Trigonometric Ratio In Olympiad Geometry

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Abstract

When solving olympiad geometry problems, we often need to simplify the hypothesis from complex things to easy ones, and ratio is a very useful tool to do this work. Furthermore, it can even make the given statement equivalent to something trivial (like the Law of Sines, etc.).

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dedicated to my family and my form teacher for their never-ending inspiration

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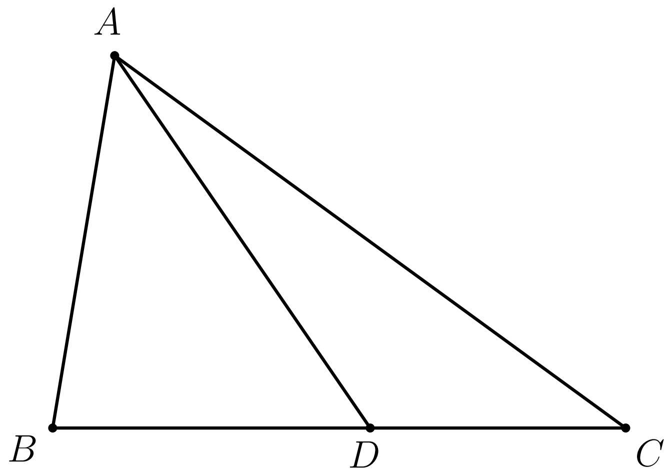
1 Theorems and properties

1. Given a triangle ABC, with $D \in BC$, we have: $\dfrac{DB}{DC} = \dfrac{AB}{AC}.\dfrac{sin\angle DAB}{sin\angle DAC}$

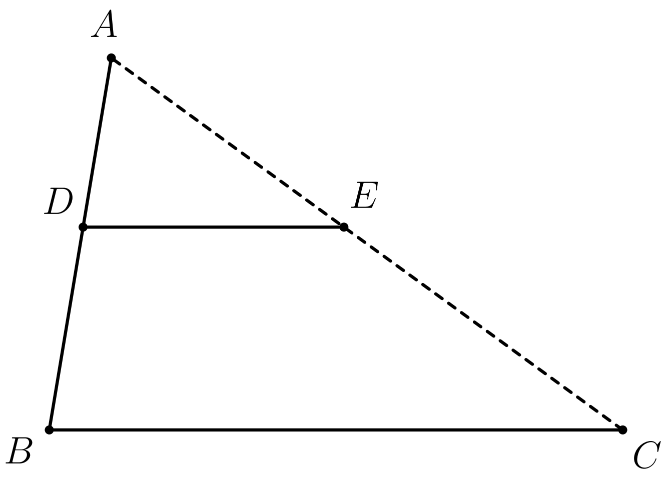


Proof. We have: $\dfrac{DB}{DC} = \dfrac{[ABD]}{[ACD]} = \dfrac{\frac{1}{2}AB.AD.sin\angle DAB}{\frac{1}{2}AC.AD.sin\angle DAC} = \dfrac{AB}{AC}.\dfrac{sin\angle DAB}{sin\angle DAC}.$

− *This is a fundamental yet crucial property if you want to utilize the ratio tool to solve a geometry problem since it links the segment lengths to the trigonometric expression of angles. If you like dealing with geometry length, think about whether there are any unique configurations, such as obtuse triangles, opposite directed segments, etc. Otherwise, your answer might not be correct.*

To establish collinearity, we have some ideas:

2. Given a triangle ABC, $D \in AB$, $DE \parallel BC$, we have: $E \in AC \Leftrightarrow \dfrac{\overline{DE}}{\overline{BC}} = \dfrac{\overline{AD}}{\overline{AB}}$

