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150 TRIANGLE IDENTITIES AND INEQUALITIES INVOLVING BROCARD'S ANGLE

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Abstract: In this paper the start point are a few known identities in triangle involving Brocard's angle.We will design new identities and triangle inequalities in an original and natural mathematical procedure-used in previous articles and mathnotes published in Romanian Mathematical Magazine.

We consider ABC triangle with usual notations.We will use the next known relationships:

$$\sin\omega = \frac{2S}{\sqrt{a^2b^2+b^2c^2+a^2c^2}} \text{ (Brocard Angle);}$$

$$\frac{m_a^2}{h_a^2}=1+\frac{(b^2-c^2)^2}{16S^2} \text{ (and analogs);}$$

$$\sum\frac{m_a^2}{h_a^2}=1+\frac{1}{2\sin^2\omega};$$

$(\frac{m_c}{h_c}+\frac{m_b}{h_b})^2\le 2(\frac{m_b^2}{h_b^2}+\frac{m_c^2}{h_c^2})$ (and analogs) $\rightarrow\left(\frac{m_c}{h_c}+\frac{m_b}{h_b}\right)^2\le 2(1+\frac{1}{2\sin^2\omega}-1-\frac{(b^2-c^2)^2}{16S^2})$ and we obtain

$$\left(\frac{m_c}{h_c}+\frac{m_b}{h_b}\right)^2\le\frac{1}{\sin^2\omega}-\frac{(b^2-c^2)^2}{8S^2} \text{ (and analogs) (1)}$$

From (1) we obtain:

$$\frac{m_c}{h_c}+\frac{m_b}{h_b}\le\frac{1}{\sin\omega} \text{ (and analogs) (2);(see also [1].)}$$

We know that: $\frac{m_b}{h_c}+\frac{m_c}{h_b}\le\frac{R}{r}$ (and analogs)[1]. and using (2) after summation we obtain :

$$\frac{R}{r}+\frac{1}{\sin\omega}\ge(m_b+m_c)(\frac{1}{h_c}+\frac{1}{h_b}) \text{ (3)}$$

From (2) after summation we obtain a new inequality:

$$\frac{3}{2\sin\omega}\ge\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \text{ (4)}$$

Now we will use this well known result:

$r_br_c\le m_al_a$ (and analogs)(Panaitopol) and $r_br_c=\frac{h_a(r_b+r_c)}{2}$ (and analogs) and we obtain : $\frac{m_a}{h_a}\ge\frac{r_b+r_c}{2l_a}$ **(and analogs) (5)**

Using (2) and (5) we obtain :

$$\frac{2}{\sin\omega} \geq \frac{r_c+r_a}{l_b} + \frac{r_a+r_b}{l_c} \textbf{(and analogs)(6)}$$

From (6) after summation and simplification we obtain :

$$\frac{1}{\sin\omega} \geq \frac{1}{3}\sum\frac{r_b+r_c}{l_a} \textbf{ (7)}$$

Now we will use : $\frac{\sin(\omega+A)}{\sin\omega} = \frac{b}{c} + \frac{c}{b}$ (and analogs)(Traian Lalescu)[2].

$$\frac{1}{\sin\omega} \geq \frac{b}{c} + \frac{c}{b} \textbf{(and analogs)(8)}$$

2S=a $h_a = bh_b = ch_c \rightarrow \frac{b}{c} = \frac{h_c}{h_b}$ (and analogs) and using (8) we obtain:

$$\frac{1}{\sin\omega} \geq \frac{h_c}{h_b} + \frac{h_b}{h_c} \textbf{ (and analogs) (9)}$$

From (2) and (9) we obtain :

$$\frac{2}{\sin\omega} \geq \frac{m_b+h_c}{h_b} + \frac{m_c+h_b}{h_c} \textbf{ (and analogs)(10)}$$

We know that : $4m_a = 2\sqrt{2(b^2+c^2)-a^2}$ (and analogs)

$\rightarrow 4m_a\sqrt{a^2+b^2+c^2} = 2\sqrt{2(b^2+c^2)-a^2}\sqrt{a^2+b^2+c^2}$ and using AM-GM inequality we obtain :

$$4m_a\sqrt{a^2+b^2+c^2} \leq 2(b^2+c^2) - a^2 + a^2 + b^2 + c^2 = 3(b^2+c^2)$$

$$\frac{4}{3}m_a\sqrt{a^2+b^2+c^2} \leq b^2+c^2 \;(11)$$

From (11) and bc=2Rh_a (and analogs) we obtain :

$$2\frac{m_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R} \leq \frac{b}{c} + \frac{c}{b} \textbf{ (and analogs) (12)}$$

From (12) and $\frac{\sin(\omega+A)}{\sin\omega} = \frac{b}{c} + \frac{c}{b}$ we obtain a new result :

$$2\frac{m_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R} \leq \frac{\sin(\omega+A)}{\sin\omega} \textbf{ (and analogs) (13)}$$

From (12) and (8) we obtain :

$$\mathbf{2max\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\}\frac{\sqrt{a^2+b^2+c^2}}{3R}\le\frac{1}{\sin\omega}\ (14)}$$

From (12) after summation we obtain the next result:

$$\mathbf{\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\le\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\ (15)}$$

From (2) and $\frac{m_b}{h_b}+\frac{m_c}{h_c}\ge 2\sqrt{\frac{m_b}{h_b}\frac{m_c}{h_c}}$ (AM-GM) after summation we obtain :

$\frac{2}{\sin\omega}\ge\frac{m_b}{h_b}+\frac{m_c}{h_c}+2\sqrt{\frac{m_b}{h_b}\frac{m_c}{h_c}}=(\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}})^2$and we obtain :

$$\mathbf{\sqrt{\frac{2}{\sin\omega}}\ge\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\ (and\ analogs)(16)}$$

From Catalan inequality :$a^2b(a-b)+b^2(b-c)+c^2(c-a)\ge 0$ [3]. we obtain:

$$a^3b+b^3c+c^3a\ge a^2b^2+b^2c^2+a^2c^2$$

$$\mathbf{\frac{a^3b+b^3c+c^3a}{4S^2}\ge\frac{a^2b^2+b^2c^2+a^2c^2}{4S^2}=\frac{1}{\sin^2\omega}\ (17)}$$

From 2S=a $h_a=bh_b=ch_c$ and (17) we obtain :

$$\mathbf{\frac{ab}{h_a^2}+\frac{bc}{h_b^2}+\frac{ac}{h_c^2}\ge\frac{1}{\sin^2\omega}(18)}$$

