GENERALIZATIONS OF ARKADY ALT INEQUALITY

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Arkady M. Alt inequality:
If $t, x, y, z > 0$ then:

$$(A,A) \qquad (x^2+t^2)(y^2+t^2)(z^2+t^2) \geq \frac{3}{4} \cdot t^4 \cdot (x+y+z)^2$$

with equality, if and only if $x = y = z = \frac{t}{\sqrt{2}}$.
This inequality generalize Hojoo Lee (A.P.M.O, 2004) inequality:
If $a, b, c > 0$ then:

$$(H,L) \qquad (a^2+2)(b^2+2)(c^2+2) \geq 3 \cdot (a+b+c)^2 \geq 9 \cdot (ab+bc+ca)$$

with equality if and only if $a = b = c = 1$.
Lemma. If $x, y, z > 0$, then:

$$(1) \qquad (x^2+1)(y^2+1)(z^2+1) \geq \frac{3}{4}(x+y+z)^2 \geq \frac{9}{4}(xy+yz+zx)$$

Proof.
We have:

$$(2) \qquad (x^2+1)(y^2+1) \geq (x+y)^2, \forall x, y > 0$$

with equality $\Leftrightarrow xy = 1$

Indeed, $(x^2+1)(y^2+1) \geq (x+y)^2 \Leftrightarrow x^2y^2 + x^2 + y^2 + 1 \geq x^2 + 2xy + y^2 \Leftrightarrow$

$$\Leftrightarrow x^2y^2 - 2xy + 1 \geq 0 \Leftrightarrow (xy-1)^2 \geq 0 \text{ obviously with equality } \Leftrightarrow xy = 1$$

Also we have:

$$(3) \qquad (x^2+1)(y^2+1) \geq \frac{3}{4}((x+y)^2+1), \forall x, y > 0$$

with equality $\Leftrightarrow x = y = \frac{1}{\sqrt{2}}$.
Indeed, we have:

$$(x^2+1)(y^2+1) \geq \frac{3}{4}((x+y)^2+1) \Leftrightarrow 4x^2y^2 + 4(x^2+y^2) + 4 \geq 3(x^2+y^2) + 6xy + 3 \Leftrightarrow$$

$$\Leftrightarrow 4x^2y^2 - 4xy + 1 + x^2 + y^2 - 2xy \geq 0 \Leftrightarrow (2xy-1)^2 + (x-y)^2 \geq 0$$

which is obviously with equality $\Leftrightarrow 2xy = 1$ and $x = y \Leftrightarrow x = y = \frac{1}{\sqrt{2}}$. We have:

$$(x^2+1)(y^2+1)(z^2+1) \overset{(3)}{\geq} \frac{3}{4}((x+y)^2+1)(z^2+1) \overset{(1)}{\geq}$$

$$\geq \frac{3}{4}((x+y)+z)^2 = \frac{3}{4}(x+y+z)^2 \geq \frac{9}{4}(xy+yz+zx)$$

with equality $\Leftrightarrow x = y = z = \frac{1}{\sqrt{2}}$. □

Theorem 1. If $a, b, c, m, n, p, t, u, v, w > 0$, then:

$$(m^2a^2 + t^2u^2)(n^2b^2 + t^2v^2)(p^2c^2 + t^2w^2) \geq$$

$$\geq \frac{3}{4}t^4(amvw + bnuw + cpuv)^2 \tag{i}$$

Proof.
We have:

$$(m^2a^2 + t^2u^2)(n^2b^2 + t^2v^2)(p^2c^2 + t^2w^2) =$$
$$= t^6u^2v^2w^2\Big(\Big(\frac{ma}{tu}\Big)^2 + 1\Big)\Big(\Big(\frac{nb}{tv}\Big)^2 + 1\Big)\Big(\Big(\frac{pc}{tw}\Big)^2 + 1\Big) \geq$$
$$\overset{(i)}{\geq} \frac{3}{4}t^6u^2v^2w^2\cdot\Big(\frac{ma}{tu} + \frac{nb}{tv} + \frac{pc}{tw}\Big)^2 =$$
$$= t^6u^2v^2w^2 \cdot \frac{3}{4} \cdot \frac{(mavw + nbuw + pcuv)^2}{t^2 \cdot u^2 \cdot v^2 \cdot w^2} =$$
$$= \frac{3}{4}t^4(mavw + nbuw + pcuv)^2$$

If $m = n = p = u = v = w = 1$ then inequality (i) becomes:

$$(a^2 + t^2)(b^2 + t^2)(c^2 + t^2) \geq \frac{3}{4}t^4(a + b + c)^2 \tag{A,A}$$

namely Arkady Alt inequality. □

Theorem 2. If $a, b, c, m, n, p, t, u, v, w, x, y, z > 0$ then:

$$(m^2a^2 + t^2u^2x^2)(n^2b^2 + t^2v^2y^2)(p^2c^2 + t^2w^2z^2) \geq$$

$$\geq \frac{3}{4}t^4(mavwyz + nbuwxz + pcuvxy)^2 \tag{ii}$$

Proof.
We have:

$$(m^2a^2 + t^2u^2x^2)(n^2b^2 + t^2v^2y^2)(p^2c^2 + t^2w^2z^2) =$$
$$= t^2u^2x^2t^2v^2y^2t^2w^2z^2\Big(\Big(\frac{ma}{tux}\Big)^2 + 1\Big)\Big(\Big(\frac{nb}{tvy}\Big)^2 + 1\Big)\cdot$$
$$\cdot\Big(\Big(\frac{pc}{twz}\Big)^2 + 1\Big) \overset{(1)}{\geq} \frac{3}{4}t^6u^2v^2w^2x^2y^2z^2\Big(\frac{ma}{tux} + \frac{nb}{tvy} + \frac{pc}{twz}\Big)^2 =$$
$$= \frac{3}{4}t^6u^2v^2w^2x^2y^2z^2 \cdot \frac{(mavwyz + nbuwxz + pcuvxy)^2}{t^2u^2v^2w^2x^2y^2z^2} =$$
$$= \frac{3}{4}t^4(mavwyz + nbuwxz + pcuvxy)^2$$

If in (ii) we take $x = y = z = 1$ we obtain the inequality (i) and if in (ii) we take $m = n = p = u = v = w = x = y = z = 1$ we obtain inequality (A,A). □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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