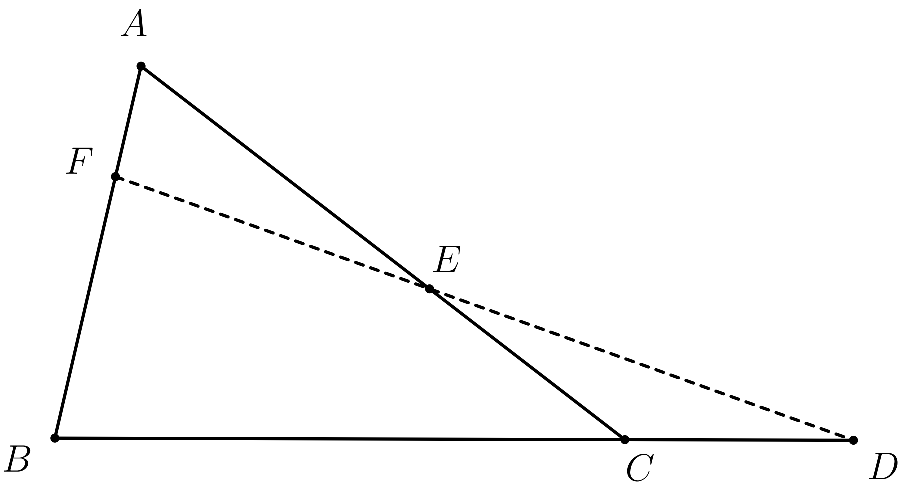


Hint. Let AE intersect BC at C'. Prove that $C \equiv C'$.

This can be used in conjunction with the Thales theorem to prove collinearity in issues where there are a number of parallel lines, which is sometimes very helpful.

3. Menelaus and **Ceva** theorem

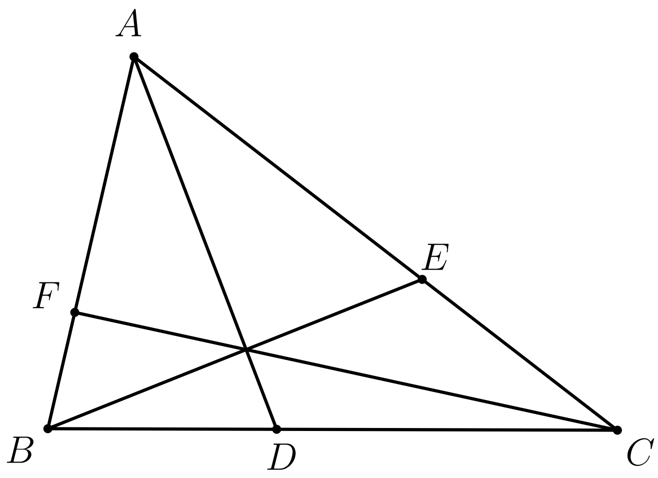
• **Menelaus theorem**



Given a triangle ABC. The points D, E, F lie on the lines BC, CA, AB, respectively. From here D, E, F are collinear if and only if:

$$\frac{\overline{FA}}{\overline{FB}} \cdot \frac{\overline{DB}}{\overline{DC}} \cdot \frac{\overline{EC}}{\overline{EA}} = 1$$

- **Ceva theorem**



Given a triangle ABC, the points D, E, F lie on BC, CA, AB, respectively. So AD, BE, CF are concurrent if and only if:

$$\frac{\overline{FA}}{\overline{FB}} \cdot \frac{\overline{DB}}{\overline{DC}} \cdot \frac{\overline{EC}}{\overline{EA}} = -1$$

Please pay close attention to the trigonometric version of the Ceva Theorem (it's important!) AD, BE, CF are concurrent if and only if:

$$\frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} = 1$$

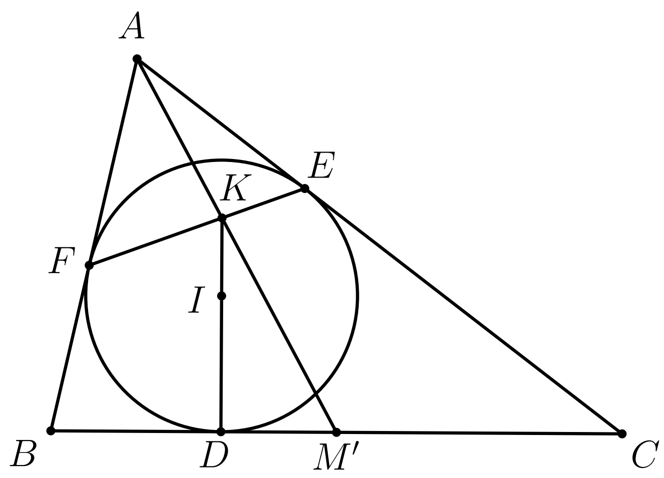
As I previously stated, ratio expressions allow us to connect the segment lengths and trig version of angles. You can review numbers 1 and 3 to have a better understanding of this connection.

