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THE AVALANCHE OF GEOMETRIC INEQUALITIES

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Note by editor: The author make a beautiful mind process to discover geometrical inequalities in triangle with Nagel and Gergonne's cevians and Brocard's angle in a given triangle. His way to write articles it is absolutely outstanding for readers because these can see the entire chains of logics.

We consider the next identity:

$$\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2r_ah_a} \text{ (and analogs) } \mathbf{[1]}\rightarrow$$

$$\mathbf{2\frac{h_a}{n_a}\left(\frac{R}{r}-1\right)=\frac{n_a}{r_a}+\frac{r_a}{n_a}\ (1)}$$

We know that : $h_a \le n_a \rightarrow 2\left(\frac{R}{r}-1\right) \ge \frac{n_a}{r_a}+\frac{r_a}{n_a}$ (and analogs)

$$\frac{2R}{r} \ge \frac{n_a}{r_a}+\frac{r_a}{n_a}+2=\left(\sqrt{\frac{n_a}{r_a}}\right)^2+\left(\sqrt{\frac{r_a}{n_a}}\right)^2+2\rightarrow$$

$$\mathbf{\sqrt{\frac{2R}{r}} \ge \sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\ \text{(and analogs)}(2)}$$

We know that :$m_a \ge \frac{b+c}{2}\cos\frac{A}{2}$ (and analogs);$bc=2Rh_a$(and analogs);

$a=4R\sin\frac{A}{2}\cos\frac{A}{2}$ (and analogs); After simple calculations we obtain a new inequality:

$$\mathbf{4m_a\sin\frac{A}{2} \ge h_b+h_c\ \text{(and analogs)}\ (3)}$$

We know that : $\sin\frac{A}{2}=\sqrt{\frac{r}{2R}\frac{r_a}{h_a}}$ (and analogs) and using (3) we obtain a new result:

$$\mathbf{4m_a\sqrt{\frac{r_a}{h_a}} \ge \sqrt{\frac{2R}{r}}(h_b+h_c)\ \text{(and analogs)}\ (4)}$$

We know that: $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs)(***Traian Lalescu***)[**2**]; ω – Brocard angle;

$2S=ah_a=bh_b=ch_c=2pr\rightarrow\frac{b}{c}=\frac{h_c}{h_b}$ (and analogs), and after summation we have

$\sum\frac{b+c}{a}=\sum\frac{h_b+h_c}{h_a}=\sum\frac{\sin(A+\omega)}{\sin\omega}$; $4\frac{m_a}{h_a}\sqrt{\frac{r_a}{h_a}} \ge \sqrt{\frac{2R}{r}}\frac{h_b+h_c}{h_a}$ and after summation we obtain:

$$4\sum\frac{m_a}{h_a}\sqrt{\frac{r_a}{h_a}}\geq\sqrt{\frac{2R}{r}}\sum\frac{\sin(A+\omega)}{\sin\omega}\ (5)$$

From (2) and (4) we get :

$$4m_a\sqrt{\frac{r_a}{h_a}}\geq(h_b+h_c)\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)(\text{and analogs})(6)$$

From (6) after summation we obtain a new inequality:

$$4\sum m_a\sqrt{\frac{r_a}{h_a}}\geq\sum(h_b+h_c)\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)(7)$$

From (2) and (4) we obtain :

$$4m_a\sqrt{\frac{r_a}{h_a}}\geq(h_b+h_c)\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)(\text{and analogs})(8)$$

From (8) after summation:

$$4\sum m_a\sqrt{\frac{r_a}{h_a}}\geq\sum(h_b+h_c)\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)(9)$$

From $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs)$\rightarrow\frac{1}{\sin\omega}\geq\frac{b}{c}+\frac{c}{b}$ (and analogs);

Also $\frac{1}{\sin\omega}=\frac{\sqrt{a^2b^2+b^2c^2+a^2c^2}}{2S}$;

We consider triangle with sides :

$a_1=\sqrt{a}$, $b_1=\sqrt{b}$, $c_1=\sqrt{c}$, $2S_1=\sqrt{r(4R+r)}$, $bc=2Rh_a$ (and analogs), $r_a+r_b+r_c=4R+r$;

$\frac{1}{\sin\omega_1}=\frac{\sqrt{2R(h_a+h_b+h_c)}}{\sqrt{r(4R+r)}}\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$, we obtain in the end a new inequality:

$$\sqrt{\frac{2R}{r}}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)(\text{and analogs})\ (10)$$

From (4) and (10) we obtain a new inequality:

$$4m_a\sqrt{\frac{r_a}{h_a}}\geq(h_b+h_c)\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)(\text{and analogs})\ (11)$$

From (11) after summation we obtain a new result :

$$4\sum m_a\sqrt{\frac{r_a}{h_a}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sum(h_b+h_c)\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)(12)$$

From (4) and (10) we have:

$$4m_a\sqrt{\frac{r_a}{h_a}} \geq (h_b+h_c)\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)\text{(and analogs) (13)}$$

From (13) after summation we obtain a new inequality:

$$4\sum m_a\sqrt{\frac{r_a}{h_a}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sum(h_b+h_c)\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)(14)$$

From (2) and (10) we obtain a new inequality:

$$\frac{2R}{r} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\text{(and analogs)}(15)$$

$$\frac{2R}{r} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{n_c}{r_c}}+\sqrt{\frac{r_c}{n_c}}\right)\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\text{(and analogs) (16)}$$

We will use

$$\frac{R}{2r} = \frac{p^2}{h_ah_b+h_ah_c+h_bh_c}\ (17)$$

(easy to prove,we use :

$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}, h_ah_b+h_ah_c+h_bh_c=\frac{h_ah_bh_c}{r}, 2S=ah_a=bh_b=ch_c$)and the well known inequality: $x^2+y^2+z^2 \geq xy+yz+xz$; x,y,z -real numbers ;

$(h_a+h_b+h_c)^2 \geq 3(h_ah_b+h_ah_c+h_bh_c)$ and after easy manipulations we obtain a new result:

$$\sqrt{\frac{R}{2r}} \geq \frac{p\sqrt{3}}{h_a+h_b+h_c}\ (18)$$

From (2) and (18) we obtain a new inequality :

$$\frac{R}{r} \geq \frac{p\sqrt{3}}{h_a+h_b+h_c}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\text{(and analogs)}(19)$$

From (10) and (18) we obtain a new result:

$$\frac{\mathbf{R}}{\mathbf{r}} \geq \sqrt{\frac{3p^2(r_a+r_b+r_c)}{(h_a+h_b+h_c)^3}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)(\text{and analogs})(20)$$