From bc=$r_br_c+rr_a$ (and analogs) and $r_br_c\le m_al_a$ (and analogs)(Panaitopol) we obtain:

bc $\le m_al_a+rr_a$(and analogs); $bc(m_al_a+rr_a)\ge b^2c^2$(and analogs)and after summations we obtain a new inequality :

$$\mathbf{\frac{\sum bc(m_al_a+rr_a)}{4S^2}\ge\frac{1}{\sin^2\omega}(19)}$$

From $m_a\ge\frac{b^2+c^2}{4R}$(and analogs)(Tereshin), bc=2Rh_a (and analogs) we obtain:

$\frac{m_a}{h_a}\ge\frac{1}{2}(\frac{b}{c}+\frac{c}{b})$(and analogs) and useing (8) we obtain a new inequality:

$$\frac{2}{\sin\omega}\frac{m_a}{h_a}\ge(\frac{b}{c}+\frac{c}{b})^2\ \text{(and analogs)}\rightarrow$$

$$\sqrt{\frac{2}{\sin\omega}}\sqrt{\frac{m_a}{h_a}} \geq \frac{b}{c}+\frac{c}{b}\ \text{(and analogs)}(20)$$

From (20) after sumation we obtain a new inequality:

$$\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}} \geq \frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\ (21)$$

LEMMA:Triangle ABC with sides a,b,c and triangle with sides m_a, m_b, m_c have the same Brocard angle[4].Using this lemma and (8) we obtain a new result:

$$\frac{1}{\sin\omega} \geq \frac{m_b}{m_c}+\frac{m_b}{m_c}\ \textbf{(and analogs) (22)}$$

We consider triangle with sides m_a, m_b, m_c and his elements:

$\overline{m_a} = \frac{3a}{4}$ (and analogs); $\overline{h_a} = \frac{3S}{2m_a}$ (and analogs); $S_m = \frac{3S}{4}$;

$\frac{\overline{m_a}}{\overline{h_a}} = \frac{am_a}{2S} = \frac{am_a}{ah_a} = \frac{m_a}{h_a}$ (and analogs)→inequality $\frac{m_a}{h_a} \geq \frac{1}{2}(\frac{b}{c}+\frac{c}{b})$(and analogs) becomes: $\frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_b}{m_c}\right)$ (and analogs); and using (22) we obtain a new inequality :

$$\sqrt{\frac{2}{\sin\omega}}\sqrt{\frac{m_a}{h_a}} \geq \frac{m_b}{m_c}+\frac{m_b}{m_c}\text{(and analogs) (23)}$$

From (23) after summation we obtain a new result:

$$\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}} \geq \sum\frac{m_b+m_c}{m_a}\ \textbf{(24)}$$

From(15) and (21) we obtain :

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right) \leq \frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c} \leq \sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}}\ \textbf{(25)}$$

From 2S=a $h_a = bh_b = ch_c = 2pr, 2p = a+b+c$→$\frac{b}{c} = \frac{h_c}{h_b}$ (and analogs); we obtain:

$\sum\frac{b+c}{a}$ =$\sum\frac{h_b+h_c}{h_a}$ and using (25) we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\le\sum\frac{h_b+h_c}{h_a}\le\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}}\ (26)$$

2S=a $h_a=bh_b=ch_c=2pr, 2p=a+b+c;\ \frac{h_a}{r}$ =1+ $\frac{b+c}{a}$ (and analogs);

Using this relationships we obtain a new result :

$$3+\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\le\frac{h_a+h_b+h_c}{r}\le 3+\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}}\ \textbf{(27)}$$

From (13) after summation we obtain :

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\le\sum\frac{\sin(\omega+A)}{\sin\omega}\ \textbf{(28)}$$

From $\sqrt{\frac{2}{\sin\omega}}\sqrt{\frac{m_a}{h_a}}\ge\frac{b}{c}+\frac{c}{b}$ (and analogs) and and $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) we obtain :

$$\sqrt{\frac{2}{\sin\omega}}\sqrt{\frac{m_a}{h_a}}\ge\frac{\sin(\omega+A)}{\sin\omega}\ \to$$

$$\sqrt{\frac{m_a}{h_a}}\ge\frac{\sin(\omega+A)}{\sqrt{2\sin\omega}}\ \textbf{(and analogs) (29)}$$

From (29) after summation we obtain :

$$\sum\sqrt{\frac{m_a}{h_a}}\ge\frac{\sin(\omega+A)+\sin(\omega+B)+\sin(\omega+C)}{\sqrt{2\sin\omega}}\ \textbf{(30)}$$

From $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) and $\frac{m_a}{h_a}\ge\frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$(and analogs) we have:

$$\frac{m_a}{h_a}\ge\frac{\sin(\omega+A)}{2\sin\omega}\to$$

$$\frac{m_a}{h_a}\frac{1}{\sin(\omega+A)}\ge\frac{1}{2\sin\omega}\textbf{(and analogs) (31)}$$

From (31) after summation we obtain :

$$\sum\frac{m_a}{h_a}\frac{1}{\sin(\omega+A)}\ge\frac{3}{2\sin\omega}\ (32)$$

From $m_b\ge\frac{a^2+c^2}{4R}$ and $m_c\ge\frac{a^2+b^2}{4R}$ after calculation we obtain :

$$16R^2m_bm_c-a^4\ge a^2b^2+b^2c^2+a^2c^2\textbf{(and analogs) (33);}$$

$$\frac{\mathbf{16R^2m_bm_c-a^4}}{\mathbf{4S^2}} \geq \frac{\mathbf{1}}{\mathbf{\sin^2\omega}}\ (\mathbf{and\ analogs})(\mathbf{34})$$

$(a^2+b^2)(a^2+c^2) \geq (ab+ac)^2$(C.B.S) and ab=2R$h_c$;ac=2Rh_b;

$$\mathbf{16R^2m_bm_c \geq 4R^2(h_c+h_b)^2 \to 2\sqrt{m_bm_c} \geq h_c+h_b}\textbf{(and analogs)(35)}$$

From 2S=a$h_a = bh_b = ch_c = 2pr, 2p = a+b+c \to \frac{b}{c} = \frac{h_c}{h_b}$ (and analogs); we have :

$\sum\frac{b+c}{a} = \sum\frac{h_b+h_c}{h_a} = \sum\frac{\sin(\omega+A)}{\sin\omega}$ .From (35) we obtain :