Let's look at some instances to see how **1,2,3** perform in order to have a better understanding of how trigonometric ratio transforming works (Practice is the key to learning a new skill).

2 Examples

Example 1. Given a triangle ABC, denote (I) by the incircle, and it is tangent to BC, CA, AB at D, E, F.

Prove that: EF, ID and the A-median of $\triangle ABC$ are concurrent.



By letting ID cross EF at K and demonstrating that AK passes through the midpoint of BC, we may determine the relationship between segment lengths and trignometry without having to prove that these three lines are concurrent directly.

Solution. Let ID intersect EF at K, and AK intersect BC at M', we will prove that M' is the midpoint of BC.

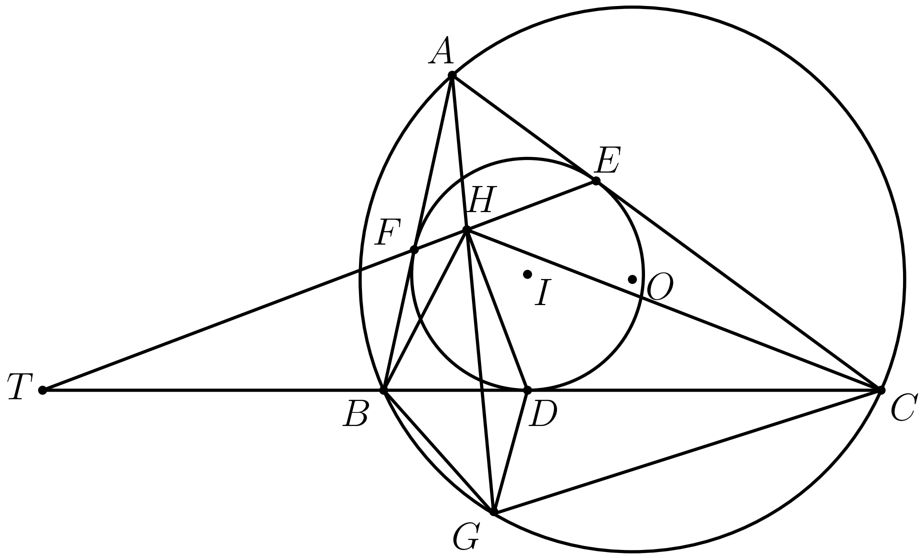
We have: $\dfrac{KF}{KE} = \dfrac{AF}{AE}.\dfrac{\sin FAK}{\sin EAK} = \dfrac{\sin FAK}{\sin EAK} = \dfrac{\sin BAM'}{\sin CAM'} = \dfrac{AC}{AB}.\dfrac{BM'}{CM'}$

On the other hand: $\dfrac{KF}{KE} = \dfrac{IE}{IF}.\dfrac{\sin FIK}{\sin EIK} = \dfrac{\sin FIK}{\sin EIK} = \dfrac{\sin ABC}{\sin ACB} = \dfrac{AC}{AB}$

So we have: $\dfrac{AC}{AB}.\dfrac{BM'}{CM'} = \dfrac{AC}{AB} \implies M'B = M'C \implies M'$ is the midpoint of BC $\square$

Example 2. Given $\triangle ABC$ inscribed in (O), (I) be the incircle. The incircle touch BC, CA, AB at D, E, F , respectively. H is a point on EF such that: DH is perpendicular to EF. The line AH intersects (O) the second time at G.

Prove that: GD is the angle bisector of $\angle BGC$.



So we need to prove that $\frac{GB}{GC} = \frac{DB}{DC}$*, and let's think,* $\frac{GB}{GC}$ *can be expressed in the form of* $\frac{\sin BAG}{\sin CAG}$*, and of course* $\frac{DB}{DC}$ *is too simple for us.*

Solution. Let EF intersect BC at T.

We have: $(TD, BC) = -1$, but $HD \perp HT$

$\implies HD$ is the internal bisector of $\angle BHC$

$\implies \frac{DB}{DC} = \frac{HB}{HC}$.

