A NEW GENERALIZATION FOR HADWIGER - FINSLER INEQUALITY

D.M. BĂTINEŢU - GIURGIU, DANIEL SITARU - ROMANIA

Abstract. In this paper is presented a new generalization for Hadwiger - Finsler's inequality in triangles.

HADWIGER - FINSLER

In ΔABC the following relationship holds:

$$a^2+b^2+c^2 \geq 4\sqrt{3}F+\frac{1}{2}\sum_{cyc}(a-b)^2$$

GENERALIZATION

If $m \geq 0$ then in ΔABC the following relationship holds:

$$a^{2m+2}+b^{2m+2}+c^{2m+2} \geq 2^{2m+2}\cdot(\sqrt{3})^{1-m}\cdot F^{m+1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 \tag{1}$$

Proof.

$$\sum_{cyc}(a^{m+1}-b^{m+1})^2 = 2\sum_{cyc}a^{2m+2}-2\sum_{cyc}(ab)^{m+1}$$

$$2\sum_{cyc}a^{2m+2} = 2\sum_{cyc}(ab)^{m+1}+\sum_{cyc}(a^{m+1}-b^{m+1})^2$$

$$\sum_{cyc}a^{2m+2} = \sum_{cyc}(ab)^{m+1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 \geq$$

$$\overset{\text{RADON}}{\geq} \frac{1}{3^m}\left(\sum_{cyc}ab\right)^{m+1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 \geq$$

$$\overset{\text{GORDON}}{\geq} \frac{1}{3^m}(4\sqrt{3}F)^{m+1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= \frac{1}{3^m}\cdot(2^2)^{m+1}\cdot(\sqrt{3})^{m+1}\cdot F^{m+1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= 2^{2m+2}\cdot 3^{\frac{m+1}{2}-m}\cdot F^{m+1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= 2^{m+2}\cdot(\sqrt{3})^{1-m}\cdot F^{m+1}+\frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2$$

Equality holds for $a=b=c$. □

Observation 1 (GOLDNER'S INEQUALITY)

In ΔABC the following relationship holds:

$$ab+bc+ca \geq 4\sqrt{3}F \tag{2}$$

Proof.

$$ab+bc+ca=s^2+r^2+4Rr \overset{\text{GERRETSEN}}{\geq}$$
$$\geq 16Rr-5r^2+r^2+4Rr=20Rr-4r^2$$

Remains to prove:

$$20Rr-4r^2 \geq 4\sqrt{3}F$$
$$20Rr-4r^2 \geq 4\sqrt{3}rs$$
$$4\sqrt{3}rs \overset{\text{MITRINOVIC}}{\geq} 4\sqrt{3}r \cdot 3\sqrt{3}r=36r^2$$

Remains to prove:

$$20Rr-4r^2 \geq 36r^2 \Leftrightarrow 20Rr \geq 40r^2$$
$$\Leftrightarrow R \geq 2r(\text{EULER})$$

□

Observation 2 (IONESCU WEITZENBOCK'S INEQUALITY)
In ΔABC the following relationship holds:

$$a^2+b^2+c^2 \geq 4\sqrt{3}F$$

Proof.

$$a^2+b^2+c^2 \geq ab+bc+ca \overset{(2)}{\geq} 4\sqrt{3}F$$

□

Observation 3.
For $m=0$ in (1):

$$a^2+b^2+c^2 \geq 4\sqrt{3}F+\frac{1}{2}\sum_{cyc}(a-b)^2$$

which is the original Hadwiger - Finsler's inequality in ΔABC.
Observation 4.
For $m=1$ in (1):

$$a^4+b^4+c^4 \geq 16F^2+\frac{1}{2}\sum_{cyc}(a^2-b^2)^2$$

Observation 5.
For $m=2$ in (1):

$$a^6+b^6+c^6 \geq \frac{64\sqrt{3}}{3}F^3+\frac{1}{2}\sum_{cyc}(a^3-b^3)^2$$

Observation 6.
For $m=3$ in (1):

$$a^8+b^8+c^8 \geq \frac{256}{3}F^4+\frac{1}{2}\sum_{cyc}(a^4-b^4)^2$$

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com