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RMM-SOLVED PROBLEMS

By Marin Chirciu – Romania

01. Solve in $\mathbb{R}$:

$$(2^x+3^x)\sqrt{6^{1-x}}=5$$

Daniel Sitaru

Solution:

$$(2^x+3^x)\sqrt{6^{1-x}}=5\Leftrightarrow(2^x+3^x)^2 6^{1-x}=25\Leftrightarrow\frac{2^{2x}+3^{2x}+2\cdot6^x}{6^x}=\frac{25}{6}\Leftrightarrow$$
$$\Leftrightarrow\frac{2^{2x}}{2^x3^x}+\frac{3^{2x}}{2^x3^x}+2=\frac{25}{6}$$
$$\Leftrightarrow\frac{2^x}{3^x}+\frac{3^x}{2^x}=\frac{13}{6}\Leftrightarrow\left(\frac{2}{3}\right)^x+\left(\frac{3}{2}\right)^x=\frac{2}{3}+\frac{3}{2}\Leftrightarrow x=\pm1$$

The set of the solutions is $S=\{-1,1\}$.

Remark: The problem can be developed.

Let $a>b>1$ fixed. Solve in $\mathbb{R}$

$$(a^x+b^x)\sqrt{(ab)^{1-x}}=a+b$$

Marin Chirciu

Solution:

$$(a^x+b^x)\sqrt{(ab)^{1-x}}=a+b\Leftrightarrow(a^x+b^x)^2(ab)^{1-x}=(a+b)^2\Leftrightarrow$$
$$\Leftrightarrow\frac{a^{2x}+b^{2x}+2\cdot(ab)^x}{(ab)^x}=\frac{(a+b)^2}{ab}\Leftrightarrow$$
$$\frac{a^{2x}}{a^xb^x}+\frac{b^{2x}}{a^xb^x}+2=\frac{a^2+b^2+2ab}{ab}\Leftrightarrow\frac{a^x}{b^x}+\frac{b^x}{a^x}=\frac{a^2+b^2}{ab}\Leftrightarrow$$
$$\Leftrightarrow\left(\frac{a}{b}\right)^x+\left(\frac{b}{a}\right)^x=\frac{a}{b}+\frac{b}{a}\Leftrightarrow x=\pm1$$

The set of the solution is $S=\{-1,1\}$.

Note:

For $a=3,b=2$ we obtain the proposed problem by Daniel Sitaru in RMM 11/2022.

02. In ΔABC holds:

$$\sum(a+2b)(a+2c)\le 81R^2$$

Daniel Sitaru

Solution: Lemma:

If $x,y,z>0$

$$(x+2y)(x+2z)\le(x+y+z)^2.$$

Proof:

$$(x+2y)(x+2z)\overset{AM-GM}{\le}\left[\frac{(x+2y)+(x+2z)}{2}\right]=(x+y+z)^2$$

with equality for $(x+2y)=(x+2z)\Leftrightarrow y=z$.

Let's get back to the main problem.
Using the Lemma for $(x,y,z)=(a,b,c)$ we obtain:

$$LHS=\sum(a+2b)(a+2c)\overset{Lemma}{\leq}\sum(a+b+c)^2=\sum(2p)^2=12p^2\overset{Mitrinovic}{\leq}$$
$$\overset{Mitrinovic}{\leq}12\cdot\frac{27R^2}{4}=81R^2=RHS$$

The equality holds if and only if the triangle is equilateral.
Remark: In the same way:
In ΔABC holds:

$$\sum(m_a+2m_b)(m_a+2m_c)\leq 3(4R+r)^2$$

Marin Chirciu

Solution:
Using the Lemma for $(x,y,z)=(m_a,m_b,m_c)$ we obtain:

$$LHS=\sum(m_a+2m_b)(m_a+2m_c)\overset{Lemma}{\leq}\sum(m_a+m_b+m_c)^2\overset{Leuenberger}{\leq}\sum(4R+r)^2$$
$$=3(4R+r)^2=RHS$$

Equality holds if and only if the triangle is equilateral.
Remark: In the same way:
In ΔABC holds:

$$\sum(h_a+2h_b)(h_a+2h_c)\leq\frac{243R^2}{4}$$

Marin Chirciu

Solution:
Using the Lemma for $(x,y,z)=(h_a,h_b,h_c)$ we obtain:

$$LHS=\sum(h_a+2h_b)(h_a+2h_c)\overset{Lemma}{\leq}\sum(h_a+h_b+h_c)^2\overset{Santalo}{\leq}\sum\left(p\sqrt{3}\right)^2=9p^2\leq$$
$$\overset{Mitrinovic}{\leq}9\cdot\frac{27R^2}{4}=\frac{243R^2}{4}=RHS$$