We also have a common properties: $\triangle BHF \sim \triangle CHE$, so by **Example 1**:

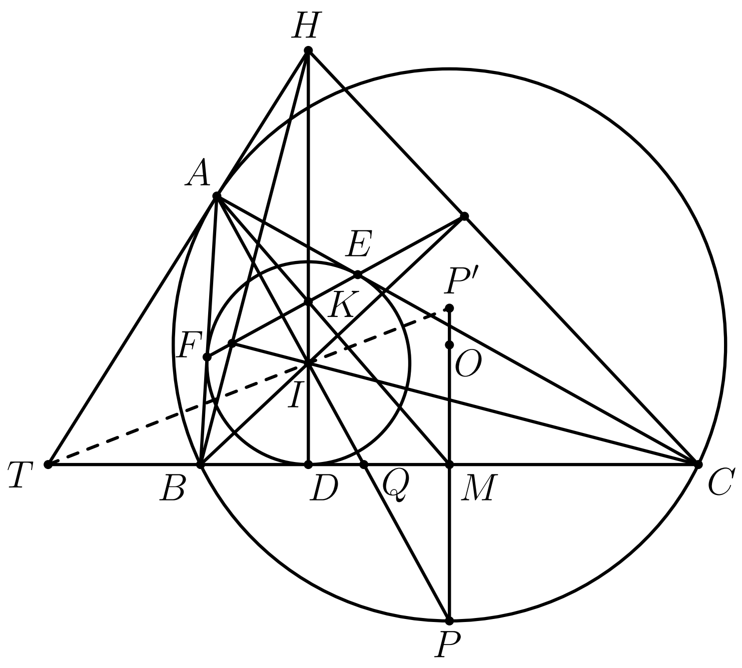
$$\frac{DB}{DC} = \frac{HB}{HC} = \frac{HF}{HE} = \frac{\sin FAH}{\sin EAH} = \frac{\sin BAG}{\sin CAG} = \frac{\sin BCG}{\sin CBG} = \frac{GB}{GC}$$

And then we are done.

Remark. To prove that $\triangle BHF \sim \triangle CHE$, we choose points M, N on EF such that BM, CN are both perpendicular to EF, then we need $\triangle BFM \sim \triangle CEN$

Then from $\frac{BF}{CE} = \frac{BM}{CN} = \frac{BD}{CD} = \frac{BH}{CH}$, $\triangle BHF \sim \triangle CHE$ is easily established by $S - A - S$ □

Example 3. Let I be the incenter of $\triangle ABC$. P is the midpoint of arc BC that does not contain A. P' is the reflection point of P through BC. H is the orthocenter of $\triangle BIC$. Prove that: AH, IP', BC are concurrent.



*Like **Example 2**, we just let AH intersects BC at T, and prove that T, I, P' are collinear, but notice that $ID \parallel P'M$, consequently, this brings to mind Property **2** as from the beginning..*

Solution. Let AH intersects BC at T, ID intersects EF at K, and AI intersects BC at Q.

We need to prove that: $\dfrac{TD}{TM} = \dfrac{ID}{P'M}$, but $\dfrac{ID}{P'M} = \dfrac{QD}{QM}$.

This means we want to have:$\dfrac{TD}{TM} = \dfrac{QD}{QM}$, or $(TQ, DM) = -1$. By the result of **Example 1**, we have: A, K, M are collinear.

Take A as the projection center: $A(TQ, DM) = A(HI, DK) = -1$ (Since BI and CH intersect on EF, and CI and BH intersect on EF).

