\frac{B}{2}\tan\frac{C}{2}+2}=\sum_{cyc}\frac{1}{\frac{4yz}{x}-x+2}=\sum_{cyc}\frac{x^2}{4xyz-x^3+2x^2}$$

$$\overset{CBS}{\geq}\frac{(x+y+z)^2}{\sum_{cyc}(4xyz-x^3+2x^2)}\overset{?}{\geq}1\ \Leftrightarrow\ 1\ge 12xyz-(x^3+y^3+z^3)+2(x^2+y^2+z^2)$$

$$\Leftrightarrow (x+y+z)^3\ge 12xyz-(x^3+y^3+z^3)+2(x+y+z)(x^2+y^2+z^2)$$

$$\Leftrightarrow x^2y+xy^2+y^2z+yz^2+z^2x+zx^2-6xyz\ge 0$$

$$\Leftrightarrow x(y-z)^2+y(z-x)^2+z(x-y)^2\ge 0,$$

which is true. The proof is completed. Equality holds iff $x=y=z$ $\Leftrightarrow$ ΔABC is equilateral.

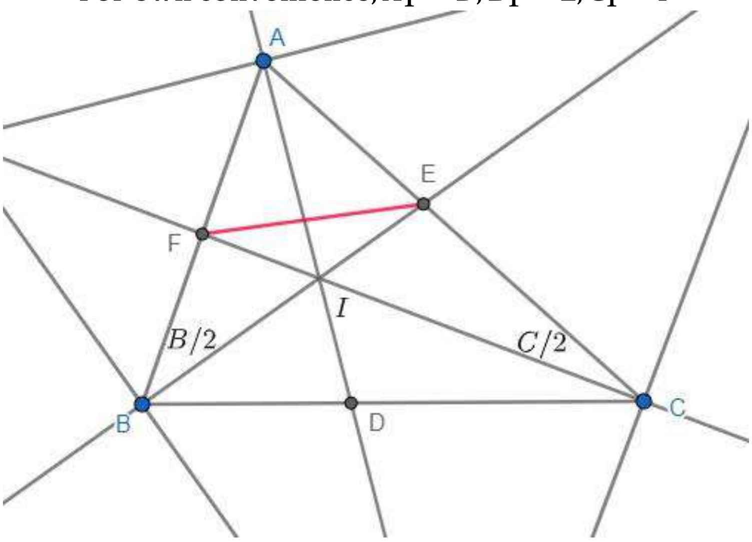
1141. **In Δ ABC, AA_1, BB_1, CC_1 → angle bisectors. Prove that :**

$$\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2+\left(\frac{AB}{A_1B_1}\right)^2\ge 12$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

For own convenience, $A_1\equiv D, B_1\equiv E, C_1\equiv F$



$$\text{Angle}-\text{bisector theorem}\Rightarrow\frac{AF}{BF}=\frac{b}{a}\Rightarrow\frac{AF+BF}{BF}=\frac{b+a}{a}\Rightarrow BF\overset{(i)}{=}\frac{ca}{a+b}$$

$and\ also, angle - \text{bisector theorem} \Rightarrow \frac{AE}{CE} = \frac{c}{a} \Rightarrow \frac{AE+CE}{CE} = \frac{c+a}{a} \Rightarrow CE \overset{(ii)}{=} \frac{ab}{c+a}$

$\text{Cosine } law \Rightarrow \left(BF^2 + w_b^2 - 2BF.w_b.\cos\frac{B}{2}\right) + \left(CE^2 + w_c^2 - 2CE.w_c.\cos\frac{C}{2}\right) = 2FE^2$

$$\overset{\text{via (i),(ii)}}{\Rightarrow} \frac{a^2b^2}{(c+a)^2} + \frac{4ca}{(c+a)^2}.s(s-b) - 2.\frac{ca}{a+b}.\frac{2ca}{c+a}.\frac{s(s-b)}{ca} + \frac{c^2a^2}{(a+b)^2}$$
$$+\frac{4ab}{(a+b)^2}.s(s-c) - 2.\frac{ab}{c+a}.\frac{2ab}{a+b}.\frac{s(s-c)}{ab} = 2FE^2$$
$$\Rightarrow 2FE^2 = \frac{a^2b^2}{(c+a)^2} + \frac{c^2a^2}{(a+b)^2} + \frac{4ca}{c+a}.s(s-b)\left(\frac{1}{c+a} - \frac{1}{a+b}\right)$$
$$+\frac{4ab}{a+b}.s(s-c)\left(\frac{1}{a+b} - \frac{1}{c+a}\right)$$
$$= \frac{a^2b^2}{(c+a)^2} + \frac{c^2a^2}{(a+b)^2} + \frac{4as(b-c)}{(c+a)(a+b)}.\left(\frac{c(s-b)}{c+a} - \frac{b(s-c)}{a+b}\right)$$
$$= \frac{a^2b^2}{(c+a)^2} + \frac{c^2a^2}{(a+b)^2}$$
$$+\frac{4as(b-c)}{(c+a)^2(a+b)^2}.\big(c(s-b)(s+s-c) - b(s-c)(s+s-b)\big)$$
$$= \frac{a^2b^2}{(c+a)^2} + \frac{c^2a^2}{(a+b)^2}$$
$$+\frac{4as(b-c)}{(c+a)^2(a+b)^2}.\big(cs(s-b) - bs(s-c) + c(s-b)(s-c) - c(s-b)(s-c)\big)$$
$$= \frac{a^2b^2}{(c+a)^2} + \frac{c^2a^2}{(a+b)^2} - \frac{4as(b-c)^2}{(c+a)^2(a+b)^2}.\big(s^2 + (s-b)(s-c)\big)$$
$$= \frac{a^2b^2}{(c+a)^2} + \frac{c^2a^2}{(a+b)^2} - \frac{4as(b-c)^2}{(c+a)^2(a+b)^2}.\big(s^2 + s^2 - s(2s-a) + bc\big)$$
$$= \frac{a^2b^2}{(c+a)^2} + \frac{c^2a^2}{(a+b)^2} - \frac{4as(b-c)^2}{(c+a)^2(a+b)^2}.(as+bc)$$
$$= \frac{a^2b^2 + c^2a^2 - a^2(b-c)^2(a+b+c)^2 - 2abc(a+b+c)(b-c)^2}{(c+a)^2(a+b)^2}$$
$$= \frac{2a^3bc(a+b+c) + 2a^2b^2c^2 - 2abc(a+b+c)(b-c)^2}{(c+a)^2(a+b)^2}$$
$$= \frac{2abc(a+b+c)\big(a^2 - (b-c)^2\big) + 2a^2b^2c^2}{(c+a)^2(a+b)^2}$$
$$\therefore FE^2 = \frac{abc(a+b+c)\left(a^2(b+c)^2 - \left(b^2-c^2\right)^2\right) + a^2b^2c^2(b+c)^2}{(c+a)^2(a+b)^2(b+c)^2}$$
$$= \frac{(b+c)^2\big(8Rrs^2.4rr_a + 16R^2r^2s^2\big)}{(c+a)^2(a+b)^2(b+c)^2} \Rightarrow \left(\frac{BC}{B_1C_1}\right)^2 = \left(\frac{BC}{FE}\right)^2$$
$$= \frac{(c+a)^2(a+b)^2(b+c)^2}{16r^2s^2}.\frac{\frac{a^2}{(b+c)^2}}{2Rr_a + R^2} \overset{\text{Cesaro}}{\geq} \frac{64.16R^2r^2s^2}{16r^2s^2}.\frac{\frac{a^2}{(b+c)^2}}{2Rr_a + R^2}$$

$$\Rightarrow\left(\frac{BC}{B_1C_1}\right)^2\geq 64R^2.\frac{\frac{a^2}{(b+c)^2}}{2Rr_a+R^2}\text{ and analogs}\therefore\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2+\left(\frac{AB}{A_1B_1}\right)^2$$

$$\overset{\text{Bergstrom}}{\geq} 64R^2.\frac{\left(\sum_{cyc}\frac{a}{b+c}\right)^2}{2R\left(\sum_{cyc}r_a\right)+3R^2}\overset{\text{Nesbitt}}{\geq}64R^2.\frac{\frac{9}{4}}{2R(4R+r)+3R^2}=\frac{144R^2}{11R^2+2Rr}$$

$$\overset{\text{Euler}}{\geq}\frac{144R^2}{11R^2+R^2}\therefore\left(\frac{BC}{B_1C_1}\right)^2+\left(\frac{CA}{C_1A_1}\right)^2+\left(\frac{AB}{A_1B_1}\right)^2\geq 12,$$

$''='' $ iff Δ ABC is equilateral (QED)

1142.

Let ABC be the triangle in which $\begin{cases}\sqrt{4\sin\widehat{A}+\cos\widehat{A}}+\sqrt{\sin\widehat{A}-\cos\widehat{A}}=3\\ 3\sin\widehat{A}+\sqrt{\sin\widehat{A}-\cos\widehat{A}}=4\end{cases}$.

Show that: $$\frac{s^2}{h_a{}^2}+\frac{R^2}{r^2}+\frac{r^2}{R^2}>7+2\sqrt{2}$$

Proposed by Daniel Văcaru, Radu Diaconu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

From the given condition, we have,

$$\sqrt{\sin\widehat{A}-\cos\widehat{A}}=4-3\sin\widehat{A}\quad\text{and}\quad\sqrt{4\sin\widehat{A}+\cos\widehat{A}}=3\sin\widehat{A}-1,$$

then,

$$5\sin\widehat{A}=\sqrt{\sin\widehat{A}-\cos\widehat{A}}^2+\sqrt{4\sin\widehat{A}+\cos\widehat{A}}^2=\left(4-3\sin\widehat{A}\right)^2+\left(3\sin\widehat{A}-1\right)^2$$

$$\Leftrightarrow 18\sin^2\widehat{A}-35\sin\widehat{A}+17=0\Leftrightarrow\left(1-\sin\widehat{A}\right)\left(17-18\sin\widehat{A}\right)=0\Leftrightarrow$$

$$\sin\widehat{A}=1\text{ or }\sin\widehat{A}=\frac{17}{18},$$

If $\sin\widehat{A}=\frac{17}{18}$, then we have,

$$\cos\widehat{A}=\pm\frac{\sqrt{35}}{18},\text{ but this pair does not satisfy the given condition.}$$

Then, $\sin\widehat{A}=1$, so the triangle ABC is right on A. In this case we have, $a=2R$ and,

$$\frac{r}{4R}=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sin\frac{\pi}{4}.\frac{1}{2}\left(\cos\frac{\widehat{B}-\widehat{C}}{2}-\cos\frac{\widehat{B}+\widehat{C}}{2}\right)\leq\frac{\sqrt{2}}{4}\left(1-\frac{\sqrt{2}}{2}\right)=\frac{1}{4\left(\sqrt{2}+1\right)}.$$

Then,

$$\frac{R}{r} \geq \sqrt{2}+1 > 2 \text{ and } \frac{s}{h_a} = \frac{a}{2r} = \frac{R}{r}.$$

Therefore,

$$\frac{s^2}{{h_a}^2} + \frac{R^2}{r^2} + \frac{r^2}{R^2} = \frac{2R^2}{r^2} + \frac{r^2}{R^2} = \frac{R^2}{r^2} + \frac{3}{4}.\frac{R^2}{r^2} + \left(\frac{R^2}{4r^2} + \frac{r^2}{R^2}\right) \overset{AM-GM}{\geq}$$

$$\geq \left(\sqrt{2}+1\right)^2 + \frac{3}{4}.2^2 + 1 = 7 + 2\sqrt{2},$$

as desired.

1143. **In any Δ ABC, the following relationship holds :**

$$\sum_{cyc} \sqrt{a}\left(\sqrt{a+|b-c|} - \sqrt{a-|b-c|}\right) \geq 2(max\{a,b,c\} - min\{a,b,c\})$$

Proposed by Hasan Mammadov-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India,

$$\sqrt{a}\left(\sqrt{a+|b-c|} - \sqrt{a-|b-c|}\right) \overset{?}{\geq} |b-c|$$

$$\Leftrightarrow \sqrt{a^2 + a.|b-c|} - \sqrt{a^2 - a.|b-c|} \overset{?}{\geq} |b-c|$$

$$\Leftrightarrow a^2 + a.|b-c| + a^2 - a.|b-c| - 2\sqrt{a^4 - a^2(b-c)^2} \overset{?}{\geq} (b-c)^2$$

$$\Leftrightarrow a^2 + a^2 - (b-c)^2 - 2a.\sqrt{a^2-(b-c)^2} \overset{?}{\geq} 0 \Leftrightarrow \left(a - \sqrt{a^2-(b-c)^2}\right)^2 \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \therefore \sqrt{a}\left(\sqrt{a+|b-c|} - \sqrt{a-|b-c|}\right) \geq |b-c| \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} \sqrt{a}\left(\sqrt{a+|b-c|} - \sqrt{a-|b-c|}\right) \overset{(\bullet)}{\geq} |b-c| + |c-a| + |a-b|$$

$\boxed{\text{Case } (*)}$ $a \geq b \geq c \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(b-c+a-c+a-b)$
$= a - c = max\{a,b,c\} - min\{a,b,c\}$

$\boxed{\text{Case } (**)}$ $a \geq c \geq b \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|) = \frac{1}{2}(c-b+a-c+a-b)$
$= a - b = max\{a,b,c\} - min\{a,b,c\}$

$\boxed{\text{Case } (***)}$ $b \geq c \geq a \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|)$
$= \frac{1}{2}(b-c+c-a+b-a) = b - a = max\{a,b,c\} - min\{a,b,c\}$

$\boxed{\text{Case } (****)}$ $b \geq a \geq c \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|)$
$= \frac{1}{2}(b-c+a-c+b-a) = b - c = max\{a,b,c\} - min\{a,b,c\}$

$\boxed{\text{Case } (*****)}$ $c \geq a \geq b \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|)$

$$=\frac{1}{2}(c-b+c-a+a-b)=c-b=max\{a,b,c\}-\min\{a,b,c\}$$

$$\boxed{Case\ (******)}\ c\ge b\ge a\ \therefore \frac{1}{2}(|b-c|+|c-a|+|a-b|)$$

$$=\frac{1}{2}(c-b+c-a+b-a)=c-a=max\{a,b,c\}-\min\{a,b,c\}$$

$$\therefore \text{combining all 6 cases, we conclude}: \frac{1}{2}(|b-c|+|c-a|+|a-b|)$$

$$=max\{a,b,c\}-\min\{a,b,c\}$$

$$\Rightarrow |b-c|+|c-a|+|a-b|\overset{(\bullet\bullet)}{=}2(max\{a,b,c\}-\min\{a,b,c\})$$

$$\therefore (\bullet)(\bullet\bullet)\Rightarrow \sum_{cyc}\sqrt{a}\left(\sqrt{a+|b-c|}-\sqrt{a-|b-c|}\right)\ge 2(max\{a,b,c\}-\min\{a,b,c\}),$$

$''=''$ iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

$$\sum_{cyc}\sqrt{a}\left(\sqrt{a+|b-c|}-\sqrt{a-|b-c|}\right)=\sum_{cyc}\frac{2\sqrt{a}|b-c|}{\sqrt{a+|b-c|}+\sqrt{a-|b-c|}}$$

$$\overset{CBS}{\ge}\sum_{cyc}\frac{2\sqrt{a}|b-c|}{\sqrt{2[(a+|b-c|)+(a-|b-c|)]}}=\sum_{cyc}|b-c|.$$

WLOG, we may assume that $a\ge b\ge c$. We have,

$$|a-b|+|b-c|+|c-a|=(a-b)+(b-c)+(a-c)=2(a-c)$$

$$=2\big(max(a,b,c)-min(a,b,c)\big).$$

$$\sum_{cyc}\sqrt{a}\left(\sqrt{a+|b-c|}-\sqrt{a-|b-c|}\right)\ge 2\big(max(a,b,c)-min(a,b,c)\big).$$

Equality holds iff ΔABC is equilateral.

1144. In any Δ ABC, the following relationship holds :

$$\frac{r_a^4}{m_a^3}+\frac{r_b^4}{m_b^3}+\frac{r_c^4}{m_c^3}\ge\frac{36r^4}{R^3-4r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$m_a^2m_b^2m_c^2=\frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now,}\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)$$

$$\overset{(3)}{=}\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2$$

$$\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\right.$$
$$\left.-18a^2b^2c^2+3a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\right.$$
$$\left.-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\le \frac{R^2s^4}{4} \Leftrightarrow s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$

$$-r^3(4R+r)^3 \overset{(\bullet)}{\le} 0$$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\le} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$

$-r^3(4R+r)^3 \overset{?}{\le} 0$

$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\ge}} 20rs^4$

Now, LHS of $(\bullet\bullet) \underset{(\blacksquare)}{\underbrace{\overset{\text{Gerretsen}}{\ge}}} s^2(16Rr - 5r^2)(8R - 16r)$

$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3$ *and*

RHS of $(\bullet\bullet) \underset{(\blacksquare\blacksquare)}{\underbrace{\overset{\text{Geretsen}}{\le}}} 20rs^2(4R^2 + 4Rr + 3r^2)$

$(\blacksquare), (\blacksquare\blacksquare) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R+r)^3$

$\ge 20rs^2(4R^2 + 4Rr + 3r^2)$

$\Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R+r)^3 \ge 0$

$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\ge} 27r^2s^2$

Now, LHS of $(\bullet\bullet\bullet) \underset{(\blacksquare\blacksquare\blacksquare)}{\underbrace{\overset{\text{Gerretsen}}{\ge}}} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet) \underset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{\underbrace{\overset{\text{Geretsen}}{\le}}} 27r^2(4R^2 + 4Rr + 3r^2)$

$(\blacksquare\blacksquare\blacksquare), (\blacksquare\blacksquare\blacksquare\blacksquare) \Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R+r)^3 \ge 27r^2(4R^2 + 4Rr + 3r^2)$

$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \ge 0 \left(\text{where } t = \frac{R}{r}\right)$

$\Leftrightarrow (t-2)\{(t-2)(224t + 309) + 648\} \ge 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (6) \Rightarrow (5) \Rightarrow (4)$

is true $\Rightarrow m_a^2m_b^2m_c^2 \le \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \le \frac{Rs^2}{2} \Rightarrow \frac{r_a^4}{m_a^3} + \frac{r_b^4}{m_b^3} + \frac{r_c^4}{m_c^3} \overset{A-G}{\ge} 3.\sqrt[3]{\prod_{cyc}\frac{r_a^4}{m_a^3}}$

$= 3.\sqrt[3]{\frac{r^4s^8}{(m_am_bm_c)^3}} \ge 3.\sqrt[3]{\frac{8r^4s^8}{R^3s^6}} \overset{?}{\ge} \frac{36r^4}{R^3 - 4r^3} \Leftrightarrow s^2(R^3 - 4r^3)^3 \underset{(*)}{\overset{?}{\ge}} 216R^3r^8$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\ge} (16Rr - 5r^2)(R^3 - 4r^3)^3 \overset{?}{\ge} 216R^3r^8$

$\Leftrightarrow 16t^{10} - 5t^9 - 192t^7 + 60t^6 + 768t^4 - 456t^3 - 1024t + 320 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$

$$\Leftrightarrow (t-2)\Big((t-2)(16t^8 + 59t^7 + 172t^6 + 260t^5 + 412t^4 + 608t^3 + 1552t^2 + 3320t$$
$$+ 7072) + 13984\Big) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$\Rightarrow$ in *any* Δ ABC, $\frac{r_a^4}{m_a^3} + \frac{r_b^4}{m_b^3} + \frac{r_c^4}{m_c^3} \ge \frac{36r^4}{R^3 - 4r^3}$, " = " iff Δ ABC is equi*lateral* (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{{r_a}^4}{{m_a}^3}+\frac{{r_b}^4}{{m_b}^3}+\frac{{r_c}^4}{{m_c}^3}\overset{Radon}{\geq}\frac{(r_a+r_b+r_c)^4}{(m_a+m_b+m_c)^3}\overset{Leuenbeger}{\geq}\frac{(4R+r)^4}{(4R+r)^3}=\frac{(4R+r)(R^3-4r^3)}{R^3-4r^3}$$

$$\overset{Euler}{\geq}\frac{(4.2r+r)\left((2r)^3-4r^3\right)}{R^3-4r^3}=\frac{36r^4}{R^3-4r^3},$$

as desired. Equality holds if and only ΔABC is equilateral.

1145. If $x,y,z\in\mathbb{R}, x+y+z=0, xy\geq 0$, then in acute Δ ABC, the following relationship holds :

$$\frac{xy}{\sec^2 A}+\frac{yz}{\sec^2 B}+\frac{zx}{\sec^2 C}\geq xy+yz+zx$$

Proposed by George Apostolopoulos-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{xy}{\sec^2 A}+\frac{yz}{\sec^2 B}+\frac{zx}{\sec^2 C}\geq xy+yz+zx$$
$$\Leftrightarrow xy.\cos^2 A+yz.\cos^2 B+zx.\cos^2 C\geq xy+yz+zx$$
$$\Leftrightarrow xy.\sin^2 A+yz.\sin^2 B+zx.\sin^2 C\leq 0$$
$$\Leftrightarrow xy.\sin^2 A+(-x-y)\left(y(1-\cos^2 B)+z(1-\cos^2 C)\right)\leq 0$$
$$\Leftrightarrow xy.\sin^2 A-(x+y)^2+x(x+y)\cos^2 C+y(x+y)\cos^2 B\leq 0$$
$$\Leftrightarrow xy-(x+y)^2+x^2\cos^2 C+y^2\cos^2 B+xy(\cos^2 B+\cos^2 C-\cos^2 A)\leq 0\Leftrightarrow$$
$$xy-(x+y)^2+x^2+y^2-x^2\sin^2 C-y^2\sin^2 B+xy\left(\sum_{cyc}\cos^2 A-2\cos^2 A\right)\leq 0$$
$$\Leftrightarrow -xy-x^2\sin^2 C-y^2\sin^2 B+xy\left(\frac{1}{2}\sum_{cyc}(1+\cos 2A)-2\cos^2 A\right)\leq 0$$
$$\Leftrightarrow -xy-x^2\sin^2 C-y^2\sin^2 B+xy\left(\frac{1}{2}\left(3-1-4\prod_{cyc}\cos A\right)-2\cos^2 A\right)\leq 0$$
$$\Leftrightarrow -x^2\sin^2 C-y^2\sin^2 B+xy\left(-1+1-2\prod_{cyc}\cos A-2\cos^2 A\right)\leq 0$$
$$\Leftrightarrow \boxed{-x^2\sin^2 C-y^2\sin^2 B+xy\left(-2\prod_{cyc}\cos A-2\cos^2 A\right)\leq 0}\rightarrow \text{true} \because xy\geq 0$$

$$and\ \Delta\ \mathrm{ABC\ being}\ acute \Rightarrow \prod_{cyc} \cos A > 0 \Rightarrow -2\prod_{cyc} \cos A - 2\cos^2 A \le 0$$

$$\therefore \frac{xy}{\sec^2 A} + \frac{yz}{\sec^2 B} + \frac{zx}{\sec^2 C} \ge xy + yz + zx\ \forall\ x,y,z \in \mathbb{R}, x+y+z=0, xy \ge 0$$
$$and\ \forall\ acute\ \Delta\ \mathrm{ABC}, '' ='' \ \text{iff}\ x = y = z = 0\ (\mathrm{QED})$$

1146. In ΔABC the following relationship holds:

$$e^{-\frac{2a}{b+c}} + e^{-\frac{2b}{c+a}} + e^{-\frac{2c}{a+b}} > 1$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Christos Tsifakis-Greece

Let be $f:[0,1) \to \mathbb{R}, f(x) = 1 + x + xe^{2x} - e^{2x} \Longrightarrow f''(x) = 4xe^{2x} \ge 0$

f' – increasing on $[0,1)$, $f'(x) = 1 - e^{2x} + 2xe^{2x} \Longrightarrow f'(x) \ge f'(0) = 0$

f –increasing on $[0,1) \Longrightarrow f(x) \ge f(0) = 0 \Longrightarrow\ 1 + x + xe^{2x} - e^{2x} \ge 0$

$$e^{2x} \le \frac{1+x}{1-x}, \forall x \in [0,1) \Longrightarrow e^x \le \frac{2+x}{2-x}, \forall x \in [0,2) \Longrightarrow e^{-x} \le \frac{2-x}{2+x} \quad (1)$$

$$a < b + c \Longrightarrow \frac{a}{b+c} < 1 \Longrightarrow \frac{2a}{b+c} < 2 \overset{(1)}{\Longrightarrow} e^{-\frac{2a}{b+c}} \le \frac{2 - \frac{2a}{b+c}}{2 + \frac{2a}{b+c}} = \frac{b+c-a}{a+b+c}$$

$$\sum_{cyc} e^{-\frac{2a}{b+c}} \le \sum_{cyc} \frac{b+c-a}{a+b+c} = \frac{1}{a+b+c}\sum_{cyc}(b+c-a) = \frac{a+b+c}{a+b+c} = 1$$

Solution 2 by Michael Sterghiou-Greece

Let be $f:(0,\infty) \to \mathbb{R}, f(x) = e^{-\frac{2x}{p-x}}, a + b + c = p.$

$$f''(x) = \frac{4pxe^{-\frac{2x}{p-x}}}{(p-x)^4} > 0 \Longrightarrow f \text{ –convex. By Jensen's inequality:}$$

$$\sum_{cyc} e^{-\frac{2a}{b+c}} = \sum_{cyc} e^{-\frac{2a}{p-a}} \ge 3e^{\frac{-\frac{2p}{3}}{p-\frac{2p}{3}}} = \frac{3}{e} > 1$$

1147. Find all values of $k \ge 0$ such that:

$$(k+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq k\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3,\ \ \forall\Delta ABC.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is equivalent to,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq -k\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\frac{b}{a}-\frac{c}{b}-\frac{a}{c}\right)=-k.\frac{(a-b)(b-c)(c-a)}{abc}$$

$$\Leftrightarrow\ ab^2+bc^2+ca^2-3abc\geq -k(a-b)(b-c)(c-a)$$

By Ravi's substitution, there are $x,y,z>0$ **such that,** $a=y+z,\ \ b=z+x,\ \ c=x+y.$

The last inequality becomes,

$$x^3+y^3+z^3-xy^2-yz^2-zx^2\geq k(x-y)(y-z)(z-x)\qquad (1)$$

Let $z\to 0$**, the inequality becomes,** $x^3+y^3-xy^2\geq kxy(y-x).$

This one is true if $x\geq y$**. Assume now that** $x<y$ **and let** $t=\frac{x}{y}\in(0,1).$

The last inequality becomes, $k\leq\frac{t^3-t+1}{t(1-t)}=f(t),\ \ t\in(0,1).$

We have, $f'(t)=\frac{-t^4+2t^3-t^2+2t-1}{t^2(1-t)^2}=\frac{-\left[t^2-(1+\sqrt{2})t+1\right]\left[t^2+(\sqrt{2}-1)t+1\right]}{t^2(1-t)^2}$

$$=\frac{-\left[2t-\left(1+\sqrt{2}-\sqrt{2\sqrt{2}-1}\right)\right]\left(2t-1-\sqrt{2}-\sqrt{2\sqrt{2}-1}\right)\left[t^2+(\sqrt{2}-1)t+1\right]}{4t^2(1-t)^2}.$$

So f is decreasing on $(0,t_0]$ **and increasing on** $[t_0,1)$**, where** $t_0=\frac{1+\sqrt{2}-\sqrt{2\sqrt{2}-1}}{2}.$

Thus, $k\leq\min_{0<t<1}\{f(t)\}=f(t_0)=k_0\approx 2,48.$

So it suffices to prove the inequality (1) **for** $k=k_0$.

WLOG, we may assume that $z=min\{x,y,z\}$**. By AM − GM inequality, we have,**

$LHS_{(1)}\geq 0,$

and if $x\geq y$ **then we have,** $RHS_{(1)}\leq 0\leq LHS_{(1)}.$

Assume now that $y\geq x\geq z$ **and let** $y=u+z,\ \ x=v+z,$ **where** $u\geq v\geq 0.$

The inequality (1) is successively equivalent to,

$$(v+z)^3+(u+z)^3+z^3-(v+z)(u+z)^2-(u+z)z^2-z(v+z)^2 \geq k_0uv(u-v)$$

$$\Leftrightarrow 2(u^2-uv+v^2)z+u^3+v^3-u^2v \geq k_0uv(u-v).$$

This one is true for $u=0, v=0$ and for $u=v$, and it is equivalent, in the other cases, to

$$2(u^2-uv+v^2)z+uv(u-v)\left(f\left(\frac{v}{u}\right)-k_0\right) \geq 0,$$

which is true because, $u^2-uv+v^2, u-v>0$ and $f\left(\frac{v}{u}\right) \geq \min_{0<t<1}\{f(t)\}=k_0$.