Equality holds if and only if the triangle is equilateral.
Remark: In the same way.
In ΔABC holds:

$$\sum(w_a+2w_b)(w_a+2w_c)\leq\frac{243R^2}{4}$$

Marin Chirciu

Solution:
Using the Lemma for $(x,y,z)=(w_a,w_b,w_c)$ we obtain:

$$LHS=\sum(w_a+2w_b)(w_a+2w_c)\overset{Lemma}{\leq}\sum(w_a+w_b+w_c)^2\overset{Santalo}{\leq}\sum\left(p\sqrt{3}\right)^2=$$
$$=9p^2\overset{Mitrinovic}{\leq}9\cdot\frac{27R^2}{4}=\frac{243R^2}{4}=RHS$$

Equality holds if and only if the triangle is equilateral.
Remark: In the same way.

In ΔABC holds:

$$\sum(s_a+2s_b)(s_a+2s_c)\le\frac{243R^2}{4}.$$

Marin Chirciu

Solution:

Using the Lemma for $(x,y,z)=(s_a,s_b,s_c)$ we obtain:

$$LHS=\sum(s_a+2s_b)(s_a+2s_c)\overset{Lemma}{\le}\sum(s_a+s_b+s_c)^2\overset{s_a\le h_a}{\le}\sum(h_a+h_b+h_c)^2\le$$

$$\overset{Santalo}{\le}\sum\left(p\sqrt{3}\right)^2=9p^2\overset{Mitrinovic}{\le}9\cdot\frac{27R^2}{4}=\frac{243R^2}{4}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

In ΔABC holds:

$$\sum(r_a+2r_b)(r_a+2r_c)\le\frac{243R^2}{4}$$

Marin Chirciu

Solution:

Using Lemma for $(x,y,z)=(r_a,r_b,r_c)$ we obtain:

$$LHS=\sum(r_a+2r_b)(r_a+2r_c)\overset{Lemma}{\le}\sum(r_a+r_b+r_c)^2=\sum(4R+r)^2=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

In ΔABC holds:

$$\sum\left(\tan\frac{A}{2}+2\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+2\tan\frac{C}{2}\right)\le3\left(1+\frac{R}{r}\right)$$

Marin Chirciu

Solution:

Using the Lemma for $(x,y,z)=\left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\tan\frac{A}{2}+2\tan\frac{B}{2}\right)\left(\tan\frac{A}{2}+2\tan\frac{C}{2}\right)\overset{Lemma}{\le}\sum\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2=$$

$$=\sum\left(\frac{4R+r}{p}\right)^2=3\cdot\frac{(4R+r)^2}{p^2}\overset{Gerretsen}{\le}3\cdot\frac{(4R+r)^2}{r(4R+r)^2}=3\cdot\frac{R+r}{r}=3\left(1+\frac{R}{r}\right)=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

In ΔABC holds:

$$\sum\left(\cot\frac{A}{2}+2\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}+2\cot\frac{C}{2}\right)\le81\left(\frac{R}{2r}\right)^2$$

Marin Chirciu

Solution:

Using the Lemma for $(x,y,z)=\left(\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\cot\frac{A}{2}+2\cot\frac{B}{2}\right)\left(\cot\frac{A}{2}+2\cot\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2=$$

$$=\sum\left(\frac{p}{r}\right)^2=3\cdot\frac{p^2}{r^2}\overset{Mitrinovic}{\leq}3\cdot\frac{\frac{27R^2}{4}}{r^2}=\frac{81R^2}{4r^2}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

In ΔABC holds:

$$\sum\left(\sin\frac{A}{2}+2\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}+2\sin\frac{C}{2}\right)\leq\frac{27}{4}$$

Marin Chirciu

Solution:

Using the Lemma for $(x,y,z)=\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\sin\frac{A}{2}+2\sin\frac{B}{2}\right)\left(\sin\frac{A}{2}+2\sin\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2\overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq}\sum\left(\frac{3}{2}\right)^2=3\cdot\left(\frac{3}{2}\right)^2=\frac{27}{4}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

In ΔABC holds:

$$\sum\left(\cos\frac{A}{2}+2\cos\frac{B}{2}\right)\left(\cos\frac{A}{2}+2\cos\frac{C}{2}\right)\leq\frac{81}{4}.$$

Marin Chirciu

Solution:

Using the Lemma for $(x,y,z)=\left(\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\cos\frac{A}{2}+2\cos\frac{B}{2}\right)\left(\cos\frac{A}{2}+2\cos\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)^2\overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq}\sum\left(\frac{3\sqrt{3}}{2}\right)^2=3\cdot\left(\frac{3\sqrt{3}}{2}\right)^2=\frac{81}{4}=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

