THE METRIC RELATIONS OF THE MIXTILINEAR INCIRCLE

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Abstract In this paper we present metric relations of the main points of the mixtilinear incircles and applications in proving distance related to this point.

Keywords: Mixtilinear Incircle, External Center of Similitude of Circumcircle and Incircle, Stewart's Theorem, Menelaus' Theorem, Isogonal Cevians, Steiner's Theorem.

1 Introduction

A mixtilinear incircle of a triangle ABC is a circle that in internally tangent to two sides of a triangle and also internally tangent to the circumcircle of the triangle (figure 1). Some interesting properties of mixtilinear incircles, as well as proof of their existence and uniqueness can be found in [1].

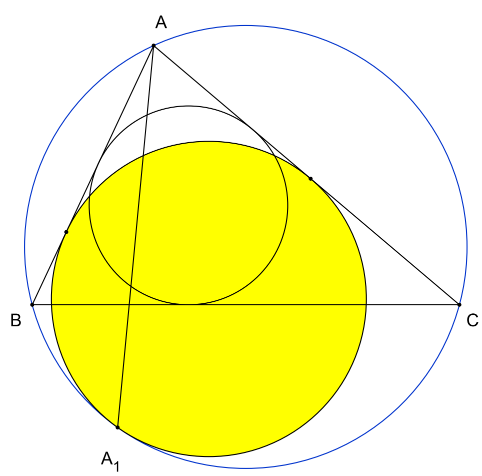

Figure 1

Every triangle has three unique mixtilinear incircles, one corresponding to each vertex. The mixtilinear incircle of a triangle ABC tangent to the two sides containing vertex A is called the A-mixtilinear incircle. Similarly, we have B-mixtilinear incircle and C-mixtilinear incircle for the vertices B and C (figure 2). The points of contact of the mixtilinear incircles with the circumcircle are A_1, B_1 and C_1.

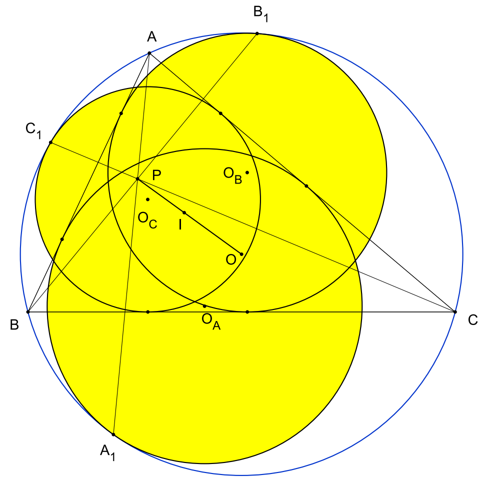

Figure 2

In this article we will deal with four special points of the mixtilinear incircles: the points of tangency of the mixtilinear incircles with the circumcircle and the point of concurrence of the lines that unite each vertex and the points of tangency of its mixtilinear incircles. We will show identities that gives us the distance between those points and any point on the plane that contains the triangle. For that, we make use of properties of the isogonal cevians of the triangle.

2 Notation

Let ABC be an acute triangle. We denote its side-lengths by $BC = a$, $AC = b$, $AB = c$, its semi perimeter by $s = \frac{1}{2}(a+b+c)$, its area by F, its circumradius by R and inradius by r. Its classical centers are the Incenter I and the Circumcenter O.

We will need also the following relations

(a) $F = \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{abc}{4R} = sr.$

(b) $-a(s-b)(s-c) + b^2(s-c) + c^2(s-b) = s[(b-c)^2 + a(s-a)]$

(c) $a^2(s-c) - b(s-a)(s-c) + c^2(s-a) = s[(a-c)^2 + b(s-b)]$

(d) $a^2(s-b) + b^2(s-a) - c(s-a)(s-b) = s[(a-b)^2 + c(s-c)]$

(e) $a^2(s-b)(s-c) + b^2(s-a)(s-c) + c^2(s-a)(s-b) = 4rs^2(R-r)$

3 Definitions

1. Isogonal Cevians

In a triangle ABC the cevians AE and AD (E, $D \in BC$) which are symmetric with respect to the angle's $\measuredangle BAC$ bisector are called isogonal cevians, otherway said, if AE and AD are isogonal cevians then $\measuredangle BAE \equiv \measuredangle CAD$ (See figure 3).