Therefore, $0 \leq k \leq k_0 \approx 2,48$.

1148. **In ΔABC the following relationship holds :**

$$\sum_{cyc}\sqrt{\frac{m_b{}^4+4m_b{}^2(m_b+m_c)(m_c+m_a)+m_c{}^4}{b^4+c^4}}<\frac{81R^2}{s^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\sqrt{\frac{m_b{}^4+4m_b{}^2(m_b+m_c)(m_c+m_a)+m_c{}^4}{b^4+c^4}}<\frac{81R^2}{s^2} \quad (1)$$

We have,

$$b^4+c^4 \overset{Jensen}{\geq} 2\left(\frac{b+c}{2}\right)^4=\frac{(s+(s-a))^4}{8} \overset{s>a}{>} \frac{s^4}{8}.$$

Then,

$$LHS_{(1)}<\sum_{cyc}\sqrt{\frac{8(m_b{}^4+4m_b{}^2(m_b+m_c)(m_c+m_a)+m_c{}^4)}{s^4}}$$

$$\overset{CBS}{\leq} \frac{2}{s^2}\sqrt{6\sum_{cyc}[m_b{}^4+4m_b{}^2(m_b+m_c)(m_c+m_a)+m_c{}^4]}$$

$$=\frac{2}{s^2}\sqrt{6\sum_{cyc}(2m_a{}^4+4m_b{}^2m_c{}^2+4m_b{}^2(m_am_b+m_bm_c+m_cm_a))}$$

$$=\frac{4}{s^2}\sqrt{3\left[\left(\sum_{cyc} m_a{}^2\right)^2+2\left(\sum_{cyc} m_a{}^2\right)\left(\sum_{cyc} m_b m_c\right)\right]}\le\frac{4\times 3}{s^2}.\sum_{cyc} m_a{}^2=\frac{12}{s^2}.\frac{3}{4}\sum_{cyc} a^2\overset{Leibniz}{\le}\frac{9}{s^2}.9R^2$$

$$=\frac{81R^2}{s^2},\ \text{as desired.}$$

1149. **In ΔABC the following relationship holds :**

$$\frac{m_a{}^4}{w_a{}^3}+\frac{m_b{}^4}{w_b{}^3}+\frac{m_c{}^4}{w_c{}^3}\ge\frac{\sqrt{3}s^4}{2s^3-81\sqrt{3}r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have,

$$\frac{m_a{}^4}{w_a{}^3}+\frac{m_b{}^4}{w_b{}^3}+\frac{m_c{}^4}{w_c{}^3}\ge\frac{\left(m_a{}^2+m_b{}^2+m_c{}^2\right)^2}{w_a{}^3+w_b{}^3+w_c{}^3},$$

and since $w_a\le\sqrt{s(s-a)}\le m_a$ (and analogs), then we have,

- $m_a{}^2+m_b{}^2+m_c{}^2\ge s(s-a)+s(s-b)+s(s-c)=s^2.$

- $w_a{}^3+w_b{}^3+w_c{}^3\le\sqrt{s(s-a)}^3+\sqrt{s(s-b)}^3+\sqrt{s(s-c)}^3$

$$\overset{AM-GM}{\le} s\left(\frac{s(s-a)+3(s-a)^2}{2\sqrt{3}}+\frac{s(s-b)+3(s-b)^2}{2\sqrt{3}}+\frac{s(s-c)+3(s-c)^2}{2\sqrt{3}}\right)$$

$$=s.\frac{12s^2-7s(a+b+c)+3\left(a^2+b^2+c^2\right)}{2\sqrt{3}}=s.\frac{12s^2-7s.2s+3.2\left(s^2-r^2-4Rr\right)}{2\sqrt{3}}$$

$$=\frac{2s^3-3sr(4R+r)}{\sqrt{3}}\overset{Mitrinovic\ \&\ Euler}{\le}\frac{2s^3-3.3\sqrt{3}r.r(4.2r+r)}{\sqrt{3}}=\frac{2s^3-81\sqrt{3}r^3}{\sqrt{3}}.$$

Therefore,

$$\frac{m_a{}^4}{w_a{}^3}+\frac{m_b{}^4}{w_b{}^3}+\frac{m_c{}^4}{w_c{}^3}\ge\frac{\sqrt{3}s^4}{2s^3-81\sqrt{3}r^3}.$$

Equality holds if and only if ΔABC is equilateral.

1150. **In ΔABC the following relationship holds:**

$$h_a h_b h_c\le\frac{1}{8}(h_a+h_b)(h_b+h_c)(h_c+h_a)\le m_a m_b m_c$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tapas Das-India

$$\frac{1}{8}(h_a+h_b)(h_b+h_c)(h_c+h_a)=F^3\prod_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{F^3}{a^2b^2c^2}\prod_{cyc}(a+b)=$$

$$=\frac{F^3}{16R^2F^2}\big((a+b+c)(ab+bc+ca)-abc\big)=$$

$$=\frac{F}{16R^2}(2s(s^2+r^2+4Rr)-4Rrs)=\frac{rs\cdot 2s}{16R^2}(s^2+r^2+2Rr)$$

$$h_ah_bh_c=\frac{8F^3}{abc}=\frac{8F^3}{4RF}=\frac{2F^2}{R}=\frac{2r^2s^2}{R}$$

$$\frac{rs\cdot 2s}{16R^2}(s^2+r^2+2Rr)\ge\frac{2r^2s^2}{R}$$

$$s^2+r^2+2Rr\ge 16Rr\Leftrightarrow s^2+r^2-14Rr\ge 0$$

$$s^2+r^2-14Rr\overset{GERRETSEN}{\ge}16Rr-5r^2+r^2-14Rr=$$

$$2Rr-4r^2\ge 0\Leftrightarrow 2R(R-2r)\ge 0$$

Lemma:

$$m_a\ge\sqrt{r_br_c}$$

Proof:

$$m_a\ge\sqrt{r_br_c}\Leftrightarrow m_a^2\ge\frac{F}{s-b}\cdot\frac{F}{s-c},\qquad \frac{2b^2+2c^2-a^2}{4}\ge\frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)}$$

$$\frac{2b^2+2c^2-a^2}{4}\ge s(s-a),\qquad \frac{2b^2+2c^2-a^2}{4}\ge\frac{b+c+a}{2}\cdot\frac{b+c-a}{2}$$

$$2b^2+2c^2-a^2\ge(b+c)^2-a^2,\qquad (b-c)^2\ge 0$$

By Lemma:

$$m_am_bm_c\ge\prod_{cyc}\sqrt{r_br_c}=s^2r,\quad \frac{rs\cdot 2s}{16R^2}(s^2+r^2+2Rr)\le s^2r$$

$$s^2+r^2+2Rr\le 8R^2$$

$$s^2+r^2+2Rr\overset{GERRETSEN}{\le}4R^2+4Rr+3r^2+r^2+2Rr\le 8R^2\Leftrightarrow$$

$$4R^2-6Rr-4r^2\ge 0\Leftrightarrow 2(R-2r)(2R+r)\ge 0$$

Equality holds for $a = b = c$.

Solution 2 by Ertan Yildirim-Turkiye

$$\frac{1}{8}(h_a+h_b)(h_b+h_c)(h_c+h_a) \overset{CESARO}{\geq} \frac{1}{8}\cdot 8h_ah_bh_c = h_ah_bh_c$$

$$\prod_{cyc} m_a \overset{TERESHIN}{\geq} \prod_{cyc}\frac{b^2+c^2}{4R} \geq \prod_{cyc}\frac{(b+c)^2}{8R} = \prod_{cyc}\frac{b+c}{4a}\cdot\frac{a(b+c)}{2R} =$$

$$= \prod_{cyc}\frac{b+c}{4a}\cdot\left(\frac{ab}{2R}+\frac{ac}{2R}\right) = \prod_{cyc}\frac{b+c}{4a}\cdot\left(\frac{ab}{2R}+\frac{ac}{2R}\right) =$$

$$= \prod_{cyc}\frac{b+c}{4a}\cdot(h_b+h_c) \overset{AM-GM}{\geq} \prod_{cyc}\frac{2\sqrt{bc}}{4a}\cdot(h_b+h_c) =$$

$$= \frac{8abc}{64abc}(h_a+h_b)(h_b+h_c)(h_c+h_a) = \frac{1}{8}(h_a+h_b)(h_b+h_c)(h_c+h_a)$$

Equality holds for $a = b = c$.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\frac{1}{8}(h_a+h_b)(h_b+h_c)(h_c+h_a) \overset{CESARO}{\geq} \frac{1}{8}\cdot 8h_ah_bh_c = h_ah_bh_c$$

$$m_a \geq \sqrt{s(s-a)} \Rightarrow \prod_{cyc} m_a \geq \prod_{cyc}\sqrt{s(s-a)} = sF$$

We will prove that:

$$\frac{1}{8}(h_a+h_b)(h_b+h_c)(h_c+h_a) \leq sF$$

$$\frac{1}{8}\prod_{cyc}(h_a+h_b) = \frac{8F^3}{8}\prod_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{F^3}{a^2b^2c^2}\prod_{cyc}(a+b) =$$

$$= \frac{F^3}{16R^2F^2}\prod_{cyc}(2RsinA+2RsinB) = \frac{F}{16R^2}\cdot 8R^3\prod_{cyc}(sinA+sinB) =$$

$$= \frac{FR}{2}\prod_{cyc}2sin\frac{A+B}{2}cos\frac{A-B}{2} \leq 4RF\prod_{cyc}sin\frac{A+B}{2} =$$

$$= abc\prod_{cyc} sin\frac{\pi - C}{2} = abc\prod_{cyc} cos\frac{C}{2} = abc\prod_{cyc}\sqrt{\frac{s(s-c)}{ab}} = abc \cdot \frac{sF}{abc} = sF$$

Equality holds for $a = b = c$.

1151. If $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ then:

$$\frac{x}{\sqrt{yz}} \cdot a + \frac{y}{\sqrt{zx}} \cdot b + \frac{z}{\sqrt{xy}} \cdot c \geq 2 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M.Bătinețu-Giurgiu, Florică Anastase-Romania

Solution by Tapas Das-India

$$\frac{x}{\sqrt{yz}} \cdot a + \frac{y}{\sqrt{zx}} \cdot b + \frac{z}{\sqrt{xy}} \cdot c \overset{AM-GM}{\geq}$$

$$\geq 3 \cdot \sqrt[3]{\frac{x}{\sqrt{yz}} \cdot a \cdot \frac{y}{\sqrt{zx}} \cdot b \cdot \frac{z}{\sqrt{xy}} \cdot c} = 3\sqrt[3]{abc} = 3 \cdot \sqrt[6]{(abc)^2} \overset{CARLITZ}{\geq}$$

$$\geq 3 \cdot \sqrt[6]{\left(\frac{4F}{\sqrt{3}}\right)^3} = 3\sqrt{\frac{4F}{\sqrt{3}}} = 3^{\frac{3}{4}} \cdot 2 \cdot \sqrt{F} = 2 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Equality holds for $x = y = z,\ a = b = c$.

1152. In $\Delta ABC, AM, BN, CP$ −medians, I −incenter. Prove that:

$$a \cdot IM^2 + b \cdot IN^2 + c \cdot IP^2 \geq \frac{abc}{4}$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Șerban-Romania

ΔBIC, **T.median,**

$$MI^2 = \frac{BI^2 + CI^2}{2} - \frac{a^2}{4} = \frac{\frac{r^2}{\sin^2\frac{B}{2}} + \frac{r^2}{\sin^2\frac{C}{2}}}{2} - \frac{a^2}{4} = \frac{r^2 ac}{2(p-a)(p-c)} + \frac{r^2 ab}{2(p-a)(p-b)} - \frac{a^2}{4},$$

$$\sum_{cyc} a \cdot IM^2 = p\sum_{cyc} a^2 - \frac{1}{2}\sum_{cyc} a^3 - \frac{abc}{p}\sum_{cyc} a - \frac{1}{4}\sum_{cyc} a^3 = p\sum_{cyc} a^2 - 2abc - \frac{3}{4}\sum_{cyc} a^3 \geq \frac{abc}{4},$$

$$4p\sum_{cyc} a^2 - 8abc - 3\sum_{cyc} a^3 \geq abc,\ 4p(2p^2 - 2r^2 - 8Rr) - 9 \cdot 4Rpr - 6p(p^2 - 3r^2 - 6Rr) \geq 0,$$

$$p(8p^2 - 8r^2 - 32Rr - 36Rr - 6p^2 + 18r^2 + 36Rr) \geq 0,\ p(2p^2 - 32Rr + 10r^2) \geq 0,$$

$p(2p^2 - 32Rr + 10r^2) \geq 0,\ p(p^2 - 16Rr + 5r^2) \geq 0,$ **true,because** $p > 0$ **and Gerretsen** inequality, $p^2 \geq 16Rr - 5r^2 \Rightarrow p^2 - 16Rr + 5r^2 \geq 0$, then $a \cdot IM^2 + b \cdot IN^2 + c \cdot IP^2 \geq \frac{abc}{4}$.

Equality for $a = b = c$.

1153. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \frac{m_a^2}{bcr_a} \leq \frac{R}{2r} \sum_{cyc} \frac{m_a^2}{bch_a}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{m_a^2}{bcr_a} = \sum_{cyc} \frac{am_a^2(s-a)}{4Rrs.rs}$$

$$= \frac{1}{16Rr^2s^2}\left(s\sum_{cyc}\left(a\left(4s(s-a)+(b-c)^2\right)\right) - \sum_{cyc}\left(a^2\left(4s(s-a)+(b-c)^2\right)\right)\right)$$

$$= \frac{4s^2\sum_{cyc} a(s-a) + s\sum_{cyc} a(b^2+c^2-2bc) - 4s\sum_{cyc} a^2(s-a) - \sum_{cyc} a^2(b^2+c^2-2bc)}{16Rr^2s^2}$$

$$= \frac{4s^2\left(s(2s) - 2(s^2-4Rr-r^2)\right) + s\left(2s(s^2+4Rr+r^2) - 3.4Rrs - 6.4Rrs\right)}{16Rr^2s^2}$$

$$+ \frac{-4s\left(2s(s^2-4Rr-r^2) - 2s(s^2-6Rr-3r^2)\right) - 2\left(\left(s^2+4Rr+r^2\right)^2 - 16Rrs^2\right) + 2.4Rrs(2s)}{16Rr^2s^2}$$

$$= \frac{(10R-5r)s^2 - r(4R+r)^2}{8Rrs^2} \therefore \sum_{cyc} \frac{m_a^2}{bcr_a} \overset{(i)}{=} \frac{(10R-5r)s^2 - r(4R+r)^2}{8Rrs^2}$$

$$\text{Again, } \frac{R}{2r}\sum_{cyc} \frac{m_a^2}{bch_a} = \frac{R}{2r}\sum_{cyc} \frac{a^2m_a^2}{4Rrs.2rs} = \frac{1}{64r^3s^2}\sum_{cyc}\left(a^2\left(4s(s-a)+(b-c)^2\right)\right)$$

$$= \frac{4s\left(2s(s^2-4Rr-r^2) - 2s(s^2-6Rr-3r^2)\right) + 2\left(\left(s^2+4Rr+r^2\right)^2 - 16Rrs^2\right) - 2.4Rrs(2s)}{64r^3s^2}$$

$$= \frac{s^4 - (8Rr-10r^2)s^2 + r^2(4R+r)^2}{64r^3s^2}$$

$$\therefore \frac{R}{2r}\sum_{cyc} \frac{m_a^2}{bch_a} \overset{(ii)}{=} \frac{s^4 - (8Rr-10r^2)s^2 + r^2(4R+r)^2}{64r^3s^2}$$

$$\therefore (i),(ii) \Rightarrow \sum_{cyc} \frac{m_a^2}{bcr_a} \leq \frac{R}{2r}\sum_{cyc} \frac{m_a^2}{bch_a}$$

$$\Leftrightarrow \frac{s^4 - (8Rr-10r^2)s^2 + r^2(4R+r)^2}{64r^3s^2} \geq \frac{(10R-5r)s^2 - r(4R+r)^2}{8Rrs^2}$$

$$\Leftrightarrow Rs^4 - rs^2(8R^2+30Rr-20r^2) + r^2(16R^3+72R^2r+33Rr^2+4r^3) \overset{(*)}{\geq} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} Rs^2(16Rr-5r^2) - rs^2(8R^2+30Rr-20r^2)$$

$$+r^2(16R^3+72R^2r+33Rr^2+4r^3) \overset{?}{\geq} 0$$

$$\Leftrightarrow (8R^2-35Rr+20r^2)s^2+r(16R^3+72R^2r+33Rr^2+4r^3)\underset{(**)}{\overset{?}{\geq}}0$$

$\boxed{Case\ 1}$ $8R^2-35Rr+20r^2\geq 0$ and then : LHS of $(**)$ $\geq r(16R^3+72R^2r+33Rr^2+4r^3)>0\Rightarrow(**)$ is true (strict inequality)

$\boxed{Case\ 2}$ $8R^2-35Rr+20r^2<0$ and then : LHS of $(**)$

$$=-\left(-(8R^2-35Rr+20r^2)\right)s^2+r(16R^3+72R^2r+33Rr^2+4r^3)$$

$$\overset{Gerretsen}{\geq}-\left(-(8R^2-35Rr+20r^2)\right)(4R^2+4Rr+3r^2)$$

$$+r(16R^3+72R^2r+33Rr^2+4r^3)\overset{?}{\geq}0\Leftrightarrow 8t^4-23t^3+9t^2+2t+16\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)(8t^2+9t+13)+18\right)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{Euler}{\geq}2\Rightarrow(**)\Rightarrow(*)$$

is true $\therefore$ in any ΔABC, $\sum_{cyc}\frac{m_a^2}{bcr_a}\leq\frac{R}{2r}\sum_{cyc}\frac{m_a^2}{bch_a}$, " = " iff ΔABC is equilateral (QED)

1154. If ω − Brocard's angle in ΔABC then,

$$max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\geq\frac{1}{2\sin\omega}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we may assume that, $a\geq b\geq c$. We have

$$\left(\frac{m_a}{h_a}\right)^2-\left(\frac{m_b}{h_b}\right)^2=\frac{a^2(2b^2+2c^2-a^2)-b^2(2c^2+2a^2-b^2)}{16F^2}=$$

$$=\frac{(a^2-b^2)(2c^2-a^2-b^2)}{16F^2}\leq 0$$

Similarly, we have

$$\left(\frac{m_b}{h_b}\right)^2-\left(\frac{m_c}{h_c}\right)^2=\frac{(b^2-c^2)(2a^2-b^2-c^2)}{16F^2}\geq 0.$$

Then, $max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}=\frac{m_b}{h_b}$, and it suffices to prove that, $\frac{m_b}{h_b}\geq\frac{1}{2\sin\omega}$.

$$\frac{m_b}{h_b}\overset{?}{\geq}\frac{1}{2\sin\omega}\Leftrightarrow\frac{b\sqrt{2c^2+2a^2-b^2}}{4F}\geq\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{4F}$$

$$\overset{squaring}{\Leftrightarrow}\quad a^2b^2+b^2c^2-c^2a^2-b^4\geq 0\Leftrightarrow(a^2-b^2)(b^2-c^2)\geq 0,$$

which is true and the proof is completes. Equality holds iff ΔABC is equilateral.

1155. In any Δ ABC, the following relationship holds :

$$\sqrt[3]{\frac{w_a^3+2w_aw_b(w_a+w_b)+w_b^3}{a^3+b^3}}+\sqrt[3]{\frac{w_b^3+2w_bw_c(w_b+w_c)+w_c^3}{b^3+c^3}}+\sqrt[3]{\frac{w_c^3+2w_cw_a(w_c+w_a)+w_a^3}{c^3+a^3}}<\frac{9\sqrt[3]{3}R}{s}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{\frac{w_a^3+2w_aw_b(w_a+w_b)+w_b^3}{a^3+b^3}}+\sqrt[3]{\frac{w_b^3+2w_bw_c(w_b+w_c)+w_c^3}{b^3+c^3}}+\sqrt[3]{\frac{w_c^3+2w_cw_a(w_c+w_a)+w_a^3}{c^3+a^3}}$$

$$=\sum_{cyc}\sqrt[3]{\frac{(w_b+w_c)(w_b^2-w_bw_c+w_c^2)+2w_bw_c(w_b+w_c)}{b^3+c^3}}$$

$$=\sum_{cyc}\sqrt[3]{\frac{(w_b+w_c)(w_b^2+w_bw_c+w_c^2)}{b^3+c^3}}\le\sum_{cyc}\sqrt[3]{\frac{(w_b+w_c)(w_b^2+w_bw_c+w_c^2)}{\frac{(b+c)^3}{4}}}$$

$$<\sqrt[3]{4}.\sum_{cyc}\sqrt[3]{\frac{(w_b+w_c)(w_b+w_c)^2}{(b+c)^3}}=\sqrt[3]{4}.\sum_{cyc}\frac{w_b+w_c}{b+c}$$

$$\le\sqrt[3]{4}.\sqrt{s}.\sum_{cyc}\frac{\sqrt{s-b}+\sqrt{s-c}}{b+c}\overset{CBS}{\le}\sqrt[3]{4}.\sqrt{2}.\sqrt{s}.\sum_{cyc}\frac{\sqrt{s-b+s-c}}{b+c}$$

$$=\sqrt[3]{4}.\sqrt{2}.\sqrt{s}.\sum_{cyc}\left(\sqrt{\frac{a}{b+c}}.\sqrt{\frac{1}{b+c}}\right)\overset{CBS}{\le}\sqrt[3]{4}.\sqrt{2}.\sqrt{s}.\sqrt{\sum_{cyc}\frac{a}{b+c}}.\sqrt{\sum_{cyc}\frac{1}{b+c}}$$

$$=\sqrt[3]{4}.\sqrt{2}.\sqrt{s}.\sqrt{\frac{\sum_{cyc}(a(c+a)(a+b))}{\prod_{cyc}(b+c)}}.\sqrt{\frac{\sum_{cyc}((c+a)(a+b))}{\prod_{cyc}(b+c)}}$$

$$=\frac{\sqrt[3]{4}.\sqrt{2}.\sqrt{s}}{2s(s^2+2Rr+r^2)}.\sqrt{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+\sum_{cyc}a^3}.\sqrt{\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab}$$

$$=\frac{\sqrt[3]{4}.\sqrt{2}.\sqrt{s}}{2s(s^2+2Rr+r^2)}.\sqrt{2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)}.\sqrt{5s^2+4Rr+r^2}$$

$$=\frac{\sqrt[3]{4}.\sqrt{2}}{s^2+2Rr+r^2}.\sqrt{s^2-Rr-r^2}.\sqrt{5s^2+4Rr+r^2}\overset{?}{<}\frac{9\sqrt[3]{3}R}{s}$$

$$\Leftrightarrow\frac{s}{R}.\frac{\sqrt{s^2-Rr-r^2}.\sqrt{5s^2+4Rr+r^2}}{s^2+2Rr+r^2}\underset{(*)}{\overset{?}{<}}\frac{9\sqrt[3]{3}}{\sqrt[3]{4}.\sqrt{2}}\approx 5.7820423$$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$$\frac{s}{R}.\frac{\sqrt{s^2-Rr-r^2}.\sqrt{5s^2+4Rr+r^2}}{s^2+2Rr+r^2}<\frac{23}{4}$$

$$\Leftrightarrow 16s^2(s^2-Rr-r^2)(5s^2+4Rr+r^2)<529R^2(s^2+2Rr+r^2)^2$$

$$\Leftrightarrow 80s^6-(529R^2+16Rr+64r^2)s^4-rs^2(2116R^3+1122R^2r+80Rr^2+16r^3)$$

$$-R^2r^2(2116R^2+2116Rr+529r^2)\overset{(**)}{<}0$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\le}\left(80(4R^2+4Rr+3r^2)-(529R^2+16Rr+64r^2)\right)s^4$$

$$-rs^2(2116R^3+1122R^2r+80Rr^2+16r^3)$$

$$-R^2r^2(2116R^2+2116Rr+529r^2)\overset{?}{<}0$$

$$\Leftrightarrow(209R^2-304Rr-176r^2)s^4+rs^2(2116R^3+1122R^2r+80Rr^2+16r^3)$$

$$+R^2r^2(2116R^2+2116Rr+529r^2)\overset{?}{>}0$$

$$\Leftrightarrow\left((R-2r)(209R+114r)+52r^2\right)s^4+rs^2(2116R^3+1122R^2r+80Rr^2+16r^3)$$

$$+R^2r^2(2116R^2+2116Rr+529r^2)\overset{?}{>}0\to\text{true}\because R-2r\overset{\text{Euler}}{\ge}0$$

$\Rightarrow(**)\Rightarrow(*)$ is true $\therefore$ in any Δ ABC,

$$\sqrt[3]{\frac{w_a^3+2w_aw_b(w_a+w_b)+w_b^3}{a^3+b^3}}+\sqrt[3]{\frac{w_b^3+2w_bw_c(w_b+w_c)+w_c^3}{b^3+c^3}}$$

$$+\sqrt[3]{\frac{w_c^3+2w_cw_a(w_c+w_a)+w_a^3}{c^3+a^3}}<\frac{9\sqrt[3]{3}R}{s}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$LHS_{(1)}\le\sqrt[3]{\sum_{cyc}[w_b{}^3+2w_bw_c(w_b+w_c)+w_c{}^3].\sum_{cyc}\frac{1}{b+c}.\sum_{cyc}\frac{1}{b^2-bc+c^2}},$$

Also, we have

$$\sum_{cyc}[w_b{}^3+2w_bw_c(w_b+w_c)+w_c{}^3]=2\sum_{cyc}w_a{}^3+2\sum_{cyc}w_bw_c(w_b+w_c)=2\sum_{cyc}w_a.\sum_{cyc}w_a{}^2$$

$$\overset{CBS}{\le}2\sqrt{3\left(\sum_{cyc}w_a{}^2\right)^3}\overset{w_a\le\sqrt{s(s-a)}}{\le}2\sqrt{3\left(\sum_{cyc}s(s-a)\right)^3}=2\sqrt{3}s^3\overset{Mitrinovic}{\le}$$

$$\leq 2\sqrt{3}\left(\frac{3\sqrt{3}R}{2}\right)^3 = \frac{243R^3}{4},$$

and since $b+c = s+(s-a) \overset{s>a}{>} s$, then we have

$$\sum_{cyc}\frac{1}{b+c} < \sum_{cyc}\frac{1}{s} = \frac{3}{s} \text{ and } \sum_{cyc}\frac{1}{b^2-bc+c^2} \overset{AM-GM}{\leq} \sum_{cyc}\frac{4}{(b+c)^2} < \sum_{cyc}\frac{4}{s^2} = \frac{12}{s^2}.$$

Therefore,

$$\sum_{cyc}\sqrt[3]{\frac{{w_b}^3+2w_bw_c(w_b+w_c)+{w_c}^3}{b^3+c^3}} < \sqrt[3]{\frac{243R^3}{4}\cdot\frac{3}{s}\cdot\frac{12}{s^2}} = \frac{9\sqrt[3]{3}R}{s}.$$

1156. AD, BE, CF **− internal bisectors in** ΔABC,

R_A, R_B, R_C **− circumradii of** $\Delta AEF, \Delta BFD, \Delta CDE$. **Prove that**

$$\frac{BC}{R_A}+\frac{CA}{R_B}+\frac{AB}{R_C} \geq 12\sqrt{3}\cdot\frac{r}{R}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a = BC, b = CA, c = AB$ be the side lengths of the triangle ABC. From the angle bisector theorem in triangle ABC, it follows that

$$AE = \frac{bc}{c+a} \text{ and } AF = \frac{bc}{a+b}.$$

By the law of cosines in triangle AEF we have

$$EF^2 = AE^2 + AF^2 - 2AE.AF.\cos A$$

$$= \left(\frac{bc}{c+a}\right)^2 + \left(\frac{bc}{a+b}\right)^2 - 2\left(\frac{bc}{a+c}\right)\left(\frac{bc}{a+b}\right).\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{bc[bc(a+b)^2+bc(c+a)^2-(b^2+c^2-a^2)(c+a)(a+b)]}{(c+a)^2(a+b)^2}$$

$$= \frac{bc\left[a^2(c+a)(a+b)+bc(b^2+c^2)+2abc(a+b+c)-(b^2+c^2)[bc+a(a+b+c)]\right]}{(c+a)^2(a+b)^2}$$

$$= \frac{bc[a^2(c+a)(a+b)-a(a+b+c)(b-c)^2]}{(c+a)^2(a+b)^2} \leq \frac{a^2bc}{(c+a)(a+b)} \leq$$

$$\leq \frac{a^2bc}{2\sqrt{ca}.2\sqrt{ab}} = \frac{\sqrt{a^2bc}}{4},$$

where we have used

the AM − GM inequality in the last step. Thus, in ΔAEF, we have,

$$R_A = \frac{EF}{2\sin\widehat{EAF}} \leq \frac{\sqrt[4]{a^2bc}}{4\sin A} = \frac{R\sqrt[4]{a^2bc}}{2a} = \frac{R}{2}\sqrt[4]{\frac{bc}{a^2}} \text{ (and analogs)}$$

Then, $$R_AR_BR_C \leq \frac{R}{2}\sqrt[4]{\frac{bc}{a^2}}.\frac{R}{2}\sqrt[4]{\frac{ca}{b^2}}.\frac{R}{2}\sqrt[4]{\frac{ab}{c^2}} = \left(\frac{R}{2}\right)^3.$$

Also, we have, $$abc = 4Rsr \overset{Euler\ \&\ Mitrinovic}{\geq} 4.2r.3\sqrt{3}r.r = \left(2\sqrt{3}r\right)^3.$$

Therefore, by the AM − GM inequality, we have,

$$\frac{BC}{R_A} + \frac{CA}{R_B} + \frac{AB}{R_C} \geq 3\sqrt[3]{\frac{abc}{R_AR_BR_C}} \geq 3.\frac{2\sqrt{3}r}{\frac{R}{2}} = 12\sqrt{3}.\frac{r}{R},$$

as desired. Equality holds if and only if the trangle ABC is equilateral.