Inequality (20) have a general form:

$$\frac{R}{r} \geq \sqrt{\frac{3p^2(r_a+r_b+r_c)}{(h_a+h_b+h_c)^3}}\max\left\{\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right),\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right),\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}\right)\right\}(21)$$

$r_a = \frac{S}{p-a}$ (and analogs); $2S=ah_a = bh_b = ch_c=2pr=(a+b+c)r$

$r_a = \frac{2S}{2p-2a} = \frac{ah_a}{a+b+c-2a} = \frac{ah_a}{b+c-a} \rightarrow \frac{r_a}{h_a} = \frac{a}{b+c-a}$ (and analogs)

$\frac{h_a}{r_a} = \frac{b+c-a}{a} = \frac{b+c}{a} - 1 \rightarrow \frac{b+c}{a} = 1 + \frac{h_a}{r_a}$ (and analogs)

Now we will use aritmetic-geometric mean inequality:

$$\frac{x+y}{2} \geq \sqrt{xy};\ x,y \geq 0, \text{ and we have: } 1 + \frac{h_a}{r_a} \geq 2\sqrt{\frac{h_a}{r_a}} \rightarrow$$

$$\frac{b+c}{a} \geq 2\sqrt{\frac{h_a}{r_a}}(\text{and analogs})(22)$$

I- the incenter of the triangle ABC; $AI = \frac{b+c}{2p} l_a$ (and analogs);

$AI = \frac{r}{\sin\frac{A}{2}}$ (and analogs); $\sin\frac{A}{2} = \sqrt{\frac{r}{2R}\frac{r_a}{h_a}}$ (and analogs);we get a new relationship:

$$\frac{AI}{r} = \sqrt{\frac{2R}{r}}\sqrt{\frac{h_a}{r_a}}\ (\text{and analogs})(23)$$

$$\frac{AI}{r} = \frac{b+c}{2pr} l_a = \frac{b+c}{a}\frac{l_a}{h_a}\ (\text{and analogs})(24)$$

From (22) and (24) we obtain :

$$\frac{AI}{r} \geq 2\frac{l_a}{h_a}\sqrt{\frac{h_a}{r_a}}\ (\text{and analogs})(25)$$

From (23) and (25) we obtain :

$$\sqrt{\frac{R}{2r}} \geq \frac{l_a}{h_a} \text{ (and analogs)(well − known inequality)} \textbf{(26)}$$

From $m_a l_a \geq p(p-a) = r_b r_c$ (and analogs)(Panaitopol) and $r_b r_c = h_a \frac{(r_b+r_c)}{2}$ (and analogs) we obtain a new result :

$$\frac{m_a l_a}{h_a} \geq \frac{r_b+r_c}{2} \textbf{ (and analogs) (27)}$$

From (26) and (27) we obtain :

$$m_a\sqrt{\frac{R}{2r}} \geq \frac{r_b+r_c}{2} \textbf{ (and analogs)(28)}$$

From (28) after summation we obtain :

$$\sqrt{\frac{R}{2r}} \geq \frac{r_a+r_b+r_c}{m_a+m_b+m_c} \textbf{ (29)}$$

From(18) and (29) we obtain a new inequality:

$$\frac{R}{2r} \geq \frac{r_a+r_b+r_c}{h_a+h_b+h_c} \frac{p\sqrt{3}}{m_a+m_b+m_c} \textbf{ (30)}$$

We rewrite (28) as : $\sqrt{\frac{2R}{r}} \geq \frac{r_b+r_c}{m_a}$ and using (2) we obtain a new inequality:

$$\frac{2R}{r} m_a \geq (r_b+r_c)\left(\sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}\right) \textbf{(and analogs)(31)}$$

From (31) after summation we obtain a new result:

$$\frac{2R}{r}(m_a+m_b+m_c) \geq \sum \geq (r_b+r_c)\left(\sqrt{\frac{n_a}{r_a}} + \sqrt{\frac{r_a}{n_a}}\right) \textbf{(32)}$$

From (2) and (28) we obtain :

$$\frac{2R}{r} m_a \geq (r_b+r_c)\left(\sqrt{\frac{n_b}{r_b}} + \sqrt{\frac{r_b}{n_b}}\right) \text{(and analogs)} \textbf{(33)}$$

From (33) after summation we obtain:

$$\frac{2R}{r}(m_a+m_b+m_c) \geq \sum \geq (r_b+r_c)\left(\sqrt{\frac{n_b}{r_b}} + \sqrt{\frac{r_b}{n_b}}\right) \textbf{ (34)}$$

From (10) and (29) we obtain a new inequality:

$$\frac{R}{r}\geq\sqrt{\frac{(r_a+r_b+r_c)^3}{(m_a+m_b+m_c)^2(h_a+h_b+h_c)}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)(\text{and analogs})(35)$$

$$\frac{R}{r}\geq\sqrt{\frac{(r_a+r_b+r_c)^3}{(m_a+m_b+m_c)^2(h_a+h_b+h_c)}}\max\left\{\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right),\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right),\left(\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}\right)\right\}$$

From (4) after summation we obtain:

$$2\sum m_a\sqrt{\frac{r_a}{h_a}}\geq\sqrt{\frac{2R}{r}}(h_a+h_b+h_c)(36)$$

From (23) after summation we obtain :

$$\frac{AI+BI+CI}{r}=\sqrt{\frac{2R}{r}}\sum\sqrt{\frac{h_a}{r_a}}\ (37)$$

From (2) and (37) after summation we obtain a new inequality :

$$\frac{AI+BI+CI}{r}\geq\frac{1}{3}\sum\sqrt{\frac{h_a}{r_a}}\left(\sum\sqrt{\frac{n_a}{r_a}}+\sum\sqrt{\frac{r_a}{n_a}}\right)(38)$$

From (16) and (37) after summation we obtain a new inequality :

$$\frac{AI+BI+CI}{r}\geq\frac{1}{3}\sqrt[4]{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sum\sqrt{\frac{h_a}{r_a}}\sum\left[\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{n_c}{r_c}}+\sqrt{\frac{r_c}{n_c}}\right)^{\frac{1}{2}}\right](39)$$

From (15) and (37) after summation we obtain a new inequality:

$$\frac{AI+BI+CI}{r}\geq\frac{1}{3}\sqrt[4]{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sum\sqrt{\frac{h_a}{r_a}}\sum\left[\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)^{\frac{1}{2}}\right](40)$$

