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100 OLD AND NEW INEQUALITIES AND IDENTITIES IN TRIANGLE

By Bogdan Fuștei-Romania

In memory of TRAN HONG-Vietnam

We consider the triangle ABC with well-known results:

$$\cos^2\frac{A}{2}=\frac{1+\cos A}{2}\text{(and analogs);}$$

$$\cos^2\frac{A}{2}=\frac{r_b+r_c}{4R}\text{(and analogs);}$$

$$r_a=\frac{S}{p-a}\text{(and analogs);}$$

$$a=2R\sin A\text{ (and analogs);}$$

$$\sin\frac{A}{2}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}=\sqrt{\frac{r_a-r}{4R}}\text{ (and analogs);}$$

$$r=\frac{S}{p}\ ;2p=a+b+c;$$

$$\frac{1+\cos A}{2}=\frac{r_b+r_c}{4R}\to\cos A=\frac{r_b+r_c-2R}{2R}=\frac{r_a+r_b+r_c-2R-r_a}{2R}$$

$$r_a+r_b+r_c=4R+r\ ;$$

We obtain the next identity : $\cos A=\frac{2R+r-r_a}{2R}$ (and analogs);

$$\to\cos B+\cos C=\frac{r_a+r}{2R}\to\frac{\cos B+\cos C}{\sin A}=\frac{r_a+r}{a}\text{ (and analogs) (1)}$$

$$r_a+r=\frac{S}{p-a}\frac{(2p-a)}{p}=r_a\frac{b+c}{p}\to\frac{r_a+r}{r_a}=\frac{b+c}{p}\text{ (and analogs) (2)}$$

$$2S=ah_a=bh_b=ch_c=2pr\ ;\ r_a=\frac{ah_a}{2(p-a)}=\frac{ah_a}{b+c-a}\to$$

$$\frac{r_a}{h_a}=\frac{a}{b+c-a}\text{ (and analogs) (3)}$$

We will obtain

$$\frac{b+c}{a}=1+\frac{h_a}{r_a}\text{ (and analogs) (4)}$$

From (2) and (4) we obtain

$$\frac{r_a+r}{a}=\frac{r_a+h_a}{p}\ \textbf{(and analogs) (5)}$$

From (1) and (5) we obtain

$$\frac{\cos B+\cos C}{\sin A}=\frac{r_a+h_a}{p}\ \text{(and analogs)}(6)$$

We know that : $\frac{R}{r}\geq\frac{n_a+h_a}{h_a}$ (and analogs)**([1])** ;

$$\frac{R}{r}=\frac{a}{2r}\,\frac{1}{\sin A}.$$

After simplification we obtain this new inequality :

$$\frac{1}{\sin A}\geq\frac{n_a+h_a}{p}\ \textbf{(7)}\times(\cos B+\cos C)\ \rightarrow\ \frac{\cos B+\cos C}{\sin A}\geq\frac{n_a+h_a}{p}\ \textbf{(}\cos B+\cos C\textbf{) (8)}$$

From (6) and (8) we obtain :

$$\frac{r_a+h_a}{n_a+h_a}\geq\cos B+\cos C\ \ (9)\ \text{(and analogs)}$$

Its easy to see that $\cos A+\cos B+\cos C=\frac{R+r}{R}$ and using (9) we will obtain a new inequality :

$$\sum\frac{r_a+h_a}{n_a+h_a}\geq\frac{2(R+r)}{R}\ \textbf{(10)}$$

Now we use the inequality: $\frac{R}{r}\geq\frac{m_b}{h_c}+\frac{m_c}{h_b}$ **([2])** and $\frac{R}{r}=\frac{a}{2r}\,\frac{1}{\sin A}$ (and analogs) $\times(\cos B++\cos C)$ and we obtain : $\frac{a}{2r}\,\frac{r_a+h_a}{p}\geq(\cos B+\cos C)\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)$

$2S=ah_a=bh_b=ch_c=2pr$,we get the next relationship :

$\frac{a}{a}\,\frac{r_a+h_a}{h_a}\geq(\cos B+\cos C)\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)$, so we will get another inequality:

$$\frac{r_a+h_a}{h_a}\geq(\cos B+\cos C)\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)\textbf{(11)}\text{(and analogs)}$$

We know that: $\sin^2\frac{A}{2}=\frac{r_a-r}{4R}=\frac{r}{2R}\,\frac{r_a}{h_a}$ (and analogs)$\rightarrow\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogs)

After sumation we obtain $\sum\frac{r_a}{h_a}=\frac{2R-r}{r}\rightarrow 3+\sum\frac{r_a}{h_a}=2\left(1+\frac{R}{r}\right)$

