A NEW PROOF FOR ERDÖS LEMMA

1. Abstract

In 1845 Bertrand postulated that there is always a prime between n and $2n$, and he verified this for $n < 3\times 10^6$ [1]. Chebyshev gave an analytic proof of the postulate in 1850. In 1932, in his first paper, Erdös gave a beautiful elementary proof using nothing more than a few easily verified facts about the middle binomial coefficient see [3]. He first proved four lemmas ([2]). In this article, we provide a new proof for one of these lemmas.

Theorem 1.1. *$\forall n \in \mathbb{N}$ we have,*

$$\prod_{p\le n} p < 4^n \tag{1}$$

where the product is over primes.

Proof. Define a function

$$a(n) = \begin{cases} 1, & \text{when n is prime.} \\ 0, & \text{otherwise.} \end{cases}$$

Then for prime p we have,

$$\vartheta(x) := \sum_{p\le x} \ln(p) = \sum_{n\le x} a(n)\ln(n)$$

Observe that

$$\sum_{n\le x} a(n) = \pi(x)$$

Summing by parts we get

$$\vartheta(x) = \left(\sum_{n\le x} a(n)\right)\cdot \ln(x) - \int_2^x \left(\sum_{n\le t} a(n)\right) d\left(\ln(t)\right)$$

$$\vartheta(x) = \pi(x)\ln(x) - \int_2^x \frac{\pi(t)}{t}\,dt$$

Now it is known that $\forall x \ge 17$ (see [4])

$$\frac{x}{\ln(x)} < \pi(x) < \frac{30\ln(113)}{113}\frac{x}{\ln(x)}$$

$$\implies \pi(x) < \frac{30\ln(113)}{113}\frac{x}{\ln(x)} \text{ and } -\pi(x) < -\frac{x}{\ln(x)}$$

Substituting these inequalities in above equation we get

$$\begin{aligned}
\vartheta(x) &< \frac{30\ln(113)}{113}\frac{x}{\ln(x)}\cdot\ln(x) - \int\limits_2^x \frac{t}{\ln(t)}\cdot\frac{1}{t}\,dt \\
&= \frac{30\ln(113)}{113}x - \mathrm{li}(x)|_2^x \\
&= \frac{30\ln(113)}{113}x - (\mathrm{li}(x) - \mathrm{li}(2)) \\
&< \frac{30\ln(113)}{113}x \quad (\because \text{since li(x)-li(2)} > 0) \\
&< \ln(4)\cdot x \qquad \left(\because \frac{30\ln(113)}{113} < \ln(4)\right)
\end{aligned}$$

Hence we have

$$\begin{gathered}
\vartheta(x) < \ln\left(4^x\right) \\
\text{or} \sum_{p\le x} \ln(p) < \ln\left(4^x\right) \\
\text{or } \ln\left(\prod_{p\le x} p\right) < \ln\left(4^x\right)
\end{gathered}$$

Manually checking $\forall x \le 17$ and combining with above we get,

$$\boxed{\prod_{p\le x} p < 4^x}$$

$\blacksquare$

References

[1] Sondow, Jonathan and Weisstein, Eric W. "Bertrand's Postulate." From MathWorld –A Wolfram Web Resource. https://mathworld.wolfram.com/BertrandsPostulate.html

[2] Hoffman, P. The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth. New York: Hyperion, 1998.

[3] M. Aigner and G. M. Ziegler, Proofs from the book (4th ed.), Springer, 2010.

[4] Rosser, J. Barkley; Schoenfeld, Lowell (1962). "Approximate formulas for some functions of prime numbers". Illinois J. Math. 6: 64–94. doi:10.1215/ijm/1255631807. ISSN 0019-2082. Zbl 0122.05001

AMRIT AWASTHI

H.no. 15, DN Enclave, Dayanand Nagar, F.G.C Road

Amritsar-143008, Punjab, India.

amrit28awasthi@gmail.com