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ROMANIAN MATHEMATICAL MAGAZINE
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PROVING THE REFLECTIVE PROPERTY OF AN ELLIPSE

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The reflective property of an ellipse states that any rays passing through a focus shall hit the boundary and reflect to the remaining focus. This property has been applied in multiple sites around the world[2]. In this article, we shall prove this geometric property.

SOLUTION

Given ellipse (E) as illustrated in Figure I with the following equation:

$$(E): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

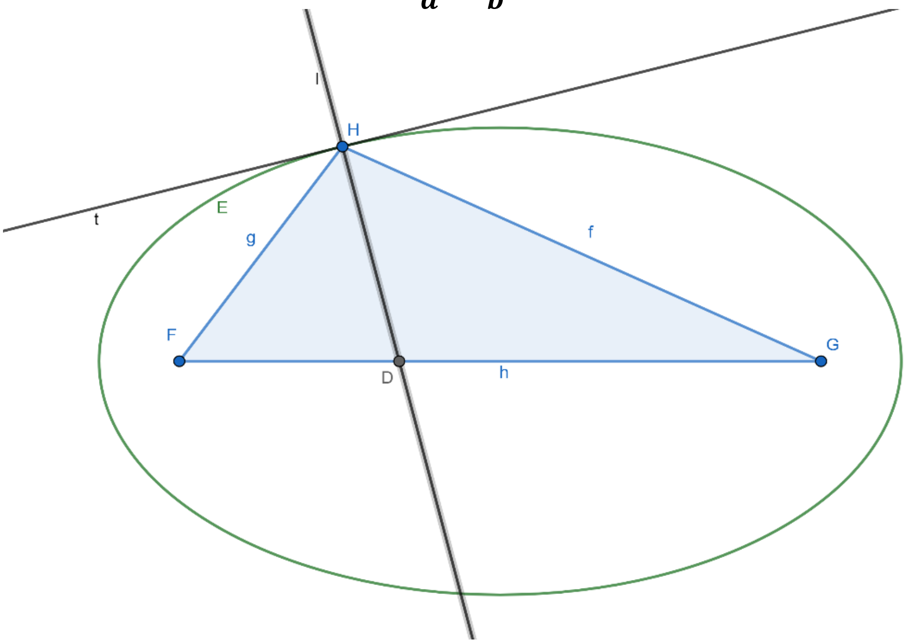

Figure I. The illustrated ellipse, constructed via GeoGebra.

In (1), a and b are semi-axes of (E) such that $0 < b < a$, and the focus is accordingly c. The foci are, respectively, $F(-c, 0)$ and $G(c, 0)$. Given point $H(j, k)$ which belongs to (E).

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[2] See: Dawson, S. (2021). The Reflective Property of an Ellipse. Retrieved February 10, 2023, from https://personal.math.ubc.ca/~cass/courses/m309-01a/dawson/index.html#:~:text=The%20Reflective%20property%20of%20an%20ellipse%20is%20simply%20this%3A%20when,pass%20through%20the%20other%20focus.

If $k=0$, the problem is straightforwardly proven. We consider $k\neq 0$. We are supposed to prove that HD is the bisector of triangle HFG.

The tangent line (t) of (E) at H and its correspondingly perpendicular line (l) are therefore:

$$(t):\frac{jx}{a^2}+\frac{ky}{b^2}=1 \quad (l):\begin{cases} x=j+\frac{ju}{a^2} \\ y=k+\frac{ku}{b^2} \\ (u\epsilon\mathbb{R}) \end{cases}$$

Let $D(d,0)$ be the intersection of (l) and FG. Since $y_D=0$, we get:

$$k+\frac{ku}{b^2}=0\Leftrightarrow u=-b^2$$

Accordingly,

$$d=j\left(1-\frac{b^2}{a^2}\right)=\frac{jc^2}{a^2}$$

We calculate DF and DG as follows:

$$DF=\frac{jc^2}{a^2}+c=\frac{c}{a^2}(a^2+jc) \quad (2)$$
$$DG=c-\frac{jc^2}{a^2}=\frac{c}{a^2}(a^2-jc) \quad (3)$$

From (2) and (3), we get:

$$\frac{DF}{DG}=\frac{a^2+jc}{a^2-jc} \quad (4)$$

In (4), it is noticeable that there exist $\varphi\epsilon[0,2\pi]$ such that $j=a\cos\varphi$. This is to ensure that $a^2\geq|jc|$, resulting in $DF>0$ and $DG>0$.

We calculate HF as follow:

$$HF^2=(j+c)^2+k^2=j^2+2jc+c^2+k^2 \quad (5)$$

As point H belongs to (E), we get the following relation:

$$\frac{j^2}{a^2}+\frac{k^2}{b^2}=1\Rightarrow\frac{k^2}{b^2}=\frac{a^2-j^2}{a^2}\Rightarrow k^2=\frac{b^2}{a^2}(a^2-j^2)=\frac{(a^2-c^2)(a^2-j^2)}{a^2} \quad (6)$$

Replacing (6) into (5), we obtain:

$$HF^2 = j^2 + 2jc + c^2 + \frac{(a^2 - c^2)(a^2 - j^2)}{a^2}$$
$$= \frac{a^2 j^2 + 2a^2 jc + a^2 c^2 + a^4 - a^2(j^2 + c^2) + j^2 c^2}{a^2} = \frac{a^4 + 2a^2 jc + j^2 c^2}{a^2}$$

Therefore,

$$HF = \frac{a^2 + jc}{a} \quad (7)$$

As point H belongs to (E), we apply the definition of an ellipse:

$$HF + HG = 2a \quad (8)$$

From (7) and (8), we get:

$$HG = 2a - HF = 2a - \frac{a^2 + jc}{a} = \frac{a^2 - jc}{a} \quad (9)$$

From (7) and (9), we get:

$$\frac{HF}{HG} = \frac{a^2 + jc}{a^2 - jc} \quad (10)$$

From (4) and (10), we obtain:

$$\frac{DF}{DG} = \frac{HF}{HG}$$

Henceforth, HD is the bisector of triangle HFG, which consequences in QED.