From (11) we obtain this inequality:

$$1+\frac{R}{r}\geq\frac{1}{2}\sum(\cos B+\cos C)\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)\ \textbf{(12)}$$

Also $\frac{r_a r_b r_c}{h_a h_b h_c} = \frac{R}{2r}$ and from (11) we obtain a new inequality:

$$\frac{R}{2r} \geq \prod\left[\left(\cos B + \cos C\right)\left(\frac{m_b}{h_c} + \frac{m_c}{h_b}\right) - 1\right] \mathbf{(13)}$$

$\frac{R}{2r} \geq \frac{m_a}{h_a}$ (and analogs)(Panaitopol); $\frac{R}{r} = \frac{a}{2r} \frac{1}{\sin A}$,after some simple manipulations we obtain a new inequality :

$$r_a + h_a \geq 2m_a(\cos B + \cos C) \textbf{(and analogs) (14)}$$

From (4) and (11) we obtain a new inequality :

$$\frac{b+c}{a} \geq \frac{2m_a}{r_a}(\cos B + \cos C)(\text{and analogs}) \mathbf{(15)}$$

From (14) we have $\frac{r_a + h_a}{2m_a} \geq \cos B + \cos C$ and after sumation and using

$\cos A + +\cos B + \cos C = \frac{R+r}{R}$,we will obtain this next inequality :

$$\sum \frac{r_a + h_a}{m_a} \geq \frac{4(R+r)}{R} \mathbf{(16)}$$

From $\frac{r_a + h_a}{2m_a} \geq \cos B + \cos C$ and (9) we have :

$$(r_a + h_a)\left(\frac{1}{2m_a} + \frac{1}{n_a + h_a}\right) \geq 2(\cos B + \cos C) \textbf{ (and analogs) (17)}$$

2S=$ah_a = bh_b$ =ch_c=2pr=(a+b+c)r→ $\frac{h_a}{r} = 1 + \frac{b+c}{a}$ (and analogs) and using (15) we obtain a new inequality:

$$\frac{h_a + h_b + h_c - 3r}{2r} \geq \sum \frac{m_a}{r_a}(\cos B + \cos C) \mathbf{(18)}$$

From $r_a + r = \frac{S}{p-a} \frac{(2p-a)}{p}$ =$r_a \frac{b+c}{p}$ and $r_a - r = \frac{r_a}{p} a$, we obtain this new relationship

$$\frac{r_a + r}{r_a - r} = \frac{b+c}{a} \textbf{ (and analogs) (19)}$$

and using (15) we obtain the next inequality:

$$\frac{r_a + r}{r_a - r} \geq \frac{2m_a}{r_a}(\cos B + \cos C)(20)$$

$\frac{h_a}{r} = 1 + \frac{b+c}{a}$ (and analogs) and using (15) we obatin another inequality:

$$\frac{h_a - r}{r} \geq \frac{2m_a}{r_a}(\cos B + \cos C)(21)$$

$$\rightarrow \frac{r_a}{2r} \geq \frac{m_a}{h_a - r}(\cos B + \cos C)\ (22)$$

and after sumation we will get a fresh inequality:

$$\frac{4R+r}{2r} \geq \sum \frac{m_a}{h_a - r}(\cos B + \cos C)(23)$$

We know that this next relation is true :$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$, $\cos B + \cos C = \frac{r_a + r}{2R} \rightarrow \frac{\cos B + \cos C}{r_a} = \frac{1}{2R}\frac{b+c}{p}$ (and analogs) and after sumation we obtain this new identity:

$$\sum \frac{\cos B + \cos C}{r_a} = \frac{2}{R}\ (24)$$

from $\frac{h_a - r}{m_a r} \geq \frac{2(\cos B + \cos C)}{r_a}$ (and analogs) and after sumation we will obtain a new relationship :

$$\sum \frac{h_a - r}{m_a} \geq \frac{4r}{R}\ (25)$$

Now we will use Tereshin inequality: $m_a \geq \frac{b^2 + c^2}{4R}$ (and analogs) and the identity $bc = 2Rh_a$ (and analogs) and we will obtain a new inequality:

$\frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{b}{c} + \frac{b}{c}\right)$ (and analogs) ,after sumation we obtain the inequality:

$\sum \frac{m_a}{h_a} \geq \frac{1}{2}\sum \frac{b+c}{a}$ and from (15) we get:

$$\frac{1}{2}\sum \frac{b+c}{a} \geq \sum \frac{m_a}{r_a}(\cos B + \cos C)\ (26)$$

