A NEW INEQUALITY OF A.P.M.O TYPE, 2004

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We know the following inequalities of A.P.M.O, 2004 type:
1. If $a, b, c > 0$, then:

$$(a^2+2)\cdot(b^2+2)\cdot(c^2+2) \ge 3\cdot(a+b+c)^2 \tag{H,L}$$

namely Ho Jo Lee inequality.

2. If $a, b, c, t > 0$ then:

$$(a^2+t^2)\cdot(b^2+t^2)\cdot(c^2+t^2) \ge \frac{3}{4}t^4(a+b+c)^2 \tag{A,A}$$

namely Arkady Alt inequality.

3. If $a, b, c > 0$ then:

$$(a^2+1)(b^2+1)(c^2+1) \ge \frac{3}{4}(a+b+c)^2 \tag{*}$$

We propose to prove the following inequalities:
Let be $m \ge 0$ and $a, b, c, x, y, z, t > 0$ then:
(**)

$$(a^{2m+2}+t\cdot x^{2m})\cdot(b^{2m+2}+t\cdot y^{2m})\cdot(c^{2m+2}+t\cdot z^{2m}) \ge \frac{3}{4}\cdot t^2(a^{m+1}\cdot y^m\cdot z^m+b^{m+1}\cdot x^m\cdot z^m+c^{m+1}\cdot x^m\cdot z^m)^2$$

Proof.
We have:

$$\prod_{cyc}(a^{2m+2}+t\cdot x^{2m}) = (xyz)^{2m}\cdot\prod_{cyc}\Big(\Big(\frac{a^{m+1}}{x^m}+t\Big) =$$

$$= (x\cdot y\cdot z)^{2m}\cdot t^3\cdot\prod_{cyc}\Big(\Big(\frac{a^{m+1}}{\sqrt{t}\cdot x^m}\Big)^2+1\Big) \overset{(*)}{\ge}$$

$$\ge (x\cdot y\cdot z)^{2m}\cdot t^3\cdot\frac{3}{4}\cdot\left(\sum_{cyc}\frac{a^{m+1}}{\sqrt{t}\cdot x^m}\right)^2 = \frac{3}{4}(xyz)^{2m}\cdot\frac{1}{t}\left(\sum_{cyc}\frac{a^{m+1}}{x^m}\right)^2 t^3 =$$

$$= \frac{3}{4}\cdot t^2(xyz)^{2m}\cdot\frac{(a^{m+1}\cdot y^m\cdot z^m+b^{m+1}\cdot x^m\cdot z^m+c^{m+1}\cdot x^m\cdot z^m)^2}{(x\cdot y\cdot z)^{2m}} =$$

$$= \frac{3}{5}\cdot t^2\cdot(a^{m+1}\cdot y^m\cdot z^m+b^{m+1}\cdot x^m\cdot z^m+c^{m+1}\cdot x^m\cdot y^m)^2$$

If instead of t we put t^2 then we obtain:

$$(a^{2m+2}+t^2\cdot x^{2m})\cdot(b^{2m+2}+t^2\cdot y^{2m})\cdot(c^{2m+2}+t^2\cdot z^{2m}) \ge$$

$$\ge \frac{3}{4}\cdot t^4(a^{m+1}\cdot y^m\cdot z^m+b^{m+1}\cdot x^m\cdot z^m+c^{m+1}\cdot x^m\cdot z^m)^2 \tag{***}$$

If in (*) we take $m = 0$ we obtain:

$$(1) \qquad (a^2 + t) \cdot (b^2 + t) \cdot (c^2 + t) \geq \frac{3}{4} \cdot t^2 \cdot (a + b + c)^2$$

and if in (***) we take $m = 0$ it follows:

$$(A,A) \qquad (a^2 + t^2) \cdot (b^2 + t^2) \cdot (c^2 + t^2) \geq \frac{3}{4} \cdot t^4 \cdot (a + b + c)^2$$

then we have obtained the inequality (A,A).
If in inequality (1) we take $t = 2$ we obtain:

$$(H.L) \qquad (a^2 + 2) \cdot (b^2 + 2) \cdot (c^2 + 2) \geq 3 \cdot (a + b + c)^2$$

namely we have obtained inequality (H.L).

Finally, if in inequality (*) we take $t = 1, m = 0$ we obtain the inequality:

$(a^2 + 1) \cdot (b^2 + 1) \cdot (c^2 + 1) \geq \frac{3}{4} \cdot (a + b + c)^2$ namely the inequality (*).

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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