1157.In any Δ ABC, the following relationship holds :

$$\frac{ab(a^2+b^2)}{a+b} + \frac{bc(b^2+c^2)}{b+c} + \frac{ca(c^2+a^2)}{c+a} < \frac{54R^4}{s}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{ab(a^2+b^2)}{a+b} + \frac{bc(b^2+c^2)}{b+c} + \frac{ca(c^2+a^2)}{c+a} \overset{G-H}{\leq} \sum_{cyc}\left(\frac{\sqrt{ab}}{2}.(a^2+b^2)\right)$$

$$= \frac{1}{2}.\sum_{cyc}\left(\sqrt{ab(a^2+b^2)}.\sqrt{a^2+b^2}\right) \overset{CBS}{\leq} \frac{1}{2}.\sqrt{\sum_{cyc} ab(a^2+b^2)}.\sqrt{\sum_{cyc}(a^2+b^2)}$$

$$= \frac{1}{2}.\sqrt{\sum_{cyc}\left(ab\left(\sum_{cyc}a^2 - c^2\right)\right)}.\sqrt{2\sum_{cyc}a^2}$$

$$= \frac{1}{2}.\sqrt{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right) - abc\sum_{cyc}a}.\sqrt{2\sum_{cyc}a^2}$$

$$<\frac{1}{2}\cdot\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2\right)}\cdot\sqrt{2\sum_{cyc}a^2}\overset{\text{Leibnitz}}{\leq}\frac{1}{2}\cdot\sqrt{2.729R^6}=\frac{27R^3.s}{\sqrt{2}s}$$

$$\overset{\text{Mitrinovic}}{\leq}\frac{27R^3.3\sqrt{3}R}{2\sqrt{2}s}\approx 49.602*\frac{R^4}{s}<\frac{54R^4}{s}\therefore \text{ in } any\ \Delta\ ABC,$$

$$\frac{ab(a^2+b^2)}{a+b}+\frac{bc(b^2+c^2)}{b+c}+\frac{ca(c^2+a^2)}{c+a}<\frac{54R^4}{s}\ (QED)$$

1158. If ω − Brocard's angle in $\Delta ABC, M\in Int(\Delta ABC)$ then :

$$\frac{MA}{h_b}+\frac{MB}{h_c}+\frac{MC}{h_a}\geq\frac{1}{\sin\omega}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : $(G.Bennett's\ inequality)$ **Ș If** $P, M\in Int(\Delta ABC)$, **then :**

$$a.PA.MA+b.PB.MB+c.PC.MC\geq abc \qquad [1]$$

For $P\equiv\Omega$, **where** Ω **is the first Brocard's point, we have**

$$a.\Omega A.MA+b.\Omega B.MB+c.\Omega C.MC\geq abc,$$

$$\text{with, }\ \Omega A=b.\frac{\sin\omega}{\sin A}=bc.\frac{\sin\omega}{h_b}\ (\text{and analogs}).\text{ Then,}$$

$$\frac{MA}{h_b}+\frac{MB}{h_c}+\frac{MC}{h_a}\geq\frac{1}{\sin\omega}.$$

References : [1] 584 G. Bennett − *Multiple Triangle Inequalities*.

(see also, Bogdan Fuștei − *About Nagel's and Gergonnes's cevian (IX)* − www.ssmrmh.ro)

1159. In ΔABC the following relationship holds :

$$\frac{(a^2+b^2+c^2)^3}{192R^3F^3}+\frac{R^3}{r^6}\geq\frac{8}{r^3}+3\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Leuenberger, Steinig and AM – GM inequalities, we have,

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\frac{1}{a}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)-\frac{1}{b}\left(\frac{1}{c^2}+\frac{1}{a^2}\right)-\frac{1}{c}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

$$\le\frac{\sqrt{3}}{2r}\cdot\frac{1}{4r^2}-\frac{6}{abc}=\frac{\sqrt{3}}{8r^3}-\frac{3}{2sRr}\overset{Mitrinovic}{\leq}\frac{\sqrt{3}}{8r^3}-\frac{3}{3\sqrt{3}R.Rr}=\frac{\sqrt{3}}{8r^3}-\frac{\sqrt{3}}{3R^2r}.$$

By Ionescu– Weitzenbock inequality, we have, $a^2+b^2+c^2\ge 4\sqrt{3}F$.
So it suffices to prove that,

$$\frac{\sqrt{3}}{R^3}+\frac{R^3}{r^6}\ge\frac{8}{r^3}+3\left(\frac{\sqrt{3}}{8r^3}-\frac{\sqrt{3}}{3R^2r}\right)\Leftrightarrow\frac{R^3-8r^3}{r^6}\ge\frac{\sqrt{3}(3R^3-8Rr^2-8r^3)}{8R^3r^3}$$

$$\Leftrightarrow (R-2r)\left[8R^3(R^2+2Rr+4r^2)-\sqrt{3}r^3(3R^2+6Rr+4r^2)\right]\ge 0,$$

which is true by Euler inequality, $R\ge 2r$, **and,**

$$8R^3(R^2+2Rr+4r^2)\ge 64r^3(R^2+2Rr+4r^2)>\sqrt{3}r^3(3R^2+6Rr+4r^2).$$

So the proof is completed. Equality holds iff ΔABC **is equilateral.**

1160. ***In ΔABC holds:***

$$\frac{27R}{4p}\le\sum_{cyc}\frac{m_a{}^2}{ar_a}\le\frac{27R^5}{64pr^4}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

$$\sum_{cyc}\frac{m_a{}^2}{ar_a}\overset{m_a\ge\sqrt{p(p-a)}}{\geq}\sum_{cyc}\frac{p(p-a)^2}{aF}\overset{CBS}{\geq}\frac{\left(\sum_{cyc}(p-a)\right)^2}{r(a+b+c)}=\frac{p}{2r}\overset{Cosnita-Turtoiu}{\geq}\frac{27R}{4p}.$$

$$\text{Also, }\sum_{cyc}\frac{m_a{}^2}{ar_a}=\sum_{cyc}\frac{(p-a)[2(a^2+b^2+c^2)-3a^2]}{4Fa}$$

$$=\frac{a^2+b^2+c^2}{2F}\cdot\sum_{cyc}\frac{p-a}{a}-\frac{3}{4F}\sum_{cyc}a(p-a)$$

$$=\frac{p^2-r^2-4Rr}{F}\cdot\frac{p^2+r^2-8Rr}{4Rr}-\frac{3}{4F}\cdot 2r(4R+r)$$

$$\overset{Gerretsen}{\leq}\frac{2(2R^2+r^2).4(R^2-Rr+r^2)}{4FRr}-\frac{3r(4R+r)}{2F}$$
$$=\frac{8R^4-8R^3r-7Rr^3+4r^4}{2pRr^2}\overset{?}{\leq}\frac{27R^5}{64pr^4}$$
$$\Leftrightarrow 27R^6-256R^4r^2+256R^3r^3+224Rr^5-128r^6\geq 0$$
$$\Leftrightarrow (R-2r)[(R-2r)(27R^4+108R^3r+68R^2r^2+96Rr^3+112r^4)+288r^4]\geq 0,$$
which is true by Euler's inequality, $R\geq 2r$.

The proof is completed. Equality holds iff ΔABC is equilateral.

1161.

Prove that in any triangle ABC are true the following inequalities:
$$\sum_{cyc}a.\sum_{cyc}\frac{a}{b+c}\leq\frac{3}{2}\sum_{cyc}\frac{a^2}{b},$$
$$\sum_{cyc}\frac{a^2}{b}\geq\frac{2}{3}\sum_{cyc}a.\sum_{cyc}\frac{a}{b+c}+\sum_{cyc}ab.\sum_{cyc}\frac{(a-b)^2}{3b(a+c)(b+c)}$$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have,

- $$\sum_{cyc}\frac{a^2}{b}-\sum_{cyc}a=\sum_{cyc}\left(\frac{a^2}{b}-2a+b\right)=\sum_{cyc}\frac{(a-b)^2}{b}.$$
- $$\sum_{cyc}\frac{a}{b+c}-\frac{3}{2}=\sum_{cyc}\left(\frac{a}{b+c}-\frac{1}{2}\right)=\sum_{cyc}\left(\frac{a-b}{2(b+c)}-\frac{c-a}{2(b+c)}\right)$$
$$=\sum_{cyc}\left(\frac{a-b}{2(b+c)}-\frac{a-b}{2(c+a)}\right)=\sum_{cyc}\frac{(a-b)^2}{2(a+c)(b+c)}.$$

Then,
$$\frac{3}{2}\sum_{cyc}\frac{a^2}{b}-\sum_{cyc}a.\sum_{cyc}\frac{a}{b+c}=\frac{3}{2}\left(\sum_{cyc}\frac{a^2}{b}-\sum_{cyc}a\right)-\sum_{cyc}a.\left(\sum_{cyc}\frac{a}{b+c}-\frac{3}{2}\right)$$
$$=\frac{3}{2}\sum_{cyc}\frac{(a-b)^2}{b}-\sum_{cyc}a.\sum_{cyc}\frac{(a-b)^2}{2(a+c)(b+c)}$$
$$=\sum_{cyc}\frac{[3(a+c)(b+c)-b(a+b+c)](a-b)^2}{2b(a+c)(b+c)}$$
$$=\sum_{cyc}\frac{[3c^2+ab+bc+3ca+b(a-b+c)](a-b)^2}{2b(a+c)(b+c)}\geq 0,$$

thus, $\frac{3}{2}\sum_{cyc}\frac{a^2}{b} \geq \sum_{cyc} a \cdot \sum_{cyc}\frac{a}{b+c}$, is true in any triangle ABC.

Using the last relation, we have,

$$\sum_{cyc}\frac{a^2}{b} - \frac{2}{3}\sum_{cyc} a \cdot \sum_{cyc}\frac{a}{b+c} = \frac{2}{3}\sum_{cyc}\frac{[3c^2+ab+bc+3ca+b(a-b+c)](a-b)^2}{2b(a+c)(b+c)}$$

$$\geq \sum_{cyc}\frac{(ab+bc+ca)(a-b)^2}{3b(a+c)(b+c)} = \sum_{cyc} ab \cdot \sum_{cyc}\frac{(a-b)^2}{3b(a+c)(b+c)},$$

thus,

$$\sum_{cyc}\frac{a^2}{b} \geq \frac{2}{3}\sum_{cyc} a \cdot \sum_{cyc}\frac{a}{b+c} + \sum_{cyc} ab \cdot \sum_{cyc}\frac{(a-b)^2}{3b(a+c)(b+c)},$$

is true in any triangle ABC.

1162. If ω– Brocard′s angle in ΔABC then

$$\prod_{cyc}\left(\frac{1}{\sin\omega} - 2\cos A\right) \geq 1$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we will prove that,

$$\frac{1}{\sin\omega} \geq \frac{b^2+c^2}{bc}, \quad \forall\Delta ABC \qquad (1)$$

$$\text{Since } \sin\omega = \frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}} \text{ and}$$

$$4F = \sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}, \text{ then,}$$

$$(1) \Leftrightarrow 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$
$$\geq (b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow} 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$$
$$\geq (2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow 0 \geq -a^4(b^2+c^2)^2 + 2(b^4+c^4)(a^2b^2+c^2a^2) - (b^4+c^4)^2$$
$$= -[a^2(b^2+c^2)-(b^4+c^4)]^2,$$

which is true. Using this result we have,

$$\frac{1}{\sin\omega} - 2\cos A \geq \frac{b^2+c^2}{bc} - \frac{b^2+c^2-a^2}{bc} = \frac{a^2}{bc} \quad \text{(and analogs)}$$

Therefore,

$$\prod_{cyc}\left(\frac{1}{\sin\omega}-2\cos A\right)\ge\prod_{cyc}\frac{a^2}{bc}=1.$$

Equality holds iff ΔABC is equilateral.

1163.

In any Δ ABC with $\omega\to$ Brocard's angle, the following relationship holds :

$$5+\frac{2}{\sin\omega}\ge\frac{h_a+h_b+h_c}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$5+\frac{2}{\sin\omega}\ge\frac{h_a+h_b+h_c}{r}\Leftrightarrow\frac{\sqrt{\sum_{cyc}a^2b^2}}{rs}\ge\frac{s^2+4Rr+r^2}{2Rr}-5$$

$$\Leftrightarrow\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{r^2s^2}\ge\frac{(s^2-6Rr+r^2)^2}{4R^2r^2}$$

$$\Leftrightarrow s^6-(4R^2+12Rr-2r^2)s^4+rs^2(32R^3+28R^2r-12Rr^2+r^3)$$

$$-4R^2r^2(4R+r)^2\overset{(*)}{\le}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where

$m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0$$

$$\Rightarrow s^6-(4R^2+20Rr-2r^2)s^4+rs^2(4R+r)^3\le 0$$

$\Rightarrow$ in order to prove $(*)$, it suffices to show :

LHS of $(*)\le s^6-(4R^2+20Rr-2r^2)s^4+rs^2(4R+r)^3$

$$\Leftrightarrow 2s^4-s^2(8R^2+5Rr+6r^2)-Rr(4R+r)^2\overset{(**)}{\le}0$$

Now, LHS of $(**)\overset{\text{Gerretsen}}{\le}s^2(8R^2+8Rr+6r^2)-s^2(8R^2+5Rr+6r^2)$

$$-Rr(4R+r)^2\overset{?}{\le}0\Leftrightarrow 3s^2\overset{?}{\le}16R^2+8Rr+r^2$$

$$\Leftrightarrow 3s^2-(12R^2+12Rr+9r^2)-4(R-2r)(R+r)\overset{?}{\le}0$$

$\to$ true $\because 3s^2-(12R^2+12Rr+9r^2)\overset{\text{Gerretsen}}{\le}0$ and $-4(R-2r)(R+r)\overset{\text{Euler}}{\le}0$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore 5+\frac{2}{\sin\omega}\ge\frac{h_a+h_b+h_c}{r},$$

" = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly we will prove that,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le 1+\frac{1}{\sin\omega}.$$

WLOG, we may assume that $c=min\{a,b,c\}$. If $a\ge b\ge c$. We have

$$\left(1+\frac{c}{a}+\frac{a}{c}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\frac{(a-b)(b-c)}{bc}\ge 0\ \Rightarrow\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le 1+\frac{c}{a}+\frac{a}{c}.$$

Similarly, if $b\ge a\ge c$ then we have, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le 1+\frac{b}{c}+\frac{c}{b}$.

Thus, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le 1+max\left\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{c}{a}+\frac{a}{c}\right\}$.

So it suffices to prove that, in any ΔABC, we have, $\frac{b}{c}+\frac{c}{b}\le\frac{1}{\sin\omega}$ **(1)**

Since $\sin\omega=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ **and** $4F=\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$, **then,**

$$(1)\ \Leftrightarrow\ 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}\ge(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow}\quad 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)\ge(2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow\ 0\ge -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2=-[a^2(b^2+c^2)-(b^4+c^4)]^2,$$

which is true, and the proof is complete.

Using this result we have

$$5+\frac{2}{\sin\omega}=3+2\left(1+\frac{1}{\sin\omega}\right)\ge 3+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

$$=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{h_a+h_b+h_c}{r},$$

as desired. Equality holds iff ΔABC is equilateral.

1164.

In any Δ ABC with ω → Brocard's angle, the following relationship holds :

$$\frac{1}{\sin\omega}\ge max\left(2+\frac{(a-c)^2}{ab+bc+ca},2+\frac{(m_a-m_c)^2}{m_am_b+m_bm_c+m_cm_a}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}a^2b^2}{4F^2}\overset{?}{\ge}\frac{(c^2+a^2)^2}{c^2a^2}\Leftrightarrow\frac{\sum_{cyc}a^4+16F^2}{8F^2}\overset{?}{\ge}\frac{c^4+a^4+2c^2a^2}{c^2a^2}$$

$$\Leftrightarrow\frac{\sum_{cyc}a^4}{8F^2}\overset{?}{\ge}\frac{c^4+a^4}{c^2a^2}\Leftrightarrow\frac{\sum_{cyc}a^4}{c^4+a^4}\overset{?}{\ge}\frac{8F^2}{c^2a^2}$$

$$\Leftrightarrow\frac{\sum_{cyc}a^4-(c^4+a^4)}{c^4+a^4}\overset{?}{\ge}\frac{2a^2b^2+2b^2c^2-(\sum_{cyc}a^4)}{2c^2a^2}$$

$$\Leftrightarrow \frac{b^4}{c^4+a^4}+\frac{b^4}{2c^2a^2}+\frac{c^4+a^4}{2c^2a^2} \overset{?}{\geq} \frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow \frac{b^4(c^2+a^2)^2}{2c^2a^2(c^4+a^4)}+\frac{c^4+a^4}{2c^2a^2} \overset{?}{\geq} \frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow b^4(c^2+a^2)^2+(c^4+a^4)^2 \overset{?}{\geq} 2b^2(c^2+a^2)(c^4+a^4) \rightarrow \text{true via AM}-\text{GM}$$

$$\therefore \frac{\sum_{cyc} a^2b^2}{4F^2} \geq \frac{(c^2+a^2)^2}{c^2a^2} \Rightarrow \frac{\sqrt{\sum_{cyc} a^2b^2}}{2F} \geq \frac{c}{a}+\frac{a}{c} \Rightarrow \frac{1}{\sin\omega} \geq \frac{c^2+a^2}{ca}$$

$$= \frac{2ca+c^2+a^2-2ca}{ca} = 2+\frac{(a-c)^2}{ca} \geq 2+\frac{(a-c)^2}{ab+bc+ca}$$

$$\therefore \frac{1}{\sin\omega} \overset{(*)}{\geq} 2+\frac{(a-c)^2}{ab+bc+ca}\,(*) \Rightarrow \frac{\sqrt{\sum_{cyc} a^2b^2}}{2F} \geq 2+\frac{(a-c)^2}{ab+bc+ca}$$

implementing which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$

whose area as a consequence of basic computation $= \frac{F}{3}$,

we arrive at : $\frac{\sqrt{\frac{16}{81}\cdot\sum_{cyc} m_a^2m_b^2}}{\frac{2F}{3}} \geq 2+\frac{\frac{4}{9}(m_a-m_c)^2}{\frac{4}{9}(m_am_b+m_bm_c+m_cm_a)}$

$$\Rightarrow \frac{\sqrt{\frac{16}{81}\cdot\frac{9}{16}\sum_{cyc} a^2b^2}}{\frac{2F}{3}} \geq 2+\frac{(m_a-m_c)^2}{m_am_b+m_bm_c+m_cm_a}$$

$$\Rightarrow \frac{1}{\sin\omega} \overset{(**)}{\geq} 2+\frac{(m_a-m_c)^2}{m_am_b+m_bm_c+m_cm_a} \therefore (*),(**) \Rightarrow$$

$$\frac{1}{\sin\omega} \geq max\left(2+\frac{(a-c)^2}{ab+bc+ca}, 2+\frac{(m_a-m_c)^2}{m_am_b+m_bm_c+m_cm_a}\right) \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly we will prove that, in any ΔABC, we have

$$\frac{1}{\sin\omega} \geq \frac{c}{a}+\frac{a}{c} \quad (1)$$

Since $\sin\omega = \frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ and $4F = \sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$, then,

$$(1) \Leftrightarrow 2ca\sqrt{a^2b^2+b^2c^2+c^2a^2} \geq (c^2+a^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow} 4c^2a^2(a^2b^2+b^2c^2+c^2a^2) \geq (2c^2a^2+c^4+a^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow 0 \geq -b^4(c^2+a^2)^2+2(c^4+a^4)(a^2b^2+b^2c^2)-(c^4+a^4)^2 = -[b^2(c^2+a^2)-(c^4+a^4)]^2,$$

which is true, and the proof of (1) is complete.

Using this result, we have

$$\frac{1}{\sin\omega}\ge\frac{c}{a}+\frac{a}{c}=2+\frac{(c-a)^2}{ca}\ge 2+\frac{(c-a)^2}{ab+bc+ca}\quad (2)$$

Now, since m_a, m_b, m_c can be the sides of triangle with, area $F_m=\frac{3F}{4}$, and,

$$m_a{}^2m_b{}^2+m_b{}^2m_c{}^2+m_c{}^2m_a{}^2=\frac{9}{16}(a^2b^2+b^2c^2+c^2a^2),\ \text{then},$$

$$\sin\omega_m=\frac{2F_m}{\sqrt{m_a{}^2m_b{}^2+m_b{}^2m_c{}^2+m_c{}^2m_a{}^2}}=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}=\sin\omega.$$

Using (2) in $\Delta m_a m_b m_c$, we get,

$$\frac{1}{\sin\omega}=\frac{1}{\sin\omega_m}\ge 2+\frac{(m_c-m_a)^2}{m_am_b+m_bm_c+m_cm_a}..$$

Therefore,

$$\frac{1}{\sin\omega}\ge max\left\{2+\frac{(c-a)^2}{ab+bc+ca},2+\frac{(m_c-m_a)^2}{m_am_b+m_bm_c+m_cm_a}\right\}$$

1165.

In any Δ ABC with $\omega\to$ Brocard's angle, the following relationship holds :

$$\frac{1}{\sin\omega}\ge\frac{a+b}{c+b}+\frac{c+b}{a+b}\ \text{and}\ \frac{1}{\sin\omega}\ge\frac{m_a+m_b}{m_c+m_b}+\frac{m_c+m_b}{m_a+m_b}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}a^2b^2}{4F^2}\overset{?}{\ge}\frac{(c^2+a^2)^2}{c^2a^2}\Leftrightarrow\frac{\sum_{cyc}a^4+16F^2}{8F^2}\overset{?}{\ge}\frac{c^4+a^4+2c^2a^2}{c^2a^2}$$

$$\Leftrightarrow\frac{\sum_{cyc}a^4}{8F^2}\overset{?}{\ge}\frac{c^4+a^4}{c^2a^2}\Leftrightarrow\frac{\sum_{cyc}a^4}{c^4+a^4}\overset{?}{\ge}\frac{8F^2}{c^2a^2}$$

$$\Leftrightarrow\frac{\sum_{cyc}a^4-(c^4+a^4)}{c^4+a^4}\overset{?}{\ge}\frac{2a^2b^2+2b^2c^2-(\sum_{cyc}a^4)}{2c^2a^2}$$

$$\Leftrightarrow\frac{b^4}{c^4+a^4}+\frac{b^4}{2c^2a^2}+\frac{c^4+a^4}{2c^2a^2}\overset{?}{\ge}\frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow\frac{b^4(c^2+a^2)^2}{2c^2a^2(c^4+a^4)}+\frac{c^4+a^4}{2c^2a^2}\overset{?}{\ge}\frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow b^4(c^2+a^2)^2+(c^4+a^4)^2\overset{?}{\ge}2b^2(c^2+a^2)(c^4+a^4)\to\text{true via AM}-\text{GM}$$

$$\therefore\frac{\sum_{cyc}a^2b^2}{4F^2}\ge\frac{(c^2+a^2)^2}{c^2a^2}\Rightarrow\frac{\sqrt{\sum_{cyc}a^2b^2}}{2F}\ge\frac{c}{a}+\frac{a}{c}\Rightarrow\frac{1}{\sin\omega}\overset{(*)}{\ge}\frac{c}{a}+\frac{a}{c}$$

$$\text{Now,}\frac{c}{a}+\frac{a}{c}\overset{?}{\ge}\frac{a+b}{c+b}+\frac{c+b}{a+b}\Leftrightarrow c\left(\frac{1}{a}-\frac{1}{a+b}\right)+a\left(\frac{1}{c}-\frac{1}{b+c}\right)\overset{?}{\ge}b\left(\frac{1}{b+c}+\frac{1}{a+b}\right)$$

$$\Leftrightarrow\frac{c}{a(a+b)}+\frac{a}{c(b+c)}\overset{?}{\ge}\frac{1}{b+c}+\frac{1}{a+b}\Leftrightarrow\frac{1}{a+b}\cdot\frac{c-a}{a}-\frac{1}{b+c}\cdot\frac{c-a}{c}\overset{?}{\ge}0$$

$$\Leftrightarrow (c-a)\left(\frac{1}{a(a+b)}-\frac{1}{c(b+c)}\right)\overset{?}{\geq}0 \Leftrightarrow \frac{(c-a)\big((c-a)(c+a)+b(c-a)\big)}{ac(a+b)(b+c)}\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{(c-a)^2(a+b+c)}{ac(a+b)(b+c)}\overset{?}{\geq}0 \to \text{true} \therefore \frac{c}{a}+\frac{a}{c}\overset{(**)}{\geq}\frac{a+b}{c+b}+\frac{c+b}{a+b}$$

$$\therefore (*),(**) \Rightarrow \boxed{\frac{1}{\sin\omega}\geq\frac{a+b}{c+b}+\frac{c+b}{a+b}} \therefore \frac{\sqrt{\sum_{cyc}a^2b^2}}{2F}\geq\frac{a+b}{c+b}+\frac{c+b}{a+b}$$

implementing which on a triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$

whose area as a consequence of basic computation $=\frac{F}{3}$, we arrive at :

$$\frac{\sqrt{\frac{16}{81}\cdot\sum_{cyc}m_a^2m_b^2}}{\frac{2F}{3}}\geq\frac{\frac{2}{3}(m_a+m_b)}{\frac{2}{3}(m_c+m_b)}+\frac{\frac{2}{3}(m_c+m_b)}{\frac{2}{3}(m_a+m_b)}$$

$$\Rightarrow \frac{\sqrt{\frac{16}{81}\cdot\frac{9}{16}\sum_{cyc}a^2b^2}}{\frac{2F}{3}}\geq\frac{m_a+m_b}{m_c+m_b}+\frac{m_c+m_b}{m_a+m_b}\Rightarrow\frac{\sqrt{\sum_{cyc}a^2b^2}}{2F}\geq\frac{m_a+m_b}{m_c+m_b}+\frac{m_c+m_b}{m_a+m_b}$$

$$\Rightarrow \boxed{\frac{1}{\sin\omega}\geq\frac{m_a+m_b}{m_c+m_b}+\frac{m_c+m_b}{m_a+m_b}}\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a+b}{b+c}+\frac{b+c}{a+b}\overset{HM-AM}{\leq}\frac{a+b}{4}\cdot\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{b+c}{4}\cdot\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{1}{4}\left(2+\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\right)$$

$$\overset{AM-GM}{\leq}\frac{1}{4}\left(\frac{1}{3}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)+\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)\right)$$

$$=\frac{1}{3abc}\sum_{cyc}bc(b+c)\overset{CBS}{\leq}\frac{1}{12RF}\sqrt{\sum_{cyc}b^2c^2\cdot\sum_{cyc}(b+c)^2}$$

$$=\frac{1}{6R\sin\omega}\cdot\sqrt{2\sum_{cyc}a^2+2\sum_{cyc}bc}\overset{Leibniz}{\leq}\frac{1}{6R\sin\omega}\cdot\sqrt{4.9R^2}=\frac{1}{\sin\omega}.$$

Now, since m_a,m_b,m_c can be the sides of triangle with, area $F_m=\frac{3F}{4}$, and,

$$m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2=\frac{9}{16}(a^2b^2+b^2c^2+c^2a^2),\ \text{then,}$$

$$\sin\omega_m=\frac{2F_m}{\sqrt{m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2}}=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}=\sin\omega.$$

Using the first result in $\Delta m_am_bm_c$, we have

$$\frac{1}{\sin\omega}=\frac{1}{\sin\omega_m}\geq\frac{m_a+m_b}{m_b+m_c}+\frac{m_b+m_c}{m_a+m_b}.$$

So the proof is complete.