From these last two inequalitys we obtain another inequality:

$$\sum \frac{m_a}{h_a} \geq \sum \frac{m_a}{r_a}(\cos B + \cos C)\ (27)$$

We know that $\cos\frac{B-C}{2} = \frac{h_a}{l_a}$ (and analogs) and $\cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2}$ (and analogs) we obtain $\frac{h_a}{l_a} = \frac{b+c}{a}\sin\frac{A}{2}$ and from $\sin\frac{A}{2} = \sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}$ (and analogs) we obtain an identity:

$$\frac{b+c}{a} = \sqrt{\frac{2R}{r}\frac{h_a}{l_a}}\sqrt{\frac{h_a}{r_a}}\text{(and analogs)}(28)$$

and from (19) we obtain the next identity :

$$\mathbf{\frac{r_a+r}{r_a-r}=\sqrt{\frac{2R}{r}\frac{h_a}{l_a}}\sqrt{\frac{h_a}{r_a}}(\text{and analogs})(29)}$$

$$\rightarrow \mathbf{\frac{r_a+r}{h_a}=\sqrt{\frac{2R}{r}}\frac{(r_a-r)}{l_a}\sqrt{\frac{h_a}{r_a}}\ (30)}$$

We will use the well known identity: $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}, \sum\frac{r_a}{h_a}=\frac{2R-r}{r}$

and we will obtain a new result:

$$\mathbf{\sum\frac{r_a+r}{h_a}=\frac{2R-r}{r}+1=\frac{2R}{r}\ (31)}$$

From (30) and (31) we get :

$$\mathbf{\sum\frac{(r_a-r)}{l_a}\sqrt{\frac{h_a}{r_a}}=\sqrt{\frac{2R}{r}}\ (32)}$$

From (4) and (28) we have :$\frac{r_a+h_a}{r_a}=\sqrt{\frac{2R}{r}\frac{h_a}{l_a}}\sqrt{\frac{h_a}{r_a}}$ and after some simple manipulation we obtain a new result :

$$\mathbf{1+\frac{r_a}{h_a}=\sqrt{\frac{2R}{r}}\frac{\sqrt{r_ah_a}}{l_a}\ (33)}\ \text{(and analogs)}$$

From (14) we get $1+\frac{r_a}{h_a}\geq\frac{2m_a}{h_a}(\cos B+\cos C)$ and from (33) we obtain:

$$\mathbf{\sqrt{\frac{R}{2r}}\frac{\sqrt{r_ah_a}}{l_a}\geq\frac{m_a}{h_a}(\cos B+\cos C)\ (34)}\ .$$

From (11) and (33) we obtain a new result:

$$\mathbf{\sqrt{\frac{2R}{r}}\frac{\sqrt{r_ah_a}}{l_a}\geq(\cos B+\cos C)\left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)\ (35)}\text{(and analogs).}$$

From $1+\frac{r_a}{h_a}\geq\frac{2m_a}{h_a}(\cos B+\cos C)$(and analogs) after sumation we obtain:

$3+\sum\frac{r_a}{h_a}=2\left(1+\frac{R}{r}\right)\geq2\sum\frac{m_a}{h_a}(\cos B+\cos C)$ and after simplification we get:

$$\mathbf{1+\frac{R}{r}\geq\sum\frac{m_a}{h_a}(\cos B+\cos C)\ (36)}\ .$$

From (33) we have :

$3+\sum\frac{r_a}{h_a}=2(1+\frac{R}{r})=\sqrt{\frac{2R}{r}}\sum\frac{\sqrt{r_ah_a}}{l_a}$ and after simplification we obtain the identity:

$$\sum\frac{\sqrt{\mathbf{r_ah_a}}}{\mathbf{l_a}}=(\mathbf{1}+\frac{\mathbf{R}}{\mathbf{r}})\sqrt{\frac{\mathbf{2r}}{\mathbf{R}}}=\sqrt{\mathbf{2}}\left(\sqrt{\frac{\mathbf{R}}{\mathbf{r}}}+\sqrt{\frac{\mathbf{r}}{\mathbf{R}}}\right)\ \mathbf{(37)}$$

From (37) and $h_a\le l_a$ (and analogs) $\sin\frac{A}{2}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}$ (and analogs), we obtain the well known inequality:

$$\sum\sin\frac{\mathbf{A}}{\mathbf{2}}\ge\cos\mathbf{A}+\cos\mathbf{B}+\cos\mathbf{C}=\mathbf{1}+\frac{\mathbf{r}}{\mathbf{R}}$$

From (4) and (15) we obtain the inequality:

$$\mathbf{3}+\sum\frac{\mathbf{h_a}}{\mathbf{r_a}}\ge\mathbf{2}\sum\frac{\mathbf{m_a}}{\mathbf{r_a}}(\cos\mathbf{B}+\cos\mathbf{C})\ \mathbf{(38)}$$