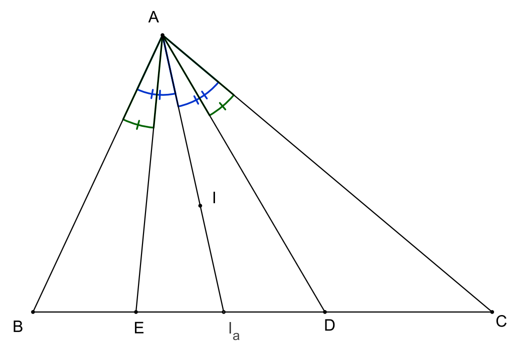

Figure 3

2. External Center of Similitude of Circumcircle and Incircle The incircle and circumcircle of a triangle ABC have two similitude centers, the internal similitude center and the external center of similitude. The external center of similitude of the circumcircle and incircle is the isogonal conjugate of the Nagel point of triangle ABC. It is Kimberling center $X(56)$ and has equivalent triangle center functions [9].

4 Basic Lemma

Lemma 4.1 *Let D the contact point of the A-excircle with BC (see figure figure 4). Then, $\measuredangle BAA_1 = \measuredangle DAC$, in the other words, AA_1 and AD are isogonal with respect to triangle ABC.*

Proof: The proof of above lemma can be found in [1].

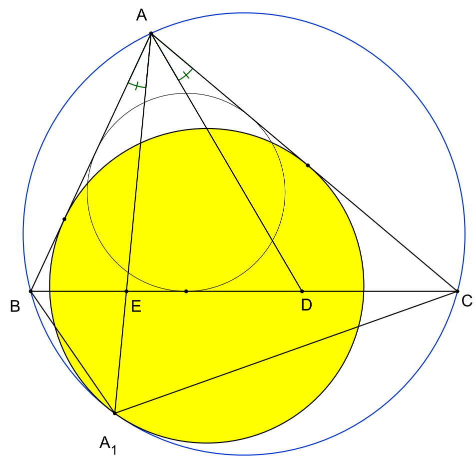

Figure 4

Lemma 4.2 *If two lines containing two chords AB and CD of a circle (O) intersect at P (see figure figure 5), then*

$$PA.PB = PC.PD \tag{1}$$

Proof: The proof of above lemma can be found in [7].

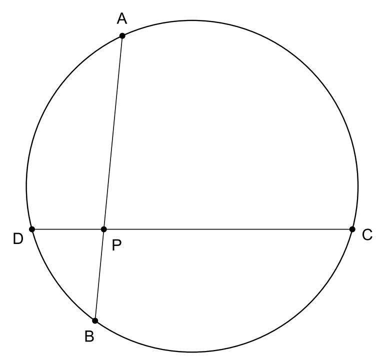

Figure 5

5 Theorems

Theorem 5.1 (Steiner) *If in the triangle ABC, AD and AE are Isogonal Cevians, D, E are points on BC then:*

$$\frac{BD}{CD} \cdot \frac{BE}{CE} = \left(\frac{AB}{AC}\right)^2.$$

Proof: Applying the law of sines, we have:

$$\frac{BD}{AB} = \frac{\sin(\measuredangle BAD)}{\sin(\measuredangle ADB)} \quad \text{and} \quad \frac{CD}{AC} = \frac{\sin(\measuredangle CAD)}{\sin(\measuredangle ADC)} = \frac{\sin(\measuredangle CAD)}{\sin(\measuredangle ADB)}.$$

From this,

$$\frac{BD}{CD} = \frac{AB}{AC}\frac{\sin(\measuredangle BAD)}{\sin(\measuredangle CAD)}. \tag{2}$$

Similarly,

$$\frac{BE}{CE} = \frac{AB}{AC}\frac{\sin(\measuredangle BAE)}{\sin(\measuredangle EAC)} \tag{3}$$

Using the expressions (2) and (3), we get

$$\frac{BD}{CD} \cdot \frac{BE}{CE} = \left(\frac{AB}{AC}\right)^2. \tag{4}$$

Hence proved

Theorem 5.2 *The lines AA_1, BB_1, CC_1 are concurrent at the external center of similitude of circumcircle and the incircle P (see figure 2).*

Proof: The proof of above lemma can be found in [8].