1166. If in ΔABC, AM, BN, CP − medians, I − incenter, then:

$$\frac{a.IM}{\sin\frac{A}{2}}+\frac{b.IN}{\sin\frac{B}{2}}+\frac{c.IP}{\sin\frac{C}{2}}\geq\frac{abc}{2r}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we will prove that, for any arbitrary point Q in the plane of ΔABC, we have

$$QA.\cos\frac{A}{2}+QB.\cos\frac{B}{2}+QC.\cos\frac{C}{2}\geq s \qquad (1)$$

Lemma : (Bottema's inequality, see [1, pp. 118, Theorem 12.56])

Let a', b', c' denote the sides of $\Delta A'B'C'$ and F' denote its area. Then

$$(a'.QA+b'.QB+c'.QC)^2\geq\frac{1}{2}\sum_{cyc}a'^2(b^2+c^2-a^2)+8FF'.$$

Let $a'=\cos\frac{A}{2}, b'=\cos\frac{B}{2}, c'=\cos\frac{C}{2}$.

Since, $(a'+b')^2=\frac{1+\cos A}{2}+\frac{1+\cos B}{2}+2\cos\frac{A}{2}.\cos\frac{B}{2}\geq 1+\frac{\cos A+\cos B}{2}$

$$=1+\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)\geq 1>c'^2\Rightarrow a'+b'>c' \text{ (and analogs)}$$

then a', b', c' can be the sides of a triangle with area F' such that:

$$16F'^2=2\sum_{cyc}\cos^2\frac{A}{2}\cos^2\frac{B}{2}-\sum_{cyc}\cos^4\frac{A}{2}$$

$$=4\sum_{cyc}\cos^2\frac{A}{2}\cos^2\frac{B}{2}-\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)^2$$

$$=4.\frac{s^2+(4R+r)^2}{16R^2}-\left(\frac{4R+r}{2R}\right)^2=\frac{s^2}{4R^2}\Rightarrow F'=\frac{s}{8R}.$$

Also, we have

$$\sum_{cyc}a'^2(b^2+c^2-a^2)=\sum_{cyc}\cos^2\frac{A}{2}.2bc\cos A=2s\sum_{cyc}(s-a)\cos A$$

$$=2s^2\sum\cos A-2s\sum a\cos A=2s^2\left(1+\frac{r}{R}\right)-2s.\frac{2sr}{R}=2s^2\left(1-\frac{r}{R}\right).$$

Then, $$QA.\cos\frac{A}{2}+QB.\cos\frac{B}{2}+QC.\cos\frac{C}{2}\geq\sqrt{\frac{1}{2}.2s^2\left(1-\frac{r}{R}\right)+8sr.\frac{s}{8R}}=s.$$

Using now this inequality in ΔMNP with $Q\equiv I$, we have

$$IM.\cos\frac{\angle PMN}{2}+IN.\cos\frac{\angle MNP}{2}+IP.\cos\frac{\angle NPM}{2}\geq\frac{MN+NP+PM}{2},$$

with, $\angle PMN = A$ and $NP = \frac{a}{2}$ (and analogs), then

$$IM.\cos\frac{A}{2} + IN.\cos\frac{B}{2} + IP.\cos\frac{C}{2} \geq \frac{a+b+c}{4} = \frac{s}{2},$$

and since $\cos\frac{A}{2} = \frac{a}{4R\sin\frac{A}{2}}$ (and analogs), then

$$\frac{a.IM}{\sin\frac{A}{2}} + \frac{b.IN}{\sin\frac{B}{2}} + \frac{c.IP}{\sin\frac{C}{2}} \geq 2Rs = \frac{abc}{2r}.$$

Equality holds iff ΔABC is equilateral.

Reference :

[1] O. BOTTEMA, R. Ž. DJORDEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ AND P. M. VASIĆ , *Geometric Inequalities*, Wolters – Noordhoff Publishing, Groningen, The Netherlands, (1969).

1167.

In any Δ ABC with $\omega \rightarrow$ Brocard's angle, the following relationship holds :

$$\frac{1}{\sin\omega} \geq 2 + \frac{2(n_b + m_b - g_b - s_b)^2}{9R(h_a + h_b + h_c)}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc} a^2b^2}{4F^2} \overset{?}{\geq} \frac{(c^2+a^2)^2}{c^2a^2} \Leftrightarrow \frac{\sum_{cyc} a^4 + 16F^2}{8F^2} \overset{?}{\geq} \frac{c^4+a^4+2c^2a^2}{c^2a^2}$$

$$\Leftrightarrow \frac{\sum_{cyc} a^4}{8F^2} \overset{?}{\geq} \frac{c^4+a^4}{c^2a^2} \Leftrightarrow \frac{\sum_{cyc} a^4}{c^4+a^4} \overset{?}{\geq} \frac{8F^2}{c^2a^2}$$

$$\Leftrightarrow \frac{\sum_{cyc} a^4 - (c^4+a^4)}{c^4+a^4} \overset{?}{\geq} \frac{2a^2b^2 + 2b^2c^2 - (\sum_{cyc} a^4)}{2c^2a^2}$$

$$\Leftrightarrow \frac{b^4}{c^4+a^4} + \frac{b^4}{2c^2a^2} + \frac{c^4+a^4}{2c^2a^2} \overset{?}{\geq} \frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow \frac{b^4(c^2+a^2)^2}{2c^2a^2(c^4+a^4)} + \frac{c^4+a^4}{2c^2a^2} \overset{?}{\geq} \frac{b^2(c^2+a^2)}{c^2a^2}$$

$$\Leftrightarrow b^4(c^2+a^2)^2 + (c^4+a^4)^2 \overset{?}{\geq} 2b^2(c^2+a^2)(c^4+a^4) \rightarrow \text{true via AM} - \text{GM}$$

$$\therefore \frac{\sum_{cyc} a^2b^2}{4F^2} \geq \frac{(c^2+a^2)^2}{c^2a^2} \Rightarrow \frac{\sqrt{\sum_{cyc} a^2b^2}}{2F} \geq \frac{c}{a} + \frac{a}{c} \Rightarrow \frac{1}{\sin\omega} \geq \frac{c}{a} + \frac{a}{c}$$

$$= \frac{c^2 + a^2 - 2ca + 2ca}{ca} = 2 + \frac{(c-a)^2}{ca} \geq 2 + \frac{(c-a)^2}{\sum_{cyc} ab}$$

$$\Rightarrow \frac{1}{\sin \omega} \overset{(*)}{\geq} 2 + \frac{(c-a)^2}{2R(h_a + h_b + h_c)}$$

$$\text{Now, } 4m_a^2.(b-c)^2 \overset{?}{<} \left(b^2+c^2\right)^2$$

$$\Leftrightarrow 4\left((b-c)^2 + (b+c)^2 - a^2\right).(b-c)^2 \overset{?}{<} \left((b-c)^2 + (b+c)^2\right)^2$$

$$\Leftrightarrow 4(b-c)^4 + 4\left(b^2-c^2\right)^2 - 4a^2(b-c)^2 \overset{?}{<} (b-c)^4 + (b+c)^4 + 2\left(b^2-c^2\right)^2$$

$$\Leftrightarrow 3(b-c)^4 + 2\left(b^2-c^2\right)^2 \underbrace{\overset{?}{<}}_{(*)} (b+c)^4 + 4a^2(b-c)^2$$

$$\because a^2 > (b-c)^2 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$(b+c)^4 + 4(b-c)^4 > 3(b-c)^4 + 2\left(b^2-c^2\right)^2$$

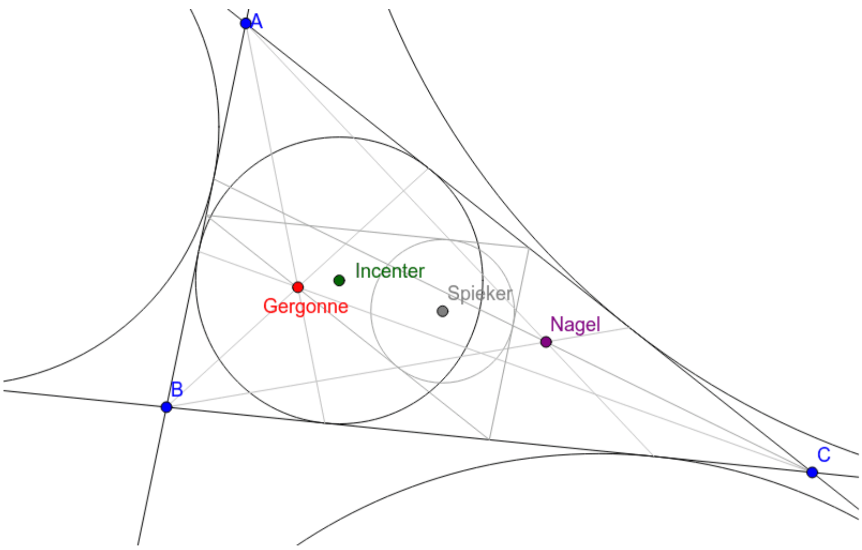
$$\Leftrightarrow (b+c)^4 + (b-c)^4 - 2(b+c)^2(b-c)^2 > 0$$

$$\Leftrightarrow \left((b+c)^2 - (b-c)^2\right)^2 > 0 \to \text{true} \Rightarrow (*) \text{ is true} \therefore 4m_a^2.(b-c)^2 \overset{(\bullet)}{<} \left(b^2+c^2\right)^2$$

$$\text{Now, } m_a - s_a = m_a - \frac{2bc}{b^2+c^2}.m_a = \frac{(b-c)^2}{b^2+c^2}.m_a \overset{?}{\leq} \frac{|b-c|}{2}$$

$$\Leftrightarrow \frac{(b-c)^4}{(b^2+c^2)^2}.4m_a^2 \overset{?}{\leq} (b-c)^2 \Leftrightarrow 4m_a^2.(b-c)^2 \overset{?}{<} \left(b^2+c^2\right)^2 \left(\because (b-c)^2 \geq 0\right)$$

$$\to \text{true via } (\bullet) \Rightarrow m_b - s_b \leq \frac{|c-a|}{2} \to (1)$$



Let Ge ≡ Gergonne point, Na ≡ N$a$gel point, $\overleftrightarrow{AGe} \cap \overleftrightarrow{BC} = \{D_1\}, \overleftrightarrow{ANa} \cap \overleftrightarrow{BC} = \{D_2\}$

$\textit{and assume } \mathrm{b} > c;\ Now, BD_1 = \mathrm{s-b}, \mathrm{BD_2 = s-c, CD_1 = s-c, CD_2 = s-b}$

$\therefore \mathrm{D_1D_2 = BD_2 - BD_1 = CD_1 - CD_2 = b - c}$ *and assuming* $\mathrm{c} > b$, *we get* :

$\mathrm{D_1D_2 = c - b} \therefore$ combining both cases, we obtain : $\mathrm{D_1D_2 = |b-c|}$

and via triangle inequality, $\mathrm{AD_1 + D_1D_2 \geq AD_2} \Rightarrow g_a + |\mathrm{b-c}| \geq \mathrm{n}_a$

$$\Rightarrow \mathrm{n_b} - g_\mathrm{b} \leq |\mathrm{c}-a| \to (2) \therefore (1),(2) \Rightarrow \mathrm{n_b + m_b} - g_\mathrm{b} - \mathrm{s_b} \leq \frac{3|\mathrm{c}-a|}{2}$$

$$\Rightarrow (\mathrm{c}-a)^2 \overset{(**)}{\geq} \frac{4(\mathrm{n_b+m_b}-g_\mathrm{b}-\mathrm{s_b})^2}{9} \therefore (*),(**) \Rightarrow \frac{1}{\sin\omega}$$

$$\geq 2 + \frac{\frac{4(\mathrm{n_b+m_b}-g_\mathrm{b}-\mathrm{s_b})^2}{9}}{2\mathrm{R}(\mathrm{h}_a+\mathrm{h_b}+\mathrm{h_c})} \Rightarrow \frac{1}{\sin\omega} \geq 2 + \frac{2(\mathrm{n_b+m_b}-g_\mathrm{b}-\mathrm{s_b})^2}{9\mathrm{R}(\mathrm{h}_a+\mathrm{h_b}+\mathrm{h_c})}\ (\mathrm{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we will prove that, $n_b + m_b - g_b - s_b \leq \frac{3}{2}|c-a|$ (1)

We have, $n_b{}^2 = s(s-b) + \frac{s(c-a)^2}{b}$ and $g_b{}^2 = s(s-b) - \frac{(s-b)(c-a)^2}{b}$, then

$$(n_bg_b)^2 = \big(s(s-b)\big)^2 + s(s-b)\left(1-\frac{(c-a)^2}{b^2}\right)(c-a)^2 \overset{b>|c-a|}{\geq} \big(s(s-b)\big)^2 \Rightarrow n_bg_b \geq s(s-b)$$

$$\Rightarrow (n_b-g_b)^2 = (n_b{}^2+g_b{}^2) - 2n_bg_b \leq \big(2s(s-b)+(c-a)^2\big) - 2s(s-b) = (c-a)^2,$$

then, $n_b - g_b \leq |c-a|$. Also, we have

$$m_b - s_b = \left(1-\frac{2ca}{c^2+a^2}\right)m_b = \frac{(c-a)^2}{c^2+a^2}\cdot\frac{\sqrt{(c+a)^2-(b^2-(c-a)^2)}}{2} \overset{b>|c-a|}{\leq} \frac{(c-a)^2(c+a)}{2(c^2+a^2)}$$

$$= \frac{|c^2-a^2||c-a|}{2(c^2+a^2)} \leq \frac{1}{2}|c-a| \Rightarrow m_b - s_b \leq \frac{1}{2}|c-a|.$$

Thus, $n_b + m_b - g_b - s_b = (n_b-g_b) + (m_b-s_b) \leq |c-a| + \frac{1}{2}|c-a| = \frac{3}{2}|c-a|.$

Now, we will prove that, $\frac{1}{\sin\omega} \geq \frac{c}{a} + \frac{a}{c}$ (2)

Since $\sin\omega = \frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ and $4F = \sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$, then,

$$(2) \Leftrightarrow 2ca\sqrt{a^2b^2+b^2c^2+c^2a^2} \geq (c^2+a^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow} 4c^2a^2(a^2b^2+b^2c^2+c^2a^2) \geq (2c^2a^2+c^4+a^4)\big[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)\big]$$

$$\Leftrightarrow 0 \geq -b^4(c^2+a^2)^2 + 2(c^4+a^4)(a^2b^2+b^2c^2) - (c^4+a^4)^2 = -\big[b^2(c^2+a^2)-(c^4+a^4)\big]^2,$$

which is true, and the proof of (2) is complete.
From (1) and (2), we get

$$\frac{1}{\sin\omega}\geq\frac{c}{a}+\frac{a}{c}=2+\frac{(c-a)^2}{ca}\geq 2+\frac{4(n_b+m_b-g_b-s_b)^2}{9(ab+bc+ca)}=2+\frac{2(n_b+m_b-g_b-s_b)^2}{9R(h_a+h_b+h_c)}.$$

1168. If ω – Brocard's angle in ΔABC then,

$$\frac{1}{r}\sum_{cyc}m_a\geq\frac{3}{\sin\omega}+\sum_{cyc}\frac{m_a}{h_a},\qquad 3\sum_{cyc}\frac{m_a}{r_a}\geq\frac{3}{\sin\omega}+\sum_{cyc}\frac{m_a}{h_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we will prove that, in ΔABC

$$bm_a+cm_b+am_c\geq\frac{3}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

We have,

$$4bm_a=\sqrt{4b^2(2b^2+2c^2-a^2)}$$
$$=\sqrt{(3b^2+c^2-a^2)^2+2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)},$$

then, $\quad bm_a=\dfrac{\sqrt{(3b^2+c^2-a^2)^2+16F^2}}{4}$ (and analogs).

By Minkowski's inequality, we have

$$\sum_{cyc}bm_a=\frac{1}{4}\sum_{cyc}\sqrt{(3b^2+c^2-a^2)^2+16F^2}\geq\frac{1}{4}\sqrt{\left[\sum_{cyc}(3b^2+c^2-a^2)\right]^2+9.16F^2}$$
$$=\frac{3}{4}\sqrt{(a^2+b^2+c^2)^2+16F^2}=\frac{3}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}.$$

Similarly, we have, $\quad\displaystyle\sum_{cyc}cm_a\geq\frac{3}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2},$

then, $\quad\displaystyle\sum_{cyc}(b+c)m_a\geq 3\sqrt{a^2b^2+b^2c^2+c^2a^2}=\frac{6F}{\sin\omega}.$

Now, if $a\leq b\leq c$ then we have, $m_a\geq m_b\geq m_c$ and

$$\frac{1}{r_a}\geq\frac{1}{r_b}\geq\frac{1}{r_c},\text{ so by Chebyshev's inequality,}$$

we have

$$3\sum_{cyc}\frac{m_a}{r_a}\geq\sum_{cyc}\frac{1}{r_a}.\sum_{cyc}m_a=\frac{1}{r}\sum_{cyc}m_a=\frac{a+b+c}{2F}\sum_{cyc}m_a=\frac{1}{2F}\sum_{cyc}(b+c)m_a+\sum_{cyc}\frac{m_a}{h_a}$$
$$\geq\frac{1}{2F}.\frac{6F}{\sin\omega}+\sum_{cyc}\frac{m_a}{h_a}=\frac{3}{\sin\omega}+\sum_{cyc}\frac{m_a}{h_a},$$

which completes the proof. Equality holds iff ΔABC is equilateral.

1169. If ω − Brocard's angle in ΔABC then,

$$1+\frac{1}{\sin\omega}\ge max\left\{\sum_{cyc}\frac{a}{c},\sum_{cyc}\frac{m_a}{m_c}\right\},\qquad 2+\frac{2}{\sin\omega}\ge max\left\{\sum_{cyc}\frac{b+c}{a},\sum_{cyc}\frac{m_b+m_c}{m_a}\right\}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we will prove that,

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\le 1+max\left\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{c}{a}+\frac{a}{c}\right\}\quad (1)$$

WLOG, we may assume that $c=min\{a,b,c\}$. If $a\ge b\ge c$. We have

$$1+\frac{c}{a}+\frac{a}{c}-\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)=\frac{(a-b)(b-c)}{ab}\ge 0,$$

then

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\le 1+\frac{c}{a}+\frac{a}{c}\le 1+max\left\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{c}{a}+\frac{a}{c}\right\}.$$

Similarly, if $b\ge a\ge c$ then we have,

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\le 1+\frac{b}{c}+\frac{c}{b}\le 1+max\left\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{c}{a}+\frac{a}{c}\right\}.$$

which completes the proof of (1).

Now, we will prove that, in any ΔABC, we have,

$$\frac{b}{c}+\frac{c}{b}\le\frac{1}{\sin\omega}\quad (2)$$

Since $\sin\omega=\dfrac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ and

$$4F=\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)},\ \text{then,}$$

$$(2)\Leftrightarrow 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$
$$\ge(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$
$$\overset{squaring}{\Leftrightarrow}\quad 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$$
$$\ge(2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$
$$\Leftrightarrow\ 0\ge -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2$$
$$=-[a^2(b^2+c^2)-(b^4+c^4)]^2,$$

which is true, and the proof of (2) is complete.

From (1) and (2), we get

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\le 1+max\left\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{c}{a}+\frac{a}{c}\right\}\le 1+\frac{1}{\sin\omega}\quad (3)$$

Now, since m_a,m_b,m_c can be the sides of triangle with, area $F_m=\dfrac{3F}{4}$, and,

$${m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2=\frac{9}{16}(a^2b^2+b^2c^2+c^2a^2),\ \text{then,}$$

$$\sin\omega_m=\frac{2\mathrm{F_m}}{\sqrt{{m_a}^2{m_b}^2+{m_b}^2{m_c}^2+{m_c}^2{m_a}^2}}=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}=\sin\omega.$$

Using (3) in $\Delta m_a m_b m_c$, we get,

$$\frac{m_a}{m_c}+\frac{m_a}{m_c}+\frac{m_a}{m_c}\le 1+\frac{1}{\sin\omega_m}=1+\frac{1}{\sin\omega}.$$

Therefore,

$$max\left\{\frac{a}{c}+\frac{b}{a}+\frac{c}{b},\frac{m_a}{m_c}+\frac{m_a}{m_c}+\frac{m_a}{m_c}\right\}\le 1+\frac{1}{\sin\omega}.$$

Similarly, we have,

$$\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\le 1+\frac{1}{\sin\omega}\quad\text{and}\quad\frac{m_c}{m_a}+\frac{m_a}{m_b}+\frac{m_b}{m_c}\le 1+\frac{1}{\sin\omega}.$$

Then,

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\le 2\left(1+\frac{1}{\sin\omega}\right)\quad\text{and}$$
$$\frac{m_b+m_c}{m_a}+\frac{m_c+m_a}{m_b}+\frac{m_a+m_b}{m_c}\le 2\left(1+\frac{1}{\sin\omega}\right).$$

Therefore,

$$max\left\{\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c},\frac{m_b+m_c}{m_a}+\frac{m_c+m_a}{m_b}+\frac{m_a+m_b}{m_c}\right\}\le 2+\frac{2}{\sin\omega}.$$

1170. **If ω − Brocard's angle in ΔABC then,**

$$2R+5r\ge 5r+4r.\,max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\ge 5r+\frac{2r}{\sin\omega}\ge h_a+h_b+h_c$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Panaitopol's inequality, we have, $\frac{m_a}{h_a}\le\frac{R}{2r}$ (and analogs) $\Rightarrow$

$$max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\le\frac{R}{2r}\quad(1)$$

Now, we will prove that, $\quad max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\ge\frac{1}{2\sin\omega}\quad(2)$

WLOG, we may assume that, $a\ge b\ge c$. We have

$$\left(\frac{m_a}{h_a}\right)^2-\left(\frac{m_b}{h_b}\right)^2=\frac{a^2(2b^2+2c^2-a^2)-b^2(2c^2+2a^2-b^2)}{16F^2}$$
$$=\frac{(a^2-b^2)(2c^2-a^2-b^2)}{16F^2}\le 0.$$

Similarly, we have, $\left(\frac{m_b}{h_b}\right)^2-\left(\frac{m_c}{h_c}\right)^2=\frac{(b^2-c^2)(2a^2-b^2-c^2)}{16F^2}\geq 0.$

Then, $max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}=\frac{m_b}{h_b}$**, and it suffices to prove that,** $\frac{m_b}{h_b}\geq\frac{1}{2\sin\omega}$.

$$\frac{m_b}{h_b}\overset{?}{\geq}\frac{1}{2\sin\omega}\Leftrightarrow\frac{b\sqrt{2c^2+2a^2-b^2}}{4F}\geq\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{4F}$$

$$\overset{squaring}{\Leftrightarrow}\quad a^2b^2+b^2c^2-c^2a^2-b^4\geq 0\Leftrightarrow (a^2-b^2)(b^2-c^2)\geq 0,$$

which is true and the proof of (2) is complete.

In this part, we will prove that, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq 1+\frac{1}{\sin\omega}\quad (3)$

WLOG, we may assume that $c=min\{a,b,c\}$. **If** $a\geq b\geq c$. **We have**

$$1+\frac{c}{a}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\frac{(a-b)(b-c)}{bc}\geq 0\ \Rightarrow\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq 1+\frac{c}{a}+\frac{a}{c}.$$

Similarly, if $b\geq a\geq c$ **then we have,** $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq 1+\frac{b}{c}+\frac{c}{b}$.

Then, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\leq 1+max\left\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{c}{a}+\frac{a}{c}\right\}$.

So it suffices to prove that, in any ΔABC**, we have,** $\frac{b}{c}+\frac{c}{b}\leq\frac{1}{\sin\omega}\quad (i)$

Since $\sin\omega=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ **and** $4F=\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$, **then,**

$$(i)\Leftrightarrow 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}\geq (b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

$$\overset{squaring}{\Leftrightarrow}\quad 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)\geq (2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]$$

$$\Leftrightarrow\ 0\geq -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2=-[a^2(b^2+c^2)-(b^4+c^4)]^2,$$

which is true, and the proof of (3) is complete.

From (1), (2) and (3), we have

$$2R+5r=5r+4r.\frac{R}{2r}\overset{(1)}{\gtrsim}5r+4r.max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}\overset{(2)}{\gtrsim}5r+\frac{2r}{\sin\omega}$$
$$=3r+r.2\left(1+\frac{1}{\sin\omega}\right)$$
$$\overset{(3)}{\gtrsim}3r+r.\left[\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\right]=r(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=h_a+h_b+h_c.$$

So the proof is done. Equality holds iff ΔABC is equilateral.

1171. If $\lambda>0$ then in ΔABC the following relationship holds:

$$\frac{a}{b+c\lambda}+\frac{b}{c+a\lambda}+\frac{c}{a+b\lambda}\geq\frac{12}{1+\lambda}\cdot\frac{r^2}{R^2}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{a}{b+c\lambda}+\frac{b}{c+a\lambda}+\frac{c}{a+b\lambda}=\frac{a^2}{ab+ac\lambda}+\frac{b^2}{bc+ba\lambda}+\frac{c^2}{ca+cb\lambda}\geq$$
$$\overset{BERGSTROM}{\geq}\frac{(a+b+c)^2}{(1+\lambda)(ab+bc+ca)}=\frac{4s^2}{(1+\lambda)(s^2+r^2+4Rr)}$$
$$\frac{4s^2}{(1+\lambda)(s^2+r^2+4Rr)}\geq\frac{12}{1+\lambda}\cdot\frac{r^2}{R^2}\Leftrightarrow$$
$$\Leftrightarrow\frac{s^2}{s^2+r^2+4Rr}\geq\frac{3r^2}{R^2}\Leftrightarrow s^2R^2\geq 3r^2(s^2+r^2+4Rr)\Leftrightarrow$$
$$\Leftrightarrow s^2(R^2-3r^2)\geq 3r^4+12Rr^3\ \text{–to prove}$$
$$s^2(R^2-3r^2)\overset{MITRINOVIC}{\gtrsim}27r^2(R^2-3r^2)\geq 3r^4+12Rr^3\Leftrightarrow$$
$$\Leftrightarrow 9r^2(R^2-3r^2)\geq r^4+4Rr^3\Leftrightarrow 9(R^2-3r^2)\geq r^2+4Rr\Leftrightarrow$$
$$\Leftrightarrow 9R^2-28r^2-4Rr\geq 0\Leftrightarrow(R-2r)(9R+14r)\geq 0$$

Equality holds for $a=b=c$.

