A GENERALIZATION OF FINSLER-HADWIGER INEQUALITY

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In any ΔABC we will denote with F the area of the triangle, with s the semiperimeter, and the other notations are the usual ones.

Theorem. If $m \in \mathbb{R}_+ = [0, \infty)$ and $x, y \in \mathbb{R}_+^* = (0, \infty)$, then in any ΔABC the following inequality holds:

(*)

$$(x^2+y^2)(a^{2m+2}+b^{2m+2}+c^{2m+2}) \geq 2^{2m+3}\cdot xy\cdot(\sqrt{3})^{1-m}\cdot F^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2$$

Proof.
We have:

$$\sum_{cyc}(xa^{m+1} - yb^{m+1})^2 = (x^2 + y^2)(a^{2m+2} + b^{2m+2} + c^{2m+2}) - 2\sum_{cyc} xy(ab)^{m+1} \Leftrightarrow$$

$$\Leftrightarrow (x^2 + y^2)\sum_{cyc} a^{2m+2} = 2xy\sum_{cyc}(a \cdot b)^{m+1} + \sum_{cyc}(xa^{m+1} - yb^{m+1})^2 \geq$$

$$\overset{\text{J. Radon}}{\geq} 2xy \cdot \frac{1}{3^m}\left(\sum_{cyc} ab\right)^{m+1} + \sum_{cyc}(xa^{m+1} - y \cdot b^{m+1})^2 \overset{\text{V.O. Gordon}}{\geq}$$

$$\geq \frac{2xy}{3^m}(4 \cdot \sqrt{3} \cdot F)^{m+1} + \sum_{cyc}(xa^{m+1} - yb^{m+1})^2 =$$

$$= \frac{2^{2m+3}}{3^m}xy(\sqrt{3})^{m+1} \cdot F^{m+1} + \sum_{cyc}(xa^{m+1} - yb^{m+1})^2 =$$

$= 2^{2m+3}\cdot xy\cdot(\sqrt{3})^{1-m}\cdot F^{m+1}+\sum_{cyc}(xa^{m+1}-yb^{m+1})^2$ which finishes the demonstration.

If $x = y$ then inequality (*) becomes:

(**) $$a^{2m+2} + b^{2m+2} + c^{2m+2} \geq 2^{2m+2} \cdot (\sqrt{3})^{1-m} \cdot F^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1} - b^{m+1})^2$$

and if in (**) we take $m = 0$ we obtain:

(F-H) $$a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot F + \frac{1}{2}\Big((a - b)^2 + (b - c)^2 + (c - a)^2\Big)$$

namely we have obtained Finsler-Hadwiger inequality.
If in inequality (**) we take $m = \frac{1}{2}$ we obtain:

$$a^3 + b^3 + c^3 \geq 8\sqrt[4]{3} \cdot (\sqrt{F})^3 + \frac{1}{2} \cdot \sum_{cyc}\Big((\sqrt{a})^3 - (\sqrt{b})^3\Big)^2$$

If in inequality (*) we take $x = 4, y = 3$ we obtain:

$$25(a^{2m+2} + b^{2m+2} + c^{2m+2}) \geq 2^{2m+5} \cdot 3 \cdot (\sqrt{3})^{1-m} \cdot F^{m+1} + \sum_{cyc}(4a^{m+1} - 3b^{m+1})^2 =$$

$$= 2^{2m+5} \cdot (\sqrt{3})^{3-m} \cdot F^{m+1} + \sum_{cyc}(4a^{m+1} - 3b^{m+1})^2 \tag{***}$$

If in (***) we take $m = 0$ we obtain:

$$25(a^2 + b^2 + c^2) \geq 96\sqrt{3} \cdot F + \sum_{cyc}(4a - 3b)^2$$

Let XYZ be a right-angled triangle with hypotenuse z.

$$z^2(a^{2m+2} + b^{2m+2} + c^{2m+2}) \geq 2^{2m+3} \cdot xy \cdot (\sqrt{3})^{1-m} \cdot F^{m+1} + \sum_{cyc}(xa^{m+1} - yb^{m+1})^2 =$$

$$= 4^{m+2} \cdot (\sqrt{3})^{1-m} \cdot F^{m+1} \cdot \text{ area } XYZ + \sum_{cyc}(xa^{m+1} - yb^{m+1})^2$$

Let's consider $t \in \mathbb{R}, \sin t, \cos t \neq 0$ and in inequality (*) we take $x = \sin t, y = \cos t$, then the inequality (*) becomes:

(1)

$$a^{2m+2} + b^{2m+2} + c^{2m+2} \geq 4^{m+1} \cdot (\sqrt{3})^{1-m} F^{m+1} \cdot \sin 2t + \sum_{cyc}(a^{m+1} \sin t - b^{m+1} \cos t)^2$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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