6 Prepositions

Theorem 6.1 *Let M be any point in the plane of a triangle ABC and A_1, B_1 and C_1 are the points of contact of the A, B and C-mixtilinear incircles, respectively, with the circumcenter. Then:*

$$\mathbf{MA_1^2} = \frac{\mathbf{1}}{\mathbf{s[(b-c)^2 + a(s-a)]}} \cdot [\mathbf{-a(s-b)(s-c)MA^2 + b^2(s-c)MB^2 + c^2(s-b)MC^2}]. \tag{5}$$

$$\mathbf{MB_1^2} = \frac{\mathbf{1}}{\mathbf{s[(c-a)^2 + b(s-b)]}} \cdot [\mathbf{a^2(s-c)MA^2 - b(s-a)(s-c)MB^2 + c^2(s-a)MC^2}]. \tag{6}$$

$$\mathbf{MC_1^2} = \frac{\mathbf{1}}{\mathbf{s[(a-b)^2 + c(s-c)]}} \cdot [\mathbf{a^2(s-b)MA^2 + b^2(s-a)MB^2 - c(s-a)(s-b)MC^2}]. \tag{7}$$

Proof: For proving the above said result, we will use of the lemma 4.1 and the theorem 5.1 in the triangle ABC (see figure 6).

We know that D is the contact point of the A-excircle, then $BD = s - c$ and $CD = s - b$. Now using (4), we have

$$\frac{BE}{CE} = \frac{c^2}{b^2} \cdot \frac{(s-b)}{(s-c)}.$$

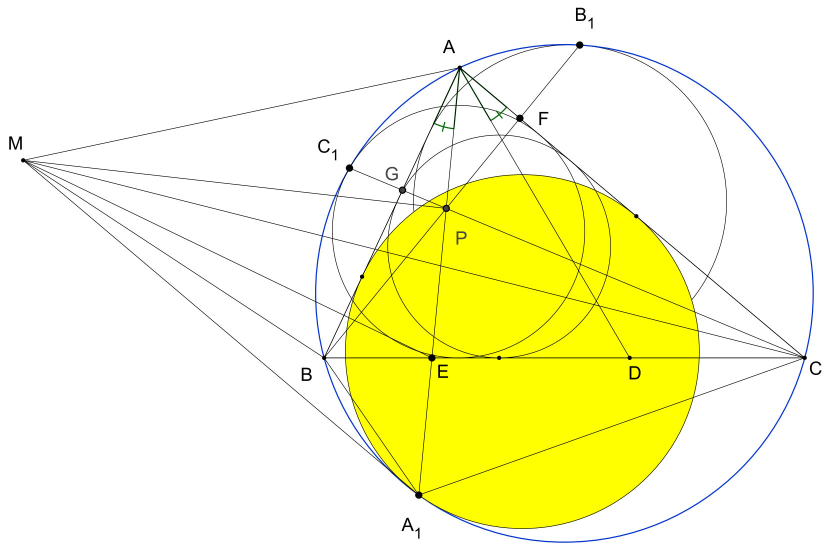

Figure 6

Now let $BE + CE = a$, it implies that

$$BE = \frac{ac^2(s-b)}{b^2(s-c)+c^2(s-b)} \text{ and } CE = \frac{ab^2(s-c)}{b^2(s-c)+c^2(s-b)}.$$

Similarly, we can prove that:

$$CF = \frac{ba^2(s-c)}{a^2(s-c)+c^2(s-a)}, AF = \frac{bc^2(s-a)}{a^2(s-c)+c^2(s-a)}, AG = \frac{cb^2(s-a)}{a^2(s-b)+b^2(s-a)} \text{ and } BG = \frac{ca^2(s-b)}{a^2(s-b)+b^2(s-a)}.$$