1172. In any Δ ABC the following relationship holds :

$$\frac{b^2+c^2}{a^2}.\cos A+\frac{c^2+a^2}{b^2}.\cos B+\frac{a^2+b^2}{c^2}.\cos C\geq 24\left(\frac{r}{R}\right)^2-3$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{b^2+c^2}{a^2}.\cos A+\frac{c^2+a^2}{b^2}.\cos B+\frac{a^2+b^2}{c^2}.\cos C$$

$$=\sum_{cyc}\left(\frac{2bc\cos A+a^2}{a^2}.\cos A\right)=\sum_{cyc}\frac{2bc.\cos^2 A}{a^2}+\sum_{cyc}\cos A$$

$$=\sum_{cyc}\frac{2bc.(1-\sin^2 A)}{a^2}+\frac{R+r}{R}=\sum_{cyc}\frac{2bc}{a^2}-\sum_{cyc}\frac{2bc.a^2}{4R^2.a^2}+\frac{R+r}{R}$$

$$\overset{A-G}{\geq} 6\sqrt[3]{\prod_{cyc}\frac{bc}{a^2}}+\frac{R+r}{R}-\frac{s^2+4Rr+r^2}{2R^2}\overset{?}{\geq}24\left(\frac{r}{R}\right)^2-3$$

$$\Leftrightarrow\frac{10R+r}{R}\overset{?}{\geq}\frac{s^2+4Rr+49r^2}{2R^2}\Leftrightarrow s^2\underset{(*)}{\overset{?}{\leq}}20R^2-2Rr-49r^2$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\leq}4R^2+4Rr+3r^2\overset{?}{\leq}20R^2-2Rr-49r^2$$

$$\Leftrightarrow 8R^2-3Rr-26r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(8R+13r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\Rightarrow(*)\text{ is true}\Rightarrow\text{in any }\Delta\,ABC,\frac{b^2+c^2}{a^2}.\cos A+\frac{c^2+a^2}{b^2}.\cos B+\frac{a^2+b^2}{c^2}.\cos C$$

$$\geq 24\left(\frac{r}{R}\right)^2-3,''='' \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1173. In ΔABC the following relationship holds:

$$\frac{a^3}{\lambda b+\mu c}+\frac{b^3}{\lambda c+\mu a}+\frac{c^3}{\lambda a+\mu b}\geq\frac{36r^2}{\lambda+\mu}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{a^3}{\lambda b+\mu c}+\frac{b^3}{\lambda c+\mu a}+\frac{c^3}{\lambda a+\mu b}=$$

$$=\frac{a^4}{\lambda ab+\mu ac}+\frac{b^4}{\lambda bc+\mu ba}+\frac{c^4}{\lambda ca+\mu cb}\overset{HOLDER}{\geq}$$

$$\geq\frac{(a+b+c)^4}{9(\lambda+\mu)(ab+bc+ca)}=\frac{16s^4}{9(\lambda+\mu)(s^2+r^2+4Rr)}\geq\frac{36r^2}{\lambda+\mu}\Leftrightarrow$$

$$\Leftrightarrow 16s^4\geq 36r^2\cdot 9(s^2+r^2+4Rr)\Leftrightarrow$$

$$\Leftrightarrow 4s^4-81s^2r^2\geq 81r^2(r^2+4Rr)$$

$$4s^4-81s^2r^2=s^2(4s^2-81r^2)\overset{MITRINOVIC}{\geq}$$

$$\geq 27r^2(4s^2-81r^2)\geq 81r^2(r^2+4Rr)\Leftrightarrow$$

$$4s^2-81r^2\geq 3(r^2+4Rr)\Leftrightarrow 4s^2\geq 84r^2+12Rr$$

$$4s^2 \overset{GERRETSEN}{\geq} 4(16Rr - 5r^2) \geq 84r^2 + 12Rr \Leftrightarrow$$

$$64Rr - 12Rr \geq 84r^2 + 20r^2 \Leftrightarrow 52Rr \geq 104r^2 \Leftrightarrow R \overset{EULER}{\geq} 2r$$

1174. In ΔABC the following relationship holds:

$$\left(\frac{m_a^5}{m_b + m_c}\right)^2 + \left(\frac{m_b^5}{m_c + m_a}\right)^2 + \left(\frac{m_c^5}{m_a + m_b}\right)^2 \geq \frac{3^9 \cdot r^8}{4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\left(\frac{m_a^5}{m_b + m_c}\right)^2 = \sum_{cyc}\frac{m_a^{10}}{(m_b + m_c)^2} \geq \sum_{cyc}\frac{m_a^{10}}{2(m_b^2 + m_c^2)} =$$

$$= \frac{1}{2}\sum_{cyc}\frac{(m_a^2)^5}{m_b^2 + m_c^2} \overset{HOLDER}{\geq} \frac{1}{2}\cdot\frac{(m_a^2 + m_b^2 + m_c^2)^5}{27 \cdot 2(m_a^2 + m_b^2 + m_c^2)} =$$

$$= \frac{1}{4 \cdot 27}\cdot(m_a^2 + m_b^2 + m_c^2)^4 = \frac{1}{4 \cdot 27}\cdot\frac{3^4}{4^4}\cdot(a^2 + b^2 + c^2)^4 \geq$$

$$\overset{IONESCU-WEITZENBOCK}{\geq} \frac{3}{4^5}\cdot(4\sqrt{3}F)^4 = \frac{3}{4}\cdot(\sqrt{3}rs)^4 \geq \frac{3}{4}\cdot(\sqrt{3}r \cdot 3\sqrt{3}r)^4 = \frac{3^9 \cdot r^8}{4}$$

Equality holds for: $a = b = c$.

1175. If $\lambda, \mu > 0$ then in ΔABC the following relationship holds:

$$\frac{\lambda a^3 + \mu b^3}{\lambda a + \mu b} + \frac{\lambda b^3 + \mu c^3}{\lambda b + \mu c} + \frac{\lambda c^3 + \mu a^3}{\lambda c + \mu a} \geq \frac{18(\sqrt{\lambda} + \sqrt{\mu})^2}{\lambda + \mu}\cdot r^2$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{\lambda a^3 + \mu b^3}{\lambda a + \mu b} = \lambda\sum_{cyc}\frac{a^3}{\lambda a + \mu b} + \mu\sum_{cyc}\frac{b^3}{\lambda a + \mu b} \overset{HOLDER}{\geq}$$

$$\geq \lambda\cdot\frac{(a+b+c)^3}{3(\lambda+\mu)(a+b+c)}+\mu\cdot\frac{(a+b+c)^3}{3(\lambda+\mu)(a+b+c)}=$$

$$=\left(\frac{\lambda}{3(\lambda+\mu)}+\frac{\mu}{3(\lambda+\mu)}\right)\cdot(a+b+c)^2=\frac{1}{3}\cdot 4s^2\overset{MITRINOVIC}{\geq}$$

$$\geq\frac{4}{3}\cdot 27r^2$$

Remains to prove:

$$\frac{4}{3}\cdot 27r^2\geq\frac{18\left(\sqrt{\lambda}+\sqrt{\mu}\right)^2}{\lambda+\mu}\cdot r^2\Leftrightarrow 2\geq\frac{\left(\sqrt{\lambda}+\sqrt{\mu}\right)^2}{\lambda+\mu}\Leftrightarrow\left(\sqrt{\lambda}-\sqrt{\mu}\right)^2\geq 0$$

Equality holds for $\lambda=\mu$ and $a=b=c$.

1176. In ΔABC the following relationship holds :

$$\frac{r_a^{10}}{r_a^3+r_b^3}+\frac{r_b^{10}}{r_b^3+r_a^3}+\frac{r_c^{10}}{r_c^3+r_a^3}\geq\frac{3(3r)^7}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{r_a^{10}}{r_a^3+r_b^3}=\sum_{cyc}\frac{r_a^{14}}{r_a^7+r_a^4r_b^3}\overset{AM-GM}{\geq}\sum_{cyc}\frac{r_a^{14}}{r_a^7+\frac{4r_a^7+3r_b^7}{7}}\overset{CBS}{\geq}\frac{(r_a^7+r_b^7+r_c^7)^2}{2(r_a^7+r_b^7+r_c^7)}$$

$$=\frac{r_a^7+r_b^7+r_c^7}{2}\overset{AM-GM}{\geq}\frac{3\sqrt[3]{(r_ar_br_c)^7}}{2}=\frac{3\sqrt[3]{(s^2r)^7}}{2}\overset{Mitrinovic}{\geq}\frac{3\sqrt[3]{(27r^2.r)^7}}{2}=\frac{3(3r)^7}{2}.$$

Equality holds iff ΔABC is equilateral.

1177. In any Δ ABC, the following relationship holds :

$$3\leq\sum_{cyc}\frac{b^2+c^2}{a^2}.\cos A\leq 3\left(\frac{R}{2r}\right)^5$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b^2+c^2}{a^2}.\cos A=\sum_{cyc}\left(\frac{2bc\cos A+a^2}{a^2}.\cos A\right)$$

$$=\sum_{cyc}\frac{2bc.\cos^2 A}{a^2}+\sum_{cyc}\cos A=\sum_{cyc}\frac{2bc.\left(1-\sin^2 A\right)}{a^2}+\frac{R+r}{R}$$

$$=\sum_{cyc}\frac{2bc}{a^2}-\sum_{cyc}\frac{2bc.a^2}{4R^2.a^2}+\frac{R+r}{R}\overset{A-G}{\leq}\sum_{cyc}\frac{b^2+c^2+a^2-a^2}{a^2}-\frac{s^2+4Rr+r^2}{2R^2}+\frac{R+r}{R}$$

$$=\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{a^2}\right)-3-\frac{s^2+4Rr+r^2}{2R^2}+\frac{R+r}{R}$$

$$=\frac{(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)}{8R^2r^2s^2}-3+\frac{2R^2-2Rr-r^2-s^2}{2R^2}$$

$$=\frac{(s^2-4Rr-r^2)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-24R^2r^2s^2+4r^2s^2(2R^2-2Rr-r^2-s^2)}{8R^2r^2s^2}$$

$$\overset{?}{\leq}3\left(\frac{R}{2r}\right)^5\Leftrightarrow 4r^3s^6-r^4s^4(48R+12r)-s^2(3R^7-128R^2r^5+20r^7)$$

$$-4r^6(4R+r)^3\underset{(*)}{\overset{?}{\leq}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\leq}4r^3s^4\left(4R^2+4Rr+3r^2-r(12R+3r)\right)$$

$$-s^2(3R^7-128R^2r^5+20r^7)-4r^6(4R+r)^3\overset{?}{\leq}0$$

$$\Leftrightarrow 4r^3s^4(4R^2-8Rr)-s^2(3R^7-128R^2r^5+20r^7)-4r^6(4R+r)^3\underset{(**)}{\overset{?}{\leq}}0$$

$$\text{Again, LHS of }(**)\overset{\text{Gerretsen}}{\leq}$$

$$\left(4r^3(4R^2-8Rr)(4R^2+4Rr+3r^2)-(3R^7-128R^2r^5+20r^7)\right)s^2$$

$$-4r^6(4R+r)^3\overset{?}{\leq}0$$

$$\Leftrightarrow(3R^7-64R^4r^3+64R^3r^4-48R^2r^5+96Rr^5+20r^7)s^2+4r^6(4R+r)^3\underset{(***)}{\overset{?}{\geq}}0$$

Case 1 $3R^7-64R^4r^3+64R^3r^4-48R^2r^5+96Rr^5+20r^7\geq 0$ *and* then :
LHS of $(***)\geq 4r^6(4R+r)^3>0\Rightarrow(***)$ is true (strict inequality)

Case 2 $3R^7-64R^4r^3+64R^3r^4-48R^2r^5+96Rr^5+20r^7<0$ *and* then :

$$\text{LHS of }(***)=-\left(-(3R^7-64R^4r^3+64R^3r^4-48R^2r^5+96Rr^5+20r^7)\right)s^2$$

$$+4r^6(4R+r)^3\overset{\text{Gerretsen}}{\geq}$$

$$-\left(-(3R^7-64R^4r^3+64R^3r^4-48R^2r^5+96Rr^5+20r^7)\right)(4R^2+4Rr+3r^2)$$

$$+4r^6(4R+r)^3\overset{?}{\geq}0$$

$$\Leftrightarrow 12t^9+12t^8+9t^7-256t^6-128t^4+640t^3+512t^2+416t+64\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)(12t^7+60t^6+201t^5+308t^4+428t^3+352t^2+336t+448)\right.$$

$$\left.+864\right)\overset{?}{\geq}0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(***)$ is true *and* combining *cases* 1 *and* 2, $(***)\Rightarrow(**)$

$$\Rightarrow(*)\Rightarrow\sum_{cyc}\frac{b^2+c^2}{a^2}.\cos A\leq 3\left(\frac{R}{2r}\right)^5\text{ is true }\forall\,\Delta\,ABC$$

$$\text{Also}, \sum_{cyc}\frac{b^2+c^2}{a^2}.\cos A = \sum_{cyc}\frac{2bc}{a^2} - \sum_{cyc}\frac{2bc.a^2}{4R^2.a^2} + \frac{R+r}{R}$$

$$\overset{A-G}{\geq} 6\sqrt[3]{\prod_{cyc}\frac{bc}{a^2}} + \frac{R+r}{R} - \frac{s^2+4Rr+r^2}{2R^2} \overset{?}{\geq} 3$$

$$\Leftrightarrow \frac{6R^2+2R(R+r)-s^2-4Rr-r^2}{2R^2} \overset{?}{\geq} 0 \Leftrightarrow s^2 \overset{?}{\leq} 8R^2-2Rr-r^2$$

$$\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-2(2R+r)(R-2r) \overset{?}{\leq} 0 \to \text{true}$$

$$\because s^2-(4R^2+4Rr+3r^2) \overset{\text{Gerretsen}}{\leq} 0 \text{ and } -2(2R+r)(R-2r) \overset{\text{Euler}}{\leq} 0$$

$$\therefore \sum_{cyc}\frac{b^2+c^2}{a^2}.\cos A \geq 3\ \forall\ \Delta\ ABC \therefore \text{in any } \Delta\ ABC,$$

$$3 \leq \sum_{cyc}\frac{b^2+c^2}{a^2}.\cos A \leq 3\left(\frac{R}{2r}\right)^5, \text{" = " iff } \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{b^2+c^2}{a^2}\cos A = \sum_{cyc}\frac{b^2+c^2}{a^2}\left(\frac{b}{2c}+\frac{c}{2b}-\frac{a^2}{2bc}\right) \overset{AM-GM}{\geq} \sum_{cyc}\left(\frac{b^2}{a^2}+\frac{c^2}{a^2}\right)\left(1-\frac{a^2}{2bc}\right)$$

$$= \sum_{cyc}\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}-\frac{b}{2c}-\frac{c}{2b}\right) = \sum_{cyc}\left[\left(\frac{b}{c}+\frac{c}{b}-2\right)\left(\frac{b}{c}+\frac{c}{b}+\frac{3}{2}\right)+1\right] \overset{AM-GM}{\geq} \sum_{cyc} 1 = 3.$$

Now, we have

$$\sum_{cyc}\frac{\cos A}{a^2} = \frac{\sum_{cyc} bc(b^2+c^2-a^2)}{2a^2b^2c^2} = \frac{(\sum_{cyc} bc)(\sum_{cyc} a^2)-2abc\sum_{cyc} a}{2(abc)^2}$$

$$= \frac{(s^2+r^2+4Rr).2(s^2-r^2-4Rr)-2.4Rsr.2s}{2(4Rsr)^2} = \frac{s^4-r^2(4R+r)^2-8s^2Rr}{16s^2R^2r^2}$$

$$= \frac{1}{16R^2r^2}\left(s^2-8Rr-\frac{r^2(4R+r)^2}{s^2}\right) \overset{Gerretsen\ \&\ Doucet}{\leq} \frac{1}{16R^2r^2}(4R^2-4Rr+3r^2-3r^2) = \frac{R-r}{4Rr^2}$$

$$\Rightarrow \sum_{cyc}\frac{b^2+c^2}{a^2}\cos A = \sum_{cyc} a^2.\sum_{cyc}\frac{\cos A}{a^2} - \sum_{cyc}\cos A \overset{Leibniz}{\leq} 9R^2.\frac{R-r}{4Rr^2} - \left(1+\frac{r}{R}\right)$$

$$\overset{x:=\frac{R}{2r}}{=} 9x^2-\frac{9}{2}x-1-\frac{1}{2x}$$

$$= 3x^5 - \frac{(x-1)}{2x}[(x-1)(6x^4+12x^3+18x^2+6x+3)+2] \overset{Euler}{\leq} 3x^5.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.

1178. In any Δ ABC, the following relationships holds :

$$11.\min\left\{\sum_{cyc}\frac{a}{b},\sum_{cyc}\frac{b}{a}\right\}+\frac{R^{20}}{r^{20}}\ge 2^{20}+11.max\left\{\sum_{cyc}\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}},\sum_{cyc}\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}\right\}$$

$$\text{and } \sqrt[3]{(a+b)(b+c)(c+a)}+\frac{R^4}{r^3}\ge 16r+\frac{2(a+b+c)}{3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{R^{20}}{r^{20}}-2^{20}=\left(\frac{R^4}{r^4}-2^4\right)\left(\frac{R^{16}}{r^{16}}+2^4.\frac{R^{12}}{r^{12}}+2^8.\frac{R^8}{r^8}+2^{12}.\frac{R^4}{r^4}+2^{16}\right)$$

$$\Rightarrow\frac{R^{20}}{r^{20}}-2^{20}-\left(\frac{R^4}{r^4}-2^4\right)$$

$$=\left(\frac{R^4}{r^4}-2^4\right)\left(\frac{R^{16}}{r^{16}}+2^4.\frac{R^{12}}{r^{12}}+2^8.\frac{R^8}{r^8}+2^{12}.\frac{R^4}{r^4}+2^{16}-1\right)\overset{\text{Euler}}{\ge}0$$

$$\Rightarrow\frac{R^4}{r^4}-2^4\overset{(\bullet)}{\le}\frac{R^{20}}{r^{20}}-2^{20}$$

$$\text{Now, }2\left(a^2-ab+b^2\right)^2-\left(a^4+b^4\right)=(a-b)^4\ge 0$$

$$\Rightarrow a^4+b^4\le 2\left(a^2-ab+b^2\right)^2\Rightarrow\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}}\le\sqrt[4]{\frac{4(a^2-ab+b^2)^2}{(b^2+c^2)^2}}$$

$$=\sqrt{\frac{2(a^2-ab+b^2)}{b^2+c^2}}\overset{A-G}{\le}\sqrt{\frac{a^2-ab+b^2}{bc}}\Rightarrow\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}}\le\sqrt{\frac{a^2-ab+b^2}{bc}}$$

$$\text{and analogs}\Rightarrow\sum_{cyc}\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}}\le\sum_{cyc}\sqrt{\frac{a^2-ab+b^2}{bc}}$$

$$\overset{CBS}{\le}\sqrt{\sum_{cyc}(a^2-ab+b^2)}.\sqrt{\sum_{cyc}\frac{1}{bc}}=\sqrt{\frac{3s^2-20Rr-5r^2}{2Rr}}$$

$$\overset{\text{Gerretsen}}{\le}\sqrt{\frac{6R^2-4Rr+2r^2}{Rr}}$$

$$\Rightarrow 11.\sum_{cyc}\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}}\le 11.\sqrt{\frac{6R^2-4Rr+2r^2}{Rr}}\overset{?}{\le}\frac{R^4}{r^4}-16+33$$

$$\Leftrightarrow\frac{\left(R^4+17r^4\right)^2}{r^8}\overset{?}{\ge}\frac{121\left(6R^2-4Rr+2r^2\right)}{Rr}$$

$$\Leftrightarrow t^9+34t^5-726t^2+773t-242\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(t^8+2t^7+4t^6+8t^5+50t^4+100t^3+37t^2+163t(t-2)+121\right) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow 11.\sum_{\text{cyc}} \sqrt[4]{\frac{a^4+b^4}{b^4+c^4}} \leq 11.3 + \frac{R^4}{r^4} - 16 \overset{\text{via A-G and }(\bullet)}{\leq}$$

$$11.\sum_{\text{cyc}} \frac{a}{b}, 11.\sum_{\text{cyc}} \frac{b}{a} + \frac{R^{20}}{r^{20}} - 2^{20}$$

$$\Rightarrow 11.\sum_{\text{cyc}} \sqrt[4]{\frac{a^4+b^4}{b^4+c^4}} \leq 11.\min\left\{\sum_{\text{cyc}} \frac{a}{b}, \sum_{\text{cyc}} \frac{b}{a}\right\} + \frac{R^{20}}{r^{20}} - 2^{20}$$

$$\Rightarrow \boxed{11.\min\left\{\sum_{\text{cyc}} \frac{a}{b}, \sum_{\text{cyc}} \frac{b}{a}\right\} + \frac{R^{20}}{r^{20}} \geq 2^{20} + 11.\sum_{\text{cyc}} \sqrt[4]{\frac{a^4+b^4}{b^4+c^4}} \to (1)}$$

$$\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} \overset{?}{\leq} \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$$

$$\Leftrightarrow \frac{1}{abc}.\sum_{\text{cyc}} a^2b \overset{?}{\geq} \frac{1}{(a+b)(b+c)(c+a)}.\sum_{\text{cyc}} (a+b)(b+c)^2$$

$$\Leftrightarrow \sum_{\text{cyc}} a^4b^2 + \sum_{\text{cyc}} a^3b^3 \underset{(\text{i})}{\overset{?}{\geq}} 3a^2b^2c^2 + abc\left(\sum_{\text{cyc}} a^2b\right)$$

Indeed, $\sum_{\text{cyc}} x^3 \overset{\text{A-G}}{\geq} \sum_{\text{cyc}} xy^2$ and choosing $x = ab, y = bc, z = ca$, we arrive at :

$\sum_{\text{cyc}} a^3b^3 \geq abc\left(\sum_{\text{cyc}} a^2b\right)$ and also, $\sum_{\text{cyc}} a^4b^2 \overset{\text{A-G}}{\geq} 3a^2b^2c^2 \Rightarrow (\text{i})$ is true

$$\therefore \frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b} \overset{(\bullet\bullet)}{\leq} \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$$

Again, $\sum_{\text{cyc}} \sqrt[3]{\frac{a^3+b^3}{b^3+c^3}} = \sum_{\text{cyc}} \sqrt[3]{\frac{a+b}{b+c}.\frac{a^2-ab+b^2}{b^2-bc+c^2}.1}$

$$\overset{\text{A-G}}{\leq} \frac{1}{3}\left(\sum_{\text{cyc}} \frac{a+b}{b+c} + \sum_{\text{cyc}} \frac{a^2-ab+b^2}{b^2-bc+c^2} + 3\right) \overset{\text{via A-G and }(\bullet\bullet)}{\leq}$$

$$\frac{1}{3}\left(\sum_{\text{cyc}} \frac{a}{c} + \sum_{\text{cyc}} \frac{a^2-ab+b^2}{bc} + 3\right) = \frac{1}{3}\left(\frac{\sum_{\text{cyc}} a^3}{4Rrs} + \sum_{\text{cyc}} \frac{b}{c} + 3\right)$$

$$\overset{\text{CBS}}{\leq} \frac{1}{3}\left(\frac{s^2-6Rr-3r^2}{2Rr} + \sqrt{\sum_{\text{cyc}} a^2}.\sqrt{\sum_{\text{cyc}} \frac{1}{a^2}} + 3\right)$$

$$\overset{\text{Leibnitz and Goldstone}}{\leq} \frac{1}{3}\left(\frac{s^2-6Rr-3r^2}{2Rr} + \sqrt{9R^2}.\sqrt{\frac{1}{4r^2}} + 3\right)$$

$$\Rightarrow 11.\sum_{cyc}\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}} \le \frac{11(s^2+3R^2-3r^2)}{6Rr} \overset{\text{Gerretsen}}{\le} \frac{11(7R+4r)}{6r}$$

$$\overset{?}{\le} \frac{R^4}{r^4} - 16 + 33 \Leftrightarrow 6t^4 - 77t + 58 \overset{?}{\ge} 0$$

$$\Leftrightarrow (t-2)\big(6t^3+12t^2+15(t-2)+9t+1\big) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow 11.\sum_{cyc}\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}} \le 11.3 + \frac{R^4}{r^4} - 16 \overset{\text{via A-G and }(\bullet)}{\le}$$

$$11.\sum_{cyc}\frac{a}{b}, 11.\sum_{cyc}\frac{b}{a} + \frac{R^{20}}{r^{20}} - 2^{20}$$

$$\boxed{\Rightarrow 11.\min\left\{\sum_{cyc}\frac{a}{b},\sum_{cyc}\frac{b}{a}\right\} + \frac{R^{20}}{r^{20}} \ge 2^{20} + 11.\sum_{cyc}\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}} \to (2) \therefore (1),(2) \Rightarrow$$

$$11.\min\left\{\sum_{cyc}\frac{a}{b},\sum_{cyc}\frac{b}{a}\right\} + \frac{R^{20}}{r^{20}} \ge 2^{20} + 11.max\left\{\sum_{cyc}\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}},\sum_{cyc}\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}\right\}$$

$$\text{Also, } \sqrt[3]{(a+b)(b+c)(c+a)} \overset{\text{G-H}}{\ge} \frac{3(a+b)(b+c)(c+a)}{\sum_{cyc}(a+b)(b+c)}$$

$$= \frac{6s(s^2+2Rr+r^2)}{5s^2+4Rr+r^2} \overset{?}{\ge} 16r - \frac{R^4}{r^3} + \frac{2(a+b+c)}{3}$$

$$\Leftrightarrow \frac{R^4-16r^4}{r^3} \underset{(*)}{\overset{?}{\ge}} \frac{2s(s^2-10Rr-7r^2)}{3(5s^2+4Rr+r^2)}$$

$$\because \frac{2s(s^2-10Rr-7r^2)}{3(5s^2+4Rr+r^2)} \overset{\text{Mitrinovic}}{\le} \frac{3\sqrt{3}R(s^2-10Rr-7r^2)}{3(5s^2+4Rr+r^2)} \overset{3\sqrt{3}<\frac{26}{5}}{\le} \frac{26R(s^2-10Rr-7r^2)}{15(5s^2+4Rr+r^2)}$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove}: \frac{R^4-16r^4}{r^3} \ge \frac{26R(s^2-10Rr-7r^2)}{15(5s^2+4Rr+r^2)}$$

$$\Leftrightarrow 315t^5 - 90t^4 - 26t^3 + 39t^2 - 5014t + 1440 \ge 0$$

$$\Leftrightarrow (t-2)\left(315t^4+540t^3+1054t^2+1787t+360(t-2)\right) \ge 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (*) \text{ is true} \therefore \sqrt[3]{(a+b)(b+c)(c+a)} + \frac{R^4}{r^3} \ge 16r + \frac{2(a+b+c)}{3},$$

equalities iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\bullet \sum_{cyc}\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}} \overset{AM-GM}{\le} \sum_{cyc}\frac{1}{4}\left(\frac{a^4+b^4}{2a^2b^2} + 2\times\frac{2ab}{\sqrt{2(b^4+c^4)}} + 1\right) \overset{AM-GM}{\le} \frac{1}{8}\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)$$

$$+\sum_{cyc}\frac{ab}{2bc} + \frac{3}{4}$$

$$=\frac{1}{8}\sum_{cyc}a^2\cdot\sum_{cyc}\frac{1}{a^2}+\frac{1}{2}\sum_{cyc}\frac{a}{c}+\frac{3}{8}\overset{Leibniz,Steinig\ \&\ Bandila}{\leq}\frac{1}{8}\cdot 9R^2\cdot\frac{1}{4r^2}+\frac{1}{2}\cdot\frac{3R}{2r}+\frac{3}{8}$$

$$\overset{Euler}{\leq}\frac{9R^2}{32r^2}\cdot\left(\frac{R}{2r}\right)^{18}+\frac{3R}{4r}\cdot\left(\frac{R}{2r}\right)^{19}+\frac{3}{8}\cdot\left(\frac{R}{2r}\right)^{20}=3\left(\frac{R}{2r}\right)^{20}.$$