From (26) and (2) we obtain:

$$\frac{R}{r}\geq\frac{l_a}{h_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)(\text{and analogs})\ (41)$$

$$\frac{R}{r}\geq\frac{l_c}{h_c}\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)(\text{and analogs})\ (42)$$

From (4) ,(41),(42) and $\frac{R}{r}\geq\frac{m_c}{h_b}+\frac{m_b}{h_c}$ (and analogs)**[3]** we obtain a new results:

$$4m_a\sqrt{\frac{r_a}{h_a}} \geq (h_b+h_c)\sqrt{\frac{m_c}{h_b}+\frac{m_b}{h_c}+\frac{l_a}{h_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)} \text{ (and analogs)(43)}$$

From (43) after summation we obtain a new inequality :

$$4\sum m_a\sqrt{\frac{r_a}{h_a}} \geq \sum(h_b+h_c)\sqrt{\frac{m_c}{h_b}+\frac{m_b}{h_c}+\frac{l_a}{h_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)} \text{ (44)}$$

$$4m_a\sqrt{\frac{r_a}{h_a}} \geq (h_b+h_c)\sqrt{\frac{m_c}{h_b}+\frac{m_b}{h_c}+\frac{l_c}{h_c}\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)} \text{ (and analogs) (45)}$$

From (45) after summation we obtain a new inequality :

$$4\sum m_a\sqrt{\frac{r_a}{h_a}} \geq \sum(h_b+h_c)\sqrt{\frac{m_c}{h_b}+\frac{m_b}{h_c}+\frac{l_c}{h_c}\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)} \text{(46)}$$

From $\frac{R}{r} \geq \frac{m_c}{h_b}+\frac{m_b}{h_c}$ (and analogs) and (35) we obtain a new inequality:

$$\frac{2R}{r} \geq \sqrt{\frac{(r_a+r_b+r_c)^3}{(m_a+m_b+m_c)^2(h_a+h_b+h_c)}}\max\left\{\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right),\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right),\left(\sqrt{\frac{b}{a}}++\sqrt{\frac{a}{b}}\right)\right\}+\max\left\{\frac{m_c}{h_b}+\frac{m_b}{h_c},\frac{m_c}{h_a}+\frac{m_a}{h_c},\frac{m_a}{h_b}+\frac{m_b}{h_a}\right\} \text{(47)}$$

From (2) and (30) we obtain a new inequality :

$$\frac{R}{r} \geq \left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\,\frac{p\sqrt{3}}{m_a+m_b+m_c}} \text{ (48)}$$

From $\frac{R}{r} \geq \frac{m_c}{h_b}+\frac{m_b}{h_c}$ (and analogs) and (48) we obtain the next result :

$$\frac{2R}{r} \geq \frac{m_c}{h_b}+\frac{m_b}{h_c}+\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\,\frac{p\sqrt{3}}{m_a+m_b+m_c}} \text{ (and analogs) (49)}$$

From (19) and (41) after summation we obtain a new result :

$$\frac{2R}{r} \geq \left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\frac{l_a}{h_a}+\frac{p\sqrt{3}}{h_a+h_b+h_c}\right) \text{(and analogs) (50)}$$

From (41) and (48) after summation we obtain a new inequality :

$$\frac{2R}{r} \geq \left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\frac{l_a}{h_a}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\frac{p\sqrt{3}}{m_a+m_b+m_c}}\right)$$ **(and analogs) (51)**

We know that: $\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=\frac{2\sqrt{r_b r_c}}{l_a}$ (and analogs) (very easy to obtain !)

Using this identity and (10) we obtain a new inequality :

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\frac{2\sqrt{r_b r_c}}{l_a}} \rightarrow \sqrt{\frac{R}{2r}}\, l_a \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sqrt{r_b r_c}$$ **(and analogs)(52)**

From (52) after summation we obtain a new inequality :

$$\sqrt{\frac{R}{2r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_b r_c}+\sqrt{r_a r_c}+\sqrt{r_a r_b}}{l_a+l_b+l_c}$$ **(53)**

From (2) and (53) we obtain a new result :

$$\frac{R}{r} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_b r_c}+\sqrt{r_a r_c}+\sqrt{r_a r_b}}{l_a+l_b+l_c}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)$$ **(and analogs)(54)**

Inequality (54) can take the next form :

$$\frac{R}{r} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_b r_c}+\sqrt{r_a r_c}+\sqrt{r_a r_b}}{l_a+l_b+l_c}\max\left\{\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right),\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right),\left(\sqrt{\frac{n_c}{r_c}}+\sqrt{\frac{r_c}{n_c}}\right)\right\}$$ **(55)**

From (48) and (54) after summation we obtain a new inequality :

$$\frac{2R}{r} \geq \left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_b r_c}+\sqrt{r_a r_c}+\sqrt{r_a r_b}}{l_a+l_b+l_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\frac{p\sqrt{3}}{m_a+m_b+m_c}}\right)$$

(and analogs)(56)

From (19) and (54) after summation we obtain a new inequality :

$$\frac{2R}{r} \geq \left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\frac{p\sqrt{3}}{h_a+h_b+h_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_b r_c}+\sqrt{r_a r_c}+\sqrt{r_a r_b}}{l_a+l_b+l_c}\right)$$ **(and analogs) (57)**

From (19) and $\frac{R}{r} \geq \frac{m_c}{h_b}+\frac{m_b}{h_c}$ (and analogs) after summation we obtain a new result :

$$\frac{2R}{r} \geq \frac{m_c}{h_b}+\frac{m_b}{h_c}+\frac{p\sqrt{3}}{h_a+h_b+h_c}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)$$ **(and analogs)(58)**

From $\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2r_a h_a}$ (and analogs) $\rightarrow \frac{R}{r}-1 \geq \frac{2n_a r_a}{2r_a h_a}=\frac{n_a}{h_a}$ .We have a new inequality :

$$\frac{R}{r}\geq 1+\frac{n_a}{h_a}\ \text{(and analogs) (59)}$$

From (59) and (35) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\frac{n_a}{h_a}+\sqrt{\frac{(r_a+r_b+r_c)^3}{(m_a+m_b+m_c)^2(h_a+h_b+h_c)}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\text{(and analogs) (60)}$$

From (59),(41) and (42) we obtain new inequalities :

$$\frac{2R}{r}\geq 1+\frac{n_a}{h_a}+\frac{l_a}{h_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\text{(and analogs) (61)}$$

$$\frac{2R}{r}\geq 1+\frac{n_a}{h_a}+\frac{l_c}{h_c}\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)\text{(and analogs) (62)}$$