By Stewart's theorem in the triangle ABC in which AE is a cevian, we get

$$AC^2 \cdot BE + AB^2 \cdot CE - AE^2 \cdot BC = BC \cdot BE \cdot CE$$

$$\frac{ab^2c^2(s-b)}{b^2(s-c)+c^2(s-b)} + \frac{ab^2c^2(s-c)}{b^2(s-c)+c^2(s-b)} - aAE^2 = \frac{a^3b^2c^2(s-b)(s-c)}{[b^2(s-c)+c^2(s-b)]^2}$$

It implies

$$AE^2 = \frac{ab^2c^2}{[b^2(s-c)+c^2(s-b)]^2} \cdot [b^2(s-c)+c^2(s-b)-a(s-b)(s-c)] \tag{8}$$

Now, using the lemma 4.2, we get

$$AE.EA_1 = BE.CE \Longrightarrow AE^2.EA_1^2 = BE^2.CE^2$$

By replacing we get

$$\frac{ab^2c^2}{[b^2(s-c)+c^2(s-b)]^2} \cdot [b^2(s-c)+c^2(s-b)-a(s-b)(s-c)] \cdot EA_1^2 = \frac{a^4b^4c^4(s-b)^2(s-c)^2}{[b^2(s-c)+c^2(s-b)]^4}$$

$$EA_1^2 = \frac{a^3b^2c^2(s-b)^2(s-c)^2}{[b^2(s-c)+c^2(s-b)]^2[b^2(s-c)+c^2(s-b)-a(s-b)(s-c)]} \tag{9}$$

Using (8) and (9), we get

$$\frac{AE}{EA_1} = \frac{[b^2(s-c)+c^2(s-b)-a(s-b)(s-c)]}{a(s-b)(s-c)} \tag{10}$$

$$\frac{AE}{EA_1}+1 = \frac{[b^2(s-c)+c^2(s-b)]}{a(s-b)(s-c)} \tag{11}$$

Now, applying the Stewart's theorem in the triangles MBC and MAA_1 (see figure 6) in which cevian ME, we have

$$MB^2 \cdot CE + MC^2 \cdot BE - ME^2 \cdot BC = BC \cdot CE \cdot BE \tag{12}$$

$$MA_1^2 \cdot AE + MA^2 \cdot EA_1 - ME^2 \cdot AA_1 = AA_1 \cdot AE \cdot EA_1 \tag{13}$$

Using (12) and replacing BE and CE, we get

$$ME^2 = \frac{b^2(s-c)\cdot MB^2 + c^2(s-b)\cdot MC^2}{[b^2(s-c)+c^2(s-b)]} - \frac{a^2b^2c^2(s-b)(s-c)}{[b^2(s-c)+c^2(s-b)]^2} \tag{14}$$

Now, using (13) and considering that $AA_1 = AE + EA_1$, we get

$$MA_1^2 \cdot AE + MA^2 \cdot EA_1 - ME^2 \cdot (AE + EA_1) = AE \cdot EA_1 \cdot (AE + EA_1)$$

$$MA^2 + MA_1^2 \cdot \frac{AE}{EA_1} - ME^2 \cdot (\frac{AE}{EA_1}+1) = AE^2 + AE \cdot EA_1 \tag{15}$$

Combining the lemma 4.2 with (8), (10), (11), (14), (15) and after simplifying a few steps we obtain,

$$MA_1^2 = \frac{1}{[-a(s-b)(s-c)+b^2(s-c)+c^2(s-b)]} \cdot [-a(s-b)(s-c)MA^2 + b^2(s-c)MB^2 + c^2(s-b)MC^2].$$

From (b), we obtain

$$MA_1^2 = \frac{1}{s[(b-c)^2+a(s-a)]} \cdot [-a(s-b)(s-c)MA^2 + b^2(s-c)MB^2 + c^2(s-b)MC^2].$$

Similarly, we can prove (6) and (7).