(Van-Aubel) If AD, BE and CF are three cevianes concurent in a point P inside to triangle ABC, then: $\frac{\mathbf{AP}}{\mathbf{PD}}=\frac{\mathbf{AF}}{\mathbf{FB}}+\frac{\mathbf{AE}}{\mathbf{EC}}$**.**

If N_a is the Nagel point in triangle ABC ,applying Van-Aubel theorem,we have:

$\frac{AN_a}{n_a-AN_a}=\frac{p-c}{p-a}+\frac{p-b}{p-a}=\frac{a}{p-a}\to\frac{n_a-AN_a}{AN_a}=\frac{p-a}{a}\to\frac{\mathbf{n_a}}{\mathbf{AN_a}}=\frac{\mathbf{p}}{\mathbf{a}}$ **(and analogs)**

$$\mathbf{AN_a}=\frac{\mathbf{an_a}}{\mathbf{p}}\ \textbf{(and analogs)}\to\mathbf{1}+\frac{\mathbf{b+c}}{\mathbf{a}}=\frac{\mathbf{2n_a}}{\mathbf{AN_a}}\ \textbf{(and analogs) (39)}$$

$\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs)and useing (39) we obtain:

$$\frac{\mathbf{n_a}}{\mathbf{h_a}}=\frac{\mathbf{AN_a}}{\mathbf{2r}}\ \textbf{(and analogs)(40)}$$

$p^2=n_a^2+2r_ah_a$(and analogs)([**1**]), $2S=ah_a=bh_b=ch_c=2pr$

$\frac{a}{2r}=\frac{p}{h_a}$ (and analogs) $\to\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+2\frac{r_a}{h_a}$ (and analogs), $\frac{r_a}{h_a}=\frac{r_a-r}{2r}$

But $(r_a-r)r=(p-b)(p-c)-r^2$

$$4p(p-a)=\frac{4(a+b+c)(b+c-a)}{4}=(a+b+c)(b+c-a)=(b+c)^2-a^2$$

$$a^2=(b+c)^2-4p(p-a)\ \text{(and analogs).}$$

$\frac{a^2}{4r^2}=\frac{n_a^2}{h_a^2}+2\frac{r_a}{h_a}$ becomes $\frac{(b+c)^2-4p(p-a)}{4r^2}=\frac{n_a^2}{h_a^2}+\frac{4(p-b)(p-c)-4r^2}{4r^2}$

$$\frac{n_a^2}{h_a^2}-1=\frac{(b+c)^2-4p(p-a)-4(p-b)(p-c)}{4r^2}\text{(and analogs)}$$

$$4(p-b)(p-c)=(a+c-b)(a+b-c)=a^2+2bc-b^2-c^2,$$

$$4p(p-a)+4(p-b)(p-c)=(b+c)^2-a^2+a^2+2bc-b^2-c^2=4bc$$

$\frac{n_a^2}{h_a^2}-1=\frac{(b+c)^2-4bc}{4r^2}=\frac{(b-c)^2}{4r^2}\rightarrow$

$$\frac{n_a^2}{h_a^2}=1+\frac{(b-c)^2}{4r^2}\ \text{(and analogs)}(41)$$

From (40) and (41) we obtain :

$$AN_a=\sqrt{4r^2+(b-c)^2}\ \text{(and analogs) (42)}$$

From (39) and (42) we obtain :

$$1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ \text{(and analogs) (43)}$$

From (40) and (42) we obtain :

$$\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}=\sqrt{1+\frac{(b-c)^2}{4r^2}}\ \text{(and analogs) (44)}$$

From (43) after sumation we obtain the identity:

$$3+\sum\frac{b+c}{a}=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ (45)$$

From (4) and (45) after sumation we obtain :

$$6+\sum\frac{h_a}{r_a}=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ (46)$$

From (4) and (43) we obtain :

$$\frac{h_a}{r_a}=\frac{b+c-a}{a}=2\left(\frac{n_a}{\sqrt{4r^2+(b-c)^2}}-1\right)\text{(and analogs) (47)}$$