$\frac{2\sqrt{m_bm_c}}{h_a} \geq \frac{h_c+h_b}{h_a}$ (and analogs)(36), from (36) after summation we get a new result:

$$\mathbf{2\sum\frac{\sqrt{m_bm_c}}{h_a} \geq \sum\frac{b+c}{a} = \sum\frac{\sin(\omega+A)}{\sin\omega}}\ \textbf{(37)}$$

We know that :$l_a = \frac{2\sqrt{bc}}{b+c}\sqrt{p(p-a)}$(and analogs) and $\sqrt{p(p-a)}=\sqrt{r_br_c}$ (and analogs);$\to\frac{1}{2}(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}) = \frac{\sqrt{r_br_c}}{l_a}$ (and analogs)$\to\frac{4r_br_c}{l_a^2}$ = 2+ $\frac{b}{c}+\frac{c}{b}$ (and analogs) (38);

From (38) :

$$\mathbf{\frac{4r_br_c}{l_a^2} = 2+\frac{\sin(\omega+A)}{\sin\omega}}\ \textbf{(and analogs) (39)}$$

From (8) and (38) we obtain a new inequality :

$$\mathbf{2+\frac{1}{\sin\omega} \geq \frac{4r_br_c}{l_a^2}}\ \textbf{(and analogs) (40)}$$

From (39) and $r_br_c \leq m_al_a$ (and analogs)(Panaitopol) we obtain :

$$\mathbf{4\frac{m_a}{l_a} \geq 2+\frac{\sin(\omega+A)}{\sin\omega}}\ (\mathbf{and\ analogs})(\mathbf{41})$$

After some simple manipulation and summation we obtain a new result from (41):

$$\mathbf{\frac{3}{2}+\frac{1}{4}\sum\frac{\sin(\omega+A)}{\sin\omega} \leq \sum\frac{m_a}{l_a}}\ \textbf{(42)}$$

Also from (41) we obtain:

$$\mathbf{\frac{\sin(\omega+A)}{2m_a-l_a} \leq \frac{2\sin\omega}{l_a}}\ \textbf{(and analogs) (43)}$$

From (43) after summation we obtain :

$$\sum\frac{\sin(\omega+A)}{2m_a-l_a}\leq 2\sin\omega\left(\frac{1}{l_a}+\frac{1}{l_b}+\frac{1}{l_c}\right)\ (44)$$

From $m_a\geq\frac{b^2+c^2}{4R}$ (and analogs)(Tereshin), bc=2Rh_a (and analogs) and $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b^2+c^2}{bc}$ (and analogs) we obtain :$\frac{h_a}{\sin\omega}\leq\frac{2m_a}{\sin(\omega+A)}$ (and analogs)(45);

From (45) after summation we obtain :

$$\frac{h_a+h_b+h_c}{2\sin\omega}\leq\frac{m_a}{\sin(\omega+A)}+\frac{m_b}{\sin(\omega+B)}+\frac{m_c}{\sin(\omega+C)}\ (46)$$

From (35) we obtain :

$$2\sqrt{\frac{m_bm_c}{h_bh_c}}\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\ \text{(and analogs)}(47)$$

From (47) and $\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)=\frac{\sqrt{r_br_c}}{l_a}$ (and analogs) we obtain:

$$\sqrt{\frac{m_bm_c}{h_bh_c}}\geq\frac{\sqrt{r_br_c}}{l_a}\ \text{(and analogs)}\ (48)$$

$\rightarrow l_a\sqrt{m_bm_c}\geq\sqrt{h_bh_cr_br_c}$ and after summation we obtain **:**

$$\sum l_a\sqrt{m_bm_c}\geq\sum\sqrt{h_bh_cr_br_c}\ (49)$$

From $r_br_c\leq m_al_a$ (and analogs)(Panaitopol)$\rightarrow$

$$\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)=\frac{\sqrt{r_br_c}}{l_a}\leq\sqrt{\frac{m_a}{l_a}}\ \text{(and analogs)}\ (50)$$

From (47) and (50) after simple manipulations we obtain :

$$2\left(2\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}-1\right)\geq\frac{b}{c}+\frac{c}{b}\ \text{(and analogs)}\ (51)$$

From (51) and $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) we obtain :

$$2\left(2\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}-1\right)\geq\frac{\sin(\omega+A)}{\sin\omega}\ \text{(and analogs)}\ (52)$$

From (51) after summation we obtain :

$$4\sum\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}} \geq 6+\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\ (53)$$

From $\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs) and (53) we obtain a new inequality :

$$4\sum\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}} \geq 3+\frac{h_a+h_b+h_c}{r}\ (54)$$

From (51) and (8) we obtain: $\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right) \geq \left(\frac{b}{c}+\frac{c}{b}\right)^2$ (and analogs)

We will obtain

$$\sqrt{\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)} \geq \frac{b}{c}+\frac{c}{b}\ \text{(and analogs)}(55)$$

From (56) after summation we obtain a new result:

$$\sum\sqrt{\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)} \geq \frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\ (56)$$

From (51) and $\frac{m_a}{h_a} \geq \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)$(and analogs) we obtain :

$\frac{m_a}{h_a}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right) \geq \frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}\right)^2$ (and analogs) and we obtain:

$$\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)} \geq \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\text{(and analogs) (57)}$$

From (58) after summation we obtain a new inequality:

$$\sum\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)} \geq \frac{1}{2}\left(\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\right)\ (58)$$

From (58) and $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) we obtain a new result :

$$\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)} \geq \frac{\sin(\omega+A)}{2\sin\omega}\text{(and analogs)(59)}$$

From (34) we obtain : $\min\{\frac{16R^2m_bm_c-a^4}{4S^2},\frac{16R^2m_am_c-b^4}{4S^2}\frac{16R^2m_am_b-c^4}{4S^2}\}\geq\frac{1}{\sin^2\omega}$ and using (2) we obtain a new inequality:

$$\min\{\frac{16R^2m_bm_c-a^4}{4S^2},\frac{16R^2m_am_c-b^4}{4S^2},\frac{16R^2m_am_b-c^4}{4S^2}\}\geq\max\{\left(\frac{m_c}{h_c}+\frac{m_b}{h_b}\right)^2,\left(\frac{m_c}{h_c}++\frac{m_a}{h_a}\right)^2,(\frac{m_a}{h_a}+\frac{m_b}{h_b})^2\}$$
(60)

