ABOUT THE FAMOUS BĂTINEŢU - GIURGIU SEQUENCES

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Abstract. In this paper we will give the definition of Bătineţu - Giurgiu's sequences, a few properties of these and some applications.

The original Bătineţu - Giurgiu's sequence is defined as:

$$(B-G)_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}; n \geq 2$$

We will prove that:

$$\lim_{n\to\infty} (B-G)_n = e$$

using only elementary methods. (without Stirling).
Main result:

$$\lim_{n\to\infty} (B-G)_n = e$$

Proof.

$$\text{Let be: } v_n = \left(\frac{n+1}{n}\right)^2 \cdot \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}; n \geq 2$$

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \left(\frac{n+1}{n} \cdot \frac{\sqrt[n]{n!}}{n} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!}}\right) =$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \cdot \lim_{n\to\infty} \sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}} =$$

$$\overset{\text{CAUCHY-D'ALEMBERT}}{=} 1 \cdot \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \cdot \lim_{n\to\infty} \frac{\frac{(n+2)^{n+2}}{(n+2)!}}{\frac{(n+1)^{n+1}}{(n+1)!}} =$$

$$= \lim_{n\to\infty} \frac{n!(n+1)}{(n+1)^n(n+1)} \cdot \frac{n^n}{n!} \cdot \lim_{n\to\infty} \frac{(n+2)^{n+1} \cdot (n+2)}{(n+1)! \cdot (n+2)} \cdot \frac{(n+1)!}{(n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^{n+1} =$$

$$= \lim_{n\to\infty} \left(1 + \frac{n}{n+1} - 1\right)^n \cdot \lim_{n\to\infty} \left(1 + \frac{n+2}{n+1} - 1\right)^{n+1} =$$

$$= \lim_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^n \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^{n+1} =$$

$$= e^{-1} \cdot e^1 = 1$$

$$\lim_{n\to\infty} v_n = 1$$

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n} \cdot \frac{n!}{(n+1)!} \cdot \sqrt[n+1]{(n+1)!} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}\cdot\lim_{n\to\infty}\frac{n!\cdot n}{(n+1)!}\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\left(\frac{n+1}{n}\right)=$$

$$= e^2\cdot\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{n+1}{n}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}=$$

$$= e^2\cdot 1\cdot 1\cdot\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}=$$

$$= e^2\cdot\lim_{n\to\infty}\frac{(n+2)!}{(n+2)^{n+2}}\cdot\frac{(n+1)^{n+1}}{(n+1)!}=$$

$$= e^2\cdot\lim_{n\to\infty}\frac{(n+1)!(n+2)}{(n+2)(n+1)!}\cdot\left(\frac{n+1}{n+2}\right)^{n+1}=$$

$$= e^2\cdot\lim_{n\to\infty}\left(1+\frac{n+1}{n+2}-1\right)^{n+1}=$$

$$= e^2\cdot\lim_{n\to\infty}\left(1+\frac{-1}{n+2}\right)^{n+1}=e^2\cdot e^{-1}=e$$

$$\lim_{n\to\infty}(B-G)_n=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)=$$

$$=\lim_{n\to\infty}\frac{n^2}{\sqrt[n]{n!}}(v_n-1)=$$

$$=\lim_{n\to\infty}\left(\frac{n^2}{\sqrt[n]{n!}}\cdot\frac{v_n-1}{\ln v_n}\cdot\ln v_n\right)=$$

$$=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{n!}}\cdot\frac{v_n-1}{\ln v_n}\cdot\ln v_n^n\right)=$$

$$=e\cdot 1\cdot\ln e=e$$

□

Definition 1:
Let be $t>0$ and $(a_n)_{n\ge1}; a_n>0$ a sequence. The sequence $(a_n)_{n\ge1}$ has the Bătineţu - Giurgiu's property if exists:

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a>0$$

The couple t and a_n define a (t,B) -sequence.
Property 1.
If $t>0$ and $(a_n)_{n\ge1}$ is a (t,B) - sequence then:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}=\frac{a}{e^t} \tag{1}$$

Proof.