- $$\sum_{cyc}\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}\overset{AM-GM}{\leq}\sum_{cyc}\frac{1}{3}\left(\frac{a+b}{2c}+\frac{a^2-ab+b^2}{ab}+\frac{2abc}{b^3+c^3}\right)\overset{AM-GM}{\leq}\frac{1}{2}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)+\sum_{cyc}\frac{2abc}{3bc(b+c)}$$

$$-1\overset{HM-AM}{\leq}\frac{1}{2}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)+\sum_{cyc}\frac{a}{6}\left(\frac{1}{b}+\frac{1}{c}\right)-1$$

$$=\frac{2}{3}\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right)-1\overset{Bandila}{\leq}\frac{2}{3}\sum_{cyc}\frac{R}{r}-1$$

$$=\frac{2R}{r}-1\overset{AM-GM}{\leq}\frac{R}{r}+\frac{R^2}{4r^2}\overset{Euler}{\leq}\frac{R}{r}\cdot\left(\frac{R}{2r}\right)^{19}+\frac{R^2}{4r^2}\cdot\left(\frac{R}{2r}\right)^{18}=3\left(\frac{R}{2r}\right)^{20}.$$

$$\Rightarrow 11.max\left\{\sum_{cyc}\frac{a}{b},\sum_{cyc}\frac{b}{a}\right\}+\frac{R^{20}}{r^{20}}\overset{AM-GM}{\geq}33+\frac{R^{20}}{r^{20}}=2^{20}-(2^{20}-33)+\frac{R^{20}}{r^{20}}$$

$$\overset{Euler}{\geq}2^{20}-(2^{20}-33)\left(\frac{R}{2r}\right)^{20}+\frac{R^{20}}{r^{20}}=2^{20}+11.3\left(\frac{R}{2r}\right)^{20}$$

$$\geq 2^{20}+11.max\left\{\sum_{cyc}\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}},\sum_{cyc}\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}\right\}.$$

- $$\sqrt[3]{(a+b)(b+c)(c+a)}+\frac{R^4}{r^3}\overset{Cesaro}{\geq}\sqrt[3]{8abc}+\frac{R^4}{r^3}=2\sqrt[3]{4Rsr}+\frac{R^4}{r^3}$$

$$\overset{Euler\ \&\ Mitrinovic}{\geq}2\sqrt[3]{4.2r.3\sqrt{3}r.r}+\frac{(2r)^3.2s}{3\sqrt{3}r^3}=4\sqrt{3}r+\frac{4(4\sqrt{3}-3)s}{9}+\frac{4s}{3}$$

$$\overset{Euler}{\geq}4\sqrt{3}r+\frac{4(4\sqrt{3}-3).3\sqrt{3}r}{9}+\frac{4s}{3}=16r+\frac{2(a+b+c)}{3}.$$

1179. In $\Delta ABC, R\in(AB), P\in(BC), Q\in(CA), AR=3, RB=1, BP=6,$ $PC=2, CQ=5, QA=4$. Prove that:

$$PQ+QR+RP>\frac{21}{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa-Romania

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos B$$

$$81 = 16 + 64 - 2 \cdot 4 \cdot 8 \cdot \cos B \Rightarrow \cos B = -\frac{1}{64}$$

$$\Delta RBP: RP^2 = RB^2 + BP^2 - 2RB \cdot BP \cdot \cos B =$$

$$= 1 + 36 - 2 \cdot 1 \cdot 6 \cdot \left(-\frac{1}{64}\right) = 37 + \frac{3}{16} \Rightarrow RP > 6$$

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos A$$

$$64 = 16 + 81 - 2 \cdot 4 \cdot 9 \cdot \cos A \Rightarrow \cos A = \frac{33}{72} = \frac{11}{24}$$

$$\Delta ARQ: RQ^2 = AR^2 + AQ^2 - 2AB \cdot AQ \cdot \cos A =$$

$$= 9 + 16 - 2 \cdot 3 \cdot 4 \cdot \frac{11}{24} = 25 - 11 = 14 \Rightarrow RQ > 3$$

$$AB^2 = AC^2 + CB^2 - 2AC \cdot BC \cdot \cos C$$

$$16 = 81 + 64 - 2 \cdot 9 \cdot 8 \cdot \cos C \Rightarrow \cos C = \frac{129}{144} = \frac{43}{46}$$

$$\Delta PQC: PQ^2 = QC^2 + PC^2 - 2QB \cdot PC \cdot \cos C =$$

$$= 25 + 4 - 2 \cdot 5 \cdot 2 \cdot \frac{43}{46} = 10,3 \ldots \Rightarrow PQ > 3$$

$$\text{So, } RP + RQ + PQ > 6 + 3 + 3 = 12 > \frac{21}{2}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\cos A = \frac{16 + 81 - 64}{2 \cdot 4 \cdot 9} = \frac{33}{8 \cdot 9} = \frac{11}{24}$$

$$(RQ)^2 = 9 + 16 - 2 \cdot 3 \cdot 4 \cdot \frac{11}{24} = 9 + 16 - 11 = 14, \qquad RQ = \sqrt{14}$$

$$\cos B = \frac{16 + 64 - 81}{2 \cdot 4 \cdot 8} = -\frac{1}{64}$$

$$(RP)^2 = 1 + 36 + 2(1)(6) \cdot \frac{1}{64} = 37 + \frac{3}{16} = \frac{595}{16}, \qquad RP = \frac{\sqrt{595}}{4}$$

$$\cos C = \frac{8 + 84 - 16}{2 \cdot 9 \cdot 8} = \frac{129}{2 \cdot 9 \cdot 8} = \frac{43}{48}$$

$$(PQ)^2 = 29 + 4 - 2 \cdot 10 \cdot \frac{43}{48}$$

$$(PQ)^2 = 29 - \frac{215}{12} = \frac{133}{12}, \qquad PQ = \frac{\sqrt{133}}{\sqrt{12}}$$

$$\sqrt{14} + \frac{\sqrt{595}}{4} + \sqrt{\frac{133}{12}} > \frac{21}{2} \text{ is true.}$$

1180. In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{2(a+c)-b}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(a+c)-b}}\right)\le\sqrt{\frac{2R}{r}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Lemma : In ΔABC, we have

$$\frac{b}{c}+\frac{c}{b}\le\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F} \qquad (*)$$

Proof : Since $16F^2=2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)$, then we have

$$(*)\Leftrightarrow(b^2+c^2)\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}\le 2bc\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

$$\overset{squaring}{\Leftrightarrow}\quad(2b^2c^2+b^4+c^4)[2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)]\le 4b^2c^2(a^2b^2+b^2c^2+c^2a^2)$$

$$\Leftrightarrow\ -a^4(b^2+c^2)^2+2(b^4+c^4)(a^2b^2+c^2a^2)-(b^4+c^4)^2=-[a^2(b^2+c^2)-(b^4+c^4)]^2\le 0,$$

which is true and the proof of the lemma is complete.

Now, since $\sqrt{a},\sqrt{b},\sqrt{c}$ can be the sides of triangle with area F',

$$F'=\frac{1}{4}.\sqrt{2(ab+bc+ca)-(a^2+b^2+c^2)}=\frac{\sqrt{4r(4R+r)}}{4}=\frac{\sqrt{r(r_a+r_b+r_c)}}{2},$$

then $m_{\sqrt{a}},m_{\sqrt{b}},m_{\sqrt{c}}$ can be the sides of triangle with, area $F_m=\frac{3F'}{4}$

$$=\frac{3}{8}\sqrt{r(r_a+r_b+r_c)},$$

$$m_{\sqrt{a}}=\frac{\sqrt{2(b+c)-a}}{2},\quad m_{\sqrt{b}}=\frac{\sqrt{2(a+c)-b}}{2},\quad m_{\sqrt{c}}=\frac{\sqrt{2(a+b)-c}}{2},\quad \text{and},$$

$$m_{\sqrt{a}}{}^2m_{\sqrt{b}}{}^2+m_{\sqrt{b}}{}^2m_{\sqrt{c}}{}^2+m_{\sqrt{c}}{}^2m_{\sqrt{a}}{}^2=\frac{9}{16}(ab+bc+ca)=\frac{9R}{8}(h_a+h_b+h_c).$$

Using the lemma in $\Delta m_{\sqrt{a}}m_{\sqrt{b}}m_{\sqrt{c}}$, we obtain

$$\frac{m_{\sqrt{b}}}{m_{\sqrt{c}}}+\frac{m_{\sqrt{c}}}{m_{\sqrt{b}}}\le\frac{\sqrt{m_{\sqrt{a}}^2m_{\sqrt{b}}^2+m_{\sqrt{b}}^2m_{\sqrt{c}}^2+m_{\sqrt{c}}^2m_{\sqrt{a}}^2}}{2F_m}=\sqrt{\frac{2R(h_a+h_b+h_c)}{r(r_a+r_b+r_c)}}.$$

Therefore,

$$\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{2(a+c)-b}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(a+c)-b}}\right)\le\sqrt{\frac{2R}{r}}.$$

1181. In ΔABC the following relationship holds:

$$\frac{\lambda a+\mu r_a}{h_a}+\frac{\lambda b+\mu r_b}{h_b}+\frac{\lambda c+\mu r_c}{h_c}\ge 2\sqrt{3}\lambda+3\mu$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{\lambda a+\mu r_a}{h_a}+\frac{\lambda b+\mu r_b}{h_b}+\frac{\lambda c+\mu r_c}{h_c}=$$

$$=\lambda\sum_{cyc}\frac{a}{h_a}+\mu\sum_{cyc}\frac{r_a}{h_a}=\lambda\sum_{cyc}\frac{a}{\frac{2F}{a}}+\mu\sum_{cyc}\frac{\frac{F}{s-a}}{\frac{2F}{a}}=$$

$$=\frac{\lambda}{2F}(a^2+b^2+c^2)+\frac{\mu}{2}\cdot\sum_{cyc}\frac{a}{s-a}=$$

$$=\frac{\lambda}{2F}(a^2+b^2+c^2)+\frac{\mu}{2}\cdot\frac{2(2R-r)}{r}\overset{IONESCU-WEITZENBOCK}{\ge}$$

$$\ge\frac{\lambda}{2F}\cdot 4\sqrt{3}F+\mu\left(2\cdot\frac{R}{r}-1\right)\overset{EULER}{\ge}\frac{\lambda}{2}\cdot 4\sqrt{3}+\mu(2\cdot 2-1)=2\sqrt{3}\lambda+3\mu$$

Equality holds for: $a=b=c$.

1182. If $\lambda>0$ then in ΔABC the following relationship holds:

$$\frac{a+b\lambda}{c}+\frac{b+c\lambda}{a}+\frac{c+a\lambda}{b}\ge\frac{6r(1+\lambda)}{R}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{a+b\lambda}{c}+\frac{b+c\lambda}{a}+\frac{c+a\lambda}{b}=$$

$$=\sum_{cyc}\frac{a}{c}+\lambda\sum_{cyc}\frac{b}{c}=\sum_{cyc}\frac{a^2}{ac}+\lambda\sum_{cyc}\frac{b^2}{bc}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{(a+b+c)^2}{ab+bc+ca}+\lambda\cdot\frac{(a+b+c)^2}{ab+bc+ca}=$$

$$=(1+\lambda)\cdot\frac{(a+b+c)^2}{ab+bc+ca}=\frac{4s^2(1+\lambda)}{s^2+r^2+4Rr}\geq\frac{6r(1+\lambda)}{R}\Leftrightarrow$$

$$\Leftrightarrow\frac{2s^2}{s^2+r^2+4Rr}\geq\frac{3r}{R}\Leftrightarrow 2s^2R\geq 3r(s^2+r^2+4Rr)\Leftrightarrow$$

$$\Leftrightarrow s^2(3R-r)\geq 3r^3+4Rr^3$$

$$s^2(3R-r)\overset{MITRINOVIC}{\geq}27r^2(3R-r)\geq 3r^3+4Rr^3\Leftrightarrow$$

$$9(2R-3r)\geq r+4R\Leftrightarrow 14R\geq 28r\Leftrightarrow R\geq 2r\ (Euler)$$

Equality holds for $a=b=c$.

1183. In ΔABC the following relationship holds:

$$\frac{\left(r_a^5+r_b^5\right)^2}{r_c^3}+\frac{\left(r_b^5+r_c^5\right)^2}{r_a^3}+\frac{\left(r_c^5+r_a^5\right)^2}{r_b^3}\geq 4\cdot 3^8\cdot r^7$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{\left(r_a^5+r_b^5\right)^2}{r_c^3}+\frac{\left(r_b^5+r_c^5\right)^2}{r_a^3}+\frac{\left(r_c^5+r_a^5\right)^2}{r_b^3}\overset{AM-GM}{\geq}$$

$$\geq\frac{4r_a^5r_b^5}{r_c^3}+\frac{4r_b^5r_c^5}{r_a^3}+\frac{4r_c^5r_a^5}{r_b^3}\overset{AM-GM}{\geq}12\sqrt[3]{\frac{(r_ar_br_c)^{10}}{(r_ar_br_c)^3}}=12\sqrt[3]{(r_ar_br_c)^7}=$$

$$=12\sqrt[3]{(rs^2)^7}\overset{MITRINOVIC}{\geq}12\sqrt[3]{(r\cdot 27r^2)^7}=$$

$$=4\cdot 3\sqrt[3]{3^{21}\cdot r^{21}}=4\cdot 3\cdot 3^7\cdot r^7=4\cdot 3^8\cdot r^7$$

Equality holds for $a=b=c$.

1184. In ΔABC the following relationship holds:

$$36\sqrt{2}\cdot r^2\le a\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}\le 9\sqrt{2}\cdot R^2$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$a\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}\overset{CBS}{\le}$$

$$\le\sqrt{\sum_{cyc}a^2\cdot\sum_{cyc}(b^2+c^2)}=\sqrt{2\cdot\sum_{cyc}a^2\cdot\sum_{cyc}a^2}=\sqrt{2}\cdot\sum_{cyc}a^2\le$$

$$\overset{LEIBNIZ}{\le}\sqrt{2}\cdot 9R^2=9\sqrt{2}\cdot R^2$$

$$a\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}+c\sqrt{a^2+b^2}\overset{AM-GM}{\ge}$$

$$\ge\sum_{cyc}a\sqrt{2bc}=\sqrt{2}\cdot\sum_{cyc}\sqrt{a^2bc}=\sqrt{2}\cdot\sqrt{abc}\cdot\sum_{cyc}\sqrt{a}\overset{AM-GM}{\ge}$$

$$\ge\sqrt{2}\cdot\sqrt{abc}\cdot 3\sqrt[3]{\sqrt{abc}}=3\sqrt{2}\cdot\sqrt[6]{(abc)^4}=3\sqrt{2}\cdot\sqrt[3]{(abc)^2}=$$

$$=3\sqrt{2}\cdot\sqrt[3]{(4Rrs)^2}\overset{EULER}{\ge}3\sqrt{2}\cdot\sqrt[3]{(4\cdot 2r\cdot rs)^2}\overset{MITRINOVIC}{\ge}$$

$$=3\sqrt{2}\cdot\sqrt[3]{\left(4\cdot 2r\cdot r\cdot 3\sqrt{3}r\right)^2}=3\sqrt{2}\cdot\sqrt[3]{\left(2\cdot\sqrt{3}r\right)^6}=3\sqrt{2}\cdot\left(2\cdot\sqrt{3}r\right)^2=36\sqrt{2}\cdot r^2$$

1185. In ΔABC the following relationship holds:

$$\frac{\lambda m_a+\mu m_b}{a+b}+\frac{\lambda m_b+\mu m_c}{b+c}+\frac{\lambda m_c+\mu m_a}{c+a}\ge 3\sqrt{3}(\lambda+\mu)\cdot\frac{r^2}{R^2}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{\lambda m_a+\mu m_b}{a+b}+\frac{\lambda m_b+\mu m_c}{b+c}+\frac{\lambda m_c+\mu m_a}{c+a}=$$

$$=\lambda\sum_{cyc}\frac{m_a}{a+b}+\mu\sum_{cyc}\frac{m_b}{a+b}\overset{TERESHIN}{\ge}\lambda\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{a+b}+\mu\sum_{cyc}\frac{\frac{c^2+a^2}{4R}}{a+b}=$$

$$=\frac{\lambda}{4R}\left(\sum_{cyc}\frac{b^2}{a+b}+\sum_{cyc}\frac{b^2}{a+b}\right)+\frac{\mu}{4R}\left(\sum_{cyc}\frac{c^2}{a+b}+\sum_{cyc}\frac{a^2}{a+b}\right)\ge$$

$$\overset{BERGSTROM}{\ge}\frac{2\lambda}{4R}\cdot\frac{(a+b+c)^2}{2(a+b+c)}+\frac{2\mu}{4R}\cdot\frac{(a+b+c)^2}{2(a+b+c)}=$$

$$=\frac{(\lambda+\mu)}{4R}\cdot(a+b+c)=\frac{(\lambda+\mu)}{4R}\cdot 2s=\frac{(\lambda+\mu)}{R}\cdot\frac{1}{2}\cdot s\overset{EULER}{\ge}\frac{(\lambda+\mu)}{R}\cdot\frac{r}{R}\cdot s\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{(\lambda+\mu)}{R}\cdot\frac{r}{R}\cdot 3\sqrt{3}r=3\sqrt{3}(\lambda+\mu)\cdot\frac{r^2}{R^2}$$

Equality holds for $a=b=c$.

1186. If $\lambda,\mu>0$ then in ΔABC the following relationship holds:

$$\frac{a}{\lambda h_b+\mu h_c}+\frac{b}{\lambda h_c+\mu h_a}+\frac{c}{\lambda h_a+\mu h_b}\ge\frac{2\sqrt{3}}{\lambda+\mu}$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{a}{\lambda h_b+\mu h_c}+\frac{b}{\lambda h_c+\mu h_a}+\frac{c}{\lambda h_a+\mu h_b}=$$

$$=\sum_{cyc}\frac{a}{\lambda\cdot\frac{2F}{b}+\mu\cdot\frac{2F}{c}}=\frac{1}{2F}\sum_{cyc}\frac{abc}{\lambda c+\mu b}=\frac{abc}{2F}\sum_{cyc}\frac{1}{\lambda c+\mu b}\ge$$

$$\overset{BERGSTROM}{\ge}\frac{4RF}{2F}\cdot\frac{(1+1+1)^2}{(\lambda+\mu)(a+b+c)}=2R\cdot\frac{9}{(\lambda+\mu)\cdot 2s}=$$

$$=\frac{9R}{(\lambda+\mu)\cdot s}\overset{MITRINOVIC}{\ge}\frac{9R}{(\lambda+\mu)\cdot\frac{3\sqrt{3}}{2}}\cdot R=\frac{2\sqrt{3}}{\lambda+\mu}$$

Equality holds for: $a=b=c$.

1187. If $a=\min\{a,b,c\}$, then in acute Δ ABC, the following relationship holds :

$$\frac{1}{r}\sum_{cyc}AI\ge\sqrt{2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)}+\sqrt{\frac{2(n_b+h_b)}{r_b}}+\sqrt{\frac{2(n_c+h_c)}{r_c}}$$

Proposed by Bogdan Fuştei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$
$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a\therefore a^2n_a^2\overset{?}{\le}4(R-r)^2s^2$$
$$\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\le}4(R-r)^2s^2\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)$$
$$\overset{?}{\le}4(R^2-2Rr+r^2)s^2\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\ge}0$$
$$\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\ge}0\Leftrightarrow(R\cos A-r)^2\overset{?}{\ge}0\to\text{true}\therefore an_a\le 2Rs-2rs$$
$$\Rightarrow\frac{n_a}{h_a}\le\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}\Rightarrow\frac{n_a}{h_a}\le\frac{R}{r}-1\text{ and analogs}\therefore$$
$$\frac{n_b}{h_c}+\frac{n_c}{h_b}\le\left(\frac{R}{r}-1\right)\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right)=\left(\frac{R}{r}-1\right)\left(\frac{a(b^2+c^2)}{4Rrs}\right)$$
$$\overset{\text{Tereshin}}{\le}\left(\frac{R}{r}-1\right)\frac{a.4Rm_a}{4Rrs}\overset{?}{\le}\frac{2m_a}{r_a}=\frac{(b+c-a)m_a}{rs}\Leftrightarrow\frac{R}{r}-1\overset{?}{\le}\frac{b+c-a}{a}$$
$$\Leftrightarrow\frac{b+c}{a}\overset{?}{\ge}\frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$
$$\Leftrightarrow(b+c)(a+b-c).(b+c-a)(c+a-b)\overset{?}{\ge}2a^2bc$$
$$\Leftrightarrow(b+c)(a+b-c).(b+c-a)(c+a-b)\overset{?}{\ge}2a^2bc$$
$$\Leftrightarrow(ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)\overset{?}{\ge}$$
$$2a^2bc\Leftrightarrow 2a^2b^2+2a^2bc+2ab(b^2-c^2)-(a^2+b^2-c^2)(ab+ac+b^2-c^2)$$
$$\overset{?}{\ge}2a^2bc$$
$$\Leftrightarrow 2a^2b^2-(a^2+b^2-c^2)(ab+ac)+2ab(b^2-c^2)-2ab(b^2-c^2).\cos C\overset{?}{\ge}0$$
$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-(b^2-c^2)(ab+ac)+2ab(b^2-c^2).2\sin^2\frac{C}{2}\overset{?}{\ge}0$$
$$\Leftrightarrow a^2(2b^2-ab-ac)-(b^2-c^2)(ab+ac)+(b^2-c^2)(c^2-(a-b)^2)\overset{?}{\ge}0$$
$$\Leftrightarrow a^2(2b^2-ab-ac)+(b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac)\overset{?}{\ge}0$$
$$\Leftrightarrow\left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$
$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac)\overset{?}{\ge}0$$
$$\Leftrightarrow(c^2+a^2-b^2)(2b^2-ab-ac)+(b^2-c^2)(b^2+c^2-a^2-2ac)\overset{?}{\ge}0$$
$$\Leftrightarrow(b^2-c^2)(b^2+c^2-2ac)-(b^2-c^2)(2b^2-ab-ac)$$
$$+a^2(2b^2-ab-ac)-a^2(b^2-c^2)\overset{?}{\ge}0$$

$$\Leftrightarrow (b^2-c^2)\left((c^2-ca)-(b^2-ab)\right)+a^2\left((c^2-ca)+(b^2-ab)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow (c^2-ca)(b^2-c^2+a^2)+(b^2-ab)(a^2+c^2-b^2)\overset{?}{\geq}0$$

$$\Leftrightarrow c(c-a)(a^2+b^2-c^2)+b(b-a)(c^2+a^2-b^2)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because \Delta\, ABC \text{ being acute} \Rightarrow (a^2+b^2-c^2),(c^2+a^2-b^2)>0$$

$$\text{and } a=\min\{a,b,c\}\Rightarrow (c-a),(b-a)\geq 0$$

$$\therefore \sqrt{2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)}\leq\sqrt{\frac{4m_a}{r_a}}\overset{\text{Panaitopol}}{\leq}\sqrt{\frac{2R}{r}\cdot\frac{h_a}{r_a}}=\sqrt{\frac{2R}{r}\cdot\frac{2rs}{4Rs\cos^2\frac{A}{2}\tan^2\frac{A}{2}}}$$

$$\Rightarrow \sqrt{2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)}\leq\frac{1}{\sin\frac{A}{2}}\Rightarrow\sqrt{2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)}+\sqrt{\frac{2(n_b+h_b)}{r_b}}+\sqrt{\frac{2(n_c+h_c)}{r_c}}$$

$$\overset{\text{via (1)}}{\leq}\frac{1}{\sin\frac{A}{2}}+\sqrt{\frac{2R}{r}\cdot\frac{h_b}{r_b}}+\sqrt{\frac{2R}{r}\cdot\frac{h_c}{r_c}}=\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}=\frac{1}{r}\sum_{cyc}AI$$

$$\Rightarrow\frac{1}{r}\sum_{cyc}AI\geq\sqrt{2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)}+\sqrt{\frac{2(n_b+h_b)}{r_b}}+\sqrt{\frac{2(n_c+h_c)}{r_c}}$$

in acute $\Delta\, ABC$ with $a=\min\{a,b,c\}$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have, $$n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}$$

$$=s^2-\frac{4s(s-b)(s-c)}{a}$$

$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_ar_a,\ \text{ and }\ r_a=s\tan\frac{A}{2}.$$

Then,

$$n_a\overset{AM-GM}{\leqq}\frac{n_a{}^2+r_a{}^2}{2r_a}=\frac{s^2\left(1+\tan^2\frac{A}{2}\right)-2h_ar_a}{2r_a}=\frac{s\sec^2\frac{A}{2}}{2\tan\frac{A}{2}}-h_a=$$

$$\frac{s}{\sin A}-h_a=\left(\frac{R}{r}-1\right)h_a.$$

Using this result, we have

$$\frac{2(n_b+h_b)}{r_b}\leq\frac{2Rh_b}{rr_b}=\frac{4RF}{rs\tan\frac{B}{2}.2R\sin B}=\frac{1}{\sin^2\frac{B}{2}}=\left(\frac{BI}{r}\right)^2\ \text{ and }\ \frac{2(n_c+h_c)}{r_c}\leq\left(\frac{CI}{r}\right)^2.$$

Now, we have

$$\frac{n_a{}^2-n_b{}^2}{s}=\left(s-a+\frac{(b-c)^2}{a}\right)-\left(s-b+\frac{(c-a)^2}{b}\right)$$
$$=b-a+\frac{b(b-c)^2-a(c-a)^2}{ab}$$
$$=b-a+\frac{(b-a)(b^2+ab+a^2-2ca-2bc+c^2)}{ab}=\frac{(b-a)(a+b-c)^2}{ab}\geq 0,$$

then $n_a\geq n_b$. Similarly, we have, $n_a\geq n_c$, then

$$2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)\leq 2\left(\frac{1}{h_c}+\frac{1}{h_b}\right)n_a\leq\frac{b+c}{F}.\left(\frac{R}{r}-1\right)h_a\overset{?}{\leq}\left(\frac{AI}{r}\right)^2\Leftrightarrow$$
$$\frac{2(R-r)(b+c)}{ra}\leq\frac{bc}{(s-b)(s-c)}$$
$$\Leftrightarrow\ (R-r)(2s-a)\leq\frac{r.4Rsr.(s-a)}{2sr^2}=2R(s-a)\ \Leftrightarrow\ a(R+r)\leq 2sr$$
$$\Leftrightarrow a\sum_{cyc}\cos A\leq\sum_{cyc}a\cos A\ \Leftrightarrow\ 0\leq(b-a)\cos B+(c-a)\cos C,$$

which is true. Therefore,

$$\sqrt{2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)}+\sqrt{\frac{2(n_b+h_b)}{r_b}}+\sqrt{\frac{2(n_c+h_c)}{r_c}}\leq\frac{AI}{r}+\frac{BI}{r}+\frac{CI}{r}=\frac{1}{r}\sum_{cyc}AI.$$

1188.