From (47) we obtain :

$$\sqrt{4r^2+(b-c)^2}=\frac{2n_ar_a}{2r_a+h_a}\text{(and analogs) (48)}$$

From (48) after sumation we obtain:

$$\sum\frac{\sqrt{4r^2+(b-c)^2}}{r_a}=\sum\frac{2n_a}{2r_a+h_a}\ \textbf{(49)}$$

We start from well known inequality :$2m_a\geq(b+c)\cos\frac{A}{2}$ and after some manipulation we obtain :$1+\frac{2m_a}{a\cos\frac{A}{2}}\geq 1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$,and we obtain:

$$\frac{1}{2}\geq\frac{n_a}{\sqrt{4r^2+(b-c)^2}}-\frac{m_a}{a\cos\frac{A}{2}}$$ **(and analogs) (50)** and we obtain:

$$\frac{1}{8}\geq\prod\left(\frac{n_a}{\sqrt{4r^2+(b-c)^2}}-\frac{m_a}{a\cos\frac{A}{2}}\right)\ \textbf{(51)}$$

From (19) we have :$1+\frac{b+c}{a}=\frac{2r_a}{r_a-r}$ and from (43) we obtain :

$$\frac{n_a}{r_a}=\frac{\sqrt{4r^2+(b-c)^2}}{r_a-r}$$ **(and analogs) (52)**

and after sumation we obtain:

$$\sum\frac{n_a}{r_a}=\sum\frac{\sqrt{4r^2+(b-c)^2}}{r_a-r}\ \textbf{(53)}$$

but $r_a-r=4R\sin^2\frac{A}{2}$ (and analogs) and we obtain:

$$\textbf{4R}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)=\sum\frac{\sqrt{4r^2+(b-c)^2}}{\sin^2\frac{A}{2}}\ \textbf{(54)}$$

$r_a-r=\frac{r_a\sqrt{4r^2+(b-c)^2}}{n_a}$ (and analogs); $r_a+r_b+r_c=4R+r$

We obtain a new relationship :

$$\textbf{2R}-\mathbf{r}=\frac{1}{2}\sum\frac{r_a\sqrt{4r^2+(b-c)^2}}{n_a}\ \textbf{(55)}$$

$1+\frac{b+c}{a}\geq 1+\frac{2m_a}{r_a}(\cos B+\cos C)$, and using (43) we obtain:

$$\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\geq 1+\frac{2m_a}{r_a}(\cos B+\cos C)$$ **(and analogs) (56)**

$\sum\frac{m_a}{h_a}\geq\frac{1}{2}\sum\frac{b+c}{a}$, and useing (43) we obtain a new inequality :

$$\frac{3}{2}+\sum\frac{m_a}{h_a}\geq\sum\frac{n_a}{\sqrt{4r^2+(b-c)^2}}\ \textbf{(57)}$$

From $\frac{m_a}{h_a}\geq\frac{1}{2}(\frac{b}{c}+\frac{b}{c})$ (and analogs) after sumation we obtain:

$2(\frac{m_b}{h_b}+\frac{m_c}{h_c})\geq\frac{b+c}{a}+\frac{a}{c}+\frac{a}{b}$ and $\frac{a}{c}=\frac{h_c}{h_a}$ (and analogs) we obtain:

$2(\frac{m_b}{h_b}+\frac{m_c}{h_c})\geq\frac{b+c}{a}+\frac{h_b+h_c}{h_a}$ and using (43) we obtain :

$$2(\frac{m_b}{h_b}+\frac{m_c}{h_c}-\frac{n_a}{\sqrt{4r^2+(b-c)^2}})\geq\frac{h_b+h_c-h_a}{h_a}\ \textbf{(and analogs) (58)}$$

$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})=3+\sum\frac{b+c}{a}$ and using (43) we obtain:

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ \textbf{(59)}$$

From $\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs), $1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$ (and analogs), $\frac{1}{h_a}+\frac{1}{h_b}++\frac{1}{h_c}=\frac{1}{r}$ we obtain a new identity:

$$(h_a+h_b+h_c)(\frac{1}{h_a}+\frac{1}{h_b}++\frac{1}{h_c})=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ \textbf{(60)}$$

From $p^2=n_a^2+2r_ah_a$(and analogs) $\rightarrow p^2-n_a^2=(p+n_a)(p-n_a)=2r_ah_a$

$\frac{a}{2r}=\frac{p}{h_a}$ (and analogs) $\rightarrow\frac{a}{2r}=\frac{n_a}{h_a}+\frac{2r_a}{n_a+p}$ and $\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs) we obtain $\frac{r_a}{n_a+p}$ $=\frac{a-\sqrt{4r^2+(b-c)^2}}{4r}$ **(and analogs) (61)**

and after sumation we obtain:

$$\frac{p}{2r}=\sum\frac{r_a}{n_a+p}+\sum\sqrt{1+\frac{(b-c)^2}{4r^2}}\ \textbf{(62)}$$

from (61) we have :

$\frac{n_a+p}{4r}=\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}$ **(and analogs)** and after sumation we obtain a new result :

$$\frac{n_a+n_b+n_c+3p}{4r}=\sum\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}\ \textbf{(63)}$$