We know that b+c=4R$\cos\frac{A}{2}\cos\frac{B-C}{2}$ (and analogs)→ $\cos\frac{A}{2}\geq\frac{b+c}{4R}$ (and analogs);

$\cos\frac{A}{2}=\sqrt{\frac{r_b+r_c}{4R}}$ (and analogs); bc=2Rh_a (and analogs);After simple manipulation we obtain :
$r_b+r_c-h_a\geq\frac{b^2+c^2}{4R}$ (and analogs) and we get a new inequality:

$$\frac{r_b+r_c-h_a}{h_a}\geq\frac{1}{2}(\frac{b}{c}+\frac{c}{b})\text{(and analogs)(61)}$$

From (61) and $\frac{\sin(\omega+A)}{\sin\omega}$=$\frac{b}{c}+\frac{c}{b}$ (and analogs) we obtain :

$$\frac{r_b+r_c-h_a}{h_a}\geq\frac{\sin(\omega+A)}{2\sin\omega}\text{(and analogs)(62)}$$

Using (61) and $\frac{m_a}{h_a}\geq\frac{1}{2}(\frac{b}{c}+\frac{c}{b})$(and analogs) we obtain :

$$\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}(\frac{b}{c}+\frac{c}{b})\text{(and analogs)(63)}$$

From (63) and $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs), we obtain a new inequality :

$$\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{\sin(\omega+A)}{2\sin\omega}\text{(and analogs)(64)}$$

From (63) after summation we obtain a new result:

$$\sum\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}(\frac{b+c}{a}++\frac{a+c}{b}+\frac{a+b}{c})\text{(65)}$$

From (64) after summation we obtain :

$$\sum\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}\sum\frac{\sin(\omega+A)}{\sin\omega}\text{(66)}$$

We proved that $b^2+c^2=n_a^2+g_a^2+2r_ar$ (and analogs)[5]., and bc=$r_br_c+rr_a$ (and analogs) and we know that $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) we obtain a new identity :

$$\frac{\sin(\omega+A)}{\sin\omega}=\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (\text{and analogs})\ \mathbf{(67)}$$

and we obtain a new inequality :

$$\frac{1}{\sin\omega}\geq\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (\text{and analogs})(68)$$

From (67) and (38) we obtain :

$$\frac{4r_br_c}{l_a^2}=2+\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (\text{and analogs})(69)$$

From (37) and (67) we obtain:

$$2\sum\frac{\sqrt{m_bm_c}}{h_a}\geq\sum\frac{\sin(\omega+A)}{\sin\omega}=\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (70)$$

From (20) and (67) we obtain :

$$\sqrt{\frac{2}{\sin\omega}}\sqrt{\frac{m_a}{h_a}}\geq\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (\text{and analogs})(71)$$

From (67) after summation we obtain :

$$\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}=\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (72)$$

From (21) and (72) we obtain:

$$\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}}\geq\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (73)$$

From (12) and (67) we obtain:

$$2\frac{m_a}{h_a}\frac{\sqrt{a^2+b^2+c^2}}{3R}\leq\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (\text{and analogs})(74)$$

From (15) and (72) we obtain :

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\leq\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (75)$$

From (25) and (72) we obtain:

$$\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\le\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\le\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}}\ (76)$$

From (51) and (67) we obtain :

$$2\left(2\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}-1\right)\ge\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\text{(and analogs) }(77)$$

From (53) and (67) we obtain :

$$4\sum\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}\ge 6+\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (78)$$

From (55) and (67) we obtain :

$$\sqrt{\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}-1\right)}\ge\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (79)$$

From (56) and (72) we obtain:

$$\sum\sqrt{\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}-1\right)}\ge\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (80)$$

From (57) and (67) we obtain :

$$\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}-1\right)}\ge\frac{n_a^2+g_a^2+2r_ar}{2(r_br_c+rr_a)}\text{ (and analogs) }(81)$$

From (58) and (72) we obtain :

$$\sum\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_am_bm_c}{l_ah_bh_c}}-1\right)}\ge\frac{1}{2}\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\ (82)$$

From (62) and (67) we obtain:

$$\frac{r_b+r_c-h_a}{h_a}\ge\frac{n_a^2+g_a^2+2r_ar}{2(r_br_c+rr_a)}\text{ (and analogs) }(83)$$

From (64) and (67) we obtain :

$$\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{n_a^2+g_a^2+2r_ar}{2(r_br_c+rr_a)}\text{ (and analogs) (84)}$$

From (65) and (72) we obtain :

$$\sum\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\frac{1}{2}\sum\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}\text{ (85)}$$

We know that: In ΔABC ,$P,P'\in(ABC)$ the following relationship holds:

$a\cdot AP\cdot AP'+b\cdot BP\cdot BP'+c\cdot CP\cdot CP'\geq abc$ (Klamkin).[6]

If $P'=\Omega$-first point of Brocard ,we have :$A\Omega=2R\frac{b}{a}\sin\omega$, $B\Omega=2R\frac{c}{b}\sin\omega$,

$C\Omega=2R\frac{a}{c}\sin\omega$.Also we know that abc=4RS, $2S=ah_a=bh_b=ch_c$. Using Klamkin theorem we obtain :$b\cdot AP+c\cdot BP+a\cdot CM\geq\sqrt{a^2b^2+b^2c^2+a^2c^2}\rightarrow$

$$\frac{AP}{h_b}+\frac{BP}{h_c}+\frac{CP}{h_a}\geq\frac{1}{\sin\omega}\text{ (86)}$$

If P= G and $AG=\frac{2}{3}m_a$, $BG=\frac{2}{3}m_b$, $CG=\frac{2}{3}m_c$ (86) become:

$$\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}\geq\frac{3}{2\sin\omega}\text{ (87)}$$

From (87) and (4) we obtain a new result :

$$\frac{m_a}{h_b}+\frac{m_b}{h_c}+\frac{m_c}{h_a}\geq\frac{3}{2\sin\omega}\geq\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\text{ (88)}$$

Using triangle with sides m_a,m_b,m_c and his elements:

$\overline{m_a}=\frac{3a}{4}$ (and analogs); $\overline{h_a}=\frac{3S}{2m_a}$ (and analogs); $S_m=\frac{3S}{4}$; we obtain :

$\frac{\overline{m_a}}{\overline{h_b}}=\frac{m_b}{h_a},\frac{\overline{m_b}}{\overline{h_c}}=\frac{m_c}{h_b},\frac{\overline{m_c}}{\overline{h_a}}=\frac{m_a}{h_c},\frac{\overline{m_a}}{\overline{h_a}}=\frac{am_a}{2S}=\frac{am_a}{ah_a}=\frac{m_a}{h_a}$ (and analogs),and from (88) we obtain a new inequality :

$$\frac{m_b}{h_a}+\frac{m_c}{h_b}+\frac{m_a}{h_c}\geq\frac{3}{2\sin\omega}\geq\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\text{ (89)}$$

NOTE :For (89) we also used LEMMA.