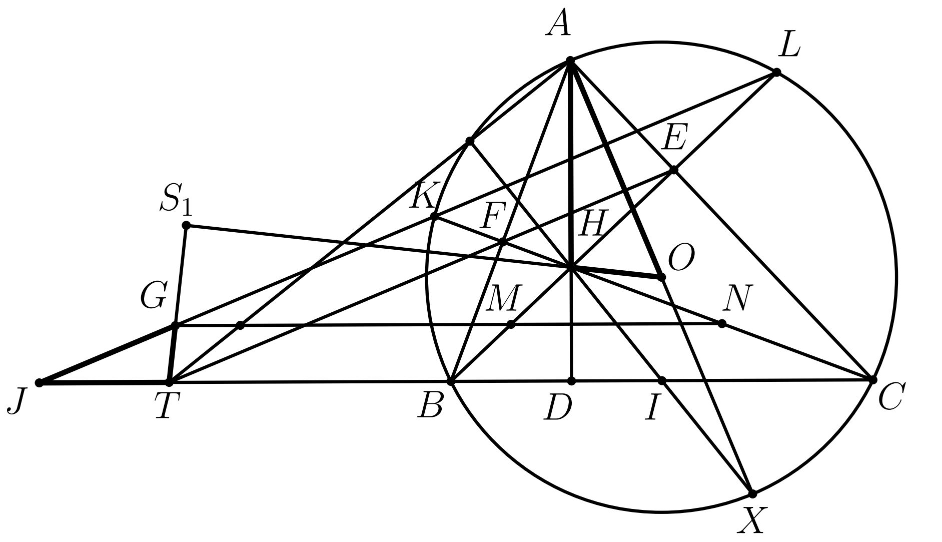
In conclusion: We have $(TQ, DM) = -1$, and that results in the desired goal.

To demonstrate the strength of this tool, let's try some more challenging problems:

Example 4. Given an acute triangle ABC inscribed in (O) with H be the orthocenter. AH, BH, CH cuts BC, CA, AB at D, E, F and I, M, N are the midpoints of BC, HB, HC. BH, CH cuts (O) at $L, K(L \neq B, K \neq C)$; KL cuts MN at G.

1. We choose a point T on EF such that $AT \perp IH$. Prove that: $GT \perp OH$.

2. DE, DF cut MN at P, Q, respectively. Let S be the intersection point of BQ, CP. Prove that: HS bisects EF.



Solution.

1) Let S_1 be the intersection point of GT and HO, according to the hypothesis, AT goes through the intersection point of (AH) and (O), and T lies on EF

So by the Radical axis theorem for $(AH), (O), (EFBC)$: $T \in BC$.

We have: $\frac{JG}{JK} = \frac{CN}{CK}; \frac{JT}{JC} = \frac{KF}{KC}$(By Thales therem)

$$\implies \frac{JG}{JT}.\frac{JC}{JK} = \frac{CN}{CK}.\frac{CK}{KF} = \frac{CN}{KF}$$

$$\implies \frac{JG}{JT} = \frac{CN}{KF}.\frac{JK}{JC} = \frac{CN}{KF}.\frac{KB}{LC} \text{ (Since } \triangle JKB \sim \triangle JCL) = \frac{CN}{KF}.\frac{KH}{LH} = \frac{CH}{KH}.\frac{KH}{LH} = \frac{CH}{LH}$$

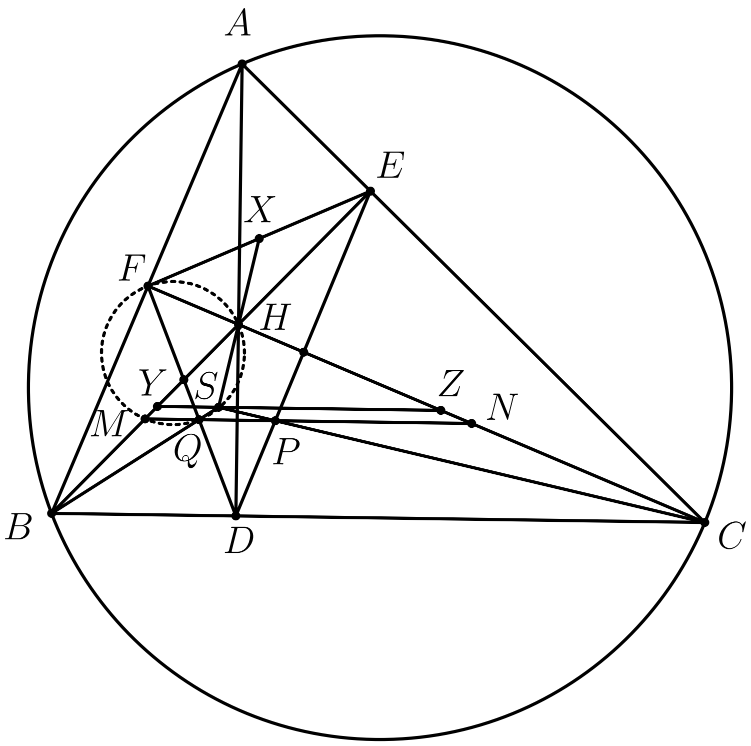
Now we will prove that: $\triangle JGT \sim \triangle AHO$

But we already have: $\angle HAO = \angle TJG$ (Because $AO \perp JL$ and $AD \perp JD$), so it seems sense to prove that: $\frac{AH}{AO} = \frac{JG}{JT}$, or $\frac{AL}{AO} = \frac{CH}{LH}$ $(AL = AH)$, and it's true since $\triangle AOL \sim \triangle HCL$.