In ΔABC holds:

$$\sum\left(\sec\frac{A}{2}+2\sec\frac{B}{2}\right)\left(\sec\frac{A}{2}+2\sec\frac{C}{2}\right)\leq\frac{9R^2}{r^2}$$

Marin Chirciu

Solution:

Using Lemma for $(x,y,z)=\left(\sec\frac{A}{2},\sec\frac{B}{2},\sec\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\sec\frac{A}{2}+2\sec\frac{B}{2}\right)\left(\sec\frac{A}{2}+2\sec\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\right)^2\overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq}\sum\left(\frac{2p}{3r}\right)^2=3\cdot\frac{4p^2}{9r^2}\overset{Mitrinovic}{\leq}3\cdot\frac{4\cdot\frac{27R^2}{4}}{9r^2}=\frac{9R^2}{r^2}=RHS.$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

In ΔABC holds:

$$\sum\left(\csc\frac{A}{2}+2\csc\frac{B}{2}\right)\left(\csc\frac{A}{2}+2\csc\frac{C}{2}\right)\leq\frac{27R^2}{r^2}$$

Marin Chirciu

Solution: Using the Lemma for $(x,y,z)=\left(\csc\frac{A}{2},\csc\frac{B}{2},\csc\frac{C}{2}\right)$ we obtain:

$$LHS=\sum\left(\csc\frac{A}{2}+2\csc\frac{B}{2}\right)\left(\csc\frac{A}{2}+2\csc\frac{C}{2}\right)\overset{Lemma}{\leq}\sum\left(\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\right)^2\overset{Jensen}{\leq}$$

$$\overset{Jensen}{\leq}\sum\left(\frac{3R}{r}\right)^2=3\cdot\frac{9R^2}{r^2}=\frac{27R^2}{r^2}=RHS.$$

Equality holds if and only if the triangle is equilateral.

03. If $a,b,c>0$ then:

$$2^{a-b}+2^{b-c}+2^{c-a}\geq\frac{2^a+2^b+2^c}{\sqrt[3]{2^{a+b+c}}}$$

Daniel Sitaru

Solution:

$$2^{a-b}+2^{b-c}+2^{c-a}\geq\frac{2^a+2^b+2^c}{\sqrt[3]{2^{a+b+c}}}\Leftrightarrow\frac{2^a}{2^b}+\frac{2^b}{2^c}+\frac{2^c}{2^a}\geq\frac{2^a+2^b+2^c}{\sqrt[3]{2^a}\sqrt[3]{2^b}\sqrt[3]{2^c}}$$

With the substitution $\left(\sqrt[3]{2^a},\sqrt[3]{2^b},\sqrt[3]{2^c}\right)=(x,y,z)$ the conclusion can be written:

$$\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}\geq\frac{x^3+y^3+z^3}{xyz}\Leftrightarrow\frac{x^6z^3+y^6x^3+z^6y^3}{x^3y^3z^3}\geq\frac{x^3+y^3+z^3}{xyz}\Leftrightarrow$$

$\Leftrightarrow x^6z^3+y^6x^3+z^6y^3\geq x^2y^2z^2(x^3+y^3+z^3)$, (*) which follows from means inequality:

$x^6z^3+x^6z^3+y^6x^3\overset{AM-GM}{\geq}3\sqrt[3]{x^6z^3\cdot x^6z^3\cdot y^6x^3}=3\sqrt[3]{x^{15}y^6z^6}=3x^5y^2z^2$ (1)

we write the other two analog inequalities $y^6x^3+y^6x^3+z^6y^3\geq 3y^5z^2x^2$ (2),

$z^6y^3+z^6y^3+z^6z^3\geq 3z^5x^2y^2$ (3).

We add the inequalities (1), (2), (3), we divide by 3 and we obtain (*).

Equality holds if and only if $x=y=z\Leftrightarrow a=b=c$.

04. If $x,y,z>0$ then in ΔABC

$$\sum\frac{x}{y+z}\cdot\frac{a\sqrt{a}}{\sqrt{h_a}}\geq\sqrt{6F}$$

D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

Solution:

$$LHS=\sum\frac{x}{y+z}\cdot\frac{a\sqrt{a}}{\sqrt{h_a}}=\sum\frac{x}{y+z}\cdot\frac{a^2}{\sqrt{ah_a}}=\sum\frac{x}{y+z}\cdot\frac{a^2}{\sqrt{2F}}=\frac{1}{\sqrt{2F}}\sum\frac{x}{y+z}\cdot a^2\geq$$

$$\overset{Tsintsifas}{\geq} \frac{1}{\sqrt{2F}} \cdot 2\sqrt{3}F = \sqrt{6F} = RHS$$