If $a=\min\{a,b,c\}$, then in acute Δ ABC, the following relationship holds :

$$\frac{2m_a}{r_a}\geq\frac{n_b}{h_c}+\frac{n_c}{h_b}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$
$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a\therefore a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2$$

$$\Leftrightarrow a^2(s^2-2h_a r_a)\overset{?}{\le}4(R-r)^2s^2\Leftrightarrow(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)$$

$$\overset{?}{\le}4(R^2-2Rr+r^2)s^2\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\ge}0$$

$$\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\ge}0\Leftrightarrow(R\cos A-r)^2\overset{?}{\ge}0\to\text{true}\therefore an_a\le 2Rs-2rs$$

$$\Rightarrow\frac{n_a}{h_a}\le\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}\Rightarrow\frac{n_a}{h_a}\le\frac{R}{r}-1\text{ and analogs }\therefore$$

$$\frac{n_b}{h_c}+\frac{n_c}{h_b}\le\left(\frac{R}{r}-1\right)\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right)=\left(\frac{R}{r}-1\right)\left(\frac{a(b^2+c^2)}{4Rrs}\right)$$

$$\overset{\text{Tereshin}}{\le}\left(\frac{R}{r}-1\right)\frac{a.4Rm_a}{4Rrs}\overset{?}{\le}\frac{2m_a}{r_a}=\frac{(b+c-a)m_a}{rs}\Leftrightarrow\frac{R}{r}-1\overset{?}{\le}\frac{b+c-a}{a}$$

$$\Leftrightarrow\frac{b+c}{a}\overset{?}{\ge}\frac{R}{r}=\frac{abcs}{4F^2}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$$

$$\Leftrightarrow(b+c)(a+b-c).(b+c-a)(c+a-b)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow(b+c)(a+b-c).(b+c-a)(c+a-b)\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow(ab+b^2-bc+ca+bc-c^2)(bc+ab-b^2+c^2+ca-bc-ca-a^2+ab)\overset{?}{\ge}$$
$$2a^2bc\Leftrightarrow 2a^2b^2+2a^2bc+2ab(b^2-c^2)-(a^2+b^2-c^2)(ab+ac+b^2-c^2)$$
$$\overset{?}{\ge}2a^2bc$$

$$\Leftrightarrow 2a^2b^2-(a^2+b^2-c^2)(ab+ac)+2ab(b^2-c^2)-2ab(b^2-c^2).\cos C\overset{?}{\ge}0$$

$$\Leftrightarrow 2a^2b^2-a^2(ab+ac)-(b^2-c^2)(ab+ac)+2ab(b^2-c^2).2\sin^2\frac{C}{2}\overset{?}{\ge}0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)-(b^2-c^2)(ab+ac)+(b^2-c^2)(c^2-(a-b)^2)\overset{?}{\ge}0$$

$$\Leftrightarrow a^2(2b^2-ab-ac)+(b^2-c^2)(c^2-a^2-b^2+2ab-ab-ac)\overset{?}{\ge}0$$

$$\Leftrightarrow\left((a^2-b^2+c^2)+(b^2-c^2)\right)(2b^2-ab-ac)$$
$$+(b^2-c^2)(c^2-a^2-b^2+ab-ac)\overset{?}{\ge}0$$

$$\Leftrightarrow(c^2+a^2-b^2)(2b^2-ab-ac)+(b^2-c^2)(b^2+c^2-a^2-2ac)\overset{?}{\ge}0$$

$$\Leftrightarrow(b^2-c^2)(b^2+c^2-2ac)-(b^2-c^2)(2b^2-ab-ac)$$
$$+a^2(2b^2-ab-ac)-a^2(b^2-c^2)\overset{?}{\ge}0$$

$$\Leftrightarrow(b^2-c^2)\left((c^2-ca)-(b^2-ab)\right)+a^2\left((c^2-ca)+(b^2-ab)\right)\overset{?}{\ge}0$$

$$\Leftrightarrow(c^2-ca)(b^2-c^2+a^2)+(b^2-ab)(a^2+c^2-b^2)\overset{?}{\ge}0$$

$$\Leftrightarrow c(c-a)(a^2+b^2-c^2)+b(b-a)(c^2+a^2-b^2)\overset{?}{\ge}0$$

$$\to\text{true}\because\Delta\,ABC\text{ being acute}\Rightarrow(a^2+b^2-c^2),(c^2+a^2-b^2)>0$$

$$\text{and }a=\min\{a,b,c\}\Rightarrow(c-a),(b-a)\ge 0$$

$$\therefore\frac{2m_a}{r_a}\ge\frac{n_b}{h_c}+\frac{n_c}{h_b}\text{ in acute }\Delta\,ABC\text{ with }a=\min\{a,b,c\}\text{ (QED)}$$

1189.

In any Δ ABC with I → incenter, the following relationship holds :

$$IA^4+IB^4+IC^4 \ge \frac{(a^2+b^2+c^2)^2}{27}$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$AI^2=bc-4Rr \Leftrightarrow \left(\frac{r}{\left(\frac{r}{4R}\right)}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2$$

$$=16R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2}-16R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Leftrightarrow \sin\frac{B}{2}\sin\frac{C}{2}$$

$$=\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{A}{2} \Leftrightarrow \cos\frac{B+C}{2}=\sin\frac{A}{2} \to \text{true} \therefore AI^2=bc-4Rr \text{ and analogs}$$

$$\therefore IA^4+IB^4+IC^4=\sum_{cyc}(bc-4Rr)^2=\sum_{cyc}a^2b^2+48R^2r^2-8Rr\sum_{cyc}ab$$

$$=\left(s^2+4Rr+r^2\right)^2-16Rrs^2+48R^2r^2-8Rr\left(s^2+4Rr+r^2\right)$$

$$\therefore IA^4+IB^4+IC^4 \ge \frac{\left(a^2+b^2+c^2\right)^2}{27}$$

$$\Leftrightarrow 27\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2+48R^2r^2-8Rr\left(s^2+4Rr+r^2\right)\right)$$

$$\ge 4\left(s^2-4Rr-r^2\right)^2$$

$$\Leftrightarrow 23s^4-\left(400Rr-62r^2\right)s^2+r^2\left(800R^2-32Rr+23r^2\right) \overset{(*)}{\ge} 0 \text{ and}$$

$$\because 23\left(s^2-16Rr+5r^2\right)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$23s^4-\left(400Rr-62r^2\right)s^2+r^2\left(800R^2-32Rr+23r^2\right) \ge 23\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow (42R-21r)s^2 \overset{(**)}{\ge} r\left(636R^2-456Rr+69r^2\right)$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\ge} (42R-21r)\left(16Rr-5r^2\right) \overset{?}{\ge} r\left(636R^2-456Rr+69r^2\right)$$

$$\Leftrightarrow 18r\left(2R^2-5Rr+2r^2\right) \overset{?}{\ge} 0 \Leftrightarrow 18r(2R-r)(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \text{in any } \Delta \text{ ABC,}$$

$$IA^4+IB^4+IC^4 \ge \frac{\left(a^2+b^2+c^2\right)^2}{27}, \text{"="} \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

1190. **If $\lambda,\mu>0$ the in ΔABC the following relationship holds:**

$$\frac{a+m_a}{\lambda b+\mu c}+\frac{b+m_b}{\lambda c+\mu a}+\frac{c+m_c}{\lambda a+\mu b} \ge \frac{12+6\sqrt{3}}{\lambda+\mu}\cdot\left(\frac{r}{R}\right)^2$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution by Daniel Sitaru-Romania

$$\frac{a+m_a}{\lambda b+\mu c}+\frac{b+m_b}{\lambda c+\mu a}+\frac{c+m_c}{\lambda a+\mu b}=\sum_{cyc}\frac{a}{\lambda b+\mu c}+\sum_{cyc}\frac{m_a}{\lambda b+\mu c}\ge$$

$$\overset{TERESHIN}{\ge}\sum_{cyc}\frac{a^2}{\lambda ab+\mu ac}+\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{\lambda b+\mu c}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(a+b+c)^2}{(\lambda+\mu)(ab+bc+ca)}+\frac{1}{4R}\left(\sum_{cyc}\frac{b^2}{\lambda b+\mu c}+\sum_{cyc}\frac{c^2}{\lambda b+\mu c}\right)\overset{BERGSTROM}{\ge}$$

$$\ge\frac{4s^2}{(\lambda+\mu)(s^2+r^2+4Rr)}+\frac{1}{2R}\cdot\frac{(a+b+c)^2}{(\lambda+\mu)(a+b+c)}\overset{EULER}{\ge}$$

$$\ge\frac{4s^2}{(\lambda+\mu)\left(s^2+\frac{R^2}{4}+4R\cdot\frac{R}{2}\right)}+\frac{2s}{2R(\lambda+\mu)}\overset{MITRINOVIC}{\ge}$$

$$\ge\frac{4\left(3\sqrt{3}r\right)^2}{(\lambda+\mu)\left(\frac{27R^2}{4}+\frac{R^2}{4}+4R\cdot\frac{R}{2}\right)}+\frac{2\cdot3\sqrt{3}r}{2R(\lambda+\mu)}=$$

$$=\frac{4\cdot27r^2}{9R^2(\lambda+\mu)}+\frac{6\sqrt{3}r}{R(\lambda+\mu)}\cdot\frac{1}{2}\overset{EULER}{\ge}\frac{12}{\lambda+\mu}\cdot\left(\frac{r}{R}\right)^2+\frac{6\sqrt{3}}{\lambda+\mu}\cdot\left(\frac{r}{R}\right)^2=\frac{12+6\sqrt{3}}{\lambda+\mu}\cdot\left(\frac{r}{R}\right)^2$$

Equality holds for: $a=b=c$.

1191. In any $\Delta\, ABC$, the following relationship holds $\forall\, k\in\mathbb{N}$:

$$\sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2\ge\frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\boxed{Case\ 1}\ \mathbf{k=0} \therefore \sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2 \geq \frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1} \Leftrightarrow \sum_{cyc}\frac{1}{(r_b+r_c)^2} \geq \frac{1}{3R^2} \overset{via\ (i)}{\Leftrightarrow}$$

$$\frac{1}{16R^2}\sum_{cyc}\sec^4\frac{A}{2} \geq \frac{1}{3R^2} \Leftrightarrow \sum_{cyc}\sec^4\frac{A}{2} \geq \frac{16}{3} \rightarrow \text{true} \because f(x)=\sec^4\frac{x}{2}$$

$$\forall x \in (0,\pi) \text{ is convex since } f''(x) = 4\sec^4\frac{x}{2}\tan^2\frac{x}{2}+\sec^6\frac{x}{2} > 0$$

$$\Rightarrow \sum_{cyc}\sec^4\frac{A}{2} \overset{Jensen}{\geq} 3\sec^4\frac{\pi}{6} = \frac{3.16}{9} = \frac{16}{3}$$

$$\Rightarrow \sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2 \geq \frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1} \text{ is true for } k=0$$

$$\boxed{Case\ 2}\ \mathbf{k=1} \therefore \sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2 \geq \frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1} \Leftrightarrow \sum_{cyc}\left(\frac{r_a}{r_b+r_c}\right)^2 \geq \frac{3}{4} \rightarrow \text{true}$$

$$\because \sum_{cyc}\left(\frac{r_a}{r_b+r_c}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc}\frac{r_a}{r_b+r_c}\right)^2 \overset{Nesbitt}{\geq} \frac{1}{3}.\frac{9}{4} = \frac{3}{4}$$

$$\Rightarrow \sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2 \geq \frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1} \text{ is true for } k=1$$

$$\boxed{Case\ 3}\ \mathbf{k} \in \mathbb{N}-\{0,1\} \therefore \sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2 \geq \frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1}$$

$$\Leftrightarrow \sum_{cyc}\left(r_a^{2k}.\frac{\sec^4\frac{A}{2}}{16R^2}\right) \geq \frac{3}{4}.\frac{4}{9R^2}.\left(\frac{9R^2}{4}\right)^k$$

$$\Leftrightarrow \frac{3}{16}.\sum_{cyc}\left(r_a^{2k}\left(1+2\tan^2\frac{A}{2}+\tan^4\frac{A}{2}\right)\right) \geq \left(\frac{9R^2}{4}\right)^k$$

$$\Leftrightarrow \frac{3}{16}.\left(\sum_{cyc}r_a^{2k}+\frac{2}{s^2}.\sum_{cyc}r_a^{2k+2}+\frac{1}{s^4}.\sum_{cyc}r_a^{2k+4}\right) \overset{(*)}{\geq} \left(\frac{9R^2}{4}\right)^k$$

$$\text{Now, } \sum_{cyc}r_a^{2k}+\frac{2}{s^2}.\sum_{cyc}r_a^{2k+2}+\frac{1}{s^4}.\sum_{cyc}r_a^{2k+4}$$

$$\overset{Chebyshev}{\geq} \sum_{cyc}r_a^{2k}+\frac{2}{3s^2}.\sum_{cyc}r_a^2.\sum_{cyc}r_a^{2k}+\frac{1}{3s^4}.\sum_{cyc}r_a^4.\sum_{cyc}r_a^{2k}$$

$$\geq \left(\sum_{cyc}r_a^{2k}\right)\left(1+\frac{2}{3s^2}.\sum_{cyc}r_ar_b+\frac{1}{9s^4}.\left(\sum_{cyc}r_a^2\right)^2\right)$$

$$\overset{Repeated\ Chebyshev}{\geq} \frac{1}{3^{k-1}}.\left(\sum_{cyc}r_a^2\right)^k\left(1+\frac{2}{3}+\frac{1}{9s^4}.\left(\sum_{cyc}r_ar_b\right)^2\right)$$

$$=\left(3+2+\frac{1}{3}\right)\left(\frac{\sum_{cyc} r_a^2}{3}\right)^k=\frac{16}{3}.\left(\frac{\sum_{cyc} r_a^2}{3}\right)^k$$

$$\Rightarrow \text{LHS of } (*) \geq \left(\frac{\sum_{cyc} r_a^2}{3}\right)^k \overset{?}{\geq} \left(\frac{9R^2}{4}\right)^k \Leftrightarrow \left(\frac{4\sum_{cyc} r_a^2}{27R^2}\right)^k \underset{(**)}{\overset{?}{\geq}} 1$$

$$\text{Again, } 4\sum_{cyc} r_a^2-27R^2=4(4R+r)^2-27R^2-8s^2 \overset{\text{Gerretsen}}{\geq}$$

$$4(4R+r)^2-27R^2-8(4R^2+4Rr+3r^2)=5(R^2-4r^2) \overset{\text{Euler}}{\geq} 0$$

$$\Rightarrow \frac{4\sum_{cyc} r_a^2}{27R^2} \geq 1 \Rightarrow \left(\frac{4\sum_{cyc} r_a^2}{27R^2}\right)^k \geq 1 \ (\because k\in\mathbb{N}-\{0,1\}) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\Rightarrow \sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2 \geq \frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1} \text{ is true } \forall\, k\in\mathbb{N}-\{0,1\}$$

$$\therefore \text{ combining } all \text{ cases}, \sum_{cyc}\left(\frac{r_a^k}{r_b+r_c}\right)^2 \geq \frac{3}{4}\left(\frac{9R^2}{4}\right)^{k-1} \quad \forall\, k\in\mathbb{N} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\sum_{cyc}\left(\frac{r_a{}^k}{r_b+r_c}\right)^2=\sum_{cyc}\frac{(r_a{}^2)^{k+1}}{(r_ar_b+r_cr_a)^2} \geq \frac{(r_a{}^2+r_b{}^2+r_c{}^2)^{k+1}}{3^{k-2}.[2(r_ar_b+r_br_c+r_cr_a)]^2}$$

$$\geq \frac{(r_a{}^2+r_b{}^2+r_c{}^2)^{k-1}}{4.3^{k-2}}.$$

So it suffices to prove that

$$r_a{}^2+r_b{}^2+r_c{}^2 \geq \frac{27R^2}{4}.$$

By Gerretsen and Euler inequalities, we have

$$r_a{}^2+r_b{}^2+r_c{}^2=(4R+r)^2-2s^2 \geq (4R+r)^2-2(4R^2+4Rr+3r^2)$$

$$=\frac{27R^2}{4}+5\left(\frac{R^2}{4}-r^2\right) \geq \frac{27R^2}{4}.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.

1192. In *any* Δ ABC, the fo*ll*owing re*la*tionship ho*l*ds :

$$\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{R^{2022}}{r^{2022}} \geq 2^{2022}+\sum_{cyc}\frac{a^3}{b^3+c^3}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{R^4}{r^4}\overset{?}{\geq}2^4+\sum_{cyc}\frac{a^3}{b^3+c^3}$$

$$\Leftrightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{R^4-16r^4}{r^4}\overset{?}{\geq}\sum_{cyc}\frac{\sum_{cyc}a^3-(b^3+c^3)}{b^3+c^3}$$

$$\Leftrightarrow 3+\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{R^4-16r^4}{r^4}\underset{(*)}{\overset{?}{\geq}}\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}\frac{1}{b^3+c^3}\right)$$

$$\text{Now,}\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}\frac{1}{b^3+c^3}\right)\overset{A-G}{\leq}\frac{2s(s^2-6Rr-3r^2)}{2}.\sum_{cyc}\frac{1}{bc\sqrt{bc}}$$

$$\overset{CBS}{\leq}s(s^2-6Rr-3r^2).\sqrt{\sum_{cyc}\frac{1}{b^2c^2}}.\sqrt{\sum_{cyc}\frac{1}{bc}}$$

$$=\frac{s(s^2-6Rr-3r^2)}{4Rrs}.\sqrt{2(s^2-4Rr-r^2).\frac{2s}{4Rrs}}$$

$$\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2-6Rr-3r^2}{4Rr}.\sqrt{\frac{4R^2+4Rr+3r^2-4Rr-r^2}{Rr}}$$

$$\Rightarrow \text{RHS of }(*)\overset{(\bullet)}{\leq}\frac{2R-r}{2r}.\sqrt{\frac{4R^2+2r^2}{Rr}}\text{ and via Nesbitt, LHS of }(*)\geq\frac{9}{2}+\frac{R^4-16r^4}{r^4}$$

$$\Rightarrow \text{LHS of }(*)\overset{(\bullet\bullet)}{\geq}\frac{2R^4-23r^4}{2r^4}\therefore(\bullet),(\bullet\bullet)\Rightarrow\text{in order to prove }(*),$$

$$\text{it suffices to prove}:\frac{2R^4-23r^4}{2r^4}\geq\frac{2R-r}{2r}.\sqrt{\frac{4R^2+2r^2}{Rr}}$$

$$\Leftrightarrow\frac{(2R^4-23r^4)^2}{r^8}\geq\frac{4R^2+2r^2}{Rr}.\frac{(2R-r)^2}{r^2}$$

$$\Leftrightarrow R(2R^4-23r^4)^2\geq(r^5)(4R^2+2r^2)(2R-r)^2$$

$$\Leftrightarrow 4t^9-92t^5-16t^4+16t^3-12t^2+537t-2\geq 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\Big((t-2)(4t^7+16t^6+48t^5+128t^4+228t^3+384t^2+640t+1012)+2025\Big)\geq 0$$

$$\rightarrow\text{true}\because t\overset{Euler}{\geq}2\Rightarrow(*)\text{ is true}\Rightarrow\frac{R^4-16r^4}{r^4}\overset{(\blacksquare)}{\geq}\sum_{cyc}\frac{a^3}{b^3+c^3}-\sum_{cyc}\frac{a^2}{b^2+c^2}$$

$$\text{Again,}\ \forall\, n\in\mathbb{N}\text{ with }n\geq 5,\frac{R^n-(2r)^n}{r^n}\overset{?}{\geq}\frac{R^4-16r^4}{r^4}\Leftrightarrow$$

$$(R-2r)\big(R^{n-1}+R^{n-2}.(2r)+\cdots+R^3(2r)^{n-4}+R^2(2r)^{n-3}+R(2r)^{n-2}+(2r)^{n-1}\big)$$

$$\overset{?}{\geq}(R-2r)(r^{n-4})\big(R^3+R^2(2r)+R(2r)^2+(2r)^3\big)$$

$$\Leftrightarrow (R-2r)\Big(R^{n-1}+R^{n-2}.(2r)+\cdots$$

$$+(2^{n-4}-1)\Big(R^3(r^{n-4})+2R^2(r^{n-3})+4R(r^{n-2})+8(r^{n-1})\Big)\Big)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because n\geq 5 \text{ and} \because R-2r\overset{\text{Euler}}{\geq}0 \therefore \frac{R^n-(2r)^n}{r^n}\geq\frac{R^4-16r^4}{r^4}$$

$$\overset{\text{via }(\blacksquare)}{\geq}\sum_{cyc}\frac{a^3}{b^3+c^3}-\sum_{cyc}\frac{a^2}{b^2+c^2}\therefore\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{R^n}{r^n}\geq 2^n+\sum_{cyc}\frac{a^3}{b^3+c^3}$$

$\forall\, n\in\mathbb{N}$ with $n\geq 5$ and choosing $n=2022$, in any Δ ABC,

$$\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{R^{2022}}{r^{2022}}\geq 2^{2022}+\sum_{cyc}\frac{a^3}{b^3+c^3}, \text{“ = ” iff } \Delta \text{ ABC is equilateral (QED)}$$

1193. In any Δ ABC with $r\geq\frac{1}{3}$, the following relationship holds :

$$\frac{(r_a^5-2r_ar_b+r_b^5)^2}{r_a^3+r_b^3}+\frac{(r_b^5-2r_br_c+r_c^5)^2}{r_b^3+r_c^3}+\frac{(r_c^5-2r_cr_a+r_a^5)^2}{r_c^3+r_a^3}\geq\frac{144r^4(27r^3-1)^2}{9R^3-64r^3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\frac{(r_a^5-2r_ar_b+r_b^5)^2}{r_a^3+r_b^3}+\frac{(r_b^5-2r_br_c+r_c^5)^2}{r_b^3+r_c^3}+\frac{(r_c^5-2r_cr_a+r_a^5)^2}{r_c^3+r_a^3}$$

$$=\sum_{cyc}\frac{|r_a^5-2r_ar_b+r_b^5|^2}{r_a^3+r_b^3}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}|r_a^5-2r_ar_b+r_b^5|\right)^2}{\sum_{cyc}(r_a^3+r_b^3)}$$

$$\geq\frac{|r_a^5-2r_ar_b+r_b^5+r_b^5-2r_br_c+r_c^5+r_c^5-2r_cr_a+r_a^5|^2}{2\sum_{cyc}r_a^3}$$

$$(\because |x|+|y|+|z|\geq|x+y+z|)\Rightarrow \text{LHS}\overset{(\bullet)}{\geq}\frac{2\left(\sum_{cyc}r_a^5-\sum_{cyc}r_ar_b\right)^2}{\sum_{cyc}r_a^3}$$

$$\text{Now,}\sum_{cyc}r_a^5-\sum_{cyc}r_ar_b\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{cyc}r_a^3\right)\left(\sum_{cyc}r_a^2\right)\overset{\text{Holder}}{\geq}\frac{1}{27}\left(\sum_{cyc}r_a\right)^3.\left(\sum_{cyc}r_a^2\right)$$

$$\geq\frac{1}{27}\left(\sum_{cyc}r_a\right)^3.\left(\sum_{cyc}r_ar_b\right)\Rightarrow\sum_{cyc}r_a^5-\sum_{cyc}r_ar_b\geq\frac{\left(\sum_{cyc}r_a\right)^3-27}{27}.s^2$$

$$\Rightarrow \sum_{cyc} r_a^5 - \sum_{cyc} r_a r_b \overset{(\bullet\bullet)}{\geq} \frac{\left((4R+r)-3\right)\left((4R+r)^2+9+3(4R+r)\right)}{27}.s^2$$

$$\text{Again}, (4R+r)-3 \overset{\text{Euler}}{\geq} 9r-3 = 9\left(r-\frac{1}{3}\right) \geq 0$$

$$\Rightarrow \frac{\left((4R+r)-3\right)\left((4R+r)^2+9+3(4R+r)\right)}{27}.s^2$$

$$\geq \frac{(3r-1)\left((4R+r)^2+9+3(4R+r)\right)}{9}.s^2$$

$$\overset{\text{via }(\bullet\bullet)}{\Rightarrow} \sum_{cyc} r_a^5 - \sum_{cyc} r_a r_b \geq \frac{(3r-1)\left((4R+r)^2+9+3(4R+r)\right)}{9}.s^2 \geq 0 \overset{\text{via }(\bullet)}{\Rightarrow} \text{LHS}$$

$$\geq \frac{2s^4(3r-1)^2\left((4R+r)^2+9+3(4R+r)\right)^2}{81\sum_{cyc} r_a^3} \overset{\text{Euler}}{\geq} \frac{2s^4(3r-1)^2\left((9r)^2+9+3(9r)\right)^2}{81\sum_{cyc} r_a^3}$$

$$= \frac{2s^4(3r-1)^2(9r^2+1+3r)^2}{81\sum_{cyc} r_a^3} = \frac{2s^4(27r^3-1)^2}{\sum_{cyc} r_a^3} \overset{?}{\geq} \frac{144r^4(27r^3-1)^2}{9R^3-64r^3}$$

$$\Leftrightarrow s^4(9R^3-64r^3) \overset{?}{\geq} 72r^4\left(\left(\sum_{cyc} r_a\right)^3 - 3\prod_{cyc}(r_b+r_c)\right)$$

$$\overset{\text{via (i)}}{=} 72r^4\left((4R+r)^3 - 3.64R^3.\frac{s^2}{16R^2}\right)$$

$$\Leftrightarrow s^4(9R^3-64r^3) + 72r^4.12Rs^2 \underset{(*)}{\overset{?}{\geq}} 72r^4(4R+r)^3$$

$$\text{Now}, 2s^2 \overset{\text{Gerretsen}}{\geq} 27Rr + 5r(R-2r) \overset{\text{Euler}}{\geq} 27Rr$$

$$\Rightarrow \text{LHS of } (*) \geq \left((9R^3-64r^3).\frac{27Rr}{2} + 72.12Rr^4\right)s^2$$

$$\overset{\text{Gerretsen}}{\geq} \left((9R^3-64r^3).\frac{27Rr}{2} + 72.12Rr^4\right)(16Rr-5r^2) \overset{?}{\geq} 72r^4(4R+r)^3$$

$$\Leftrightarrow 432t^5 - 135t^4 - 1024t^3 - 768t^2 - 192t - 16 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(432t^4+729t^3+434t^2+100t+8) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true} \Rightarrow \text{in any } \Delta\, ABC \text{ with } r \geq \frac{1}{3},$$

$$\sum_{cyc} \frac{(r_a^5 - 2r_a r_b + r_b^5)^2}{r_a^3 + r_b^3} \geq \frac{144r^4(27r^3-1)^2}{9R^3-64r^3}, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc} \frac{(r_b{}^5 - 2r_b r_c + r_c{}^5)^2}{r_b{}^3 + r_c{}^3} \overset{CBS}{\geq} \frac{\left(\sum_{cyc}(r_b{}^5 - 2r_b r_c + r_c{}^5)\right)^2}{\sum_{cyc}(r_b{}^3 + r_c{}^3)} = \frac{2\left(\sum_{cyc} r_a{}^5 - \sum_{cyc} r_b r_c\right)^2}{\sum_{cyc} r_a{}^3}.$$

and, $r_a{}^3 + r_b{}^3 + r_c{}^3 \overset{AM-GM}{\leq} (r_a + r_b + r_c)^3 - 24 r_a r_b r_c = (4R+r)^2 - 24 s^2 r$
$\overset{Euler\ \&\ Mitrinovic}{\leq} \left(\frac{9R}{2}\right)^3 - 24.27r^2.r = \frac{81(9R^3 - 64r^3)}{8}.$

and,

$$\sum_{cyc} r_a{}^5 - \sum_{cyc} r_b r_c \overset{AM-GM}{\geq} 3\sqrt[3]{(r_a r_b r_c)^5} - s^2 = 3\sqrt[3]{(s^2 r)^5} - s^2 = s^2\left(3\sqrt[3]{s^4 r^5} - 1\right)$$

$$\overset{Mitrinovic}{\geq} s^2(27r^3 - 1) \overset{Mitrinovic}{\geq} 27r^2(27r^3 - 1) \geq 0\,.$$

Using these results we have

$$\sum_{cyc} \frac{\left(r_b{}^5 - 2r_b r_c + r_c{}^5\right)^2}{r_b{}^3 + r_c{}^3} \geq \frac{16[27r^2(27r^3-1)]^2}{81(9R^3 - 64r^3)} = \frac{144r^4(27r^3-1)^2}{9R^3 - 64r^3}.$$

Equality holds iff ΔABC is equilateral.

1194. In ΔABC the following relationship holds :

$$\left(\sum_{cyc} a^2 m_b\right)\left(\sum_{cyc} a^2 m_c\right) \geq \left(\sum_{cyc} m_a m_b\right)\left(\sum_{cyc} a^2 b^2\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is successively equivalent to

$$\sum_{cyc} a^4 m_b m_c + \sum_{cyc} b^2 c^2 m_a{}^2 \geq \sum_{cyc} a^2(b^2 + c^2) m_b m_c$$

$$\Leftrightarrow \sum_{cyc} (a^2 - b^2)(a^2 - c^2) m_b m_c + \sum_{cyc} b^2 c^2 m_a{}^2 - \sum_{cyc} b^2 c^2 m_b m_c \geq 0.$$

The inequality is symmetrical, WLOG we may assume that
$a \geq b \geq c$. We have $m_a \leq m_b \leq m_c$,
$a^2 - c^2 \geq b^2 - c^2$, and, $(c^2 - a^2)(c^2 - b^2) m_a m_b \geq 0$, then

$$\sum_{cyc} (a^2 - b^2)(a^2 - c^2) m_b m_c \geq (a^2 - b^2)(b^2 - c^2) m_b m_c + (b^2 - a^2)(b^2 - c^2) m_c m_a$$

$$= (a^2 - b^2)(b^2 - c^2)(m_b - m_a) m_c \geq 0.$$

Since $b^2 c^2 \leq c^2 a^2 \leq a^2 b^2$, $m_b m_c \geq m_c m_a \geq m_a m_b$
then by Chebyshev's inequality, we have

$$\sum_{cyc} b^2c^2m_bm_c \le \frac{1}{3}\sum_{cyc} b^2c^2 . \sum_{cyc} m_bm_c \le \frac{1}{3}\sum_{cyc} b^2c^2 . \sum_{cyc} m_a{}^2 \le \sum_{cyc} b^2c^2m_a{}^2.$$

Adding these two results yields the desired inequality.
Equality holds iff ΔABC is equilateral.