So: $\triangle JGT \sim \triangle AHO \implies \angle AHO = \angle JTG \implies TS_1HD$ is a cyclic quadrilateral $\square$

2) This appears to be an impossible question to answer using a trigonometric ratio, right?

But let's give it a try, though.



To support your ideas, you may occasionally need to construct objects like parallel and perpendicular lines. Keep in mind that ratios are not always obvious for you to transform.

Solution.

2) Let X be the intersection point of HS and EF. We will prove that X is the midpoint of EF.

We have: $\dfrac{XE}{XF} = \dfrac{HE}{HF}.\dfrac{\sin EHX}{\sin FHX} = \dfrac{HE}{HF}.\dfrac{\sin BHS}{\sin CHS}$, and $\dfrac{SY}{SZ} = \dfrac{\sin BHS}{\sin CHS}.\dfrac{HY}{HZ}$

$\implies \dfrac{XE}{XF} = \dfrac{SY}{SZ}.\dfrac{HE}{HF}.\dfrac{HZ}{HY} = \dfrac{SY}{SZ}.\dfrac{HE}{HF}.\dfrac{HC}{HB}$

So all we need to do is to prove that:

$$\frac{SY}{SZ}.\frac{HE}{HF}.\frac{HC}{HB} = 1$$

Indeed, $\dfrac{SY}{MQ} = \dfrac{BS}{BQ}; \dfrac{SZ}{NP} = \dfrac{CS}{CP}$ (By Thales Theorem) $\implies \dfrac{SY}{SZ}.\dfrac{NP}{MQ} = \dfrac{BS}{BQ}.\dfrac{CP}{CS} = 1$ (Thales theorem again) $\implies \dfrac{SY}{SZ} = \dfrac{MQ}{NP}$

Similar to the "trick" above: $\dfrac{MQ}{MH} = \dfrac{\sin MHQ}{\sin MQH}; \dfrac{NH}{NP} = \dfrac{\sin NPH}{\sin NHP}$

$\implies \dfrac{MQ}{NP}.\dfrac{NH}{MH} = \dfrac{\sin MHQ}{\sin MQH}.\dfrac{\sin NPH}{\sin NHP}$

We easily have: $\triangle HPQ$ is isosceles $\implies \angle MQH = \angle NPH$

$\implies \dfrac{MQ}{NP}.\dfrac{NH}{MH} = \dfrac{\sin MHQ}{\sin NHP}$, but notice that: $\angle MHQ = \angle MDF = \angle BDF - \angle MDB = \angle BHF - \angle MBD = \angle BHF - \angle HFE = \angle HEF$

And similarly: $\angle NHP = \angle HFE$

So $\dfrac{MQ}{NP}.\dfrac{NH}{MH} = \dfrac{\sin HEF}{\sin HFE} \implies \dfrac{SY}{SZ} = \dfrac{MQ}{NP} = \dfrac{MH}{NH}.\dfrac{\sin HEF}{\sin HFE} = \dfrac{HB}{HC}.\dfrac{\sin HEF}{\sin HFE}$

So the thing that we need for this problem is deduced to:

$$\frac{HB}{HC}.\frac{\sin HEF}{\sin HFE}.\frac{HE}{HF}.\frac{HC}{HB} = 1$$

which is trivial by the Law of Sines.

Remark on section 2.

– The skill you must master is computing the difficult-to-bash terms using the way in which they can cancel one another out while multiplying.

– Sound ethereal? Try to solve more problems using ratio transforming, and after each problem you solve, think about the sentence above again!