Preposition 6.2 *The external center of similitude of circumcircle and incircle P of the triangle ABC divides each cevian in the ratio given by*

$$\frac{AP}{PE} = \frac{(s-a)[b^2(s-c)+c^2(s-b)]}{a^2(s-b)(s-c)} \tag{16}$$

$$\frac{BP}{PF} = \frac{(s-b)[a^2(s-c)+c^2(s-a)]}{b^2(s-a)(s-c)} \tag{17}$$

$$\frac{CP}{PG} = \frac{(s-c)[a^2(s-b)+b^2(s-a)]}{c^2(s-a)(s-b)} \tag{18}$$

Proof: Let AE, CG and BF are cevians of triangle ABC. In the triangle ABE the line CG as transversal. Applying Menelaus' Theorem we have

$$\frac{AG}{BG} \cdot \frac{BC}{CE} \cdot \frac{PE}{AP} = 1$$

By replacing the all known relations and by little algebra, we get the conclusion (16).

In the similar manner we can prove the conclusions (17) and (18).

Theorem 6.3 *Let M be any point in the plane of a triangle ABC and P the concurrence point of the lines AA_1, BB_1 and CC_1, then*

$$\mathbf{MP^2} = \frac{\mathbf{1}}{\mathbf{4rs^2(R-r)}} \cdot [\mathbf{a^2(s-b)(s-c)MA^2} + \mathbf{b^2(s-a)(s-c)MB^2} + \mathbf{c^2(s-a)(s-b)MC^2}] - \frac{\mathbf{R^2r^2}}{\mathbf{(R-r)^2}}. \tag{19}$$

Proof: Using the expression (16) and considering that $AE = AP + PE$, we get

$$\frac{PE}{AE} = \frac{a^2(s-b)(s-c)}{a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b)} \tag{20}$$

$$\frac{AP}{AE} = \frac{(s-a)[b^2(s-c)+c^2(s-b)]}{a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b)} \tag{21}$$

Applying the Stewart's theorem in the triangles MAE (see figure 6) in which cevian MP, we have

$$MA^2 \cdot PE + ME^2 \cdot AP - MP^2 \cdot AE = AE \cdot AP \cdot PE$$

$$MA^2 + ME^2 \cdot \frac{AP}{PE} - MP^2 \cdot (\frac{AP}{PE} + 1) = AP^2 + AP \cdot PE$$

Now, using (8), (14), (16), (20), (21), (a) and algebraic manipulation, we get

$$MP^2 = \frac{1}{a^2(s-b)(s-c)+b^2(s-a)(s-c)+c^2(s-a)(s-b)} \cdot [a^2(s-b)(s-c)MA^2+b^2(s-a)(s-c)MB^2+c^2(s-a)(s-b)MC^2] - \frac{R^2r^2}{(R-r)^2}.$$

And by using (e) we can prove the conclusion of (19)

7 Main Result

Corollary 7.1 *Let A_1, B_1 and C_1 are the Points of contact of the A, B and C-mixtilinear incircles, respectively, with the circumcircle of the triangle ABC, then*

$$\mathbf{AA_1^2} = \frac{\mathbf{ab^2c^2}}{\mathbf{s[(b-c)^2+a(s-a)]}}, \quad \mathbf{BA_1^2} = \frac{\mathbf{ac^2(s-b)^2}}{\mathbf{s[(b-c)^2+a(s-a)]}} \quad \textit{and} \quad \mathbf{CA_1^2} = \frac{\mathbf{ab^2(s-c)^2}}{\mathbf{s[(b-c)^2+a(s-a)]}}. \tag{22}$$

$$\mathbf{AB_1^2} = \frac{\mathbf{bc^2(s-a)^2}}{\mathbf{s[(c-a)^2+b(s-b)]}}, \quad \mathbf{BB_1^2} = \frac{\mathbf{a^2bc^2}}{\mathbf{s[(c-a)^2+b(s-b)]}} \quad \textit{and} \quad \mathbf{CB_1^2} = \frac{\mathbf{a^2b(s-c)^2}}{\mathbf{s[(c-a)^2+b(s-b)]}}. \tag{23}$$