From (88) and (89) we obtain a new inequality:

$$\frac{m_b+m_c}{h_a}+\frac{m_a+m_c}{h_b}+\frac{m_a+m_b}{h_c}\geq\frac{3}{\sin\omega}\text{ (90)}$$

We know :$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$,and using Cebyshev inequality twice we obtain :

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\le\frac{m_a+m_b+m_c}{3r}\le\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}.$$

We obtain two new results ,using (90):

$$\mathbf{\frac{m_a+m_b+m_c}{r}\ge\frac{3}{\sin\omega}+\sum\frac{m_a}{h_a}\ (91)}$$

$$\mathbf{3\sum\frac{m_a}{r_a}\ge\frac{3}{\sin\omega}+\sum\frac{m_a}{h_a}\ (92)}$$

We consider without losing generality, $a\ge b\ge c$.We will show that :$\frac{m_b}{h_b}\ge\frac{m_a}{h_a}\to m_b h_a\ge m_a h_b$;

$2S=a\,h_a=bh_b=ch_c$.$\frac{m_b}{a}\ge\frac{m_a}{b}\to b^2 4m_b^2\ge a^2 4m_a^2$

$b^2[2(a^2+c^2)-b^2]\ge a^2[2(b^2+c^2)-a^2]\to 2b^2c^2-2a^2c^2\ge b^4-a^4\to$

$2c^2(b^2-a^2)\ge(b^2-a^2)(b^2+a^2)\to(b^2-c^2)(2c^2-b^2-a^2)\ge 0.$

Now we show that : $\frac{m_b}{h_b}\ge\frac{m_c}{h_c}\to m_b h_c\ge m_c h_b$; $2S=a\,h_a=bh_b=ch_c$

$\frac{m_b}{c}\ge\frac{m_c}{b}\to bm_b\ge cm_c\to b^2 4m_b^2\ge c^2 4m_c^2$, we obtain :

$b^2[2(a^2+c^2)-b^2]\ge c^2[2(b^2+a^2)-c^2]\to 2a^2b^2-2a^2c^2\ge b^4-c^4$

$2a^2(b^2-c^2)\ge(b^2-c^2)(b^2+c^2)\to(b^2-c^2)(2a^2-b^2-c^2)\ge 0$

Deci $\max\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\}=\frac{m_b}{h_b}$ pentru $a\ge b\ge c$

Vom arăta că: $\frac{m_b}{h_b}\ge\frac{1}{2\sin\omega}$, $\sin\omega=\frac{2S}{\sqrt{a^2b^2+b^2c^2+a^2c^2}}$, $4\frac{m_b^2}{h_b^2}\ge\frac{1}{\sin^2\omega}$

$2S=a\,h_a=bh_b=ch_c$,we will obtain :$\frac{b^2[2(a^2+c^2)-b^2]}{4S^2}\ge\frac{a^2b^2+b^2c^2+a^2c^2}{4S^2}$

$2b^2c^2+2b^2a^2-b^4\ge a^2b^2+b^2c^2+a^2c^2\to b^2c^2+a^2b^2-b^4\ge a^2c^2\to$

$\to(c^2-b^2)(b^2-a^2)\ge 0$ wich is true because $a\ge b\ge c$

In the end we obtain a new inequality:

$$\mathbf{\max\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\}\ge\frac{1}{2\sin\omega}\ (93)}$$

From (2) and (93) we obtain a new inequality:

$$2\max\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\}\geq\frac{1}{\sin\omega}\geq\max\{\frac{m_c}{h_c}+\frac{m_b}{h_b},\frac{m_c}{h_c}+\frac{m_a}{h_a},\frac{m_a}{h_a}+\frac{m_b}{h_b}\}\ \mathbf{(94)}$$

We consider x,y,z real and positive numbers.We note : $\dot{\alpha}=\{\frac{x}{y}+\frac{y}{x},\frac{y}{z}+\frac{z}{y},\frac{x}{z}+\frac{z}{x}\}$

We will show that : $\max\dot{\alpha}\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-1\geq\min\dot{\alpha}$

Without losing generality we consider $x\geq y\geq z\rightarrow\frac{x}{z}+\frac{z}{x}\geq\frac{x}{y}+\frac{y}{x}\rightarrow$

$x^2y+z^2y\geq x^2z+y^2z\rightarrow(x^2-yz)(y-z)\geq 0$, in the same way we show that :

$\frac{x}{z}+\frac{z}{x}\geq\frac{y}{z}+\frac{z}{y}\rightarrow\max\dot{\alpha}=\frac{x}{z}+\frac{z}{x}$;Now we have: $\frac{x}{z}+\frac{z}{x}\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-1\rightarrow$

$\frac{x+z}{z}\geq\frac{x}{y}+\frac{y}{z}\rightarrow(y-x)(y-z)\leq 0$

$2(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-1)\geq(\frac{x}{y}+\frac{y}{x})+(\frac{y}{z}+\frac{z}{y})\rightarrow(x-z)[y(y-z)+z(x-y)]\geq 0$

We obtain : $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-1\geq\min\left\{\left(\frac{x}{y}+\frac{y}{x}\right),\left(\frac{y}{z}+\frac{z}{y}\right)\right\}=\min\dot{\alpha}$

We prove $\max\dot{\alpha}\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-1\geq\min\dot{\alpha}$ in same way for $x\geq z\geq y$

For a,b,c $\rightarrow\max\dot{\alpha}=\max\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{a}{c}+\frac{c}{a}\}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1$ and using (8)

we obtain a new inequality:

$$1+\frac{1}{\sin\omega}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ \mathbf{(95)}$$

Using LEMMA and (95) we obtain:

$$1+\frac{1}{\sin\omega}\geq\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\ \mathbf{(96)}$$