3 Training problems

1.(MEMO 2016) Let ABC be an acute triangle for which $AB \neq AC$, and let O be its circumcenter. Line AO meets the circumcircle of ABC again in D, and the line BC in E. The circumcircle of CDE meets the line CA again in P. The lines PE and AB intersect in Q. Line passing through O parallel to the line PE intersects the A-altitude of ABC at F.
Prove that: $FP = FQ$.

2.(ISL 2015) Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J.
Prove that: $IJ = AH$.

3.(CHKMO 2021) Let $ABCD$ be a cyclic quadrilateral inscribed in Γ such that $AB = AD$. Let E be a point on the segment CD such that $BC = DE$. The line AE intersect Γ again at F. The chords AC and BF meet at M. Let P be the symmetric point of C about M.
Prove that: PE and BF are parallel.

4.(Adapted from VIASM Summer Camp 2022) Given an acute not isosceles triangle ABC inscribed in (O). Choose points M, N, P on AB, BC, CA such that $AMNP$ is a parallelogram. The segment CM cuts NP at E, the segment BP cuts NM at F. The segment BE cuts CF at D.
Prove that: A, D, N are collinear.

5.(Iranian TST 2020) Given a triangle ABC with circumcircle Γ. Points E and F are the foot of angle bisectors of B and C, I is incenter and K is the intersection of AI and EF. Suppose that T be the midpoint of arc BAC. Circle Γ intersects the A-median and circumcircle of AEF for the second time at X and S. Let S' be the reflection of S across AI and J be the second intersection of circumcircle of $AS'K$ and AX.

Prove that: the quadrilateral $TJIX$ is cyclic.

6.(Baltic Way 2021) Let D be the foot of the A-altitude of an acute triangle ABC. The internal bisector of the angle DAC intersects BC at K. Let L be the projection of K onto AC. Let M be the intersection point of BL and AD. Let P be the intersection point of MC and DL.

Prove that: PK is perpendicular to AB.

7.(Vietnamese Northern Delta And Coastal Area Olympiad 2022) Given a triangle ABC inscribed in (O), $AB < AC$ and the internal angle bisector of $\angle BAC$ intersects $BC, (O)$ at D, E, respectively. M is the midpoint of AD. BM intersects (O) at P ($P \neq B$). EP cuts AC at N.

1. Prove that: N is the midpoint of AC
2. Assume that (EMN) cuts BM at R ($R \neq M$).

   Prove that: $RA \perp RC$.

8.(Vietnamese HSGS Olympiad 2022) Given a non-isosceles triangle ABC, (I) is its incircle. (I) touch BC, CA, AB at D, E, F, respectively. The lines IB, IC cuts EF at M, N, respectively. Choose points P, Q on IC, IB such that MP and NQ are perpendicular to BC.

1. Prove that: BP, CQ, ID are concurrent at a point, let its name T.
2. Let ID intersects PQ at Y, EP intersects FQ at Z.

   Prove that: EQ, FP, AT, YZ are concurrent.

9.(Iranian TST 2021) In acute scalene triangle ABC the external angle bisector of $\angle BAC$ meet BC at point X.Lines l_b and l_c which tangents of B and C with respect to (ABC).The line pass through X intersects l_b and l_c at points Y and Z respectively. Let $(AYB) \cap (AZC) = N$ and $l_b \cap l_c = D$.

Prove that: ND is angle bisector of $\angle YNZ$.

10.(Iranian TST 2017) In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E, F lie on AB, AC respectively such that $BE = PC$,$CF = PB$. Let K be the intersection point of AP, OH.

Prove that: $\angle EKF = 90°$

11.(IMO 2010) Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S.

Prove that: if $SC = SP$, then $MK = ML$.

12.(ISL 2012) Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, C to sides BC, CA, AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively.

Prove that: $MP = NQ$.

13.(Sharygin 2013) The altitudes AA_1, BB_1, CC_1 of an acute triangle ABC concur at H. The perpendicular lines from H to B_1C_1, A_1C_1 meet rays CA, CB at P, Q respectively.

Prove that: the line from C perpendicular to A_1B_1 passes through the midpoint of PQ.

14.(ISL 2017) Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$.

Prove that: the perpendicular line from E to BC and the lines AC and BD are concurrent.

Bibliography

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[2] Trinh Quoc Khanh's Math Olympiad Blog: https://llwys.blogspot.com

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