From (48) and (59) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\frac{n_a}{h_a}+\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\,\frac{p\sqrt{3}}{m_a+m_b+m_c}}\ \text{(and analogs) (63)}$$

Inequality (63) can be written as :

$$\frac{2R}{r}\geq 1+\max\left(\frac{n_a}{h_a},\frac{n_b}{h_b},\frac{n_c}{h_c}\right)+\max\left\{\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right),\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right),\left(\sqrt{\frac{n_c}{r_c}}+\sqrt{\frac{r_c}{n_c}}\right)\right\}\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}\,\frac{p\sqrt{3}}{m_a+m_b+m_c}}\ \text{(64)}$$

From (59) and (55) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\max\left(\frac{n_a}{h_a},\frac{n_b}{h_b},\frac{n_c}{h_c}\right)+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\,\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\max\left\{\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right),\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right),\left(\sqrt{\frac{n_c}{r_c}}+\sqrt{\frac{r_c}{n_c}}\right)\right\}\ \text{(65)}$$

From (59) after a simple manipulation we obtain a new result :

$$\frac{R}{r}\geq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}\ \text{(66)}$$

and using $\frac{R}{r}\geq\frac{m_c}{h_b}+\frac{m_b}{h_c}$ (and analogs) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\frac{m_c}{h_b}+\frac{m_b}{h_c}+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}\ \text{(and analogs) (67)}$$

$\left(\frac{R}{r}-1\right)^2 h_ah_b\geq n_an_b$ (and analogs) ,after summation we obtain :

$$\left(\frac{R}{r}-1\right)^2(h_ah_b+h_ah_c+h_bh_c)\geq n_an_b+n_an_c+n_bn_c$$

$$\left(\frac{R}{r}-1\right)^2\frac{h_ah_b+h_ah_c+h_bh_c}{p^2}\geq\frac{n_an_b+n_an_c+n_bn_c}{p^2},\ \frac{R}{2r}=\frac{p^2}{h_ah_b+h_ah_c+h_bh_c}$$

We obtain a new result :

$$\frac{R}{r}\geq 1+\sqrt{\frac{R}{2r}}\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}\ \textbf{(68)}$$

$$\frac{2R}{r}\geq 2+\sqrt{\frac{2R}{r}}\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}$$

From (2) and (68) we obtain a new inequality :

$$\frac{2R}{r}\geq 2+\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}\ \text{(and analogs) (69)}$$

From (10) and (68) we obtain a new result :

$$\frac{2R}{r}\geq 2+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}\ \text{(and analogs) (70)}$$

From (18) and (68) we obtain a new inequality :

$$\frac{R}{r}\geq 1+\frac{\sqrt{3(n_an_b+n_an_c+n_bn_c)}}{h_a+h_b+h_c}\ \textbf{(71)}$$

From (29) and (68) we obtain a new inequality:

$$\frac{R}{r}\geq 1+\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}\ \textbf{(72)}$$

From (28) and (68) we obtain a new result :

$$\frac{R}{r}\geq 1+\frac{r_b+r_c}{m_a}\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{a+b+c}\ \text{(and analogs) (73)}$$

From (26) and (68) we obtain a new inequality :

$$\frac{R}{r}\geq 1+\frac{l_a}{h_a}\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}\ \text{(and analogs) (74)}$$

From (53) and (68) we obtain a new inequality :

$$\frac{R}{r}\geq 1+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}\ \textbf{(75)}$$

$$\frac{R}{r}\geq 1+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\,\frac{\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}}{l_a+l_b+l_c}\,\sqrt{\frac{n_an_b+n_an_c+n_bn_c}{p}}\ \textbf{(76)}$$

From (55) and (75) after summation we obtain :

$$\frac{2R}{r}\geq 1+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\,\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\left(\frac{\sqrt{n_an_b+n_an_c+n_bn_c}}{p}\max\left\{\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right),\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right),\left(\sqrt{\frac{n_c}{r_c}}+\sqrt{\frac{r_c}{n_c}}\right)\right\}\right)\ \textbf{(77)}$$

From (19) and (66) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}+\frac{p\sqrt{3}}{h_a+h_b+h_c}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\ (\text{and analogs})\textbf{(78)}$$

From (20) and (66) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}+\sqrt{\frac{3p^2(r_a+r_b+r_c)}{(h_a+h_b+h_c)^3}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)(\text{and analogs})\ \textbf{(79)}$$

From (41) and (66) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}+\frac{l_a}{h_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\textbf{(and analogs) (80)}$$

From (42) and (66) after summation we obtain a new inequality:

$$\frac{2R}{r}\geq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}+\frac{l_c}{h_c}\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)\textbf{(and analogs) (81)}$$

From (54) and (66) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\,\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\textbf{(and analogs) (82)}$$

From (66) and $\frac{R}{r}\geq\frac{m_c}{h_b}+\frac{m_b}{h_c}$ (and analogs) after summation we obtain a new inequality :

$$\frac{2R}{r}\geq 1+\frac{m_c}{h_b}+\frac{m_b}{h_c}+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}}\ \textbf{(and analogs) (83)}$$

We shown : $\frac{b+c}{a}=\sqrt{\frac{2R}{r}\frac{h_a}{l_a}}\sqrt{\frac{h_a}{r_a}}$ (and analogs)**[4]** and using (2) we obtain a new inequality :

$$\frac{b+c}{a}\geq\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(and analogs) (84)}$$

From (84) after summation we obtain a new inequality :

$\sum\frac{b+c}{a}\geq\sum\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$,but $\sum\frac{\sin(A+\omega)}{\sin\omega}=\sum\frac{b+c}{a}$, we obtain this result :

$$\sum\frac{\sin(A+\omega)}{\sin\omega}\geq\sum\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(85)}$$

Using $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs) and useing (2) we obtain :

$$\frac{b+c}{a}\geq\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(and analogs)(86)}$$

From (86) after summation we obtain a new inequality :

$$\sum\frac{\sin(A+\omega)}{\sin\omega}\geq\sum\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(87)}$$

Using $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs) and (15) we obtain :

$$\frac{b+c}{a}\geq\sqrt[4]{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sqrt{\frac{h_a}{r_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\frac{h_a}{l_a}\ \textbf{(and analogs) (88)}$$

From (88) after summation we obtain :

$$\sum\frac{\sin(A+\omega)}{\sin\omega}\geq\sqrt[4]{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}\ \textbf{(89)}$$

Useing $\frac{b+c}{a}=\sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs) and (16) we obtain a new inequality :

$$\frac{b+c}{a}\geq\sqrt[4]{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sqrt{\frac{h_a}{r_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}\frac{h_a}{l_a}\ \textbf{(and analogs) (90)}$$

From (90) after summation we obtain a new inequality :

$$\sum\frac{\sin(A+\omega)}{\sin\omega}\geq\sqrt[4]{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)}\ \textbf{(91)}$$

From (18) and (29) after summation ,we obtain :

$$\sqrt{\frac{2R}{r}} \geq \frac{r_a+r_b+r_c}{m_a+m_b+m_c} + \frac{p\sqrt{3}}{h_a+h_b+h_c} \textbf{ (92)}$$

From (92) and $\frac{b+c}{a} = \sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs) we obtain a new inequality :

$$\frac{b+c}{a} \geq \left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c} + \frac{p\sqrt{3}}{h_a+h_b+h_c}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}} \textbf{ (and analogs)(93)}$$

From (93) we obtain a new inequality :

$$\sum\frac{\sin(A+\omega)}{\sin\omega} \geq \left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c} + \frac{p\sqrt{3}}{h_a+h_b+h_c}\right)\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}} \textbf{ (94)}$$

From (92) and (2) we obtain a new inequality :

$$\frac{2R}{r} \geq \left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c} + \frac{p\sqrt{3}}{h_a+h_b+h_c}\right)\left(\sqrt{\frac{n_b}{r_b}} + \sqrt{\frac{r_b}{n_b}}\right) \textbf{ (and analogs) (95)}$$

From (53) and (18) after summation we obtain :

$$\sqrt{\frac{2R}{r}} \geq \frac{p\sqrt{3}}{h_a+h_b+h_c} + \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c} \textbf{ (96)}$$

Using (96) and $\frac{b+c}{a} = \sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs)we obtain a new inequality :

$$\frac{b+c}{a} \geq \left(\frac{p\sqrt{3}}{h_a+h_b+h_c} + \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}} \textbf{ (and analogs) (97)}$$

From (97) after summation we obtain a new inequality:

$$\sum\frac{\sin(A+\omega)}{\sin\omega} \geq \left(\frac{p\sqrt{3}}{h_a+h_b+h_c} + \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}} \textbf{ (98)}$$

From (53) and (29) after summation we obtain a new inequality :

$$\sqrt{\frac{2R}{r}} \geq \frac{r_a+r_b+r_c}{m_a+m_b+m_c} + \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c} \textbf{ (99)}$$

Using (99) and $\frac{b+c}{a} = \sqrt{\frac{2R}{r}}\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}$ (and analogs) we obtain a new inequality :

$$\frac{b+c}{a}\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(and analogs) (100)}$$

From (100) after summation we obtain a new inequality :

$$\sum\frac{\sin(A+\omega)}{\sin\omega}\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\sum\frac{h_a}{l_a}\sqrt{\frac{h_a}{r_a}}\ \textbf{(101)}$$

From (2) and (99) we obtain a new inequality :

$$\frac{2R}{r}\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\textbf{(and analogs)(102)}$$

From (2) and (96) we obtain a new inequality :

$$\frac{2R}{r}\geq\left(\frac{p\sqrt{3}}{h_a+h_b+h_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\ \textbf{(and analogs) (103)}$$

From (96) and (29) we obtain a new inequality :

$$\frac{R}{r}\geq\left(\frac{p\sqrt{3}}{h_a+h_b+h_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\ \textbf{(104)}$$

From (99) and (18) we obtain a new inequality:

$$\frac{R}{r}\geq\left(\frac{r_a+r_b+r_c}{m_a+m_b+m_c}+\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\frac{\sqrt{r_br_c}+\sqrt{r_ar_c}+\sqrt{r_ar_b}}{l_a+l_b+l_c}\right)\frac{p\sqrt{3}}{h_a+h_b+h_c}\ \textbf{(105)}$$

We consider x,y,z >0 ,real numbers.Then we have :

$$\frac{x}{y}+\frac{y}{x}\geq\frac{x+z}{y+z}+\frac{y+z}{x+z}\rightarrow\frac{x}{y}-\frac{x+z}{y+z}\geq\frac{y+z}{x+z}-\frac{y}{x}\rightarrow\frac{x(y+z)-y(x+z)}{y(y+z)}\geq\frac{x(y+z)-y(x+z)}{x(x+z)}\rightarrow$$

$$\frac{z(x-y)}{y(y+z)}\geq\frac{z(x-y)}{x(x+z)}\rightarrow z(x-y)\left[\frac{1}{y(y+z)}-\frac{1}{x(x+z)}\right]\geq 0\ ;x^2-y^2=(x-y)(x+y)$$

After simple calculations ,we obtain : $\frac{z(x-y)^2(x+y+z)}{xy(x+z)(y+z)}\geq 0$, which is true !

We have a new inequality : x,y,z >0 ,real numbers.

$$\frac{x}{y}+\frac{y}{x}\geq\frac{x+z}{y+z}+\frac{y+z}{x+z}\ \textbf{(and analogs) (106)}$$

From (106) we obtain : $\frac{b}{c}+\frac{c}{b}\geq\frac{b+x}{c+x}+\frac{c+x}{b+x}$ (and analogs) ,x-real number ,x>0

$$\frac{\sin(A+\omega)}{\sin\omega}\geq\frac{b+x}{c+x}+\frac{c+x}{b+x}\ \textbf{(and analogs) ,x-real number ,x>0 (107)}$$

From (107) we obtain a new inequality :

$$\frac{1}{\sin\omega} \geq \frac{b+x}{c+x} + \frac{c+x}{b+x} \text{ (and analogs) ,x-real number ,x>0 (108)}$$

LEMMA:Triangle ABC with sides a,b,c and triangle with sides ma, mb, mc have the same Brocard angle.[5]

Useing this result ,from (108) we obtain a new inequality:

$$\frac{1}{\sin\omega} \geq \frac{m_b+x}{m_c+x} + \frac{m_c+x}{m_b+x} \text{ (and analogs) ,x-real number ,x>0 (109)}$$

From (107) we obtain a new result :

$$\frac{\sin(A+\omega)}{\sin\omega} \geq \frac{b+a}{c+a} + \frac{c+a}{b+a} \text{ (and analogs) (110)}$$

From (108) we obtain a new result :

$$\frac{1}{\sin\omega} \geq \frac{b+a}{c+a} + \frac{c+a}{b+a} \text{ (and analogs) (111)}$$

From (109) we obtain a new result :

$$\frac{1}{\sin\omega} \geq \frac{m_b+m_a}{m_c+m_a} + \frac{m_c+m_a}{m_b+m_a} \text{ (and analogs) (112)}$$

We consider $\Delta A_1B_1C_1$ with: $a_1 = \sqrt{a}$, $b_1 = \sqrt{b}$, $c_1 = \sqrt{c}$ and $2S_1=\sqrt{r(4R+r)}$,

bc=$2Rh_a$(and analogs), $r_a+r_b+r_c$= 4R+r.From (111) we obtain a new inequality:

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{c}} + \frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}+\sqrt{b}}\right) \text{(and analogs) (113)}$$

From (1) and (106) we have : $2\frac{h_a}{n_a}\left(\frac{R}{r}-1\right) \geq \frac{n_a+x}{r_a+x} + \frac{r_a+x}{n_a+x}$,x-real number ,x>0

$$2\left(\frac{R}{r}-1\right) \geq \frac{n_a}{h_a}\left(\frac{n_a+x}{r_a+x} + \frac{r_a+x}{n_a+x}\right) \text{(and analogs) ,x-real number ,x>0 (114)}$$

Because $n_a \geq h_a$ from (114) we obtain a new inequality:

$$2\left(\frac{R}{r}-1\right) \geq \frac{n_a+x}{r_a+x} + \frac{r_a+x}{n_a+x} \text{ (and analogs), x-real number ,x>0 (115)}$$

Also $n_a g_a \geq m_a l_a$(and analogs)→ $\frac{n_a}{h_a} \geq \frac{m_a}{h_a}\frac{l_a}{g_a}$ (and analogs) and using (114) we obtain a new inequality :

$$2\left(\frac{R}{r}-1\right) \geq \frac{m_a}{h_a}\frac{l_a}{g_a}\left(\frac{n_a+x}{r_a+x} + \frac{r_a+x}{n_a+x}\right) \text{(and analogs) x-real number ,x>0 (116)}$$

We consider $\Delta A_1B_1C_1$ with: $a_1=\sqrt{a}$, $b_1=\sqrt{b}$, $c_1=\sqrt{c}$ and $2S_1=\sqrt{r(4R+r)}$,

bc=2Rh_a(and analogs), $r_a+r_b+r_c$= 4R+r ,from (108) we obtain a new result :

$$\sqrt{\frac{2R}{r}}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\frac{\sqrt{b}+x}{\sqrt{c}+x}+\frac{\sqrt{c}+x}{\sqrt{b}+x}\right)$$ **(and analogs) x-real number ,x>0 (117)**

From (106) $\rightarrow\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\geq\frac{\sqrt{b}+x}{\sqrt{c}+x}+\frac{\sqrt{c}+x}{\sqrt{b}+x}$, x-real number ,x>0 but $\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=\frac{2\sqrt{r_br_c}}{l_a}$

We obtain a new inequality :

$$\frac{2\sqrt{r_br_c}}{l_a}\geq\frac{\sqrt{b}+x}{\sqrt{c}+x}+\frac{\sqrt{c}+x}{\sqrt{b}+x}$$ **, x-real number ,x>0 (118)**

From (106) $\rightarrow\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\geq\frac{\sqrt{n_a}+x}{\sqrt{r_a}+x}+\frac{\sqrt{r_a}+x}{\sqrt{n_a}+x}$ and using (2) we obtain a new inequality : $\sqrt{\frac{2R}{r}}\geq\frac{\sqrt{n_a}+x}{\sqrt{r_a}+x}+\frac{\sqrt{r_a}+x}{\sqrt{n_a}+x}$ **(and analogs) ,x-real number ,x>0 (119)**

From $m_al_a\geq p(p-a)=r_br_c$ (and analogs)(Panaitopol) and (118) we obtain a new inequality :

$$\sqrt{\frac{m_a}{l_a}}\geq\frac{1}{2}\left(\frac{\sqrt{b}+x}{\sqrt{c}+x}+\frac{\sqrt{c}+x}{\sqrt{b}+x}\right)$$ **, x-real number ,x>0 (and analogs) (120)**

From (106) $\frac{h_b}{h_c}+\frac{h_c}{h_b}\geq\frac{h_b+x}{h_c+x}+\frac{h_c+x}{h_b+x}$ (and analogs) ,x-real number ,x>0

2S=$ah_a=bh_b=ch_c\rightarrow\frac{h_b}{h_c}=\frac{c}{b}$ (and analogs) , $\frac{\sin(A+\omega)}{\sin\omega}=\frac{b}{c}+\frac{c}{b}$ (and analogs) we obtain :

$$\frac{b}{c}+\frac{c}{b}\geq\frac{h_b+x}{h_c+x}+\frac{h_c+x}{h_b+x}$$ **(and analogs) ,x-real number ,x>0 (121)**

$$\frac{\sin(A+\omega)}{\sin\omega}\geq\frac{h_b+x}{h_c+x}+\frac{h_c+x}{h_b+x}$$ **(and analogs) ,x-real number ,x>0 (122)**

From (122) we obtain a new inequality : $\frac{1}{\sin\omega}\geq\frac{h_b+x}{h_c+x}+\frac{h_c+x}{h_b+x}$ (and analogs) ,x-real number ,x>0 (123)

From (106) we obtain : $\sqrt{\frac{h_b}{h_c}}+\sqrt{\frac{h_c}{h_b}}\geq\frac{\sqrt{h_b}+x}{\sqrt{h_c}+x}+\frac{\sqrt{h_c}+x}{\sqrt{h_b}+x}$, x-real number ,x>0

and $\frac{h_b}{h_c}=\frac{c}{b}$ (and analogs) ,we obtain :

$$\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\geq\frac{\sqrt{h_b}+x}{\sqrt{h_c}+x}+\frac{\sqrt{h_c}+x}{\sqrt{h_b}+x}$$ **, x-real number ,x>0 (and analogs) (124)**

We use (124) and $\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=\frac{2\sqrt{r_b r_c}}{l_a}$ (and analogs) and we obtain :

$$\frac{\sqrt{r_b r_c}}{l_a}\geq\frac{1}{2}\left(\frac{\sqrt{h_b}+x}{\sqrt{h_c}+x}+\frac{\sqrt{h_c}+x}{\sqrt{h_b}+x}\right)$$ **, x-real number ,x>0 (and analogs) (125)**

From (125) and $m_a l_a\geq p(p-a)=r_b r_c$ (and analogs)(Panaitopol), we obtain a new inequality :

$$\sqrt{\frac{m_a}{l_a}}\geq\frac{1}{2}\left(\frac{\sqrt{h_b}+x}{\sqrt{h_c}+x}+\frac{\sqrt{h_c}+x}{\sqrt{h_b}+x}\right)$$ **, x-real number ,x>0 (and analogs) (126)**

From (124) and (10) we obtain a new result :

$$\sqrt{\frac{2R}{r}}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\frac{\sqrt{h_b}+x}{\sqrt{h_c}+x}+\frac{\sqrt{h_c}+x}{\sqrt{h_b}+x}\right)$$ **, x-real number ,x>0 (and analogs) (127)**

From (2),(117),(127) we obtain two new inequalitys:

$$\frac{2R}{r}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\frac{\sqrt{b}+x}{\sqrt{c}+x}+\frac{\sqrt{c}+x}{\sqrt{b}+x}\right)\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)$$ **, x-real number ,x>0 (and analogs) (128)**

$$\frac{2R}{r}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}}\right)\left(\frac{\sqrt{h_b}+x}{\sqrt{h_c}+x}+\frac{\sqrt{h_c}+x}{\sqrt{h_b}+x}\right)$$ **,x-real number ,x>0 (and analogs)(129)**

In same manner we obtain :

$$\frac{2R}{r}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\frac{\sqrt{b}+x}{\sqrt{c}+x}+\frac{\sqrt{c}+x}{\sqrt{b}+x}\right)\left(\sqrt{\frac{n_b}{r_b}}+\sqrt{\frac{r_b}{n_b}}\right)$$ **,x-real number ,x>0 (and analogs) (130)**

$$\frac{2R}{r}\geq\sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\frac{\sqrt{h_b}+x}{\sqrt{h_c}+x}+\frac{\sqrt{h_c}+x}{\sqrt{h_b}+x}\right)\left(\sqrt{\frac{n_c}{r_c}}+\sqrt{\frac{r_c}{n_c}}\right)$$ **,x-real number ,x>0 (and analogs)(131)**

From (106) $\rightarrow\frac{l_a}{h_a}+\frac{h_a}{l_a}\geq\frac{l_a+x}{h_a+x}+\frac{h_a+x}{l_a+x}$, x-real number ,x>0 (and analogs)

Using (26) we obtain :

$$\sqrt{\frac{R}{2r}}+\frac{h_a}{l_a}\geq\frac{l_a+x}{h_a+x}+\frac{h_a+x}{l_a+x}$$ **, x-real number ,x>0 (and analogs) (132)**

We consider x,y,z >0 ,real numbers

$$\frac{x}{y}+\frac{y}{x}\geq\frac{x+z}{y+z}+\frac{y+z}{x+z}\rightarrow 2+\frac{x}{y}+\frac{y}{x}\geq\frac{x+z}{y+z}+\frac{y+z}{x+z}+2\rightarrow$$

$$\left(\sqrt{\frac{x}{y}}\right)^2+\left(\sqrt{\frac{y}{x}}\right)^2+2\sqrt{\frac{x}{y}}\sqrt{\frac{y}{x}}\geq\left(\sqrt{\frac{x+z}{y+z}}\right)^2+\left(\sqrt{\frac{y+z}{x+z}}\right)^2+2\sqrt{\frac{x+z}{y+z}}\sqrt{\frac{y+z}{x+z}}$$

$$\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)^2 \geq \left(\sqrt{\frac{x+z}{y+z}}+\sqrt{\frac{y+z}{x+z}}\right)^2 \rightarrow \sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}} \geq \sqrt{\frac{x+z}{y+z}}+\sqrt{\frac{y+z}{x+z}}$$

We obtain a new inequality : x,y,z >0 ,real numbers

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}} \geq \sqrt{\frac{x+z}{y+z}}+\sqrt{\frac{y+z}{x+z}} \text{ (and analogs) (133)}$$

From (133) we obtain :

$$\frac{b}{c}+\frac{c}{b} \geq \sqrt{\frac{x+b^2}{x+c^2}}+\sqrt{\frac{c^2+x}{b^2+x}} \text{ (and analogs) , x-real number ,x>0 (134)}$$

From (134) we obtain :

$$\frac{\sin(A+\omega)}{\sin\omega} \geq \sqrt{\frac{x+b^2}{x+c^2}}+\sqrt{\frac{c^2+x}{b^2+x}} \text{ (and analogs) , x-real number ,x>0 (135)}$$

From (135) we obtain:

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{x+b^2}{x+c^2}}+\sqrt{\frac{c^2+x}{b^2+x}} \text{ (and analogs) , x-real number ,x>0 (136)}$$

We use now :

LEMMA:Triangle ABC with sides a,b,c and triangle with sides ma, mb, mc have the same Brocard angle. And from (136) we obtain a new inequality:

$$\frac{1}{\sin\omega} \geq \sqrt{\frac{x+{m_b}^2}{x+{m_c}^2}}+\sqrt{\frac{{m_c}^2+x}{{m_b}^2+x}} \text{ (and analogs) , x-real number ,x>0 (137)}$$

From (133) we obtain :

$$\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \geq \sqrt{\frac{x+b}{x+c}}+\sqrt{\frac{c+x}{b+x}} \text{ (and analogs) , x-real number ,x>0 (138)}$$

$\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=\frac{2\sqrt{r_b r_c}}{l_a}$ (and analogs) and using (138) we obtain :

$$\frac{\sqrt{r_b r_c}}{l_a} \geq \frac{1}{2}\left(\sqrt{\frac{x+b}{x+c}}+\sqrt{\frac{c+x}{b+x}}\right) \text{ (and analogs) , x-real number ,x>0 (139)}$$

From (139) and $m_a l_a \geq p(p-a)=r_b r_c$ (and analogs)(Panaitopol) we obtain a new inequality :

$$\sqrt{\frac{m_a}{l_a}} \geq \frac{1}{2}\left(\sqrt{\frac{x+b}{x+c}}+\sqrt{\frac{c+x}{b+x}}\right) \text{ (and analogs) , x-real number ,x>0 (140)}$$

From $\frac{h_b}{h_c}=\frac{c}{b}$ (and analogs) and (133) we obtain :

$$\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \geq \sqrt{\frac{x+h_b}{x+h_c}}+\sqrt{\frac{h_c+x}{h_b+x}} \text{ (and analogs) , x-real number ,x>0 (141)}$$

Fom (141) and $\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}=\frac{2\sqrt{r_b r_c}}{l_a}$ (and analogs) we obtain a new inequality :

$$\frac{\sqrt{r_b r_c}}{l_a} \geq \frac{1}{2}\left(\sqrt{\frac{x+h_b}{x+h_c}}+\sqrt{\frac{h_c+x}{h_b+x}}\right) \text{(and analogs) , x-real number ,x>0 (142)}$$

From (142) and $m_a l_a \geq p(p-a)=r_b r_c$ (and analogs)(Panaitopol) we obtain:

$$\sqrt{\frac{m_a}{l_a}} \geq \frac{1}{2}\left(\sqrt{\frac{x+h_b}{x+h_c}}+\sqrt{\frac{h_c+x}{h_b+x}}\right) \text{(and analogs) , x-real number ,x>0 (143)}$$

From (138) and (10) we obtain a new inequality:

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{x+b}{x+c}}+\sqrt{\frac{c+x}{b+x}}\right) \text{(and analogs) , x-real number ,x>0 (144)}$$

From (141) and (10) we obtain a new inequality :

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{r_a+r_b+r_c}{h_a+h_b+h_c}}\left(\sqrt{\frac{x+h_b}{x+h_c}}+\sqrt{\frac{h_c+x}{h_b+x}}\right) \text{(and analogs) , x-real number ,x>0 (145)}$$

From (133) $\rightarrow \sqrt{\frac{n_a}{r_a}}+\sqrt{\frac{r_a}{n_a}} \geq \sqrt{\frac{x+n_a}{x+r_a}}+\sqrt{\frac{r_a+x}{n_a+x}}$ (and analogs) , x-real number ,x>0

and useing (2) we obtain a new inequality :

$$\sqrt{\frac{2R}{r}} \geq \sqrt{\frac{x+n_a}{x+r_a}}+\sqrt{\frac{r_a+x}{n_a+x}} \text{ (and analogs) , x-real number ,x>0 (146)}$$

From (133) $\rightarrow \frac{n_a}{r_a}+\frac{r_a}{n_a} \geq \sqrt{\frac{x+n_a{}^2}{x+r_a{}^2}}+\sqrt{\frac{r_a{}^2+x}{n_a{}^2+x}}$ and using (1) we obtain a new inequality :

$$2\left(\frac{R}{r}-1\right) \geq \frac{n_a}{h_a}\left(\sqrt{\frac{x+n_a{}^2}{x+r_a{}^2}}+\sqrt{\frac{r_a{}^2+x}{n_a{}^2+x}}\right) \text{ , x-real number ,x>0 (147)}$$

From (147) and $\frac{n_a}{h_a} \geq \frac{m_a}{h_a}\frac{l_a}{g_a}$ (and analogs) we obtain a new inequality :

$$\mathbf{2\left(\frac{R}{r}-1\right)\geq\frac{m_a}{h_a}\frac{l_a}{g_a}\left(\sqrt{\frac{x+{n_a}^2}{x+{r_a}^2}}+\sqrt{\frac{{r_a}^2+x}{{n_a}^2+x}}\right)}$$ **, x-real number ,x>0 (148)**

Now we write $\frac{R}{r}\geq\frac{m_c}{h_b}+\frac{m_b}{h_c}$ (and analogs) for triangle with sides ma, mb, mc

$R_m=\frac{m_am_bm_c}{3S}$,$r_m=\frac{3S}{2(m_a+m_b+m_c)}$, $m_{a_1}=\frac{3}{4}$ a (and analogs) , $h_{a_1}=\frac{3S}{2m_a}$ (and analogs)(We used triangle with sides: $a_1=m_a$, $b_1=m_b$, $c_1=m_c$), $2S=ah_a=bh_b=ch_c=2pr=(a+b+c)r$

After simple manipulations: $\frac{m_{b_1}}{h_{c_1}}=\frac{m_c}{h_b}$, $\frac{m_{c_1}}{h_{b_1}}=\frac{m_b}{h_c}$

We obtain a new inequality :

$$\mathbf{\frac{R_m}{r_m}=\frac{2(m_a+m_b+m_c)m_am_bm_c}{9S^2}\geq\frac{m_c}{h_b}+\frac{m_b}{h_c}}$$ **(and analogs) (149)**

Now we use identity in triangle 160(www.ssmrmh.ro): $\sum\frac{n_a^2+g_a^2}{h_a^2}=2\left(1+\frac{1}{(\sin\omega)^2}-\frac{R}{r}\right)$**[6]** and

LEMMA:Triangle ABC with sides a,b,c and triangle with sides ma, mb, mc have the same Brocard angle.

$$\frac{1}{(\sin\omega)^2}=\frac{R}{r}-1+\frac{1}{2}\sum\frac{n_a^2+g_a^2}{h_a^2}\ (A)$$

$$\frac{1}{(\sin\omega_m)^2}=\frac{R_m}{r_m}-1+\frac{1}{2}\sum\frac{n_{a_1}^2+g_{a_1}^2}{h_{a_1}^2}\ (B),\ a_1=m_a\ ,\ b_1=m_b\ ,\ c_1=m_c$$

$$\frac{1}{(\sin\omega)^2}=\frac{1}{(\sin\omega_m)^2}\rightarrow\frac{R}{r}-1+\frac{1}{2}\sum\frac{n_a^2+g_a^2}{h_a^2}=\frac{R_m}{r_m}-1+\frac{1}{2}\sum\frac{n_{a_1}^2+g_{a_1}^2}{h_{a_1}^2}$$

$$\frac{R}{r}=\frac{R_m}{r_m}+\frac{1}{2}\sum\left(\frac{n_{a_1}^2+g_{a_1}^2}{h_{a_1}^2}-\frac{n_a^2+g_a^2}{h_a^2}\right),\ a_1=m_a\ ,\ b_1=m_b\ ,\ c_1=m_c$$

$$\mathbf{\frac{R}{r}=\frac{2(m_a+m_b+m_c)m_am_bm_c}{9S^2}+\frac{1}{2}\sum\left(\frac{n_{a_1}^2+g_{a_1}^2}{h_{a_1}^2}-\frac{n_a^2+g_a^2}{h_a^2}\right),\ a_1=m_a\ ,\ b_1=m_b\ ,\ c_1=m_c}$$ **(150)**

This last identity can be used to obtain new inequalitys from all results which have $\mathbf{\frac{R}{r}}$ as component.

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