We use the well know inequality $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$ and $\frac{n_a+p}{r_a}=\frac{4r}{a-\sqrt{4r^2+(b-c)^2}}$ **(and analogs)** we obtain a new result:

$$\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}+\frac{p}{r}=4\sum\frac{r}{a-\sqrt{4r^2+(b-c)^2}}\ (64)$$

$$\frac{(n_a+p)(n_b+p)(n_c+p)}{64r^3}=\prod\frac{r_a}{a-\sqrt{4r^2+(b-c)^2}}\ (65)$$

but $\sqrt{r_b r_c}\geq l_a$ (and analogs)→$r_a r_b r_c\geq l_a l_b l_c$ and we obtain:

$$\frac{(n_a+p)(n_b+p)(n_c+p)}{64r^3}\geq\prod\frac{l_a}{a-\sqrt{4r^2+(b-c)^2}}\ (66)$$

$p\geq 3\sqrt{3}r$ (Mitrinovic inequality) and with (64) we obtain a new inequality:

$$3\sqrt{3}+\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\leq 4\sum\frac{r}{a-\sqrt{4r^2+(b-c)^2}}\ (67)$$

$\frac{p}{r}\geq\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$(***Ion Cristian Miu-rafinament of Mitrinovic inequality***)

and from (64) we obtain a new inequality:

$$\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}+\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\leq 4\sum\frac{r}{a-\sqrt{4r^2+(b-c)^2}}\ (68)$$

Now we use :

$m_a l_a\geq r_b r_c=p(p-a)$(and analogs)(Panaitopol)→$\sqrt{m_a m_b m_c l_a l_b l_c}\geq r_a r_b r_c$ and with (65) we obtain a new inequality :

$$\frac{(n_a+p)(n_b+p)(n_c+p)}{64r^3}\leq\prod\frac{\sqrt{m_a l_a}}{a-\sqrt{4r^2+(b-c)^2}}\ (69)$$

$\frac{(n_b+p)(n_c+p)}{16r^2}=\frac{r_b r_b}{[b-\sqrt{(a-c)^2+4r^2}][c-\sqrt{(a-b)^2+4r^2}}$ (and analogs) and using $m_a l_a\geq r_b r_c$ we obtain **:**

$$\frac{(n_b+p)(n_c+p)}{16r^2}\leq\frac{m_a}{[b-\sqrt{(a-c)^2+4r^2}]}\frac{l_a}{[c-\sqrt{(a-b)^2+4r^2}]}\ (70)$$

Is easy to proof that :$(b-c)^2=4(m_a^2-r_b r_c)$ (and analogs)

$AN_a=\sqrt{4r^2+(b-c)^2}$ (and analogs)→ $AN_a=\sqrt{4r^2+4(m_a^2-r_b r_c)}$

$$AN_a=2\sqrt{m_a^2+r^2-r_b r_c}\text{(and analogs)(71)}$$

$$1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\to 1+\frac{b+c}{a}=\frac{n_a}{\sqrt{m_a^2+r^2-r_b r_c}}\text{ (and analogs) (72)}$$

$$\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}\to\frac{n_a}{h_a}=\frac{\sqrt{m_a^2+r^2-r_b r_c}}{r}\text{ (and analogs) (73)}$$

$$3+\sum\frac{b+c}{a}=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\rightarrow 3+\sum\frac{b+c}{a}=\sum\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}} \text{ (and analogs) (74)}$$

$$6+\sum\frac{h_a}{r_a}=\sum\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\rightarrow 6+\sum\frac{h_a}{r_a}=\sum\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}} \text{ (and analogs) (75)}$$

$$\frac{h_a}{r_a}=\frac{b+c-a}{a}=2\left(\frac{n_a}{\sqrt{4r^2+(b-c)^2}}-1\right)$$

$$\frac{h_a}{r_a}=\frac{b+c-a}{a}=\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}}-2 \text{ (and analogs)(76)}$$

$$\sqrt{4r^2+(b-c)^2}=\frac{2n_ar_a}{2r_a+h_a}\rightarrow\sqrt{m_a^2+r^2-r_br_c}=\frac{n_ar_a}{2r_a+h_a} \text{(and analogs) (77)}$$

$$\sum\frac{\sqrt{4r^2+(b-c)^2}}{r_a}=\sum\frac{2n_a}{2r_a+h_a}\rightarrow\sum\frac{\sqrt{m_a^2+r^2-r_br_c}}{r_a}=\sum\frac{n_a}{2r_a+h_a} \text{ (and analogs) (78)}$$

If triangle ABC is acuteangled then we have ERDOS Inequality :