$(a_n)_{n\ge1}$ is a (t,B) - sequence hence exists:

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a>0$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{nt}}}\overset{\text{CAUCHY-D'ALEMBERT}}{=}$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{(n+1)t}}\cdot\frac{n^{nt}}{a_n}=$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} \cdot \left(\frac{n}{n+1}\right)^{(n+1)t} = a \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{(n+1)t} = \frac{a}{e^t}$$

□

Property 2.
If $t > 0$ and $(a_n)_{n\geq 1}$ is a (t, B) - sequence and $u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}; n \geq 2$ then:

(2) $$\lim_{n\to\infty} u_n^n = e^t$$

Proof.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} \cdot \frac{n^t}{\sqrt[n]{a_n}} \cdot \left(\frac{n+1}{n}\right)^t \overset{(1)}{=}$$

$$= \frac{a}{e^t} \cdot \frac{e^t}{a} \cdot 1 = 1. \text{ Hence:}$$

$$\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1 \text{ and:}$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} \cdot \frac{(n+1)^t}{\sqrt[n+1]{a_{n+1}}} \cdot \left(\frac{n}{n+1}\right)^t = a \cdot \frac{e^t}{a} \cdot 1 = e^t$$

□

Definition 2.
If $t > 0$ and $(a_n)_{n\geq 1}$ is a (t, B) - sequence we define:

$$B_n = \frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}} - \frac{n^{t+1}}{\sqrt[n]{a_n}}; n \geq 2$$

and the name of this sequence is the t - Bătineţu - Giurgiu's sequence noted as $(t, B - G)$.
Theorem 1.
If $(t, B - G)$ is a Bătineţu - Giurgiu's sequence then:

(4) $$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \left(\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}} - \frac{n^t}{\sqrt[n]{a_n}}\right) = \frac{e^t}{a}$$

Proof.

$$\text{Let be: } v_n = \left(\frac{n+1}{n}\right)^{t+1} \cdot \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}; n \geq 2$$

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \left(\frac{n+1}{n} \cdot \frac{(n+1)^t}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{\sqrt[n]{a_n}}{n^t}\right) = 1 \cdot \frac{e^t}{a} \cdot \frac{a}{e^t} = 1$$

$$\lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1$$

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n(t+1)} \cdot \frac{a_n}{a_{n+1}} \cdot \sqrt[n+1]{a_{n+1}} =$$

$$= e^{t+1} \cdot \lim_{n\to\infty} \frac{a_n \cdot n^t}{a_{n+1}} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} \cdot \left(\frac{n+1}{n}\right)^t =$$

$$= e^{t+1} \cdot \frac{1}{a} \cdot \frac{a}{e^t} \cdot 1 = e$$

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty}\left(\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}} - \frac{n^{t+1}}{\sqrt[n]{a_n}}\right) =$$

$$= \lim_{n\to\infty} \frac{n^{t+1}}{\sqrt[n]{a_n}}(v_n - 1) = \lim_{n\to\infty}\left(\frac{n^{t+1}}{\sqrt[n]{a_n}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n\right) =$$

$$= \lim_{n\to\infty}\left(\frac{n^t}{\sqrt[n]{a_n}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n\right) = \frac{e^t}{a} \cdot 1 \cdot \ln e = \frac{e^t}{a}$$

□

Definition 3.
If $t > 0$ and $(a_n)_{n\geq 1}$ is a $(t+1, B)$ sequence we define:

$$G_n = \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{a_n}}{n^t}; n \geq 2$$

and the name of this sequence is the $t+1$ - Bătineţu - Giurgiu's sequence noted $(t+1, B-G)$.
Theorem 2.
If $(t+1, B-G)$ is a Bătineţu - Giurgiu's sequence then:

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{a_n}}{n^t}\right) = \frac{a}{e^{t+1}} \tag{5}$$

Proof.

$$\text{Let be: } w_n = \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} \cdot \frac{n^t}{\sqrt[n]{a_n}}; n \geq 2$$

$$\text{By (1): } \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^{t+1}} = \frac{a}{e^{t+1}}$$