$$\mathbf{AC_1^2} = \frac{\mathbf{b^2c(s-a)^2}}{\mathbf{s[(a-b)^2+c(s-c)]}}, \quad \mathbf{BC_1^2} = \frac{\mathbf{a^2c(s-b)^2}}{\mathbf{s[(a-b)^2+c(s-c)]}} \quad \textit{and} \quad \mathbf{CC_1^2} = \frac{\mathbf{a^2b^2c}}{\mathbf{s[(a-b)^2+c(s-c)]}}. \tag{24}$$

Proof: For proving (22) we using the theorem 6.1, replacing M by the A and consider $AA = 0$, $AB = c$ and $AC = b$, then

$$MA_1^2 = \frac{1}{s[(b-c)^2+a(s-a)]} \cdot [-a(s-b)(s-c)AA^2 + b^2(s-c)AB^2 + c^2(s-b)AC^2].$$

$$MA_1^2 = \frac{1}{s[(b-c)^2+a(s-a)]} \cdot [c^2b^2(s-c) + b^2c^2(s-b)].$$

Hence

$$AA_1^2 = \frac{ab^2c^2}{s[(b-c)^2+a(s-a)]}$$

By replacing M by B and C in (5) we can arrive at the required conclusions of (22).

In the similar manner, using (6) and (7), we can prove the conclusion (23) and (24).

Corollary 7.2 *Be I the Incenter of the triangle ABC and A_1, B_1 and C_1 are the points of contact of the A, B and C-mixtilinear incircles, respectively, with the circumcircle, then*

$$\mathbf{IA_1^2} = \frac{\mathbf{abc(s-b)(s-c)}}{\mathbf{s[(b-c)^2+a(s-a)]}}, \quad \mathbf{IB_1^2} = \frac{\mathbf{abc(s-a)(s-c)}}{\mathbf{s[(c-a)^2+b(s-b)]}} \quad \textit{and} \quad \mathbf{IC_1^2} = \frac{\mathbf{abc(s-a)(s-b)}}{\mathbf{[(a-b)^2+c(s-c)]}}. \tag{25}$$

Proof: In Theorem 6.1, replace in (22) M by the incenter I. We get

$$IA_1^2 = \frac{1}{s[(b-c)^2 + a(s-a)]} \cdot [-a(s-b)(s-c)IA^2 + b^2(s-c)IB^2 + c^2(s-b)IC^2].$$

Now, we know that

$$IA^2 = \frac{bc(s-a)}{s}, \qquad IB^2 = \frac{ac(s-b)}{s} \quad \text{and} \quad IC^2 = \frac{ab(s-c)}{s}.$$

Then,

$$IA_1^2 = \frac{1}{s[(b-c)^2 + a(s-a)]} \cdot [-a(s-b)(s-c) \cdot \frac{bc(s-a)}{s} + b^2(s-c) \cdot \frac{ac(s-b)}{s} + c^2(s-b) \cdot \frac{ab(s-c)}{s}].$$

$$IA_1^2 = \frac{abc(s-b)(s-c)}{s^2[(b-c)^2 + a(s-a)]} \cdot [-(s-a) + b + c].$$

Hence,

$$IA_1^2 = \frac{abc(s-b)(s-c)}{s[(b-c)^2 + a(s-a)]}$$

In the similar manner, using (6) and (7), we can prove the relations IB_1 and IC_1.

Corollary 7.3 *Be O the circumcenter of the triangle ABC and A_1, B_1 and C_1 are the points of contact of the A, B and C-mixtilinear incircles, respectively, with the circumcircle, then*

$$\mathbf{OA_1 = OB_1 = OC_1 = R} \tag{26}$$

Proof: In Theorem 6.1, replace in (5), (6) and (7) M by the circumcenter O, and consider that $OA = OB = OC = R$, we get conclusion (26).