$\dot{\alpha}=\{\frac{a}{b}+\frac{b}{a},\frac{b}{c}+\frac{c}{b},\frac{a}{c}+\frac{c}{a}\},\frac{\sin(\omega+A)}{\sin\omega}=\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a}=\frac{b}{c}+\frac{c}{b}$ (and analogs) ,we obtain:

$$\max\{\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a},\frac{n_b^2+g_b^2+2r_br}{r_ar_c+rr_b},\frac{n_c^2+g_c^2+2r_cr}{r_ar_b+rr_c}\}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\geq$$
$$\min\{\frac{n_a^2+g_a^2+2r_ar}{r_br_c+rr_a},\frac{n_b^2+g_b^2+2r_br}{r_ar_c+rr_b},\frac{n_c^2+g_c^2+2r_cr}{r_ar_b+rr_c}\}\ \mathbf{(97)}$$

Using same method used in proving (95) and (96), we obtain new results:

$$1+\frac{1}{\sin\omega}\geq\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\ \mathbf{(98)}$$

$$1+\frac{1}{\sin\omega}\ge\frac{m_a}{m_c}+\frac{m_c}{m_b}+\frac{m_b}{m_a}\ (99)$$

Using (95),(96),(98),(99) after summation we obtain new results :

$$1+\frac{1}{\sin\omega}\ge\frac{1}{2}\sum\frac{b+c}{a}\ (100)$$

$$1+\frac{1}{\sin\omega}\ge\frac{1}{2}\sum\frac{m_b+m_c}{m_a}\ (101)$$

We know $\frac{h_a}{r}=1+\frac{b+c}{a}$ (and analogs);$\sum\frac{b+c}{a}=\frac{h_a+h_b+h_c-3r}{r}$ and using (100) we obtain :

$$5+\frac{2}{\sin\omega}\ge\frac{h_a+h_b+h_c}{r}\ (102)$$

We consider ΔABC acute. We will prove this inequality:

$\frac{2m_a^2}{r_b+r_c}\le\frac{b^2+c^2}{4R}$ (and analogs) ; We can write : $\frac{b^2+c^2}{2R}(r_b+r_c)\ge 4m_a^2$

$4m_a^2=2(b^2+c^2)-a^2$(and analogs); $2bc\cos A=b^2+c^2-a^2$ (and analogs);

$r_b+r_c=4R\cos^2\frac{A}{2}$ (and analogs); $2\cos^2\frac{A}{2}=1+\cos A$ (and analogs);

We get : $r_b+r_c=2R(1+\cos A)$ (and analogs) ;

$4m_a^2=b^2+c^2+2bc\cos A$; $\frac{b^2+c^2}{2R}(r_b+r_c)=(b^2+c^2)(1+\cos A)$

We need to prove : $(b^2+c^2)(1+\cos A)\ge b^2+c^2+2bc\cos A$

We obtain :$(b-c)^2\cos A\ge 0$ -true because $\cos A>0$ (since ΔABC acute)

Easy can be proved that : $h_a(r_b+r_c)=2r_br_c$ (and analogs)

$\frac{2m_a^2}{r_b+r_c}\le\frac{b^2+c^2}{4R}\rightarrow\frac{2m_a^2}{h_a(r_b+r_c)}\le\frac{b^2+c^2}{4Rh_a}$; $bc=2Rh_a$(and analogs)→

$$\frac{m_a^2}{r_br_c}\le\frac{b^2+c^2}{2bc}\ \textbf{(and analogs) for ΔABC acute(103)}$$

From (103) and $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) we obtain :

$$\frac{2m_a^2}{r_br_c}\le\frac{\sin(\omega+A)}{\sin\omega}\ \textbf{(and analogs) for ΔABC acute (104)}$$

$$\frac{2m_a^2}{r_br_c}\le\frac{1}{\sin\omega}\ \textbf{(and analogs) for ΔABC acute (105)}$$

From (105) after summation we obtain:

$$\frac{3}{2\sin\omega} \geq \sum \frac{m_a^2}{r_b r_c}$$ **(106) for ΔABC acute**

We know that: $4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c$ (and analogs)[5]. and $n_a g_a \geq m_a l_a$ (and analogs)[7]., $r_b r_c \leq m_a l_a$ (and analogs)(Panaitopol),we obtain :

$$1 \leq \frac{m_a l_a}{r_b r_c} \leq \frac{n_a g_a}{r_b r_c} \leq \frac{n_a^2 + g_a^2}{r_b r_c} \leq \frac{b}{c} + \frac{c}{b} - 1 = \frac{\sin(\omega + A)}{\sin\omega} - 1 \leq \frac{1}{\sin\omega} - 1$$ **(and analogs)**

for ΔABC acute (107)

If triangle ABC is acuteangled then we have ERDOS Inequality :

$R+r \leq \max(h_a, h_b, h_c)$ (RMM-Famous Inequalitys Marathon 1-100,inequality 31)[8].

If $h_a = \max(h_a, h_b, h_c) \rightarrow h_a \geq R+r \rightarrow \frac{h_a}{r} = 1 + \frac{b+c}{a} \geq \frac{R+r}{r} \rightarrow \frac{b+c}{a} \geq \frac{R}{r}$

$\frac{b+c}{a} = 1 + \frac{h_a}{r_a}$ (and analogs);We know that: $\frac{R}{r} \geq 1 + \frac{n_a}{h_a}$ (and analogs);

We obtain : $\frac{h_a}{r_a} \geq \frac{n_a}{h_a}$; $\frac{h_a}{r_a} \geq \frac{n_b}{h_b}$; $\frac{h_a}{r_a} \geq \frac{n_c}{h_c}$. $\rightarrow h_a h_b \geq r_a n_b$ and $h_a h_c \geq r_a n_c$

After summation we obtain: $h_a(h_b + h_c) \geq r_a(n_b + n_c)$

$$\frac{h_a}{r_a} \geq \frac{n_b + n_c}{h_b + h_c} \quad (108)$$ **, for ΔABC acute and a=min(a,b,c)**

We can prove easy that :$a^2 = 2R\frac{h_b h_c}{h_a}$ (and analogs), using this identity we obtain:

$\frac{h_a h_b}{h_c} \geq \frac{r_a n_b}{h_c}$ and $\frac{h_a h_c}{h_b} \geq \frac{r_a n_c}{h_b}$ and after summation we obtain :

$\frac{b^2+c^2}{2R} \geq r_a\left(\frac{n_b}{h_c} + \frac{n_c}{h_b}\right)$ (109), for ΔABC acute and a=min(a,b,c)

From (109) we obtain :

$$\frac{b^2+c^2}{bc} \geq \frac{r_a}{h_a}\left(\frac{n_b}{h_c} + \frac{n_c}{h_b}\right)$$ **(110) for ΔABC acute and a=min(a,b,c)**