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} \cdot \frac{n^t}{\sqrt[n]{a_n}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} \cdot \left(\frac{n}{n+1}\right)^t =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{t+1}} \cdot \frac{n^{t+1}}{\sqrt[n]{a_n}} \cdot \frac{n+1}{n} = \frac{a}{e^{t+1}} \cdot \frac{e^{t+1}}{a} \cdot 1 = 1$$

$$\lim_{n\to\infty} \frac{w_n - 1}{\ln w_n} = 1$$

$$\lim_{n\to\infty} w_n^n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \left(\frac{n}{n+1}\right)^{nt} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{n^{t+1} a_n} \cdot \frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}} \cdot \left(\frac{n}{n+1}\right)^{nt} \cdot \left(\frac{n}{n+1}\right)^{t+1} =$$

$$= a \cdot \frac{1}{e^t} \cdot \frac{e^{t+1}}{a} \cdot 1 = e$$

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{a_n}}{n^t}\right) =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^t} \cdot (w_n - 1) = \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^t} \cdot \frac{w_n - 1}{\ln w_n} \cdot \ln w_n =$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^{t+1}} \cdot \frac{w_n - 1}{\ln w_n} \cdot \ln w_n^n =$$

$$= \frac{a}{t+1} \cdot 1 \cdot \ln e = \frac{a}{e^{t+1}}$$

□

Theorem 3.
If $t \geq 0; (a_n)_{n\geq 1}; a_n > 0; n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \frac{n^t \cdot a_{n+1}}{a_n} = a > 0$ then:

(6)
$$\lim_{n\to\infty} n^{t+1}(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = -ate^t$$

Proof.

$$\text{Denote } d_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}; n \geq 2$$

$$\lim_{n\to\infty} n^t \cdot \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n \cdot n^{nt}} \overset{\text{CAUCHY-D'ALEMBERT}}{=}$$

$$= \lim_{n\to\infty} \frac{a_{n+1} \cdot (n+1)^{t(n+1)}}{a_n \cdot n^{tn}} = \lim_{n\to\infty} \frac{a_{n+1} \cdot n^t}{a_n} \cdot \left(\frac{n+1}{n}\right)^{t(n+1)} = a \cdot e^t$$

(7)
$$\lim_{n\to\infty} n^t \cdot \sqrt[n]{a_n} = a \cdot e^t$$

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}} \cdot (n+1)^t}{\sqrt[n]{a_n} \cdot n^t} \cdot \left(\frac{n}{n+1}\right)^t =$$

$$= \frac{a \cdot e^{t+1}}{a \cdot e^t} \cdot \frac{1}{e} = 1$$

$$\lim_{n\to\infty} \frac{d_n - 1}{\ln d_n} = 1$$

$$\lim_{n\to\infty} d_n^n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{n^t \cdot a_{n+1}}{a_n} \cdot \frac{1}{(n+1)^t \cdot \sqrt[n+1]{a_{n+1}}} \cdot \left(\frac{n+1}{n}\right)^t =$$

$$= a \cdot \frac{1}{a \cdot e^t} = e^{-t}$$

$$\lim_{n\to\infty} n^{t+1}(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) =$$

$$= \lim_{n\to\infty} n^{t+1} \cdot \sqrt[n]{a_n} \cdot (d_n - 1) =$$

$$= \lim_{n\to\infty} n^{t+1} \cdot \sqrt[n]{a_n} \cdot \frac{d_n - 1}{\ln d_n} \cdot \ln d_n =$$

$$= \lim_{n\to\infty} n^t \cdot \sqrt[n]{a_n} \cdot \frac{d_n - 1}{\ln d_n} \cdot \ln d_n^n =$$

$$\overset{(7)}{=} a \cdot e^t \cdot 1 \cdot \ln e^{-1} = -a \cdot t \cdot e^t$$

□

Application 1.
For $t = 1$ in (4):

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)=\frac{e}{a} \tag{8}$$

For $a_n = n!$ in (8):

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}\right)=e$$

Application 2.
For $t = 1$ in (6):

$$\lim_{n\to\infty} n^2(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}) = -a\cdot e \tag{9}$$

For $a_n = \frac{1}{n!}$ in (9):

$$\lim_{n\to\infty} n^2\left(\frac{1}{\sqrt[n+1]{(n+1)!}}-\frac{1}{\sqrt[n]{n!}}\right) = -e$$

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