$R+r\le\max(h_a,h_b,h_c)$ (RMM-Famous Inequalitys Marathon 1-100,inequality 31)**[3]**

If $h_a=\max(h_a,h_b,h_c)\rightarrow\frac{h_a}{r}=1+\frac{b+c}{a}\ge\frac{R+r}{r}$ and $1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$ we obtain a new inequality :

$$\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}\ge\frac{R+r}{r} \text{ (79)}$$

and using (72) we obtain an equivalent form :

$$\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}}\ge\frac{R+r}{r} \text{ (80)}$$

Now we will use this inequality,true for every triangle ABC:

$m_a+m_b+m_c\le 2(R-2r)+h_a+h_b+h_c$ (Jian Liu **[4]**)

$\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs) $\rightarrow\frac{m_a+m_b+m_c}{r}\le 2\left(\frac{R}{r}-2\right)+3+\sum\frac{b+c}{a}$

$1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}$ (analogs) $\rightarrow\frac{m_a+m_b+m_c}{r}\le 2\left(\frac{R}{r}-2\right)+2\sum\frac{n_a}{\sqrt{4r^2+(b-c)^2}}$ and we obtain a new inequality :

$$\frac{m_a+m_b+m_c}{2r}\le\frac{R}{r}-2+\sum\frac{n_a}{\sqrt{4r^2+(b-c)^2}} \text{ (81)}$$

Now we start from the next inequality: $n_a + g_a \geq 2m_a$ (and analogs)**[5]**

And we obtain: $2n_a \geq 2(2m_a - g_a)$

$$\rightarrow 1+\frac{b+c}{a} \geq \frac{2(2m_a-g_a)}{\sqrt{4r^2+(b-c)^2}} \text{ (and analogs) (82)}$$

$$1+\frac{b+c}{a} = \frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}} \rightarrow 1+\frac{b+c}{a} \geq \frac{2m_a-g_a}{\sqrt{m_a^2+r^2-r_br_c}} \text{ (and analogs) (83)}$$

We know that : $l_a = \frac{2\sqrt{bcp(p-a)}}{b+c}$, $r_br_c = p(p-a) \rightarrow l_a = \frac{2\sqrt{bc}}{b+c}\sqrt{r_br_c}$ (and analogs)

and we obtain : $\frac{r_ar_br_c}{l_al_bl_b} = \frac{(a+b)(b+c)(a+c)}{8abc}$, $\frac{b+c}{a} = \frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1$(and analogs) so we obtain : $\frac{(a+b)(b+c)(a+c)}{abc} = \prod\left(\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1\right)$

From what we prove ,we obtain a new identity:

$$\frac{8r_ar_br_c}{l_al_bl_b} = \prod\left(\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1\right) (84)$$

$1+\frac{b+c}{a} = \frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}}$ (and analogs) $\rightarrow \frac{8r_ar_br_c}{l_al_bl_b} = \prod\left(\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}} - 1\right)$ **(85)**

From $m_al_a \geq r_br_c$ (and analogs)$\rightarrow m_am_bm_cl_al_bl_c \geq (r_ar_br_c)^2$

$\sqrt{\frac{m_am_bm_c}{l_al_bl_c}} \geq \frac{r_ar_br_c}{l_al_bl_c}$,and useing (84) and (85) we obtain:

$$\sqrt{\frac{m_am_bm_c}{l_al_bl_c}} \geq \frac{1}{8}\prod\left(\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1\right) \text{(86)}$$

$$\sqrt{\frac{m_am_bm_c}{l_al_bl_c}} \geq \frac{1}{8}\prod\left(\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}} - 1\right) \text{(87)}$$

If ABC is acuteangled triangle with a=min(a,b,c) $\rightarrow r_a = \min(r_a, r_b, r_c)$

$\frac{r_a+r}{r_a-r} = \frac{b+c}{a} \geq \frac{R}{r}$ (from ERDOS INEQUALITY)$\rightarrow r_a + r \geq \frac{R}{r}(r_a - r) \rightarrow$

$R + r \geq (\frac{R}{r} - 1)\, r_a \rightarrow \frac{R+r}{R-r} \geq \frac{r_a}{r}$, in the end we have the next result:

If ABC is acuteangled triangle then:

$$\frac{\mathbf{R+r}}{\mathbf{R-r}} \geq \frac{\mathbf{min(r_a,r_b,r_c)}}{\mathbf{r}} \textbf{ (88)}$$

$R+r \geq \left(\frac{R}{r}-1\right) r_a$, but $\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2r_ah_a}$ (and analogs)([1]) $R+r \geq \frac{n_a^2+r_a^2}{2r_ah_a} r_a \rightarrow \mathbf{2h_a(R+r) \geq n_a^2+r_a^2}$ **(89)**