From (110) and $\frac{\sin(\omega+A)}{\sin\omega} = \frac{b}{c} + \frac{c}{b}$ we obtain :

$$\frac{\sin(\omega+A)}{\sin\omega} \geq \frac{r_a}{h_a}\left(\frac{n_b}{h_c} + \frac{n_c}{h_b}\right)$$ **(111) for ΔABC acute and a=min(a,b,c)**

From (111) and (8) we obtain :

$$\frac{h_a}{r_a}\frac{1}{\sin\omega} \geq \frac{n_b}{h_c}+\frac{n_c}{h_b}$$ **(112) for ΔABC acute and a=min(a,b,c)**

From $m_a \geq \frac{b^2+c^2}{4R}$ (Tereshin) and (109) we obtain :

$$2\frac{m_a}{r_a} \geq \frac{n_b}{h_c}+\frac{n_c}{h_b}\ (113)$$ **for ΔABC acute and a=min(a,b,c)**

We consider $\Delta A_1B_1C_1$ with $a_1=\sqrt{a}$, $b_1=\sqrt{b}$, $c_1=\sqrt{c}$, $2S_1=\sqrt{r(4R+r)}$

(8) becomes: $\sqrt{\frac{ab+bc+ac}{r(4R+r)}} \geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$ (and analogs) ab+bc+ac=2R$(h_a+h_b+h_c)$

$r_a+r_b+r_c=4R+r$,we obtain :

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\max\left\{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}},\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}},\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right\}\ (114)$$

$2m_{a_1}=\sqrt{2(b+c)-a}$ (and analogs) useing (22) we obtain a new inequality:

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\max\left\{\sqrt{\frac{2(b+c)-a}{2(a+c)-b}}+\sqrt{\frac{2(a+c)-b}{2(b+c)-a}},\sqrt{\frac{2(a+c)-b}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(a+c)-b}},\sqrt{\frac{2(b+c)-a}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(b+c)-a}}\right\}\ (115)$$

From (95),(96),(98),(99) we obtain:

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}-1\right)(116)$$

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}-1\right)(117)$$

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{2(b+c)-a}{2(a+c)-b}}+\sqrt{\frac{2(a+c)-b}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(b+c)-a}}-1\right)(118)$$

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{2(b+c)-a}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(a+c)-b}}+\sqrt{\frac{2(a+c)-b}{2(b+c)-a}}-1\right)(119)$$

From (100) we obtain a new result :

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\frac{1}{2}\sum\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}-1\right)(120)$$

We show that: $3+\sum\frac{b+c}{a}=\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}$ [9].,from (15) we obtain :

$$3+\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\leq\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (121)$$

From (21) we obtain a new inequality :

$$3+\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}}\geq\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (122)$$

From (25) we obtain a new result :

$$3+\frac{2\sqrt{a^2+b^2+c^2}}{3R}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\leq\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\leq 3+\sqrt{\frac{2}{\sin\omega}}\sum\sqrt{\frac{m_a}{h_a}}\ (123)$$

From $\frac{m_a}{h_a}\geq\frac{1}{2}(\frac{b}{c}+\frac{c}{b})$(and analogs) after summation we obtain:

$$\frac{3}{2}+\sum\frac{m_a}{h_a}\geq\sum\frac{n_a}{\sqrt{(b-c)^2+4r^2}}\ (124)$$

From $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$(and analogs) we obtain :

$$3+\sum\frac{\sin(\omega+A)}{\sin\omega}=\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (125)$$

From (37) we obtain a new inequality:

$$\frac{3}{2}+\sum\frac{\sqrt{m_b m_c}}{h_a}\geq\sum\frac{n_a}{\sqrt{(b-c)^2+4r^2}}\ (126)$$

From (53) we obtain a new result:

$$4\sum\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}\geq 3+\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (127)$$

From (56) we obtain a new inequality :

$$3+\sum\sqrt{\frac{2}{\sin\omega}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)}\geq\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (128)$$

From (58) we obtain a new result:

$$3+2\sum\sqrt{\frac{m_a}{h_a}\left(2\sqrt{\frac{m_a m_b m_c}{l_a h_b h_c}}-1\right)}\geq\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (129)$$

From (65) we obtain a new inequality :

$$3+2\sum\frac{\sqrt{m_a(r_b+r_c-h_a)}}{h_a}\geq\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (130)$$

From (100) we obtain :

$$5+\frac{2}{\sin\omega}\geq\sum\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}\ (131)$$

From $\sqrt{\frac{ab+bc+ac}{r(4R+r)}}\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$ (and analogs)→ $\sqrt{\frac{2R}{r}}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)$

$\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)=\frac{\sqrt{r_br_c}}{l_a}$(and analogs)→ $\sqrt{\frac{2R}{r}}\geq 2\frac{\sqrt{r_br_c}}{l_a}\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}$ (and analogs), we obtain a new result :

$$\sqrt{\frac{R}{2r}}l_a\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sqrt{r_br_c}\ \text{(and analogs) (132)}$$

From (132) after summation we obtain a new inequality:

$$\sqrt{\frac{R}{2r}}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}(133)$$

We proved : $m_a\leq h_a+R\left(\frac{b-c}{a}\right)^2$ (and analogs)[1].,using $\frac{b}{c}=\frac{h_c}{h_b}$ (and analogs) and $\frac{\sin(\omega+A)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) after a easy manipulation we obtain a new inequality:

$\frac{m_a}{h_b}+\frac{m_b}{h_a}\leq\frac{a}{b}+\frac{b}{a}+\frac{R}{h_b}\left(\frac{b-c}{a}\right)^2+\frac{R}{h_a}\left(\frac{a-c}{b}\right)^2$

$$\frac{m_a}{h_b}+\frac{m_b}{h_a}\leq\frac{\sin(\omega+C)}{\sin\omega}+\frac{R}{h_b}\left(\frac{b-c}{a}\right)^2+\frac{R}{h_a}\left(\frac{a-c}{b}\right)^2\ \text{(and analogs) (134)}$$

From (134) we obtain a new inequality:

$$\frac{m_a}{h_b}+\frac{m_b}{h_a}\leq\frac{1}{\sin\omega}+\frac{R}{h_b}\left(\frac{b-c}{a}\right)^2+\frac{R}{h_a}\left(\frac{a-c}{b}\right)^2\ \text{(and analogs) (135)}$$