Corollary 7.4 *Be I the Incenter of the triangle ABC and P the external center of similitude of circumcircle and the incircle, then*

$$\mathbf{IP^2 = \frac{Rr^2(R-2r)}{(R-r)^2}} \tag{27}$$

Proof: In Theorem 6.3, replace M by the incenter I. We get

$$IP^2 = \frac{1}{4rs^2(R-r)} \cdot [a^2(s-b)(s-c)IA^2 + b^2(s-a)(s-c)IB^2 + c^2(s-a)(s-b)IC^2] - \frac{R^2r^2}{(R-r)^2}.$$

$$IP^2 = \frac{1}{4rs^2(R-r)} \cdot [a^2(s-b)(s-c)\frac{bc(s-a)}{s} + b^2(s-a)(s-c)\frac{ac(s-b)}{s} + c^2(s-a)(s-b)\frac{ab(s-c)}{s}] - \frac{R^2r^2}{(R-r)^2}.$$

$$IP^2 = \frac{abc(s-a)(s-b)(s-c)}{2rs^2(R-r)} - \frac{R^2r^2}{(R-r)^2}.$$

Now using (a), we get

$$IP^2 = \frac{2Rr^2}{(R-r)} - \frac{R^2r^2}{(R-r)^2}.$$

Hence,

$$IP^2 = \frac{Rr^2(R-2r)}{(R-r)^2}.$$

Corollary 7.5 *Be O the circumcenter of the triangle ABC and P is center of similitude of the circumcircle and the incircle, then*

$$\mathbf{OP^2} = \frac{\mathbf{R^3(R-2r)}}{\mathbf{(R-r)^2}} \tag{28}$$

Proof: In Theorem 6.3, replace M by the circumcenter O, and consider $OA = OB = OC = R$. We get

$$OP^2 = \frac{R^2}{4rs^2(R-r)} \cdot [a^2(s-b)(s-c) + b^2(s-a)(s-c) + c^2(s-a)(s-b)] - \frac{R^2r^2}{(R-r)^2}.$$

Using the expression (e), then

$$OP^2 = R^2 - \frac{R^2r^2}{(R-r)^2}.$$

Hence,

$$OP^2 = \frac{R^3(R-2r)}{(R-r)^2}.$$

8 Conclusion

In this the current paper we proved metric relations of the main points of the mixtilinear circles. To arrive at the result, we use a propertie of the isogonal cevians of the triangle as the main tool. Using these metric relations we can find the distance between the these points and other notable centers of the triangle, as well as investigate interesting properties of the mixtilinear circles. The proofs presented here only require basic knowledge of geometry and its manipulation and application.

References

[1] BACA, Jafet. "On Mixtilinear Incircles". Mathematical Reflections, 2020.

[2] DE QUEIROZ FILHO, José Ferreira.***GERGONNE'S POINT AND OUTSTANDING DISTANCES***.2021. www.ssmrmh.ro.

[3] DE QUEIROZ FILHO, José Ferreira.***THE BEVAN'S POINT AND ITS SPECIAL RELATIONSHIPS***.2021. www.ssmrmh.ro.

[4] HONSBERGER, R. ***Episodes in Nineteenth and Twentieth Century Euclidean Geometry***. The Mathematical Association of America, 1995.

[5] KIMBERLING, C. ***Encyclopedia of Triangle Centers***. Available online: http://faculty.evansville.edu/ck6/encyclopedia/ETC.html (accessed on 20/08/2021).

[6] ***ROMANIAN MATHEMATICAL MAGAZINE*** - Interactive Journal, www.ssmrmh.ro.

[7] YIU, Paul. "Notes on Euclidean Geometry". Version of 2005, http://math.fau.edu/yiu/PSRM2015/yiu/New%20Folder%20(4)/geometry2005/g2005notes.pdf

[8] YIU, Paul. "Mixtilinear Incircles".*The American Mathematical Monthly*, Vol 106, N° 10 (Dec., 1999) pp 952-955.

[9] WEISSTEIN, Eric W. "***External Similitude Center.***" From MathWorld-A Wolfram Web Resource.https://mathworld.wolfram.com/ExternalSimilitudeCenter.html