Lemma (G. Tsintsifas).
In ΔABC holds:

$$\frac{x}{y+z}a^2 + \frac{y}{z+x}b^2 + \frac{z}{x+y}c^2 \geq 2\sqrt{3}S, \text{ where } x, y, z > 0$$

G. Tsintsifas

Solution:

We have $\sum \frac{x}{y+z}a^2 = \sum\left(\frac{x}{y+z} + 1 - 1\right)a^2 = \sum \frac{x+y+z}{y+z}a^2 - \sum a^2 \overset{Bergstrom}{\geq}$

$$\geq (x+y+z)\frac{(\sum a)^2}{\sum(y+z)} - \sum a^2 = (x+y+z)\frac{(2p)^2}{2(x+y+z)} - 2(p^2 - r^2 - 4Rr) =$$
$$= 2p^2 - 2(p^2 - r^2 - 4Rr) = 2(r^2 + 4Rr)$$

Above, we've used the known identities in triangle:
$\sum a = 2p$ and $\sum a^2 = 2(p^2 - r^2 - 4Rr)$.
It remains to prove that $2(r^2 + 4Rr) \geq 2\sqrt{3}S \Leftrightarrow r^2 + 4Rr \geq \sqrt{3}rp \Leftrightarrow 4R + r \geq p\sqrt{3}$, witch is Doucet's inequality.Equality holds if and only if $a = b = c$ and $x = y = z$.

05. In ΔABC holds:

$$\sum \frac{a^3}{b+c} \geq 2\sqrt{3}F$$

D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

Solution:

$$LHS = \sum \frac{a^3}{b+c} \overset{Holder}{\geq} \frac{(\sum a)^3}{3\sum(b+c)} = \frac{(\sum a)^3}{3 \cdot 2\sum a} = \frac{(\sum a)^2}{6} = \frac{(2p)^2}{6} = \frac{4p^2}{6} \overset{Mitrinovic}{\geq} 2\sqrt{3}pr$$
$$= 2\sqrt{3}F = RHS.$$

Equality holds if and only if the triangle is equilateral.
Remark: The problem can be developed.
In ΔABC holds:

$$\sum \frac{a^3}{b+\lambda c} \geq \frac{4\sqrt{3}}{\lambda+1}F, \lambda \geq 0$$

Marin Chirciu

Solution:

$$LHS = \sum \frac{a^3}{b+\lambda c} \overset{Holder}{\geq} \frac{(\sum a)^3}{3\sum(b+\lambda c)} = \frac{(\sum a)^3}{3(\lambda+1)\sum a} = \frac{(\sum a)^2}{3 \cdot (\lambda+1)} = \frac{(2p)^2}{3 \cdot (\lambda+1)} =$$
$$= \frac{4p^2}{3 \cdot (\lambda+1)} \overset{Mitrinovic}{\geq} \frac{4\sqrt{3}}{\lambda+1}pr = \frac{4\sqrt{3}}{\lambda+1}F = RHS.$$

Equality holds if and only if the triangle is equilateral
Note: For $\lambda = 1$ we obtain the problem J.2104 from RMM-40 Spring Edition2024, proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți.

In ΔABC holds:

$$\sum \frac{a^3}{b+c} \ge 2\sqrt{3}F.$$

D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

Remark: The problem can be developed.

In ΔABC holds:

$$\sum \frac{a^n}{b+\lambda c} \ge \left(\frac{2p}{3}\right)^{n-1} \cdot \frac{3}{\lambda+1}, \lambda \ge 0.$$

Marin Chirciu

Solution:

$$LHS = \sum \frac{a^n}{b+\lambda c} \overset{Holder}{\ge} \frac{(\sum a)^n}{3^{n-2}\sum(b+\lambda c)} = \frac{(\sum a)^n}{3^{n-2}(\lambda+1)\sum a} = \frac{(\sum a)^{n-1}}{3^{n-2}(\lambda+1)} =$$

$$= \frac{(2p)^{n-1}}{3^{n-2}(\lambda+1)} = \left(\frac{2p}{3}\right)^{n-1} \frac{3}{\lambda+1} = \frac{2^{n-1}p^{n-1}}{3^{n-2}(\lambda+1)} = \frac{2^{n-1}p^{n-3}p^2}{3^{n-2}(\lambda+1)} \overset{Mitrinovic}{\ge}$$

$$\overset{Mitrinovic}{\ge} \frac{2^{n-1}p^{n-3} \cdot 3\sqrt{3}pr}{3^{n-2}(\lambda+1)} = \frac{2^{n-1}p^{n-3} \cdot 3\sqrt{3}F}{3^{n-2}(\lambda+1)} = RHS$$

Equality holds if and only if the triangle is equilateral.

Note: For $\lambda = 1$ and $n = 3$ we obtain the problem J.2104 from RMM-40 Spring Edition 2024, proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți.

Reference:

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