In ΔABC with usual notations:

$8(a^2b^2+b^2c^2+a^2c^2)\geq(a+b+c)(a+b)(b+c)(c+a)$([10]-RMM-SP380)

$l_a=\frac{2\sqrt{bc}}{b+c}\sqrt{r_br_b}$ (and analogs)→ $l_al_bl_c=\frac{8abc}{(a+b)(b+c)(c+a)}r_ar_br_c$; abc=4RS

We obtain:

$$\frac{a^2b^2+b^2c^2+a^2c^2}{4S^2} \geq \frac{(a+b+c)abc}{4S^2}\frac{r_ar_br_c}{l_al_bl_c} \rightarrow \frac{1}{\sin^2\omega} \geq \frac{2R}{r}\frac{r_ar_br_c}{l_al_bl_c} \text{ (136)}$$

From (114) and (136) we obtain a new inequality:

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{(r_a+r_b+r_c)r_ar_br_c}{(h_a+h_b+h_c)l_al_bl_c}}\max\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}, \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}, \sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right) \text{ (137)}$$

From (115) and (136) we obtain a new result :

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{(r_a+r_b+r_c)r_ar_br_c}{(h_a+h_b+h_c)l_al_bl_c}}\max\left\{\sqrt{\frac{2(b+c)-a}{2(a+c)-b}}+\sqrt{\frac{2(a+c)-b}{2(b+c)-a}}, \sqrt{\frac{2(a+c)-b}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(a+c)-b}}, \sqrt{\frac{2(b+c)-a}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(b+c)-a}}\right\} \text{ (138)}$$

From (116) and (136) we obtain a new inequality:

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{(r_a+r_b+r_c)r_ar_br_c}{(h_a+h_b+h_c)l_al_bl_c}}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}-1\right) \text{(139)}$$

From (117) and (136) we obtain a new result :

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{(r_a+r_b+r_c)r_ar_br_c}{(h_a+h_b+h_c)l_al_bl_c}}\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}-1\right) \text{(140)}$$

From (118) and (136) we obtain a new inequality:

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{(r_a+r_b+r_c)r_ar_br_c}{(h_a+h_b+h_c)l_al_bl_c}}\left(\sqrt{\frac{2(b+c)-a}{2(a+c)-b}}+\sqrt{\frac{2(a+c)-b}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(b+c)-a}}-1\right) \text{(141)}$$

From(119) and (136) we obtain a new inequality:

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{(r_a+r_b+r_c)r_ar_br_c}{(h_a+h_b+h_c)l_al_bl_c}}\left(\sqrt{\frac{2(b+c)-a}{2(a+b)-c}}+\sqrt{\frac{2(a+b)-c}{2(a+c)-b}}+\sqrt{\frac{2(a+c)-b}{2(b+c)-a}}-1\right) \text{(142)}$$

From (120) and (136) we obtain a new inequality:

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{(r_a+r_b+r_c)r_ar_br_c}{(h_a+h_b+h_c)l_al_bl_c}}\left(\frac{1}{2}\sum\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}-1\right) \text{(143)}$$

We can prove easy that: $h_a(r_a-r) = 2r_ar$ (and analogs),

$$b^2+c^2 = n_a^2+g_a^2+2r_ar \geq 2n_ag_a+2r_ar=2n_ag_a+h_a(r_a-r)$$

$$\frac{\sin(\omega+A)}{\sin\omega} = \frac{b}{c}+\frac{c}{b} \geq \frac{2n_ag_a+h_a(r_a-r)}{2Rh_a} = \frac{1}{R}\left(\frac{n_ag_a}{h_a}+\frac{r_a-r}{2}\right) \text{ (and analogs)(144)}$$

From (144) after summation we obtain a new result:

$\sum\frac{b+c}{a}\geq\frac{1}{R}\left(\sum\frac{n_ag_a}{h_a}+2R-r\right)=2+\frac{1}{R}\sum\frac{n_ag_a}{h_a}-\frac{r}{R}$

$\sum\frac{b+c}{a}=\frac{h_a+h_b+h_c-3r}{r}\geq2+\frac{1}{R}\sum\frac{n_ag_a}{h_a}-\frac{r}{R}\to\frac{h_a+h_b+h_c}{r}\geq5+\frac{1}{R}\sum\frac{n_ag_a}{h_a}-\frac{r}{R}=\frac{5R-r}{R}+\frac{1}{R}\sum\frac{n_ag_a}{h_a}$

We obtain a new inequality :

$$\frac{R}{r}\geq\frac{5R-r+\frac{n_ag_a}{h_a}+\frac{n_bg_b}{h_b}+\frac{n_cg_c}{h_c}}{h_a+h_b+h_c}\ (145)$$

From (136) and (145) we obtain a new inequality:

$$\frac{1}{2\sin^2\omega}\geq\frac{5R-r+\frac{n_ag_a}{h_a}+\frac{n_bg_b}{h_b}+\frac{n_cg_c}{h_c}}{h_a+h_b+h_c}\frac{r_ar_br_c}{l_al_bl_c}\ (146)$$

$5R-r\geq4R+r=r_a+r_b+r_c$ (true because $R\geq2r$ – Euler) .From (146) we obtain a new result:

$$\frac{1}{2\sin^2\omega}\geq\frac{r_a+r_b+r_c+\frac{n_ag_a}{h_a}+\frac{n_bg_b}{h_b}+\frac{n_cg_c}{h_c}}{h_a+h_b+h_c}\frac{r_ar_br_c}{l_al_bl_c}\ (147)$$

$g_a\geq h_a$ (and analogs)$\to$

$$\frac{1}{2\sin^2\omega}\geq\frac{5R-r+n_a+n_b+n_c}{h_a+h_b+h_c}\frac{r_ar_br_c}{l_al_bl_c}\ (148)$$

$$\frac{1}{2\sin^2\omega}\geq\frac{r_a+r_b+r_c+n_a+n_b+n_c}{h_a+h_b+h_c}\frac{r_ar_br_c}{l_al_bl_c}\ (149)$$

From $n_ag_a\geq m_al_a$ (and analogs)$\to$

$$\frac{1}{2\sin^2\omega}\geq\frac{5R-r+\frac{m_al_a}{h_a}+\frac{m_bl_b}{h_b}+\frac{m_cl_c}{h_c}}{h_a+h_b+h_c}\frac{r_ar_br_c}{l_al_bl_c}\ (150)$$

Any future inequalities will be presented as problems in Romanian Mathematical Magazine.

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