$n_a^2+r_a^2 \geq 2n_ar_a$ and useing (89)$\rightarrow \sqrt{\mathbf{h_a(R+r)}} \geq \frac{\mathbf{n_a+r_a}}{\mathbf{2}}$ **(90)**

We know that : $\frac{b+c}{a}=\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$ (and analogs) (Mollweide's formula)

$1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}=\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}}$ (and analogs) ,se obtain a new result:

$$\frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}=\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_br_c}}}=\mathbf{1}+\frac{\mathbf{\cos\frac{B-C}{2}}}{\mathbf{\sin\frac{A}{2}}} \textbf{ (and analogs)(91)}$$

Also $\cos\frac{B-C}{2}=\frac{h_a}{l_a}$ (and analogs) and $AI=\frac{r}{\sin\frac{A}{2}}$ (and analogs) $\rightarrow$

$\frac{AI}{r}=\frac{b+c}{a}\frac{l_a}{h_a}$ (and analogs) , $1+\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}=\frac{n_a}{\sqrt{m_a^2+r^2-r_br_c}}$ (and analogs)

We obtain a new identity:

$$\frac{\mathbf{AI}}{\mathbf{r}}=\left(\frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}\right)\frac{\mathbf{l_a}}{\mathbf{h_a}}=\left(\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_br_c}}}-\mathbf{1}\right)\frac{\mathbf{l_a}}{\mathbf{h_a}} \textbf{ (and analogs) (92)}$$

$\frac{AI}{r} \geq \frac{b+c}{a}$ (and analogs)$\rightarrow \frac{\mathbf{AI}}{\mathbf{r}} \geq \frac{\mathbf{2n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}=\frac{\mathbf{n_a}}{\sqrt{\mathbf{m_a^2+r^2-r_br_c}}}-\mathbf{1}$ **(93)**

Also from $AI=\frac{r}{\sin\frac{A}{2}}$ (and analogs) , $\sin\frac{A}{2}=\sqrt{\frac{r}{2R}}\sqrt{\frac{r_a}{h_a}}$ and

$\frac{h_a}{r_a}=\frac{b+c-a}{a}=2\left(\frac{n_a}{\sqrt{4r^2+(b-c)^2}}-1\right)$ we obtain a new relationship :

$$\frac{\mathbf{AI}}{\mathbf{2r}}=\sqrt{\frac{\mathbf{R}}{\mathbf{r}}}\sqrt{\frac{\mathbf{n_a}}{\sqrt{\mathbf{4r^2+(b-c)^2}}}-\mathbf{1}} \textbf{ (and analogs) (94)}$$

From $\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2r_ah_a}$ (and analogs) $\rightarrow 2\frac{h_a}{n_a}\left(\frac{R}{r}-1\right)=\frac{n_a}{r_a}+\frac{r_a}{n_a}$ (and analogs)

$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogs)→

$$\frac{n_a}{r_a} + \frac{r_a}{n_a} = \frac{4(R-r)}{\sqrt{4r^2+(b-c)^2}} \text{ (and analogs) (95)}$$

From R ≥ 2r (Euler) →

$$\frac{\sqrt{R^2+(b-c)^2}}{2r} \geq \frac{n_a}{h_a} \text{ (and analogs) (96)}$$

From Euler inequality R ≥ 2r and (95) we obtain a new inequality:

$$\frac{n_a}{r_a} + \frac{r_a}{n_a} \geq \frac{4(R-r)}{\sqrt{R^2+(b-c)^2}} \text{ (and analogs) (97)}$$

$\frac{n_a}{r_a} = \frac{\sqrt{4r^2+(b-c)^2}}{r_a-r}$ (and analogs) and R≥ 2r →

$$\frac{\sqrt{R^2+(b-c)^2}}{r_a-r} \geq \frac{n_a}{r_a} \text{ (and analogs) (98)}$$

From $\frac{(a+b)(b+c)(a+c)}{abc} = \prod\left(\frac{2n_a}{\sqrt{4r^2+(b-c)^2}} - 1\right)$ and R≥ 2r we obtain a new result:

$$\frac{(a+b)(b+c)(a+c)}{abc} \geq \prod\left(\frac{2n_a}{\sqrt{R^2+(b-c)^2}} - 1\right) \text{ (99)}$$

$$\text{and } \frac{8r_ar_br_c}{l_al_bl_b} \geq \prod\left(\frac{2n_a}{\sqrt{R^2+(b-c)^2}} - 1\right) \text{(100)}$$

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