1195. In any Δ ABC, the following relationship holds :

$$\frac{8(2Rr - r^2)^2}{R} \le \sum_{cyc} AI^2 . h_a \le 8\left(R^3 - R^2r + \frac{r^4}{R}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$AI^2 = bc - 4Rr \Leftrightarrow \left(\frac{r}{\left(\frac{r}{4R}\right)}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2$$

$$= 16R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 16R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Leftrightarrow \sin\frac{B}{2}\sin\frac{C}{2} = \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{A}{2} \Leftrightarrow \cos\frac{B+C}{2} = \sin\frac{A}{2} \to \text{true}$$

$$\therefore AI^2 = bc - 4Rr \text{ and analogs} \therefore \sum_{cyc} AI^2 . h_a = \frac{1}{2R} . \sum_{cyc}(bc - 4Rr). bc$$

$$= \frac{\left(\sum_{cyc} ab\right)^2 - 4Rr\sum_{cyc} ab - 16Rrs^2}{2R} = \frac{(s^2 + 4Rr + r^2)(s^2 + r^2) - 16Rrs^2}{2R}$$

$$\therefore \sum_{cyc} AI^2 . h_a \overset{(\bullet)}{=} \frac{s^4 - (12Rr - 2r^2)s^2 + r^3(4R + r)}{2R}$$

$$\therefore \sum_{cyc} AI^2 . h_a \le 8\left(R^3 - R^2r + \frac{r^4}{R}\right)$$

$$\Leftrightarrow \frac{s^4 - (12Rr - 2r^2)s^2 + r^3(4R + r)}{2R} \le \frac{8(R^4 - R^3r + r^4)}{R}$$

$$\Leftrightarrow s^4 - (12Rr - 2r^2)s^2 + r^3(4R + r) \overset{(*)}{\le} 16(R^4 - R^3r + r^4)$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\le} (4R^2 - 8Rr + 5r^2)s^2 + r^3(4R + r)$$

$$\overset{\text{Gerretsen}}{\le} (4R^2 - 8Rr + 5r^2)(4R^2 + 4Rr + 3r^2) + r^3(4R + r) = 16(R^4 - R^3r + r^4)$$

$$\Rightarrow (*) \text{ is true} \therefore \sum_{cyc} AI^2 . h_a \le 8\left(R^3 - R^2r + \frac{r^4}{R}\right)$$

$$\text{Again, } \frac{8(2Rr - r^2)^2}{R} \le \sum_{cyc} AI^2 . h_a$$

$$\overset{\text{via } (\bullet)}{\Leftrightarrow} \frac{s^4 - (12Rr - 2r^2)s^2 + r^3(4R + r)}{2R} \ge \frac{8(2Rr - r^2)^2}{R}$$

$$\Leftrightarrow s^4-(12Rr-2r^2)s^2+r^3(4R+r)\overset{(**)}{\geq}16(2Rr-r^2)^2$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\geq}(4Rr-3r^2)s^2+r^3(4R+r)$$

$$\overset{\text{Gerretsen}}{\geq}(4Rr-3r^2)(16Rr-5r^2)+r^3(4R+r)=16(2Rr-r^2)^2$$

$$\Rightarrow (**)\text{ is true }\therefore \frac{8(2Rr-r^2)^2}{R}\leq\sum_{cyc}AI^2.h_a\ \ (QED)$$

1196. In any Δ ABC, the following relationship holds :

$$\frac{h_a^2}{r_a}+\frac{h_b^2}{r_b}+\frac{h_c^2}{r_c}\leq\frac{9R^3}{8r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a^2}{r_a}+\frac{h_b^2}{r_b}+\frac{h_c^2}{r_c}=\sum_{cyc}\frac{b^2c^2(s-a)}{4R^2rs}$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2-4Rr(s^2+4Rr+r^2)}{4R^2r}$$

$$=\frac{(s^2+4Rr+r^2)(s^2+r^2)-16Rrs^2}{4R^2r}\leq\frac{9R^3}{8r^2}$$

$$\Leftrightarrow 2rs^4-r^2s^2(24R-4r)-9R^5+8Rr^4+2r^5\overset{(*)}{\leq}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\leq}\left(2r(4R^2+4Rr+3r^2)-r^2(24R-4r)\right)s^2-9R^5+8Rr^4$$

$$+2r^5=r(8R^2-16Rr+10r^2)s^2-9R^5+8Rr^4+2r^5$$

$$\overset{\text{Gerretsen}}{\leq}r(8R^2-16Rr+10r^2)(4R^2+4Rr+3r^2)-9R^5+8Rr^4+2r^5\overset{?}{\leq}0$$

$$\Leftrightarrow 9t^5-32t^4+32t^3-32\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(7t^3(t-2)+2t^4+4t^2+8t+16)\overset{?}{\geq}0$$

$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}\Rightarrow\text{in any }\Delta\text{ ABC},$$

$$\frac{h_a^2}{r_a}+\frac{h_b^2}{r_b}+\frac{h_c^2}{r_c}\leq\frac{9R^3}{8r^2},''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

1197. In any Δ ABC, the following relationship holds :

$$\left(\frac{w_a^5+w_b^5}{w_b+w_c}\right)^2+\left(\frac{w_b^5+w_c^5}{w_c+w_a}\right)^2+\left(\frac{w_c^5+w_a^5}{w_a+w_b}\right)^2\geq\frac{4.3^9.r^{10}}{R^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \ge 32Rr\cos^2\frac{A}{2} \overset{\text{via (i)}}{=} 8r(r_b + r_c) = 8r^2 s\left(\frac{1}{s-b} + \frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \ge 0 \Leftrightarrow (b+c-2a)^2 \ge 0 \to \text{true}$$

$$\therefore b + c \ge \sqrt{32Rr}.\cos\frac{A}{2} \Rightarrow \frac{w_a}{h_a} \le \frac{2abc\cos\frac{A}{2}.}{\sqrt{32Rr}.\cos\frac{A}{2}.2rs} = \frac{4R}{\sqrt{32Rr}}$$

$$\Rightarrow w_a \le \sqrt{\frac{R}{2r}}.h_a \text{ and analogs} \Rightarrow (w_b + w_c)^2 \le \frac{R}{2r}.(h_b + h_c)^2 = \frac{R}{2r}.\frac{4r^2s^2(b+c)^2}{b^2c^2}$$

$$\text{and power} - \text{mean inequality} \Rightarrow \sqrt[5]{\frac{w_a^5 + w_b^5}{2}} \ge \sqrt{\frac{w_a^2 + w_b^2}{2}}$$

$$\Rightarrow \left(w_a^5 + w_b^5\right)^2 \ge \frac{1}{8}\left(w_a^2 + w_b^2\right)^5$$

$$\Rightarrow \left(\frac{w_a^5 + w_b^5}{w_b + w_c}\right)^2 \ge \frac{1}{8}\left(w_a^2 + w_b^2\right)^5.\frac{2r}{R}.\frac{16R^2r^2s^2}{4r^2s^2a^2(b+c)^2} \text{ and analogs}$$

$$\Rightarrow \left(\frac{w_a^5 + w_b^5}{w_b + w_c}\right)^2 + \left(\frac{w_b^5 + w_c^5}{w_c + w_a}\right)^2 + \left(\frac{w_c^5 + w_a^5}{w_a + w_b}\right)^2$$

$$\ge Rr.\sum_{cyc}\frac{\left(w_a^2 + w_b^2\right)^5}{\left(\sqrt{a(b+c)}\right)^4} \overset{\text{Radon}}{\ge} \frac{32Rr\left(\sum_{cyc} w_a^2\right)^5}{\left(\sum_{cyc}\sqrt{a(b+c)}\right)^4} \overset{\text{CBS}}{\ge} \frac{32Rr\left(\sum_{cyc} w_a^2\right)\left(\sum_{cyc} h_a^2\right)^4}{\left(\left(\sqrt{\sum_{cyc} a}\right)\left(\sqrt{\sum_{cyc}(b+c)}\right)\right)^4}$$

$$\ge \frac{32Rr\left(\sum_{cyc} w_a^2\right)\left(\frac{1}{3}\left(\sum_{cyc} h_a\right)^2\right)^4}{\left(\sum_{cyc} a\right)^2.\left(\sum_{cyc}(b+c)\right)^2} \overset{\substack{\sum_{cyc} h_a \ge 9r \\ + \\ \text{CBS}}}{\ge} \frac{32Rr\left(\sum_{cyc} w_a^2\right).3^{12}.r^8}{4s^2.16s^2} \overset{?}{\ge} \frac{4.3^9.r^{10}}{R^2}$$

$$\Leftrightarrow 27R^3\left(\sum_{cyc} w_a^2\right) \underset{(*)}{\overset{?}{\ge}} 8rs^4$$

$$\text{Now, } AI^2 = bc - 4Rr \Leftrightarrow \left(\frac{r}{\left(\frac{r}{4R}\right)}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2$$

$$= 16R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 16R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Leftrightarrow \sin\frac{B}{2}\sin\frac{C}{2} = \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{A}{2} \Leftrightarrow \cos\frac{B+C}{2} = \sin\frac{A}{2} \to \text{true}$$

$$\therefore AI^2 = bc - 4Rr \text{ and analogs} \rightarrow (1)$$

$$\text{Again}, \frac{2bc.\cos\frac{A}{2}}{b+c} \overset{?}{\geq} \frac{r\left(\sin\frac{A}{2}+1\right)}{\sin\frac{A}{2}} \Leftrightarrow \frac{a}{2R}.\frac{bc}{b+c} \overset{?}{\geq} r\left(\sin\frac{A}{2}+1\right)$$

$$\Leftrightarrow \frac{4Rrs}{2R(b+c)} \overset{?}{\geq} r\left(\sin\frac{A}{2}+1\right) \Leftrightarrow \frac{a+b+c}{b+c} \overset{?}{\geq} \sin\frac{A}{2}+1$$

$$\Leftrightarrow \frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\sin\frac{A}{2}} \overset{?}{\geq} 4R\cos\frac{A}{2}4R\cos\frac{B-C}{2} \rightarrow \text{true} \because 0 < \cos\frac{B-C}{2} \leq 1$$

$$\Rightarrow w_a^2 \geq AI^2 + 2r.AI + r^2 \overset{\text{via (1)}}{=} bc - 4Rr + 2r.AI + r^2 \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} w_a^2 \geq s^2 + 4Rr + r^2 - 12Rr + 2r^2.\sum_{cyc}\frac{1}{\sin\frac{A}{2}} + 3r^2$$

$$\overset{\text{Jensen}}{\geq} s^2 - 8Rr + 16r^2 \text{ and } 27R^2 \overset{\text{Mitrinovic}}{\geq} 4s^2$$

$$\Rightarrow 27R^3\left(\sum_{cyc} w_a^2\right) \geq 4Rs^2(s^2 - 8Rr + 16r^2) \overset{?}{\geq} 8rs^4 \Leftrightarrow (R-2r)s^2 \overset{?}{\geq} 8Rr(R-2r)$$

$$\Leftrightarrow (R-2r)(s^2 - 16Rr + 5r^2 + 8r(R-2r) + 11r^2) \overset{?}{\geq} 0$$

$$\rightarrow \text{true via Gerretsen and Euler} \Rightarrow (*) \text{ is true}$$

$$\therefore \text{in any } \Delta\, ABC, \left(\frac{w_a^5 + w_b^5}{w_b + w_c}\right)^2 + \left(\frac{w_b^5 + w_c^5}{w_c + w_a}\right)^2 + \left(\frac{w_c^5 + w_a^5}{w_a + w_b}\right)^2 \geq \frac{4.3^9.r^{10}}{R^2},$$

'' ='' iff Δ ABC is equilateral (QED)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{(b+c)^2} = \sum_{cyc}\frac{(a-2s)+2s}{(b+c)^2} = 2s\frac{\sum_{cyc}(c+a)^2(a+b)^2}{\prod_{cyc}(b+c)^2} - \sum_{cyc}\frac{1}{b+c}$$

$$= \frac{\left(\sum_{cyc}(c+a)(a+b)\right)^2 - 2.2s(s^2+2Rr+r^2)(4s)}{2s(s^2+2Rr+r^2)^2} - \frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)} =$$

$$\frac{\left(\left(\sum_{cyc}a^2 + 2\sum_{cyc}ab\right) + \sum_{cyc}ab\right)^2 - 16s^2(s^2+2Rr+r^2) - (s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\Rightarrow \sum_{cyc}\frac{a}{(b+c)^2} \overset{(\bullet)}{=} \frac{(5s^2+4Rr+r^2)^2 - 16s^2(s^2+2Rr+r^2) - (s^2+2Rr+r^2)(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)^2}$$

$$\text{Also}, r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \ge 32Rr\cos^2\frac{A}{2} \overset{\text{via (i)}}{=} 8r(r_b+r_c) = 8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \ge 0 \Leftrightarrow (b+c-2a)^2 \ge 0 \to \text{true}$$

$$\therefore b+c \ge \sqrt{32Rr}.\cos\frac{A}{2} \Rightarrow \frac{w_a}{h_a} \le \frac{2abc\cos\frac{A}{2}.}{\sqrt{32Rr}.\cos\frac{A}{2}.2rs} = \frac{4R}{\sqrt{32Rr}}$$

$$\Rightarrow w_a \le \sqrt{\frac{R}{2r}}.h_a \text{ and analogs} \Rightarrow (w_b+w_c)^2 \le \frac{R}{2r}.(h_b+h_c)^2 = \frac{R}{2r}.\frac{4r^2s^2(b+c)^2}{b^2c^2}$$

$$\text{and power} - \text{mean inequality} \Rightarrow \sqrt[5]{\frac{w_a^5+w_b^5}{2}} \ge \sqrt{\frac{w_a^2+w_b^2}{2}}$$

$$\Rightarrow \left(w_a^5+w_b^5\right)^2 \ge \frac{1}{8}\left(w_a^2+w_b^2\right)^5$$

$$\Rightarrow \left(\frac{w_a^5+w_b^5}{w_b+w_c}\right)^2 \ge \frac{1}{8}\left(w_a^2+w_b^2\right)^5.\frac{2r}{R}.\frac{16R^2r^2s^2}{4r^2s^2a^2(b+c)^2} \text{ and analogs}$$

$$\Rightarrow \left(\frac{w_a^5+w_b^5}{w_b+w_c}\right)^2 + \left(\frac{w_b^5+w_c^5}{w_c+w_a}\right)^2 + \left(\frac{w_c^5+w_a^5}{w_a+w_b}\right)^2$$

$$\ge Rr.\sum_{\text{cyc}}\frac{\left(w_a^2+w_b^2\right)^5}{\left(\sqrt{a(b+c)}\right)^4} \overset{\text{Radon}}{\ge} \frac{32Rr\left(\sum_{\text{cyc}}w_a^2\right)^5}{\left(\sum_{\text{cyc}}\sqrt{a(b+c)}\right)^4} \overset{\text{CBS}}{\ge} \frac{32Rr\left(\sum_{\text{cyc}}w_a^2\right)\left(\sum_{\text{cyc}}h_a^2\right)^4}{\left(\sqrt{3}.\sqrt{2\sum_{\text{cyc}}ab}\right)^4}$$

$$\ge \frac{32Rr\left(\sum_{\text{cyc}}w_a^2\right)\left(\frac{1}{3}\left(\sum_{\text{cyc}}h_a\right)^2\right)^4}{\left(\sqrt{3}.\sqrt{2\sum_{\text{cyc}}a^2}\right)^4} \overset{\substack{\sum_{\text{cyc}}h_a \ge 9r \\ + \\ \text{Leibnitz}}}{\ge} \frac{32Rr\left(\sum_{\text{cyc}}w_a^2\right).3^{12}.r^8}{9.4.81R^4} \overset{?}{\ge} \frac{4.3^9.r^{10}}{R^2}$$

$$\Leftrightarrow 2\sum_{\text{cyc}}w_a^2 \overset{?}{\ge} 27Rr \Leftrightarrow 2\sum_{\text{cyc}}\left(bc - \frac{a^2bc}{(b+c)^2}\right) \overset{?}{\ge} 27Rr$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow} 2\left(s^2+4Rr+r^2\right)$$

$$+4Rr.\frac{\left(s^2+2Rr+r^2\right)\left(5s^2+4Rr+r^2\right)+16s^2\left(s^2+2Rr+r^2\right)-\left(5s^2+4Rr+r^2\right)^2}{\left(s^2+2Rr+r^2\right)^2} \overset{?}{\ge} 27Rr$$

$$\Leftrightarrow 2s^6 - \left(27Rr-6r^2\right)s^4 - r^2s^2\left(44R^2-26Rr-6r^2\right)$$

$$-r^3\left(108R^3+76R^2r+11Rr^2-2r^3\right) \underset{(*)}{\overset{?}{\ge}} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\ge} \left(5Rr-4r^2\right)s^4 - r^2s^2\left(44R^2-26Rr-6r^2\right)$$

$$-r^3\left(108R^3+76R^2r+11Rr^2-2r^3\right)$$

$$\overset{\text{Gerretsen}}{\ge} \left(\left(5Rr-4r^2\right)\left(16Rr-5r^2\right)-r^2\left(44R^2-26Rr-6r^2\right)\right)s^2$$

$$-r^3\left(108R^3+76R^2r+11Rr^2-2r^3\right)$$

$$= r^2\left(36R^2-63Rr+26r^2\right)s^2 - r^3\left(108R^3+76R^2r+11Rr^2-2r^3\right)$$

$$\overset{Gerretsen}{\geq} r^2(36R^2-63Rr+26r^2)(16Rr-5r^2)$$
$$-r^3(108R^3+76R^2r+11Rr^2-2r^3)\overset{?}{\geq}0$$
$$\Leftrightarrow 117t^3-316t^2+180t-32\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(76t^2+41t(t-2)+16)\overset{?}{\geq}0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (*) \text{ is true}$$
$$\therefore \text{in any } \Delta\, ABC, \left(\frac{w_a^5+w_b^5}{w_b+w_c}\right)^2+\left(\frac{w_b^5+w_c^5}{w_c+w_a}\right)^2+\left(\frac{w_c^5+w_a^5}{w_a+w_b}\right)^2 \geq \frac{4.3^9.r^{10}}{R^2},$$
$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1198. In ΔABC the following relationship holds:

$$\sqrt[3]{(a+b)(b+c)(c+a)} \geq 4\sqrt{3}r$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\sqrt[3]{(a+b)(b+c)(c+a)} \overset{HOLDER}{\geq} \sqrt[3]{\left(\sqrt[3]{abc}+\sqrt[3]{abc}\right)^3} = 2\sqrt[3]{abc}$$
$$2\sqrt[3]{abc} \geq 4\sqrt{3}r \Leftrightarrow \sqrt[3]{4Rrs} \geq 2\sqrt{3}r \Leftrightarrow 4Rrs \geq 24\sqrt{3}r^3 \Leftrightarrow$$
$$\Leftrightarrow Rs \geq 6\sqrt{3}r^2$$
$$Rs \overset{EULER}{\geq} 2rs \overset{MITRINOVIC}{\geq} 6\sqrt{3}r^2$$

Equality holds for $a=b=c$.

Solution 2 by Tapas Das-India

$$(a+b)(b+c)(c+a) \overset{AM-GM}{\geq} 8(abc)^2$$
$$\sqrt[3]{(a+b)(b+c)(c+a)} \geq \sqrt[3]{8(abc)^2} = 2\sqrt[3]{(abc)^2} \overset{CARLITZ}{\geq}$$
$$\geq 2\left(\frac{4F}{\sqrt{3}}\right)^{3\cdot\frac{1}{3}} = \frac{2\sqrt{3}}{3}\cdot F = \frac{2\sqrt{3}}{3}\cdot rs \overset{MITRINOVIC}{\geq}$$
$$\geq \frac{2\sqrt{3}}{3}\cdot r\cdot 3\sqrt{3}r = 6\sqrt{3}r^2$$

Equality holds for $a=b=c$.

Solution 3 by Hikmat Mammadov-Azerbaijan

$$a=x+y, b=y+z, c=z+x, x,y,z>0, r=\sqrt{\frac{xyz}{x+y+z}}$$

We must prove that:

$$(x+y+z)^3(2x+y+z)^2(x+2y+z)^2(x+y+2z)^2 \geq 4^6\cdot 3^3\cdot (xyz)^3$$

$$(x+y+z)^3 \overset{AM-GM}{\geq} \left(3\sqrt[3]{xyz}\right)^3 = 27xyz$$
$$(2x+y+z)^2 \overset{AM-GM}{\geq} \left(4\sqrt[4]{x^2yz}\right)^2$$

$$(x+2y+z)^2 \overset{AM-GM}{\gtrsim} \left(4\sqrt[4]{xy^2z}\right)^2$$

$$(x+y+2z)^2 \overset{AM-GM}{\gtrsim} \left(4\sqrt[4]{xyz^2}\right)^2$$

By multypling:

$$(x+y+z)^3(2x+y+z)^2(x+2y+z)^2(x+y+2z)^2 \geq$$

$$\geq 27xyz\cdot\left(4\sqrt[4]{x^2yz}\right)^2\cdot\left(4\sqrt[4]{xy^2z}\right)^2\cdot\left(4\sqrt[4]{xyz^2}\right)^2 = 4^6\cdot 3^3\cdot(xyz)^3$$

Equality holds for $a=b=c$.

Solution 4 by George Florin Șerban-Romania

$$\sqrt[3]{(a+b)(b+c)(c+a)} \geq 4\sqrt{3}r$$

$$(a+b)(b+c)(c+a) \geq 64\cdot 3\sqrt{3}r^3$$

$$2s(s^2+r^2+2Rr)) \geq 64\cdot 3\sqrt{3}r^3\ (to\ prove)$$

$$2s(s^2+r^2+2Rr)) \overset{GERRETSEN}{\gtrsim} 2s(16Rr-5r^2+r^2+2Rr) \geq$$

$$\overset{MITRINOVIC}{\gtrsim} 2\cdot 3\sqrt{3}r(18Rr-4r^2)) \geq 64\cdot 3\sqrt{3}r^3 \Leftrightarrow$$

$$\Leftrightarrow 18Rr-4r^2 \geq 32r^2 \Leftrightarrow 18Rr \geq 36r^2 \Leftrightarrow R \geq 2r\ (EULER)$$

Equality holds for $a=b=c$.

1199. In acute ΔABC the following relationship holds:

$$\frac{1}{m_a^5}+\frac{1}{m_b^5}+\frac{1}{m_c^5} \geq \frac{2^{12}r^7}{81R^{12}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

Lemma:

In acute ΔABC the following relationship holds:

$$m_a+m_b+m_c \leq \frac{9R}{2}$$

Proof:

O –circumcenter, $AM=m_a$ –median. In ΔAOM:

$$AM \leq AO+OM$$

$$m_a \leq R+R\cos A = R\left(1+2\cos^2\frac{A}{2}-1\right) = 2R\cos^2\frac{A}{2} = \frac{2Rs(s-a)}{bc}$$

$$m_a+m_b+m_c \leq \sum_{cyc}\frac{2Rs(s-a)}{bc} = \frac{2Rs}{abc}\sum_{cyc}a(s-a) =$$

$$= \frac{2Rs}{abc}\cdot 2r(4R+r) = \frac{4Rrs}{4Rrs}\cdot(4R+r) \overset{EULER}{\lesssim} 4R+\frac{R}{2} = \frac{9R}{2}$$

Back to the problem:

$$\frac{1}{m_a^5}+\frac{1}{m_b^5}+\frac{1}{m_c^5}=\frac{1^6}{m_a^5}+\frac{1^6}{m_b^5}+\frac{1^6}{m_c^5}\overset{RADON}{\geq}\frac{(1+1+1)^6}{(m_a+m_b+m_c)^5}=$$

$$=\frac{3^6}{(m_a+m_b+m_c)^5}\overset{LEMMA}{\geq}\frac{3^6}{\left(\frac{9R}{2}\right)^5}=\frac{3^6\cdot 2^5}{3^{10}\cdot R^5}=\frac{2^5}{81R^5}=$$

$$=\frac{2^{12}}{81R^5}\cdot\left(\frac{1}{2}\right)^7\overset{EULER}{\geq}\frac{2^{12}}{81R^5}\cdot\left(\frac{r}{R}\right)^7=\frac{2^{12}r^7}{81R^{12}}$$

Equality holds for: $a=b=c$.

1200. In any Δ ABC, the following relationship holds :

$$\frac{a}{b}\cdot\sqrt{\cot\frac{A}{2}}+\frac{b}{c}\cdot\sqrt{\cot\frac{B}{2}}+\frac{c}{a}\cdot\sqrt{\cot\frac{C}{2}}\leq 3\sqrt[4]{3}.\frac{R}{2r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b}\cdot\sqrt{\cot\frac{A}{2}}+\frac{b}{c}\cdot\sqrt{\cot\frac{B}{2}}+\frac{c}{a}\cdot\sqrt{\cot\frac{C}{2}}=\sum_{cyc}\left(a.\sqrt{\cot\frac{A}{2}}\right)\left(\frac{1}{b}\right)$$

$$\overset{CBS}{\leq}\sqrt{\sum_{cyc}\frac{a^2s}{r_a}}.\sqrt{\sum_{cyc}\frac{1}{a^2}}=\sqrt{\frac{s}{rs}.\sum_{cyc}a^2(s-a)}.\sqrt{\frac{\sum_{cyc}a^2b^2}{16R^2r^2s^2}}$$

$$\overset{Goldstone}{\leq}\sqrt{\frac{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{r}}.\sqrt{\frac{4R^2s^2}{16R^2r^2s^2}}=\sqrt{\frac{2s(2R+2r)}{4r^2}}$$

$$\overset{Mitrinovic+Euler}{\leq}\sqrt{\frac{3\sqrt{3}R(2R+R)}{4r^2}}=3.\sqrt[4]{3}.\frac{R}{2r},''=''\text{ iff } a=b=c\text{ (QED)}$$

Solution 2 by George Florin Șerban-Romania

$$\sum_{cyc}\frac{a}{b}\sqrt{\cot\frac{A}{2}}=\sum_{cyc}\frac{\sin A}{\sin B}\sqrt{\frac{1+\cos A}{\sin A}}=\sum_{cyc}\frac{\sqrt{(\sin A)(1+\cos A)}}{\sin B}\leq\sqrt{\left(\sum_{cyc}(\sin A)(1+\cos A)\right)\sum_{cyc}\frac{1}{\sin^2 B}}=$$

(C.B.S inequality)

$=\sqrt{(\sum_{cyc}\sin A+\frac{1}{2}\sum_{cyc}\sin 2A)\cdot 4R^2\sum_{cyc}\frac{1}{b^2}}\le\sqrt{(\frac{p}{R}+\frac{pr}{R^2})\cdot 4R^2\cdot\frac{1}{4r^2}}=\sqrt{\frac{p(R+r)}{r^2}}\le$ **(Mitrinovic inequality)** $\le\sqrt{\frac{3\sqrt{3}R(R+r)}{2r^2}}\le 3\sqrt[4]{3}\cdot\frac{R}{2r}$, **then** $\frac{27R^2(R+r)^2}{4r^4}\le 243\cdot\frac{R^4}{16r^4}$, $9R^2\ge 4(R+r)^2$,

$3R\ge 2R+2r,\ \ R\ge 2r$, **true,Euler inequality,then** $\sum_{cyc}\frac{a}{b}\sqrt{\cot\frac{A}{2}}\le 3\sqrt[4]{3}\cdot\frac{R}{2r}$.

Equality is if $a=b=c$.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru