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SOLUTIONS

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JP.001. Let a, b, c, d be non-negative real numbers such that:

$$a + b + c + d = 4. \text{ Prove that:}$$

$$2 + \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \ge ab + ac + ad + bc + bd + cd$$

Proposed by Hung Nguyen Viet – Hanoi – Vietnam

Solution by Soumitra Mukherjee-Chandar Nagore-India

$$\text{Let } f(x) = x^2 + 2\sqrt{x} - 3x, \forall\, x \ge 0;\ f'(x) = 2x + \frac{1}{\sqrt{x}} - 3 \ge 0, \forall x \ge 0$$

$f(x)$ *is continuous on* $[0,\infty)$ *and* $f'(x) \ge 0, \forall\, x \in [0,\infty)$; f *is increasing on* $[0,\infty)$.

$$f(x) \ge f(0) \Rightarrow x^2 + 2\sqrt{x} - 3x \ge 0 \Rightarrow x^2 + 2\sqrt{x} \ge 3x, \forall x \ge 0$$

$$\sum_{cyc} a^2 + 2\sum_{cyc} \sqrt{a} \ge 3\sum_{cyc} a \Rightarrow (a + b + c + d)^2 - 2\sum_{cyc} ab + 2\sum_{cyc} \sqrt{a} \ge 3\sum_{cyc} a$$

$$\Rightarrow 16 + 2\sum_{cyc} \sqrt{a} \ge 12 + 2\sum_{cyc} ab \Rightarrow 4 + 2\sum_{cyc} \sqrt{a} \ge 2\sum_{cyc} ab$$

$$\Rightarrow 2 + \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} \ge ab + ac + ad + bc + bd + cd$$

JP.002. Determine all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + a - 1) - x|f(x + a - 1)| \le x \le f(x) - (x - a + 1)|f(x)| + a - 1$$

for all $x \in \mathbb{R}$, when $a \in \mathbb{R}$.

Proposed by Mihály Bencze – Romania

Solution by Mihály Bencze – Romania

$$\text{The inequalities are equivalent with: } \frac{f(x+a-1)}{1+|f(x+a-1)|} \le x \le \frac{f(x)}{1+|f(x)|} + a - 1$$

$$\text{Denote } g(x) = \frac{f(x)}{1+|f(x)|} \Rightarrow g(x + a - 1) \le x \le g(x) + a - 1$$

$$\text{In } g(x + a - 1) \le x \text{ we take } x \to x - a + 1 \Rightarrow$$

$$g(x) \le x - a + 1 \qquad (1)$$

$$\text{but from } x \le g(x) + a - 1 \Rightarrow$$

$$g(x) = x - a + 1 \qquad (2)$$

$$(1) \wedge (2) \Rightarrow g(x) = x - a + 1 \Rightarrow f(x) = \frac{x-a+1}{1+|x-a+1|}$$

JP. 003. If $a,b>0$ then:

$$4\sqrt{a^4+a^2b^2+b^4}+(a^2+b^2)\sqrt{3}\geq 2a\sqrt{2a^2+ab}+2b\sqrt{2b^2+ab}+$$
$$+a\sqrt{2a^2+b^2}+b\sqrt{a^2+2b^2}$$

Proposed by Mihály Bencze – Romania

Solution by Mihály Bencze – Romania

$$(a^2-b^2)^2\Rightarrow 4a^4+4a^2b^2+4b^2\geq 3a^4+6a^2b^2+3b^4\Rightarrow$$
$$\Rightarrow\sqrt{a^4+a^2b^2+b^4}\geq\frac{\sqrt{3}}{2}(a^2+b^2)$$

If $a,b,c>0\Rightarrow\left(\sum\sqrt{a^4+a^2b^2+b^4}\right)^2\geq 3\left(\sum a^2\right)^2$

$$\left(\sum a\sqrt{2a^2+bc}\right)^2\leq\left(\sum a^2\right)^2\left(\sum(2a^2+bc)\right)\leq 3\left(\sum a^2\right)^2\Rightarrow$$
$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^2}\geq\quad (1)$$
$$a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}$$

In (1) we take $c=a$ and $c=b$ therefore

$$\begin{cases}2\sqrt{a^4+a^2b^2+b^4}+a^2\sqrt{3}\geq 2a\sqrt{2a^2+ab}+b\sqrt{a^2+b^2}\\ 2\sqrt{a^4+a^2b^2+b^4}+b^2\sqrt{3}\geq 2b\sqrt{2b^2+ab}+a\sqrt{b^2+2a^2}\end{cases}$$

After addition the conclusion follows.

JP.004. Let be $n\in\mathbb{N}^*\setminus\{1\}$ și $a_k\in\mathbb{R};\ k\in\overline{1,n}$. Prove that:

$$\sum_{k=1}^{n}\sqrt{a_k^2-a_ka_{k+1}+a_{k+1}^2}\geq\sum_{k=1}^{n}a_k\,;\ a_{n+1}=a_1$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Abhay Chandra – India

$$\sqrt{a_k^2-a_ka_{k+1}+a_{k+1}^2}=\sqrt{\frac{3}{4}(a_k-a_{k+1})^2+\frac{1}{4}(a_k+a_{k+1})^2}\geq\frac{a_k+a_{k+1}}{2}$$

And the result follows after summation. Equality at $a_1=a_2=\cdots=a_{n+1}$.

JP.005. Prove that if $a, b, x, y, z \in (0, \infty)$ then:

$$\frac{yz(a^2y+b^2z)}{x}+\frac{zx(a^2z+b^2x)}{y}+\frac{xy(a^2x+b^2y)}{z}\geq\frac{2}{3}ab(x+y+z)^2$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mukherjee-Chandar Nagore-India

$$\frac{yz}{x}(a^2y+b^2z)+\frac{zx}{y}(a^2z+b^2x)+\frac{xy}{z}(a^2x+b^2y)=$$

$$=y^2\left(\frac{a^2z}{x}+\frac{b^2x}{z}\right)+z^2\left(\frac{b^2y}{x}+\frac{a^2x}{y}\right)+x^2\left(\frac{b^2z}{y}+\frac{a^2y}{x}\right)$$

$$\geq 2ab(x^2+y^2+z^2)\ \textit{(Applying AM}\geq GM)\geq\frac{2ab}{3}(x+y+z)^2$$

JP.006. Prove that if $a, b, c \in \mathbb{R}, a+b+c=2$ then:

$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2)>5(a^3+b^3+c^3)+1$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$2a^4+10a^2-5a^3-8a+5=$$

$$=2a^4-3a^3+5a^2-2a^3+3a^2-5a+2a^2-3a+5=$$

$$=a^2(2a^2-3a+5)-a(2a^2-3a+5)+(2a^2-3a+5)=$$

$$=(2a^2-3a+5)(a^2-a+1)=\left[2\left(a-\frac{3}{4}\right)^2+\frac{31}{4}\right]\left[\left(a-\frac{1}{2}\right)^2+\frac{3}{4}\right]>0$$

$$2a^4+10a^2-5a^3-8a+5>0$$

$$2b^4+10b^2-5b^3-8b+5>0$$

$$2c^4+10c^2-5c^3-8c+5>0$$

$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2)+1>5(a^3+b^3+c^3)+8(a+b+c)$$

$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2)>5(a^3+b^3+c^3)+16-15$$

$$2(a^4+b^4+c^4)+10(a^2+b^2+c^2)>5(a^3+b^3+c^3)+1$$

JP.007. Prove that if: $a,b,c>0;\ a+b+c=3$ then:

$$\sum a\left(\frac{1}{b^3}+\frac{1}{c^3}\right)\geq\frac{18}{a^3+b^3+c^3}$$

Proposed by Daniel Sitaru – Romania

Solution by Hung Nguyen Viet – HaNoi – VietNam

Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$a\left(\frac{1}{b^3}+\frac{1}{c^3}\right)+b\left(\frac{1}{c^3}+\frac{1}{a^3}\right)+c\left(\frac{1}{a^3}+\frac{1}{b^3}\right)\geq\frac{18}{a^3+b^3+c^3}$$

By Cauchy – Schwarz inequality we obtain:

$$\sum_{cyc} a\left(\frac{1}{b^3}+\frac{1}{c^3}\right)=\frac{b+c}{a^3}+\frac{c+a}{b^3}+\frac{a+b}{c^3}$$

$$=\frac{(b+c)^2}{(b+c)a^3}+\frac{(c+a)^2}{(c+a)b^3}+\frac{(a+b)^3}{(a+b)c^3}\geq\frac{4(a+b+c)^2}{(b+c)a^3+(c+a)b^3+(a+b)c^3}=$$

$$=\frac{36}{(a+b+c)(a^3+b^3+c^3)-(a^4+b^4+c^4)}=\frac{36}{3(a^3+b^3+c^3)-(a^4+b^4+c^4)}$$

It suffices to show that: $a^4+b^4+c^4\geq a^3+b^3+c^3$

Indeed, this is true by Cauchy – Schwarz inequality as follows:

$\frac{a^4+b^4+c^4}{a^3+b^3+c^3}\geq\frac{a^3+b^3+c^3}{a^2+b^2+c^2}\geq\frac{a^2+b^2+c^2}{a+b+c}\geq\frac{a+b+c}{3}=1$ *and we are done.*

JP.008. If a,b,c are the length's sides in any triangle the following relationship doesn't holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{2}{3}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

Proposed by Redwane El Mellas – Morroco

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: a,b,c son lados de un triángulo, la siguiente relación no se mantiene.

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{2}{3}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$. *Multiplicando* $(\times\ abc)$

$$a^2c+b^2a+c^2b=\frac{2}{3}(b^2c+c^2a+a^2b)$$

$\frac{2}{3}=\frac{b^2c+c^2a+a^2b}{a^2c+b^2a+c^2b}\rightarrow$ *Llevando a razones y proporciones*

$$-\frac{5}{1}=\frac{ab(a+b)+bc(b+c)+ac(a+c)}{ab(b-a)+bc(c-b)+ca(a-c)}$$

Por desigualad triangular, sebemos:

$$b+c>a \Leftrightarrow bc(b+c)>abc,$$

$$a+c>b \Leftrightarrow ac(a+c)>abc,$$

$$a+b>c \Leftrightarrow ab(a+b)>abc$$

$$ab(a+b)+bc(b+c)+ac(a+c)>3abc \quad \textbf{(1)}$$

$$b>a-c \Leftrightarrow ac(a-c)<abc,$$

$$c>b-a \Leftrightarrow ab(b-a)<abc,$$

$$a>c-b \Leftrightarrow bc(c-b)<abc$$

$$ab(b-a)+bc(c-b)+ca(a-c)<3abc \rightarrow \frac{1}{ab(b-a)+bc(c-b)+ca(c-a)}>\frac{1}{3abc} \quad \textbf{(2)}$$

Multiplicando (1) × (2):

$$\frac{ab(a+b)+bc(b+c)+ac(a+c)}{ab(a-b)+bc(b-c)+ca(c-a)}>1 \rightarrow -5>1 \ \textit{(Es falso)}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=2\left(\frac{b}{a}-\frac{a}{b}+\frac{c}{b}-\frac{b}{c}+\frac{a}{c}-\frac{c}{a}\right)$$

$$=2\left(\frac{b^2-a^2}{ab}+\frac{c^2-b^2}{bc}+\frac{a^2-c^2}{ca}\right)$$

$$=\frac{2}{bc}\left(c(b^2-a^2)+a(c^2-b^2)+b(a^2-c^2)\right)=\frac{2}{abc}(b-a)(b-c)(c-a) \ \textbf{(1)}$$

If any 2 sides are equal, RHS of (1) = 0

Also, if all sides are equal, RHS of (1) = 0

But LHS of (1) $=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$ ***(AM≥GM)***

all sides can't be equal. Also 2 sides can't be equal.

(1) $\Rightarrow (b-a)(b-c)(c-a)>0\Rightarrow a>b>c$ **or** $b>c>a$ ***or*** $c>a>b$

Case 1 $a>b>c$

$$a-b<c,\ b-c<a,\ a-c<b$$

$$\Rightarrow (a-b)(b-c)(a-c)<abc\Rightarrow (b-a)(b-c)(c-a)<abc$$

$$\Rightarrow \frac{2}{abc}(b-a)(b-c)(c-a)<2\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}<2, \textit{ false}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$$

Case 2 $b > c > a$

$$b-c < a, c-a < b, b-a < c$$

$$\Rightarrow (b-a)(b-c)(c-a) < abc \Rightarrow \frac{2}{abc}(b-a)(b-c)(c-a) < 2,$$

$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} < 2$ ***which is false,*** $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$

Case 3 $c > a > b$

$$a-b < c, c-b < a, c-a < b$$

$$\Rightarrow (a-b)(c-b)(c-a) < abc \Rightarrow (b-a)(b-c)(c-a) < abc$$

$\Rightarrow \frac{2}{abc}(b-a)(b-c)(c-a) < 2 \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} < 2$, ***false,*** $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$

Hence, in any Δ, $3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) = 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$ ***is impossible.***

JP.009. Prove that if $a, b, c \in \mathbb{R}; 0 < c \leq b \leq a$ **then:**

$$(a+2b)(a+2c)(b+2c) \leq 8\prod\left(\frac{a^2+ab+b^2}{a+b}\right) \leq (2a+b)(2a+c)(2b+c)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash- New Delhi-India

For $0 < y \leq x$

$$2(x^2+xy+y^2)-(x+2y)(x+y) = 2x^2+2xy+2y^2-(x^2+3xy+2y^2)$$

$= x^2-xy = x(x-y) \geq 0$. ***Also,***

$$(2x+y)(2+y)-2(x^2+xy+y^2) = 2x^2+3xy+y^2-2x^2-2xy-2y^2 = (x-y)y \geq 0$$

for $0 < y \leq 2x, x+2y \leq \frac{2(x^2+xy+y^2)}{x+y} \leq 2x+y$.

As $0 < c \leq b \leq a$, ***the desired inequality follows.***

JP.010. Prove that:

$$\tan 78^\circ = \frac{4\cos 12^\circ + 4\cos 36^\circ + 1}{\sqrt{3}}$$

Proposed by Kevin Soto Palacios – Huarmey - Peru

Solution by Hamza Mahmood- Lahore – Pakistan

First we show that $4\cos 12^\circ + 4\cos 36^\circ = \frac{2\sin 48^\circ}{\sin 12^\circ}$:

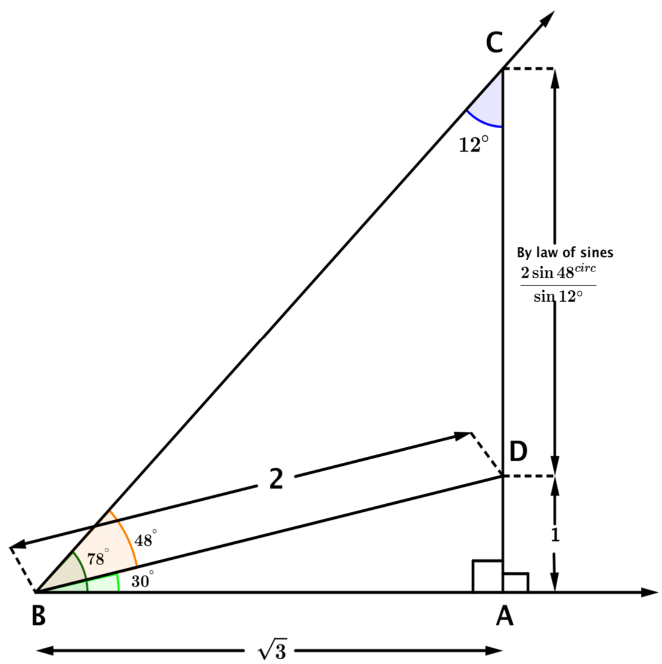
Using identities $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{B-A}{2}\right)$ & $\cos\theta = \frac{\sin 2\theta}{2\sin\theta}$, ***we have:***

$$4\cos 12^\circ + 4\cos 36^\circ = 8\cos(24^\circ)\cos(12^\circ) = 8\cdot\frac{\sin 48^\circ}{2\sin 24^\circ}\cdot\frac{\sin 24^\circ}{2\sin 12^\circ} = \frac{2\sin 48^\circ}{\sin 12^\circ}$$

Now consider a right angled triangle ABC ***with***

$$m\angle BAC = 90^\circ, m\angle ABC = 78^\circ \ \& \ m\overline{AB} = \sqrt{3}$$

as shown in the figure below (not drawn to scale):



In right angled triangle BAD: $\tan 30^\circ = \frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AD}}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}} = \frac{m\overline{AD}}{\sqrt{3}} \Rightarrow m\overline{AD} = 1$

and $\sin 30^\circ = \frac{m\overline{AD}}{m\overline{BD}} = \frac{1}{m\overline{BD}} \Rightarrow \frac{1}{2} = \frac{1}{m\overline{BD}} \Rightarrow m\overline{BD} = 2$. ***Now in*** ΔBCD: ***By law of sines:***

$$\frac{m\overline{BD}}{\sin 12^\circ} = \frac{m\overline{CD}}{\sin 48^\circ} \Rightarrow m\overline{CD} = \frac{2\sin 48^\circ}{\sin 12^\circ}$$

Since we have already shown that $\frac{2\sin 48^\circ}{\sin 12^\circ} = 4\cos 12^\circ + 4\cos 36^\circ$,

$$\Rightarrow m\overline{CD} = \frac{2\sin 48^\circ}{\sin 12^\circ} = 4\cos 12^\circ + 4\cos 36^\circ$$

Finally from the figure, $\tan 78^\circ = \frac{m\overline{AC}}{m\overline{AB}} = \frac{m\overline{CD}+m\overline{DA}}{m\overline{AB}}$

$$\Rightarrow \tan 78^\circ = \frac{4\cos 12^\circ + 4\cos 36^\circ + 1}{\sqrt{3}}$$

JP.011. If a, b, c are the length sides in any triangle ABC then:

$$\frac{a}{\sqrt{s-a}} + \frac{b}{\sqrt{s-b}} + \frac{c}{\sqrt{s-c}} \geq 3\sqrt{s}$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by 1 Kevin Soto Palacios – Huarmey – Peru

Si: a, b, c *son lados de un triángulo* ABC*. Probar que:* $\frac{a}{\sqrt{s-a}} + \frac{b}{\sqrt{s-b}} + \frac{c}{\sqrt{s-c}} \geq 3\sqrt{s}$

La desigualdad se puede expresar como: $\frac{a}{\sqrt{s(s-a)}} + \frac{b}{\sqrt{s(s-b)}} + \frac{c}{\sqrt{s(s-c)}} \geq 3$ *(A)*

El area de un triángulo $[ABC] = sr = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)} \to (R \wedge r) \to$

$\to$ *(Circunradio e Inradio). Es bien conocido que en un triángulo* ABC

$\operatorname{sen} A + \operatorname{sen} B + \operatorname{sen} C \leq \frac{3\sqrt{3}}{2} \to$ *Por Ley de senos equivale:*

$a + b + c \leq 3\sqrt{3}R \to 2s \leq 3\sqrt{3}R$*. En (A)* $\to$ *Por* $MA \geq MG$

$$\frac{a}{\sqrt{s(s-a)}} + \frac{b}{\sqrt{s(s-b)}} + \frac{c}{\sqrt{s(s-c)}} \geq 3\sqrt[3]{\frac{abc}{s\sqrt{s(s-a)(s-b)(s-c)}}} =$$

$= 3\sqrt[3]{\frac{4R}{s}} = 3\sqrt[3]{\frac{8}{3\sqrt{3}}} = 2\sqrt{3}$*. Por transitividad:* $\frac{a}{\sqrt{s(s-a)}} + \frac{b}{\sqrt{s(s-b)}} + \frac{c}{\sqrt{s(s-c)}} \geq 2\sqrt{3} \geq 3$

Solution 2 by Anas Adlany –El Jadida- Morroco

Let $s = \frac{a+b+c}{2} = 3$ *(the inequality is homogenuous), so we need to prove that*

$\sum \frac{a}{\sqrt{3-a}} \geq 3\sqrt{3}$*. Consider* $f(x) = \frac{x}{\sqrt{3-x}} \Rightarrow f''(x) = -\frac{x-12}{4\sqrt{(3-x)^5}} > 0$ *[since* $0 < x < 3$*], so* f *is*

convex on $(0,3)$ *hence by Jensen inequality we get*

$$\sum f(a) \geq 3f\left(\sum \frac{a}{3}\right) = 3f(2) = 6 \geq 3\sqrt{3}$$

JP.012. Prove that if: $a,b,c,d>0$ then:

$$a^2+b^2+c^2+d^2=1;\ abc+bcd+cda+dab=\frac{1}{2}$$

$$\sum\frac{a^2}{1+2bcd}\ge\frac{4}{5}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mukherjee-Chandar Nagore-India

$$\sum_{cyc}a^2=1\text{ and }\sum_{cyc}abc=\frac{1}{2}$$

Let $a\ge b\ge c\ge d$ *then* $\frac{1}{bcd}\ge\frac{1}{acd}\ge\frac{1}{abd}\ge\frac{1}{abc}$. *Applying Cebyshev's Inequality,*

$$\sum_{cyc}\frac{a^2}{1+bcd}\ge\frac{1}{4}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{1+2bcd}\right)=\frac{1}{4}\left(\sum_{cyc}\frac{1}{1+2bcd}\right)\ge\frac{4}{4+2\sum_{cyc}bcd}=\frac{4}{5}$$

Equality at $a=b=c=d=\frac{1}{2}$.

Solution 2 Omar Raza – Lahore – Pakistan

$$\sum\frac{a^4}{a^2+2abcd(a)}\ge\frac{(a^2+b^2+c^2+d^2)^2}{a^2+b^2+c^2+d^2+2abcd(a+b+c+d)}=$$

$$=\frac{1}{1+2abcd(a+b+c+d)}$$

From $RMS\ge AM$ *inequality* $\left(\frac{1}{4}\right)^{\frac{1}{2}}\ge\frac{a+b+c+d}{4}$; $a+b+c+d\le2$ *and similarly*

$$\left(\frac{1}{4}\right)^{\frac{1}{2}}\ge(abcd)^{\frac{1}{4}}\text{ so }abcd\le\frac{1}{16}$$

thus $\frac{1}{1+2abcd(a+b+c+d)}\ge\frac{1}{(1+2abcd(2))}\ge\frac{1}{1+2\left(\frac{1}{16}\right)\cdot2}=\frac{1}{1+\frac{1}{4}}\ge\frac{4}{5}$

Solution 3 by Redwane El Mellas – Morocoo

Using:

$$\begin{cases}Bergstrom\ inequality: (\forall x,y,z,t\in\mathbb{R})(\forall u,v,w,l>0): \frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}+\frac{t^2}{l}\ge\frac{(x+y+z+t)^2}{u+v+w+l} & (1)\\ Mac\ Laurian\ inequality: \sum_{cyclic}x_1x_2\dots x_k\le\frac{\binom{n}{k}}{n^k}\left(\sum_{i=1}^n x_i\right)^k\ for\ all\ x_i>0\ (i=1,2,\dots,n)\ and\ k\in\{1,2,..,n\} & (2)\end{cases}$$

We get from (1) $\sum \frac{a^2}{1+2bcd} \geq \frac{(a+b+c+d)^2}{\sum 1+2bcd} = \frac{1+2\sum ab}{5}$. *And from (2)*

$$\sum_{cyclic} abc \leq \frac{\binom{4}{3}}{4^3}(a+b+c+d)^3 \Leftrightarrow (a+b+c+d)^3 \geq \frac{1}{2} \cdot \frac{4^3}{\binom{4}{3}} = 2^3$$

$\Leftrightarrow (a+b+c+d)^2 \geq 4 \Leftrightarrow 1 + 2\sum ab \geq 4$. *Finally* $\sum \frac{a^2}{1+2bcd} \geq \frac{4}{5}$ *for the equality it should be* $a+b+c+d = 2$*: So, by AM – QM* $\sqrt{\frac{a^2+b^2+c^2+d^2}{4}} = \frac{1}{2} = \frac{a+b+c+d}{4} \Leftrightarrow a = b = c = d = \frac{1}{2}$

JP.013. Prove that if $a > 0, a \neq 1$, then it does exists an infinity of pairs of numbers real strictly positive (x, y) such that:

a. $\log_a(x+y) = \log_a x + \log_a y$.

b. $\log_a(x+y) = (\log_a x) \cdot (\log_a y)$.

Proposed by Dana Heuberger – Romania

Solution by Dana Heuberger – Romania

Conditions of existence: $x, y \in (0, \infty)$.

a. $\log_a(x+y) = \log_a x + \log_a y \Leftrightarrow x + y = x \cdot y \Leftrightarrow (x-1)(y-1) = 1$ *and any pair* (x, y), *with* $\begin{cases} x > 1 \\ y = \frac{x}{x-1} \end{cases}$ *is solution.*

b. *We choose* $y = a^k$, *with* $k \in \mathbb{N}, k \geq 2$. *We obtain*

$$\log_a(x + a^k) = \log_a(x^k) \Leftrightarrow x + a^k = x^k \text{ and then}$$

$$1 + \frac{1}{x} \cdot a^k = x^{k-1} \qquad (1)$$

Let be $f, g: (0, \infty) \to \mathbb{R}, f(x) = 1 + \frac{1}{x} \cdot a^k, g(x) = x^{k-1}$.

Because f *is strictly decreasing and* g *is strictly increasing, the equation (1) has at least a solution.*

I. $a > 1$. $f(1) > g(1)$ *and* $f(2a) < g(2a) \Rightarrow$ *the equation has a unique solution,* x_k, *which belongs to the interval* $(1, 2a)$.

II. $a < 1$. $f(1) > g(1)$ **and** $f\left(\frac{2}{a}\right) < g\left(\frac{2}{a}\right) \Rightarrow$ ***the equation has a unique solution,*** x_k***, which belongs to the interval*** $\left(1, \frac{2}{a}\right)$.

JP.014. Let a, b, c be a non-negative real numbers such that: $a + b + c = 3$. Prove that:

$$11 + \frac{2}{3}\left(\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c}\right) \geq 13abc$$

Proposed by Hung Nguyen Viet – Hanoi – Vietnam

Solution 1 by Manish Tayal-New Delhi-India

Given $a, b, c \in [0, \infty)$:

$$a + b + c = 3, \frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}} \to (abc) \leq 1 \quad (1)$$

Also: we know for $x, y, z \in R$

$$x^4 + y^4 + z^4 \geq xyz(x + y + z)$$

$$(a + b + 1) \geq (abc)^{\frac{1}{4}}\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right) \Rightarrow 3 \geq (abc)^{\frac{1}{4}}\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right) \quad (2)$$

$$\text{Also: } \frac{a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}}{3} \geq (abc)^{\frac{1}{12}}$$

$$\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right)^3 \geq 27(abc)^{\frac{1}{4}} \quad (3)$$

Be the sides of inequality (2), (3) are non-negative: multiplying them

$$3\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right)^3 \geq 27(abc)^{\frac{1}{2}}\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right)$$

$$\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right)^2 \geq 9(abc)^{\frac{1}{2}}$$

$$\frac{9}{3}\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right) \geq 2(abc)^{\frac{1}{4}} \Rightarrow 11 + \frac{2}{3}\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right) \geq 11 + 2(abc)^{\frac{1}{4}} \quad (4)$$

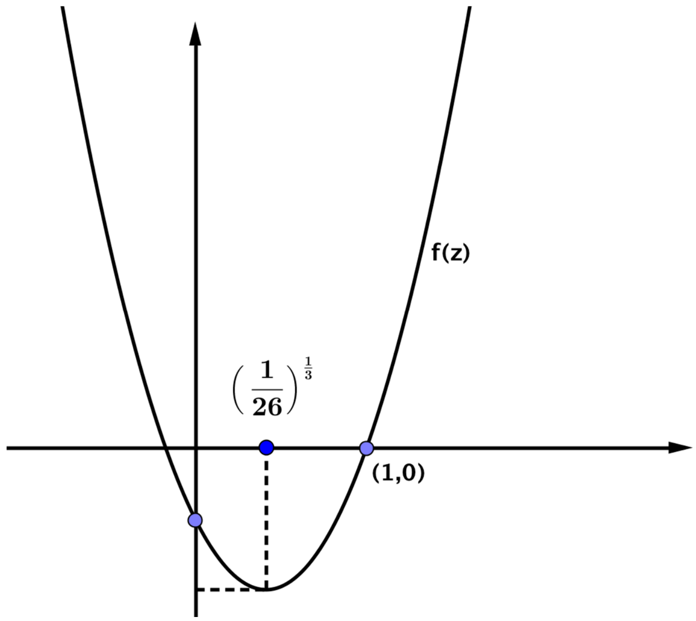
Consider the expression: $13(abc) - 2(abc)^{\frac{1}{4}} - 11$

$$abc \in [0, 1], (abc)^{\frac{1}{4}} = z \Rightarrow 13z^4 - 2z - 11$$

$$f(z) = 13z^4 - 2z - 11; f'(z) > 0$$

$$f'(z) = 53z^3 - 2 \Rightarrow z^3 > \frac{2}{52}, f''(z) = 156z^2 \geq 0; z^3 > \frac{1}{26}$$

$$f(1) = 0$$



For $z \in [0,1], f(z) \le 0 \Rightarrow 13z^5 \le 2z + 11$

$$\Rightarrow 13(abc) \le 2(abc)^{\frac{1}{4}} + 11 \textbf{ (4)}$$

from (4),(3)

$$11 + \frac{2}{3}\left(a^{\frac{1}{4}} + b^{\frac{1}{4}} + c^{\frac{1}{4}}\right) \ge 13abc \textbf{ (2)}$$

Solution 2 by Soumitra Mukherjee-Chandar Nagore-India

Let $f(x) = 11x + 2\sqrt[4]{x} - 13x^3; \forall x \in (0,1)$

$$f'(x) = 11 + \frac{1}{2x^{\frac{3}{4}}} - 39x^2; \forall x \in (0,1), f'(x) > 0; \forall x \in (0,1)$$

$f(x)$ ***is continuous on*** $(0,1)$ ***again*** $f'(x) > 0; \forall x \in (0,1)$

$f(x)$ ***is increasing on*** $(0,1)$

$$f(x) \ge f(0) = 0 \Rightarrow 11x + 2\sqrt[4]{x} \ge 13x^3; (\forall)x \in (0,1)$$

for $a, b, c \in (0,1)$ ***and*** $a + b + c = 3$,

$$11\sum_{cyc} a + 2\sum_{cyc}\sqrt[4]{a} \ge 13\sum_{cyc} a^3 \Rightarrow 33 + 2\sum_{cyc}\sqrt[4]{a} \ge 39abc \Rightarrow 11 + \frac{2}{3}\sum_{cyc}\sqrt[4]{a} \ge 13abc$$

Solution 3 by Kevin Soto Palacios –Huarmey-Peru

Sean: a,b,c***: números reales no negativos. Si:*** $a+b+c=3$***. Probar que:***

$11+\frac{2}{3}(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c})\ge 13abc$***. Supongamos sin pérdida de generalidad:***

$c\ge b\ge a\ge 0\to c+b+a\ge 3a\to 1\ge a\ge 0$***, por la tanto:*** $1\ge abc\ge 0$

Sea: $a=x^4, b=y^4, c=z^4\Leftrightarrow x^4+y^4+z^4=3\to x+y+z\ge 3\sqrt[3]{xyz}$

Multiplicando ($\times 3$***) y llevando en function a las variables, se tiene que:***

$$33+2(x+y+z)\ge 39x^4y^4z^4$$

$33+6\sqrt[3]{xyz}-39x^4y^4z^4\ge 0\to$ **Sea:** $xyz=m^3$, **si:** $1\ge abc\Leftrightarrow 1\ge m\ge 0\to$

$\to 1-m\ge 0\Rightarrow 33+6m-39m^{12}\ge 0\to$ ***(Factorizando se obtiene que)***

$$3(1-m)(13(m^{11}+m^{10}+m^9+\cdots+m)+11)\ge 0 \textit{ (LQQD)}$$

La igualidad se alcanza cuando: $a=b=c=1$

JP.015. Prove that if $a,b,c\in(0,\infty);\sqrt{a}+\sqrt{b}+\sqrt{c}=3$ **then:**

$$\frac{a\sqrt{b}+b\sqrt{a}}{a-\sqrt{ab}+b}+\frac{b\sqrt{c}+c\sqrt{b}}{b-\sqrt{bc}+c}+\frac{c\sqrt{a}+a\sqrt{c}}{c-\sqrt{ca}+a}\le 6$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ngo Dinh Tuan-Quang Nam-Da Nang-VietNam

$$x=\sqrt{a}, y=\sqrt{b}, z=\sqrt{c}\Rightarrow\begin{cases}x,y,z>0\\x+y+z-3\end{cases}$$

$$\sum\frac{x^2y+xy^2}{x^2-xy+y^2}\le 6$$

$$\sum\frac{x^2y+xy^2}{(x^2+y^2)-xy}\le\sum\frac{x^2y+xy^2}{2xy-xy}=\sum\frac{x^2y+xy^2}{xy}=2\sum x=6$$

Solution 2 by Soumitra Mukherjee-Chandar Nagore-India

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=3$$

$$\sum_{cyc}\frac{a\sqrt{b}+b\sqrt{a}}{a-\sqrt{ab}+b}\le\sum_{cyc}\frac{a\sqrt{b}+b\sqrt{a}}{\sqrt{ab}}=\sum_{cyc}(\sqrt{a}+\sqrt{b})=2\sum_{cyc}\sqrt{a}=6$$

Equality at $a=b=c=1$.

Solution 3 by Le Van-Ho Chi Min City-VietNam

$$LHS \le \sum\left[\left(a\cdot\sqrt{a}+b\cdot\sqrt{b}\right)/\left(a-\sqrt{b}\right)\right]$$

$$LHS \le \sum\left(\sqrt{a}+\sqrt{b}\right)=6$$

Equality holds when $a=b=c$

Solution 4 by Seyran Ibrahimov – Maasilli – Azerbaidian

$$a,b,c\in(0,\infty)$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=3$$

$$\frac{a\sqrt{b}+b\sqrt{a}}{a-\sqrt{ab}+b}+\frac{b\sqrt{c}-c\sqrt{b}}{b-\sqrt{bc}+c}+\frac{c\sqrt{a}+a\sqrt{c}}{c-\sqrt{ac}+c}\le 6$$

$$x+y+z=3$$

$$\frac{x^2y+y^2x}{x^2-xy+y^2}+\frac{y^2z+z^2y}{y^2-yz+z^2}+\frac{z^2x+x^2z}{z^2-xz+x^2}\le 6$$

$$x^2+y^2\ge 2xy$$

$$y^2+z^2\ge 2yz$$

$$z^2+x^2\ge 2xz$$

$$\max---\frac{x^2y+y^2x}{xy}+\frac{y^2z+z^2y}{yz}+\frac{z^2x+x^2z}{xz}\le 6$$

$$2x+2y+2z\le 6$$

$$6=6.$$

SP.001. Prove that if: $a,b,c\in(0,\infty)$ **then:**

$$\sum\frac{2a+3c}{a+2b+5c}<\frac{273\sum ab+87\sum a^2}{64\left(\sum\sqrt{ab}\right)^2}$$

Proposed by Mihály Bencze – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, a,b,c\in(0,\infty),\ \sum\frac{2a+3c}{a+2b+5c}\le\frac{273\sum ab+87\sum a^2}{64\left(\sum\sqrt{ab}\right)^2}$$

$$\because \left(\sum\sqrt{ab}\right)^2 \overset{CBS}{\le} 3\left(\sum ab\right), \therefore \frac{273\sum ab + 87\sum a^2}{64\left(\sum\sqrt{ab}\right)^2} \ge \frac{87\sum a^2 + 273\sum ab}{129\sum ab} \ge$$

$$\overset{?}{\ge} \sum \frac{2a+3c}{a+2b+5c} \Leftrightarrow \left(87\sum a^2 + 273\sum ab\right)(a+2b+5c)(b+2c+5a)(c+2a+5b) -$$

$$-192\left(\sum ab\right)\left\{\begin{matrix}(2a+3c)(b+2c+5a)(c+2a+5b)+\\(2b+3a)(c+2a+5b)(a+2b+5c)+\\(2c+3b)(a+2b+5c)(b+2c+5a)\end{matrix}\right\} \overset{?}{\ge} 0 \Leftrightarrow$$

$$\Leftrightarrow 870\sum a^5 + 1827\sum a^4b + 2871\sum ab^4 + 3594\sum a^2b^3 + 4272abc\left(\sum a^2\right) \underset{(1)}{\overset{?}{\ge}}$$

$\ge 366\sum a^3b^3 + 13068abc(\sum ab)$. **Now,** $\sum ab^4 = abc\left(\frac{b^2}{c}+\frac{c^3}{a}+\frac{a^3}{b}\right) = abc\left(\frac{b^4}{bc}+\frac{c^4}{ca}+\frac{a^4}{ab}\right) \ge$

$$\overset{Bergstrom}{\ge} abc\cdot\frac{\left(\sum a^2\right)^2}{\left(\sum ab\right)} \ge abc\frac{\left(\sum ab\right)^2}{\sum ab} = abc\left(\sum ab\right) \therefore 2871\sum ab^4 \overset{(a)}{\ge} 2871abc\left(\sum ab\right)$$

Also, $\sum a^4b = abc\left(\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\right) = abc\left(\frac{a^4}{ca}+\frac{b^4}{ab}+\frac{c^4}{bc}\right) \overset{Bergstrom}{\ge} abc\frac{\left(\sum a^2\right)^2}{\sum ab} \ge abc(\sum ab)$

$\therefore 1827\sum a^4b \overset{(b)}{\ge} 1827abc(\sum ab)$. **Again,** $4272abc(\sum a^2) \overset{(c)}{\ge} 4272abc(\sum ab)$

(a)+(b)+(c) ⇒ LHS of (1)

$$\ge 870\sum a^5 + 3594\sum a^2b^3 + 8970abc\left(\sum ab\right) \overset{?}{\ge} 366\sum a^3b^2 + 13068abc\left(\sum ab\right) \Leftrightarrow$$

$$\Leftrightarrow 870\sum a^5 + 3594\sum a^2b^3 \underset{(2)}{\overset{?}{\ge}} 366\sum a^3b^2 + 4098abc\left(\sum ab\right)$$

Now, $\sum(a^5+b^5) \overset{Chebyshev}{\ge} \frac{1}{2}\sum(a^2+b^2)(a^3+b^3) \ge \frac{1}{2}\sum(2ab)ab(a+b) = \sum a^2b^2(a+b) =$

$$= \sum a^3b^2 + \sum a^2b^3 \Rightarrow 2\sum a^5 \ge \sum a^3b^2 + \sum a^2b^3 \Rightarrow 732\sum a^5 \overset{(d)}{\ge} 366\sum a^3b^2 +$$

$$+366\sum a^2b^3$$

(d)⇒ LHS of (2) $\ge 138\sum a^5 + 366\sum a^3b^2 + 366\sum a^2b^3 + 3594\sum a^2b^3 \overset{?}{\ge}$

$$\ge 366\sum a^3b^2 + 4098abc\left(\sum ab\right) \Leftrightarrow 138\sum a^5 + 3960\sum a^2b^3 \overset{?}{\ge} 4098abc\left(\sum ab\right)$$

$\Leftrightarrow 23\sum a^5 + 660\sum a^2b^3 \underset{(3)}{\overset{?}{\ge}} 683abc(\sum ab)$. **Now,** $\sum a^2b^3 = abc\left(\frac{ab^2}{c}+\frac{bc^2}{a}+\frac{ca^2}{b}\right) =$

$$= abc\left(\frac{a^2b^2}{ca}+\frac{b^2c^2}{ab}+\frac{c^2a^2}{bc}\right) \overset{Bergstrom}{\ge} abc\cdot\frac{(\sum ab)^2}{\sum ab} = abc\left(\sum ab\right) \Rightarrow$$

$$\Rightarrow 660\sum a^2b^3 \Rightarrow \overset{(e)}{\geq} 660abc\left(\sum ab\right)$$. **Now,** $\sum(a^5+b^5) \geq \sum a^3b^2 + \sum a^2b^3$ **(proved earlier)**

$$= \sum a^3(b^2+c^2) \overset{A-G}{\geq} 2abc\left(\sum a^2\right) \geq 2abc\left(\sum ab\right) \Rightarrow \sum a^5 \geq abc\left(\sum ab\right) \Rightarrow$$

$$\Rightarrow 23\sum a^5 \overset{(f)}{\geq} 23abc\left(\sum ab\right)$$

(e)+(f)⇒ (3) is true (proved)

SP.002. Prove that in any acute-angled ΔABC the following relationship holds:

$$\cos A + \cos B + \cos C + \ln\big((A+1)(B+1)(C+1)\big) \leq 3 + \pi$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

$$f:(0,\infty) \to \mathbb{R}, f(x) = \cos x - x - \ln(x+1)$$

$$f'(x) = \sin x - 1 + \frac{1}{x+1} = \sin x - \frac{x}{x+1} \leq 0, \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$f - decreasing \to f(A) \leq f(0) = 1, f(B) \leq 1, f(C) \leq 1$$

By adding: $f(A) + f(B) + f(C) \leq 3$

$$\cos A - A + \ln(A+1) + \cos B - B + \ln(B+1) + \cos C - C + \ln(C+1) \leq 3$$

$$\cos A + \cos B + \cos C + \ln\big((A+1)(B+1)(C+1)\big) \leq 3 + (A+B+C) = 3 + \pi$$

SP.003. Let be $f:[0,1] \to (0,\infty)$ a differentiable function, convex and $a, b, c \in [0,1]$ such that:

$$f'(a) + f'(b) + f'(c) = 1; af'(a) + bf'(b) + cf'(c) = 2$$

Prove that:

$$\frac{1}{2} + \int_0^1 f(x)\,dx \geq \frac{1}{3}\big(f(a) + f(b) + f(c)\big)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tin Lu-Binh Son-Quang Ngai-VietNam

$\forall x_0 \in [0,1]$ ***and*** $f(x)$ ***is a differentiable; convex, we have:***

$$f(x) \geq f'(x)(x - x_0) + f(x_0)$$

$$f(x) \geq f'(a)(x - a) + f(a)$$

$$f(x) \geq f'(b)(x-b) + f(b)$$
$$f(x) \geq f'(c)(x-c) + f(c)$$
$$3f(x) \geq x\sum f'(a) - af(a) + \sum f(a) = x - q + \sum f(a)$$
$$\Rightarrow 3\int_0^1 f(x)\,dx \geq \int_0^1 \left[(x-2) + \sum f(a)\right] dx \Leftrightarrow 3\int_0^1 f(x)\,dx \geq -\frac{3}{2} + \sum f(a)$$
$$\Leftrightarrow \frac{1}{2} + \int_0^1 f(x)\,dx \geq \frac{1}{3}\sum f(a)$$

Solution 2 by Soumitra Mukherjee-Chandar Nagore-India

$f'(a) + f'(b) + f'(c) = 1$ **and** $af'(a) + bf'(b) + cf'(c) = 2.$

$$\Rightarrow \sum_{cyc} af'(a) = 2\sum_{cyc} f'(a) \Rightarrow \sum_{cyc} f'(a)\,(a-2) = 0$$

Now, $f: [0,1] \to (0,\infty)$ ***is a differentiable, convex function and*** $a, b, c \in [0,1]$.

$$f(2) \geq f(x) + f'(x)(2-x) \Rightarrow f'(x)(x-2) \geq f(x) - f(2)$$

for $a, b, c \in [0,1]$, $f'(a)(a-2) \geq f(a) - f(2)$; $f'(b)(b-2) \geq f(b) - f(2)$

and $f'(c)(c-2) \geq f(c) - f(2)$. ***From*** $\sum_{cyc} f'(a)(a-2) = 0$

$$\Rightarrow 0 \geq \sum_{cyc} f(a) - 3f(2) \Rightarrow 3f(2) \geq \sum_{cyc} f(a)$$

Let $\phi(x) = x + f(x) - f(2)\ \forall\, x \in (0, f(2)] \cup [f(2), \infty)$; $\phi'(x) = 1 + f'(x)$

Now, $\phi'(a) = 1 + f'(a) = 2f'(a) + f'(b) + f'(c) > 0$.

Again, $\phi'(b) = 2f'(b) + f'(a) + f'(c) > 0$ ***and*** $\phi'(c) = 2f'(c) + f'(a) + f'(b) > 0$

$\phi'(a), \phi'(b)$ ***and*** $\phi'(c) > 0\ \ \forall\, a, b, c \in [0,1]$. ***Since*** a, b ***and*** c ***are arbitrary elements from***

$[0,1]$ $\phi'(x) > 0\ \ \forall\, x \in (0, f(2)] \cup [f(2), \infty)$. ***Now,*** $\phi(x)$ ***is continuos on***

$(0, f(2)] \cup [f(2), \infty)$. ***And*** $\phi'(x) > 0\ \ \forall\, x \in (0, f(2)] \cup [f(2), \infty)$

$\phi(x)$ ***is increasing on*** $(0, f(2)] \cup [f(2), \infty)$

$\phi(x) \geq \phi\{f(2)\}$, ***since,*** $f(2)$ ***is the point of inflextions.***

$$\phi'(x) > 0$$
$$x + f(x) > f(2)$$

$$\int_0^1 x\,dx + \int_0^1 f(x)\,dx \geq f(2) = \frac{f(a)+f(b)+f(c)}{3}$$

since, $3f(2) \geq \sum_{cyc} f(a)$.

$$\frac{1}{2} + \int_0^1 f(x)\,dx \geq \frac{1}{3}\left(f(a)+f(b)+f(c)\right)$$

SP.004. If $x, y, z \in (0, \infty)$ then: $x + y + z + \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \geq \frac{12}{\sqrt{3\sqrt{3}}}$

Proposed by Daniel Sitaru – Romania

Solution by Bao Ngo Minh Ngoc – Gia Lai Province– VietNam

Use $AM - GM$ *we have:* $x + \frac{1}{x^3} = \frac{x}{3} + \frac{x}{3} + \frac{x}{3} + \frac{1}{x^3} \geq 4\sqrt[4]{\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{1}{x^3}} = \frac{4}{\sqrt{3\sqrt{3}}}$

$$\Rightarrow x + y + z + \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \geq 4\sum \sqrt[4]{\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{1}{x^3}} = \frac{12}{\sqrt{3\sqrt{3}}}$$

SP.005. Prove that:

$$\sum_{k=1}^{\infty}\left(\frac{1+(k^2-1)^{\frac{1}{2}}}{1+(k^2-1)^{\frac{3}{4}}}\right)^4 \leq \frac{\pi^2}{3}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

First we show that $2(x^3+1)^4 \geq (x^4+1)(x^2+1)^4$ *for all* $x \geq 0$*. But*
$(x^2+1)^4 \leq (x+1)^2(x^3+1)^2$ *and we are left with the inequality*
$2(x^3+1)^2 \geq (x+1)^2(x^4+1) \Leftrightarrow 2(x^2-x+1)^2 \geq x^4+1 \Leftrightarrow (x-1)^4 \geq 0$ *which follows.*

Therefore $\frac{2}{x^4+1} \geq \left(\frac{x^2+1}{x^3+1}\right)^4$*. If* $x = \sqrt[4]{k^2-1}$ *then* $\left(\frac{1+(k^2-1)^{\frac{1}{2}}}{1+(k^2-1)^{\frac{3}{4}}}\right)^4 \leq \frac{2}{k^2}$

therefore $\sum_{k=1}^{\infty}\left(\frac{1+(k^2-1)^{\frac{1}{2}}}{1+(k^2-1)^{\frac{3}{4}}}\right)^4 \leq 2\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{3}$

SP.006. If $f:\mathbb{R}\to\mathbb{R}$ continuous and convexe then:

$$\int_0^e f(x)\,dx \ge \int_0^1 (x^3+nx^{n-1})f\left(\frac{e^{2x}+nx^{2n-1}}{e^x+nx^{n-1}}\right)dx$$

where $n\ge 1$.

Proposed by Mihály Bencze – Romania

Solution by proposer

By Jensen's inequality: $\frac{e^x f(e^x)+nx^{n-1}f(x^n)}{e^x+nx^{n-1}} \ge f\left(\frac{e^{2x}+nx^{2n-1}}{e^x+nx^{n-1}}\right)$

$$\int_0^1 e^x f(e^x)\,dx = \int_1^e f(t)\,dt$$

$$\int_0^1 nx^{n-1}f(x^n)\,dx = \int_0^1 f(t)\,dt$$

$$\int_0^1 f(t)\,dt + \int_1^e f(t)\,dt = \int_0^e f(t)\,dt \Rightarrow \int_0^e f(x)\,dx \ge \int_0^1 (e^x+nx^{n-1})\,f\left(\frac{e^{2x}+nx^{2n-1}}{e^x+nx^{n-1}}\right)dx$$

$$\int_0^e f(x)\,dx \ge \int_0^1 (e^x+nx^{n-1})\,f\left(\frac{e^{2x}+nx^{2n}-1}{e^x+nx^{n-1}}\right)dx$$

SP.007. If $x_k>1\ (k=1,2,\dots,n)$ and $S=\sum_{k=1}^n x_k$ then:

$$\prod_{k=1}^n \log_{x_k}\frac{s-k}{n-1} \ge 1$$

for all $n\ge 3$.

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\prod_{k=1}^n \log_{x_k}\frac{s-x_k}{n-1} \ge \prod_{k=1}^n \log_{x_k}\sqrt[n-1]{x_1\cdot x_{k-1}\cdot x_{k+1}\cdot\ldots\cdot x_n} =$$

$$= \prod_{k=1}^{n} \frac{1}{n-1}\left(\log_{x_k} x_1 + \cdots + \log_{x_k} x_{k-1} + \log_{x_k} x_{k+1} + \cdots + \log_{x_k} x_n\right) \geq$$

$$\geq \prod_{k=1}^{n} \sqrt[n-1]{\log_{x_k} x_1 \cdot \ldots \cdot \log_{x_k} x_{k-1} \log_{x_k} x_{k+1} \cdot \ldots \cdot \log_{x_k} x_n} =$$

$$= \prod_{cyclic} \log_{x_1} x_2 \log_{x_2} x_1 = 1$$

SP.008. Prove that if $a, b, c \in (0, \infty)$ then:

$$12 \sum \frac{c}{a^2 + b^2 + 9} \leq \frac{1}{abc} \sum c^2 \sqrt{a^2 + b^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Henry Ricardo – New York – USA

First we note that the AM – GM inequality gives us

$a^2 + b^2 + 9 = (a^2 + b^2) + 9 \geq 6\sqrt{a^2 + b^2}$ *and* $a^2 + b^2 \geq 2ab$. *Thus*

$$\frac{c}{a^2 + b^2 + 9} \leq \frac{c}{6\sqrt{a^2 + b^2}} = \frac{c\sqrt{a^2 + b^2}}{6(a^2 + b^2)} \leq \frac{c\sqrt{a^2 + b^2}}{12ab} = \frac{c^2\sqrt{a^2 + b^2}}{12abc},$$

Which implies the desired inequality.

Solution 2 by Ngô Minh Ngọc Bảo – Gia Lai Province – VietNam

Use AM – GM, we have:

$$\frac{\sum c^2\sqrt{a^2 + b^2}}{abc} = \sum \frac{c\sqrt{a^2 + b^2}}{ab} \geq 2 \sum \frac{c}{\sqrt{a^2 + b^2}} =$$

$$= 6 \sum \frac{c}{3\sqrt{a^2 + b^2}} \geq 12 \sum \frac{c}{a^2 + b^2 + 9}$$

SP.009. Let be $a, b, c \in (0, \infty); a < b < c; f: [0, a] \to [0, b]; g: [0, b] \to [0, c]$ continuous, bijectifs and strictly increasing functions. Prove that:

$$\frac{1}{c}\int_0^a (g \circ f)^2 (x)dx + \frac{1}{a}\int_0^c (f^{-1} \circ g^{-1})^2 (x)dx < ac$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$(g\circ f)(x)\in[0,c]\Rightarrow(g\circ f)(x)\le c;(\forall)x\in[0,a]$$

$$\frac{1}{c}\int_0^a(g\circ f)^2(x)\,dx\le\frac{1}{c}\int_0^a c\cdot(g\circ f)(x)dx=\int_0^a(g\circ f)(x)dx$$

$$(f^{-1}\circ g^{-1})(x)\in[0,a]\Rightarrow(f^{-1}\circ g^{-1})(x)\le a;(\forall)x\in[0,c]$$

$$\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})^2(x)\,dx\le\frac{1}{a}\int_0^c a(f^{-1}\circ g^{-1})(x)dx=\int_0^c(f^{-1}\circ g^{-1})(x)\,dx$$

$$\frac{1}{c}\int_0^a(g\circ f)^2(x)dx\le\int_0^a(g\circ f)(x)dx$$

$$\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})(x)\,dx\le\int_0^c(f^{-1}\circ g^{-1})(x)dx$$

$$\frac{1}{c}\int_0^a(g\circ f)^2(x)dx+\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})(x)dx\le$$

$$\le\int_0^a(g\circ f)(x)\,dx+\int_0^c(f^{-1}\circ g^{-1})(x)dx=ac$$

SP.010. In all acute – angle triangle ABC holds:

$$\sum\left(\frac{\cosh\frac{A}{2}}{\cos\frac{A}{2}}\right)^2\le\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

If $x\in\left(0,\frac{\pi}{2}\right)\Rightarrow\cos\frac{x}{2}\le\cosh\frac{x}{2}\Rightarrow\tanh\frac{x}{2}\le\tan\frac{x}{2}\Rightarrow\cosh\frac{x}{2}\le\frac{1}{\sqrt{1-\tan^2\frac{x}{2}}}=$

$$=\frac{\cos\frac{x}{2}}{\sqrt{\cos x}}\Rightarrow\left(\frac{\cosh\frac{x}{2}}{\cos\frac{x}{2}}\right)^2\le\frac{1}{\cos x}\Rightarrow\sum\left(\frac{\cosh\frac{A}{2}}{\cos\frac{A}{2}}\right)^2\le\sum\frac{1}{\cos A}=\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}$$

SP.011. Prove that: $\frac{\pi^2}{6}-\frac{1}{n}<\sum_{k=1}^{n}\frac{1}{k^2}<\frac{\pi^2}{6}-\frac{1}{n+1}$ **for all** $n\geq 1$ **posive integers.**

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\sum_{k=n}^{\infty}\frac{1}{k^2}<\sum_{k=n}^{\infty}\frac{1}{k(n-1)}=\sum_{k=n}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k}\right)=\frac{1}{n-1}$$

$$\sum_{k=n}^{\infty}\frac{1}{k^2}>\sum_{n=n}^{\infty}\frac{1}{k(k+1)}=\sum_{k=n}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{1}{n},so,\frac{1}{n}<\sum_{k=n}^{\infty}\frac{1}{k^2}<\frac{1}{n-1}$$

$$\frac{1}{n}+\sum_{k=1}^{n-1}\frac{1}{k^2}<\sum_{k=1}^{\infty}\frac{1}{k^2}<\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{n-1},or,\frac{1}{n}+\sum_{k=1}^{n-1}\frac{1}{k^2}<\frac{\pi^2}{6}<\sum_{k=1}^{n-1}\frac{1}{k^2}+\frac{1}{n-1}$$

Or $\frac{\pi^2}{6}-\frac{1}{n-1}<\sum_{k=1}^{n-1}\frac{1}{k^2}<\frac{\pi^2}{6}-\frac{1}{n},or,\frac{\pi^2}{6}-\frac{1}{n}<\sum_{k=1}^{n}\frac{1}{k^2}<\frac{\pi^2}{6}-\frac{1}{n+1}$

SP.012. If $A,B\in M_2(C)$ **then:**

$$\sum_{k=1}^{n}(\det(A+kB)+\det(A-kB))=2n\det A+\frac{n(n+1)(2n+1)}{3}\det B$$

Proposed by Mihály Bencze – Romania

Solution by proposer

Let be $f(t)=\det(A+tB)=t^2\det B+\alpha t+\det A\Rightarrow$

$$\sum_{k=1}^{n}(\det(A+kB)+\det(A-kB))=\sum_{k=1}^{n}\left(k^2\det B+\alpha k+\det A+k^2\det B-\alpha k+\det A\right)=$$

$$=2n\det A+\frac{n(n+1)(2n+1)}{3}\det B$$

SP.013. Let be $a,b\in\mathbb{R},a<b$**. Find:**

$$\lim_{n\to\infty}\int_a^b\sin x\cdot\arctan(nx)\,dx$$

Proposed by Dan Nedeianu – Romania

Solution by Francis Fregeaux-Quebec-Canada

$$\lim_{n\to\infty}\int_a^b \sin(x)\arctan(nx)\,dx = \alpha$$

$$\lim_{n\to\infty}\arctan(n) = \frac{\pi}{2}$$

For any $x\neq 0$:

$$\lim_{n\to\infty} nx = \lim_{n\to\infty} \pm n = \pm\infty,$$

depending on the sign of "x".

$$\arctan(-x) = -\arctan(x)$$

And since both $\sin(x)$ ***and*** $\arctan(x)$ ***share the same limit when*** $x\to 0$

$$\alpha = \int_a^b \sin(x)\cdot\lim_{n\to\infty}\arctan(nx)\,dx \to \frac{\pi}{2}\int_a^b \sin(x)\,dx;\ 0\le a<b$$

$$\alpha = \int_a^b \sin(x)\cdot\lim_{n\to\infty}\arctan(nx)\,dx \to \frac{-\pi}{2}\int_a^b \sin(nx)\,dx;\ a<b\le 0$$

$$\alpha = \frac{\pi}{2}[\cos(a)-\cos(b)]\ \textbf{for: } 0\le a<b$$

$$\alpha = \frac{\pi}{2}[\cos(b)-\cos(a)]\ \textbf{for: } a<b\le 0$$

And if $a<0, b>0, a<b$***:***

$$\alpha = \frac{\pi}{2}[\cos(0)-\cos(a)] + \frac{\pi}{2}[\cos(0)-\cos(b)] = \pi - \frac{\pi}{2}[\cos(a)+\cos(b)]$$

SP.014. Prove that if $a,b,c\in(0,\infty)$ and $b\ge a$, then:

$$2\sqrt{2}\left(e^{bc}-e^{ac}\right)\le c(b-a)\sqrt{e^{2ac}+e^{2bc}}$$

Proposed by Daniel Sitaru – Romania

Solution by Ngô Minh Ngọc Bảo –Gia Lai Province-VietNam

Let $x=e^{bc}, y=e^{ac}, (x,y>0)$. ***We need to prove that:***

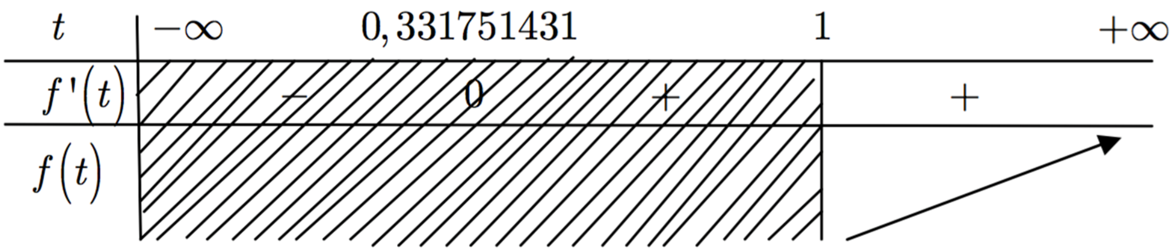
$$2\sqrt{2}(x-y)\le(\ln x-\ln y)\sqrt{x^2+y^2} \Leftrightarrow \sqrt{\left(\frac{x}{y}\right)^2+1}\cdot\ln\frac{x}{y}\ge 2\sqrt{2}\left(\frac{x}{y}-1\right)\ \textbf{(*)}$$

Indeed, let $t = \frac{x}{y} \geq 1$***, we have: (*)*** $\Leftrightarrow \sqrt{t^2+1} \cdot \ln t - 2\sqrt{2}(t-1) \geq 0.$

Considering function: $f(t) = \sqrt{t^2+1} \cdot \ln t - 2\sqrt{2}t + 2\sqrt{2}, \forall t \geq 1.$

$$f'(t) = \frac{t \ln t}{\sqrt{t^2+1}} + \frac{\sqrt{t^2+1}}{t} - 2\sqrt{2},\ f''(t) = \frac{\ln t}{\left(\sqrt{t^2+1}\right)^3} + \frac{t^2-1}{t^2\sqrt{t^2+1}} > 0$$

Therefore, the equation $f'(t) = 0$ ***has a unique solution.***



$$\Rightarrow f(t) \geq f(1) = \sqrt{1+1} \cdot \ln 1 - 2\sqrt{2} + 2\sqrt{2} = 0, (!)$$

SP.015. Prove that if $a, b, c \in [0, \infty)$ **then:**

$$25\sum a^2\sqrt[5]{a} + 11\sum ab^6 \geq 33\sum a^2 b$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

By Young's inequality:

$$px^q + qx^p \geq pqxy; p > 1; \frac{1}{p} + \frac{1}{q} = 1; x, y \geq 0$$

$$\frac{1}{q} = 1 - \frac{1}{p} \Rightarrow \frac{1}{q} = \frac{p-1}{p} \Rightarrow q = \frac{p}{p-1}$$

$$px^{\frac{p}{p-1}} + \frac{p}{p-1}y^p \geq \frac{p^2}{p-1}xy$$

$$p\int_0^x x^{\frac{p}{p-1}}\,dx + \frac{p}{p-1}y^p\int_0^x dx \geq \frac{p^2}{p-1}y\int_0^x x\,dx$$

$$p\frac{x^{\frac{p}{p-1}+1}}{\frac{p}{p-1}+1} + \frac{p}{p-1}y^p x \geq \frac{p^2}{p-1}y \cdot \frac{x^2}{2}$$

$$\frac{x^{\frac{2p-1}{p-1}}}{\frac{2p-1}{p-1}}+\frac{1}{p-1}xy^p\geq\frac{p}{2(p-1)}x^2y$$

For $p=6, x=a, y=b$**:**

$$\frac{x^{\frac{11}{5}}}{\frac{11}{5}}+\frac{1}{5}xy^6\geq\frac{6}{10}x^2y\to\frac{5}{11}a^{\frac{11}{5}}+\frac{1}{5}ab^6\geq\frac{3}{5}a^2b$$

$$\frac{5}{11}\sum a^{\frac{11}{5}}+\frac{1}{5}\sum ab^6\geq\frac{3}{5}\sum a^2b$$

$$25\sum a^2\sqrt[5]{a}+11\sum ab^6\geq 33\sum a^2b$$

UP.001. Prove that if $\alpha\in[2,7]$ then:

$$\int_2^{\alpha}\arctan^5 x\cdot dx\leq\frac{\alpha-2}{5}\int_2^7\arctan^5 x\cdot dx$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Quang Minh – Nguyen Thi Linh – Ho Chi Minh – Vietnam

We denote $f(\alpha)=\frac{\alpha-2}{5}\int_2^7\arctan^5 x\cdot dx-\int_2^{\alpha}\arctan^5 x\cdot dx$ ***with*** $\alpha\in[2,7]$***, we have:***

$$f''(\alpha)=-\frac{5\arctan^4\alpha}{\alpha^2+1}<0$$

for all $\alpha\in[2,7]$***, so for all*** $\alpha\in[2,7]$ ***we have inequality:***

$$f(\alpha)\geq\min\{f(2),f(7)\}=0$$

Or

$$\frac{\alpha-2}{5}\int_2^7\arctan^5 x\cdot dx\geq\int_2^{\alpha}\arctan^5 x\cdot dx$$

for all $\alpha\in[2,7]$***.***

UP.002. Let a,b,c be positive real numbers. Prove that:

$$\sum\frac{1}{5a}+\sum\frac{1}{3a+b+c}\geq\sum\left(\frac{1}{4a+b}+\frac{1}{4a+c}\right)$$

Proposed by Soumitra Mukherjee - Chandar Nagore – India

Solution by Tran Quang Minh – Nguyen Thi Linh – Ho Chi Minh – Vietnam

We have one lemma.

Lemma 1. If $x, y, z \in (0, +\infty)$ then:

$$x^5 + y^5 + z^5 + x^3yz + xy^3z + xyz^3 \geq x^4(y+z) + y^4(z+x) + z^4(x+y) \quad (1)$$

Proof. We normalize $x + y + z = 1$ and denote $xy + yz + zx = q$, $xyz = r$ then:

$$(1) \Leftrightarrow (-12q + 7)r + 8q^2 - 6q + 1 \geq 0$$

Use $r \geq \max\left\{0, \frac{4q-1}{9}\right\}$ we will have $(-12q + 7)r + 8q^2 - 6q + 1 \geq 0$

Back to the problem:

From Lemma, denote $x = t^a, y = t^b, z = t^c$, we have:

$$\sum t^{5a} + \sum t^{3a+b+c} \geq \sum\left(t^{4a+b} + t^{4a+c}\right)$$

$$\text{or } \sum t^{5a-1} + \sum t^{3a+b+c-1} \geq \sum\left(t^{4a+b-1} + t^{4a+c-1}\right)$$

Take integral from 0 to 1 we have:

$$\int_0^1 \sum t^{5a-1}\,dt + \int_0^1 \sum t^{3a+b+c-1}\,dt \geq \int_0^1 \sum\left(t^{4a+b-1} + t^{4a+c-1}\right)dt$$

$$\text{Or } \sum \frac{1}{5a} + \sum \frac{1}{3a+b+c} \geq \sum\left(\frac{1}{4a+b} + \frac{1}{4a+c}\right)$$

UP.003. If $A, B \in M_2(C)$ then:

$$(\det(A+B))^2 + (\det(A-B))^2 \geq 2\det(AB+BA)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

If $X, Y \in M_2(C)$ and $f: C \to C, f(t) = \det(X + tY) = t^2 \det Y + at + \det X; a \in C$

$$f(1) + f(-1) = 2(\det X + \det Y) \Rightarrow \det(X+Y) + \det(X-Y) = 2(\det X + \det Y)$$

Let be $X = A^2 + B^2, Y = AB + BA \Rightarrow A^2 + B^2 + AB + BA = (A+B)^2$

$$A^2 + B^2 + AB + BA = (A+B)^2$$

$$(\det(A+B))^2 + (\det(A-B))^2 = 2\det(A^2+B^2) + 2\det(AB+BA) \Rightarrow$$

$$\frac{1}{2}((\det(A+B))^2+(\det(A-B))^2-2\det(AB+BA))=\det(A^2+B^2)=$$
$$=\det(A+iB)(A-iB)=\det(A+iB)\det(A-iB)=\det(A+iB)\overline{\det(A+\imath B)}=$$
$$=(\alpha+i\beta)(\alpha-i\beta)=\alpha^2+\beta^2\geq 0\quad(\alpha,\beta\in R)$$

UP.004. Let $f:[a,c]\to\mathbb{R}, 0<a<c$ be a continuous and convex function on $[a,c]$. Prove that if $b\in[a,c]$ then:

$$2\int_a^c f(x)\,dx\leq(b-a)[f(b)+f(a)]+(c-b)[f(b)+f(c)]$$

Proposed by Daniel Sitaru – Romania

Solution by Tran Quang Minh – Nguyen Thi Linh – Ho Chi Minh – Vietnam

If $b\in[a,c]$ we will have:

$$\begin{cases} f(x)\leq g(x)=\dfrac{f(b)-f(a)}{b-a}(x-a)+f(a), x\in[a,b] \\ f(x)\leq h(x)=\dfrac{f(c)-f(b)}{c-b}(x-b)+f(b), x\in[b,c]\end{cases}$$

then

$$2\int_a^c f(x)\,dx\leq 2\left[\int_a^b g(x)\,dx+\int_b^c h(x)\,dx\right]=(b-a)[f(b)+f(a)]+(c-b)[f(c)+f(b)]$$

UP.005. If $x\geq 1$ then:

$$ex\ln\left(1+\frac{1}{x}\right)\leq\left(1+\frac{1}{x}\right)^x\leq\frac{2x}{\ln 2}\ln\left(1+\frac{1}{x}\right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

Let be $g(x)=\frac{y}{\ln y}, y\in[2,3], g'(y)=\frac{\ln y-1}{\ln^2 y}\Rightarrow$

y	2	e	3
$g'(x)$	$- - - - - - - - -$	0	$- - - - - - - - - -$

$g(y)$	$\frac{2}{\ln 2}$ ↘ e ↗ $\frac{3}{\ln 3}$

But $9 > 8 \Rightarrow 3^2 > 2^3 \Rightarrow 2\ln 3 > 3\ln 2 \Rightarrow \frac{2}{\ln 2} > \frac{3}{\ln 3}$

$$\Rightarrow Im(g) = \left[e, \frac{2}{\ln 2}\right] \Rightarrow e \le g(y) \le \frac{2}{\ln 2} \Rightarrow e \le \frac{y}{\ln y} \le \frac{2}{\ln 2} \Rightarrow$$

$\Rightarrow e\ln y \le y \le \frac{2}{\ln 2}\ln y$. *In these we take* $y = \left(1+\frac{1}{x}\right)^x, x \ge 1 \Rightarrow$

$$ex\ln\left(1+\frac{1}{x}\right) \le \left(1+\frac{1}{x}\right)^x \le \frac{2x}{\ln 2}\ln\left(1+\frac{1}{x}\right)$$

UP.006. If $a, b, c > 0$ and $x, y, z \ge 1$ then:

$$\left(\frac{xz}{y}\right)^{2a}\left(\frac{yx}{z}\right)^{2b}\left(\frac{zy}{x}\right)^{2c} \le x^{\frac{a^2+b^2}{c}}y^{\frac{b^2+c^2}{a}}z^{\frac{c^2+a^2}{b}}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

We have: $2a+2b-2c \le \frac{a^2+b^2}{c} \Leftrightarrow (a-c)^2+(b-c)^2 \ge 0$ *therefore*

$$\begin{cases}(2a+2b-2c)\ln x \le \frac{a^2+b^2}{c}\ln x\\(2b+2c-2a)\ln y \le \frac{b^2+c^2}{a}\ln y\\(2c+2a-2b)\ln z \le \frac{c^2+a^2}{b}\ln z\end{cases}$$ *. After addition we obtain:*

$$\sum(2a+2b-2c)\ln x = \sum 2a(\ln x - \ln y + \ln z) = \sum \ln\left(\frac{xy}{y}\right)^{2a} \le$$

$$\le \sum \frac{a^2+b^2}{c}\ln x = \sum \ln x^{\frac{a^2+b^2}{c}}$$

UP.007. Let be $a, r \in (0,\infty); (a_n)_{n\ge 1}; a_1 = a; a_{n+1} = a_n + r, n \in \mathbb{N}^*;$

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$$b_n = \prod_{k=1}^{n} a_k, c_n = \prod_{k=1}^{n} b_k^2.$$

Find:

$$\lim_{n\to\infty} \frac{n}{\sqrt[n^2]{c_n}}.$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by George – Florin Șerban – Romania

$$1 = \lim_{n\to\infty} \frac{n}{\sqrt[n^2]{c_n}}$$

$$\ln 1 = \lim_{n\to\infty} \sqrt[n^2]{\frac{n^{n^2}}{c_n}} = \lim_{n\to\infty} \frac{\ln \frac{n^{n^2}}{c_n}}{n^2} = \lim_{n\to\infty} \frac{\ln \frac{(n+1)^{(n+1)^2}}{c_{n+1}} - \ln \frac{n^{n^2}}{c_n}}{(n+1)^2 - n^2} =$$

$$= \lim_{n\to\infty} \frac{\ln \frac{(n+1)^{(n+1)^2} c_n}{c_{n+1} n^{n^2}}}{2n+1}, \qquad (Cesaro\ Stolz)$$

$$\ln l = \lim_{n\to\infty} \frac{\ln \frac{(n+1)^{(n+1)^2} c_n}{c_{n+1} n^{n^2}}}{2n+1} = \lim_{n\to\infty} \frac{\ln \frac{(n+2)^{(n+2)^2} c_{n+1}}{c_{n+2}(n+1)^{(n+1)^2}} - \ln \frac{(n+1)^{(n+1)^2} c_n}{c_{n+1} n^{n^2}}}{(2n+3) - (2n+1)} =$$

$$= \lim_{n\to\infty} \frac{\ln \frac{n^{n^2}(n+2)^{(n+2)^2} c_{n+1}^2}{(n+1)^{2(n+1)^2} c_n c_{n+2}}}{2},$$

$$\frac{c_{n+1}^2}{c_n c_{n+2}} = \frac{b_{n+1}^4}{b_{n+1}^2 b_{n+2}^2} = \frac{b_{n+1}^2}{b_{n+2}^2} = \frac{1}{a_{n+2}^2} = \frac{1}{[a+(n+1)r]^2},$$

$$\frac{n^{n^2}(n+2)^{(n+2)^2} c_{n+1}^2}{(n+1)^{2(n+1)^2} c_n c_{n+2}} = \frac{n^{n^2}(n+2)^{n^2}(n+2)^{4n}(n+2)^4}{(n+1)^{2n^2}(n+1)^{4n}(n+1)^2[a+(n+1)r]^2} =$$

$$= \left(\frac{n^2+2n}{n^2+2n+1}\right)^{n^2} \left(\frac{n+2}{n+1}\right)^{4n+2} \left(\frac{n+2}{a+(a+1)r}\right)^2,$$

$$\lim_{n\to\infty} \left(\frac{n^2+2n}{n^2+2n+1}\right)^2 = \lim_{n\to\infty} \left\{\left(1 + \frac{-1}{n^2+2n+1}\right)^{\frac{n^2+2n+1}{-1}}\right\}^{\frac{-n^2}{n^2+2n+1}} = e^{-1},$$

$$\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{4n+2}=\lim_{n\to\infty}\left\{\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{1}}\right\}^{\frac{4n+2}{n+1}}=e^4,$$

$$\lim_{n\to\infty}\left(\frac{n+2}{a+(n+1)r}\right)^2=\left(\frac{1}{r}\right)^2=r^{-2},$$

$$\ln l=\lim_{n\to\infty}\frac{\ln\frac{n^{n^2}(n+2)^{(n+2)^2}c_{n+1}^2}{(n+1)^{2(n+1)^2}c_nc_{n+2}}}{2}=\frac{\ln e^{-1}\cdot e^4\cdot r^{-2}}{2}=\ln\sqrt{e^3\cdot r^{-2}},$$

$$l=\sqrt{e^3\cdot r^{-2}}=\frac{e\sqrt{e}}{r},\ \lim_{n\to\infty}\frac{n}{\sqrt[n^2]{c_n}}=\frac{e\sqrt{e}}{r}.$$

UP.008. Find:

$$\int\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}dx$$

Proposed by Mihály Bencze – Romania

Solution 1 by Hamza Mahmood – Lahore – Pakistan:

First we factorize the denominator of the integrand:

$$e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1$$

$$=(e^x)^2+(1+\cos x+1-\sin x)e^x+\cos x(1-\sin x)+1-\sin x$$

$$=(e^x)^2+(1+\cos x+1-\sin x)e^x+(1+\cos x)(1-\sin x)$$

Using the identity, $x^2+(a+b)x+ab=(x+a)(x+b)$, *we have:*

$$=(e^x+1+\cos x)(e^x+1-\sin x)$$

$$e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1=(e^x+1+\cos x)(e^x+1-\sin x)\quad (A)$$

& we observe the following:

$$(e^x+1+\cos x)(e^x-\cos x)-(e^x+1-\sin x)(e^x-\sin x)$$

$$=e^{2x}+e^x+e^x\cos x-e^x\cos x-\cos x-\cos^2 x-e^{2x}-e^x+e^x\sin x+e^x\sin x+\sin x-\sin^2 x$$

$$=\left(e^{2x}-e^{2x}\right)+(e^x-e^x)+(e^x\cos x-e^x\cos x)+2e^x\sin x+\sin x-\cos x-\left(\cos^2 x+\sin^2 x\right)$$

$$=2e^x\sin x+\sin x-\cos x-1=(2e^x+1)\sin x-\cos x-1$$

$$(e^x+1+\cos x)(e^x-\cos x)-(e^x+1-\sin x)(e^x-\sin x)=(2e^x+1)\sin x-\cos x-1\quad (B)$$

From (A) and (B), we have

$$\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}=$$

$$=\frac{(e^x+1+\cos x)(e^x-\cos x)-(e^x+1-\sin x)(e^x-\sin x)}{(e^x+1+\cos x)(e^x+1-\sin x)}$$

$$\Rightarrow\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}$$

$$=\frac{e^x-\cos x}{e^x+1-\sin x}-\frac{e^x-\sin x}{e^x+1+\cos x}$$

$$\Rightarrow\int\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}dx$$

$$=\int\frac{e^x-\cos x}{e^x+1-\sin x}dx-\int\frac{e^x-\sin x}{e^x+1+\cos x}dx$$

Since $\int\frac{e^x-\cos x}{e^x+1-\sin x}dx=\int\frac{1}{e^x+1-\sin x}d(e^x+1-\sin x)=\ln(e^x+1-\sin x)$

$\&\int\frac{e^x-\sin x}{e^x+1+\cos x}dx=\int\frac{1}{e^x+1+\cos x}d(e^x+1+\cos x)=\ln(e^x+1+\cos x)$

$$\int\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}dx$$

$$=\ln(e^x+1-\sin x)-\ln(e^x+1+\cos x)$$

$$\int\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}dx=\ln\left(\frac{e^x+1-\sin x}{e^x+1+\cos x}\right)+c$$

Solution 2 by Ravi Prakash – New Delhi – India

$$e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1$$

$$=e^{2x}+[(1+\cos x)+(1-\sin x)]+(1+\cos x)(1-\sin x)$$

$=(e^x+1+\cos x)(e^x+1-\sin x)$. ***Also,***

$$(e^x+\cos x+1)(e^x-\cos x)-(e^x-\sin x)(e^x-\sin x+1)=e^{2x}-\cos^2 x+e^x-\cos x$$

$$-[e^{2x}-2e^x\sin x+\sin^2 x+e^x-\sin x]=2e^x\sin x+\sin x-\cos x-1$$

$=(2e^x+1)\sin x-\cos x-1=$ ***Numerator. Thus,***

$$I=\int\left[\frac{e^x-\cos x}{e^x+1-\sin x}-\frac{e^x-\sin x}{e^x+1+\cos x}\right]dx=\ln\left(\frac{e^x+1-\sin x}{e^x+1+\cos x}\right)+c$$

Solution 3 by Yen Tung Chung – Tainan – Taiwan

$$\int\frac{(2e^x+1)\sin x-\cos x-1}{e^{2x}+(\cos x-\sin x+2)e^x+\cos x-\sin x\cos x-\sin x+1}dx$$

$$= \int \frac{(2e^x + 1)\sin x - \cos x - 1}{e^{2x} + (\cos x + 1)(1 - \sin x)e^x + (\cos x + 1)(1 - \sin x)} dx$$

$$= \int \frac{(2e^x + 1)\sin x - \cos x - 1}{(e^x + \cos x + 1)(e^x - \sin x + 1)} dx = \left(\int \frac{e^x - \sin x}{e^x + \cos x + 1} - \frac{e^x - \cos x}{e^x - \sin x + 1}\right) dx$$

$$= \ln|e^x + \cos x + 1| - \ln|e^x - \sin x + 1| + C = \ln\left|\frac{e^x + \cos x + 1}{e^x - \sin x + 1}\right| + C$$

Solution 4 by Soumitra Mukherjee - Chandar Nagore – India

$$e^{2x} + (\cos x - \sin x + 2)e^x + \cos x - \sin x \cos x - \sin x + 1$$

$$= e^{2x} + (\cos x - \sin x + 2)e^x + (\cos x + 1)(1 - \sin x)$$

$$= e^{2x} + (\cos x + 1)e^x + (1 - \sin x)e^x + (\cos x + 1)(1 - \sin x)$$

$$= (e^x + \cos x + 1)(e^x + 1 - \sin x)$$

Now, $(2e^x + 1)\sin x - \cos x - 1$

$$= (e^x + \cos x + 1)(e^x + 1 - \sin x)' - (e^x + 1 - \sin x)(e^x + \cos x + 1)'$$

$$\int \frac{(2e^x + 1)\sin x - \cos x - 1}{e^{2x} + (\cos x - \sin x + 2)e^x + \cos x - \sin x \cos x - \sin x + 1} dx$$

$$= \int \frac{(2e^x + 1)\sin x - \cos x - 1}{(e^x + \cos x + 1)(e^x + 1 - \sin x)} dx$$

$$= \int \frac{d(e^x + 1 - \sin x)}{e^x + 1 - \sin x} - \int \frac{d(e^x + \cos x + 1)}{e^x + \cos x + 1} = \ln \frac{e^x + 1 - \sin x}{e^x + \cos x + 1} + C$$

Solution 5 by Igor Soposki – Skopje – Macedonia

$$I = \int \frac{(2e^x + 1)\sin x - \cos x - 1}{e^{2x} + (\cos x - \sin x + 2)e^x + \cos x - \sin x \cdot \cos x - \sin x + 1} dx =$$

$$= \int \frac{(2e^x + 1)\sin x - \cos x - 1}{(e^x - \sin x + 1)(e^x + \cos x + 1)} dx$$

$$e^{2x} + e^x \cdot \cos x - e^x \cdot \sin x + e^x + e^x + \cos x - \sin x \cdot \cos x - \sin x + 1 =$$

$$= e^x(e^x + \cos x) + (e^x + \cos x) - \sin x\,(e^x + \cos x) + e^x - \sin x + 1 =$$

$$= (e^x + \cos x)(e^x - \sin x + 1) + e^x - \sin x + 1 = (e^x - \sin x + 1)(e^x + \cos x + 1)$$

$$(2e^x + 1)\sin x - \cos x - 1 = e^x \cdot \sin x + e^x \sin x + \sin x - \cos x - 1 =$$

$$= e^x \cdot \sin x + e^x \cdot \sin x + \sin x - \cos x - \sin^2 x - \cos^2 x + e^{2x} - e^{2x} + e^x \cdot \cos x + e^x - e^x =$$

$$= (e^{2x} + e^x \cdot \cos x + e^x - e^x \cdot \cos x - \cos^2 x - \cos x) - (e^{2x} - e^x \sin x - e^x \cdot \sin x + e^x - \sin x + \sin^2 x) =$$

$$= [e^x \cdot (e^x + \cos x + 1) - \cos x\,(e^x + \cos x + 1)] - [e^x(e^x - \sin x + 1) - \sin x\,(e^x - \sin x + 1)] =$$

$$= [(e^x + \cos x + 1)(e^x - \cos x)] - [(e^x - \sin x + 1)(e^x - \sin x)]$$

$$I = \int \frac{(e^x + \cos x + 1)(e^x - \cos x)}{(e^x - \sin x + 1)(e^x + \cos x + 1)}dx \cdot \int \frac{(e^x - \sin x + 1)(e^x - \sin x)}{(e^x - \sin x + 1)(e^x + \cos x + 1)}dx =$$

$$= \ln(e^x - \sin x + 1) - \ln(e^x + \cos x + 1) = \ln\frac{e^x - \sin x + 1}{e^x + \cos x + 1} + C$$

Solution 6 by Omar Raza- Lahore – Pakistan

$$\int \frac{[(2e^x + 1)\sin x - \cos x - 1]}{e^{2x} + (\cos x - \sin x + 2)e^x + \cos x - \sin x \cos x - \sin x + 1}$$

considering the denominator,

$$e^{2x} + (\cos x - \sin x + 2)e^x + 1 - \frac{\sin x \cos x}{2} + \frac{1}{4} - \frac{1}{4} - \frac{\sin x \cos x}{2}$$

$$= e^{2x} + (\cos x - \sin x + 2)e^x + \left(\frac{\cos x - \sin x + 2}{2}\right)^2 - \frac{1}{4}\cdot(1 + 2\sin x \cos x)$$

$$= \left(e^x + \frac{(\cos x - \sin x + 2)}{2}\right)^2 - \frac{1}{4}\cdot(\sin x + \cos x)^2$$

$$= \left(e^x + \frac{\cos x - \sin x + 2}{2}\right)^2 - \left(\frac{\sin x + \cos x}{2}\right)^2$$

$$= (e^x + \cos x + 1)(e^x - \sin x + 1)$$

from $a^2 - b^2 = (a + b)(a - b)$ ***so***

$$\int \frac{[(2e^x + 1)\sin x - \cos x - 1]}{e^{2x} + (\cos x - \sin x + 2)e^x + \cos x - \sin x \cos x - \sin x + 1} =$$

$$\int \frac{[(2e^x + 1)\sin x - \cos x - 1]}{(e^x + \cos x + 1)(e^x - \sin x + 1)} = \int -\frac{e^x - \sin x}{e^x + \cos x + 1} + \frac{e^x - \cos x}{e^x - \sin x + 1} =$$

$$= -\ln(e^x + \cos x + 1) + \ln(e^x - \sin x + 1) + C$$

UP.009. Prove that if $n \in \mathbb{N}; n \geq 3$ then:

$$\left(\frac{n!}{2}\right)^{2e} \leq e^{n^2+n-6}$$

Proposed by Mihály Bencze – Romania

Solution 1 by Soumitra Mukherjee-Chandar Nagore-India

For $n = 3; \left(\frac{3!}{2}\right)^{2e} < e^6 \Leftrightarrow 3^{2e} < e^6 \Leftrightarrow 3^e < e^3$***, which is true,***

For $n = 4; \left(\frac{4!}{2}\right)^{2e} < e^{14} \Leftrightarrow 12^{2e} < e^{14} \Leftrightarrow 12^e < e^7$***, which is also true,***

Let us assume that the statement is true for $n = k; \left(\frac{k!}{2}\right)^{2e} \le e^{k^2+k-6}$ ***holds true.***

$$\text{Now, } \left\{\frac{(k+1)!}{2}\right\}^{2e} = \left(\frac{k\times k!}{2}\right)^{2e} = \left(\frac{k}{2}\right)^{2e}\left(\frac{k!}{2}\right)^{2e} \le \left(\frac{k}{2}\right)^{2e}\cdot e^{k^2+k-6}$$

we need to prove, $\left(\frac{k}{2}\right)^{2e}\cdot e^{k^2+k-6} \le e^{(k+1)^2+(k+1)-6}$

$$\Leftrightarrow \left(\frac{k}{2}\right)^{2e} \le e^{(k+1)^2-k+1} = e^2(k+1) \Leftrightarrow \left(\frac{k}{2}\right)^{e} \le e^{k+1} \qquad (1)$$

We need to prove statement (1);

$$\text{Let } f(x) = e^{x+1} - \left(\frac{x}{2}\right)^e \ \forall\, x \ge 3$$

$$f'(x) = e^{x+1} - \frac{e}{2}\left(\frac{x}{2}\right)^{e-1} \ge 0 \ \forall\, x \ge 3.$$

f increasing- on $[3,\infty)$ ***and*** $f'(x) \ge 0\ \forall x \ge 3, f(x) \ge f(3) > 0$

$$e^{x+1} > \frac{e}{2}\left(\frac{x}{2}\right)^{e-1} \quad \forall x \ge 3$$

hence, statement (1) is prove $\left\{\frac{(k+1)!}{2}\right\}^{2e} \le e^{(k+1)^2+(k+1)-6}$ ***(proved).***

When $n = k$ ***is true then*** $n = k+1$ ***is also true. So, by theory of Induction, we have,***

$$\left(\frac{n!}{2}\right)^{2e} \le e^{n^2+n-6} \text{ (proved).}$$

Solution 2 by Francis Fregeau – Quebec – Canada:

We will prove for any natural number $n \ge 3$:

$$\left(\frac{n!}{2}\right)^{2e} \le e^{n^2+n-6}$$

Let $a_n = \left(\frac{n!}{2}\right)^{2e}$ ***and*** $b_n = e^{n^2+n-6}$

Lemma 1: $\left(\frac{3!}{2}\right)^{2e} \le e^6 \Rightarrow a_3 \le b_3$

Next: $((n+1)^2 + (n+1) - 6) - (n^2 + n - 6) = 2(n+1)$

$\Rightarrow$ ***Lemma 2:*** $b_{n+1} = b_n \cdot e^{2(n+1)}; a_{n+1} = a_n \cdot (n+1)^{2e}$

Now, consider the function: $g(x) = 2(x+1) - 2e\ln(x+1); x \geq 3$

$$g(3) > 0; g'(x) = 2 - \frac{2e}{x+1} > 0 \text{ for } x \geq 3$$

$$2(x+1) > 2e\ln(x+1) \text{ for } x \geq 3.$$

$\Rightarrow$ ***Lemma 3:*** $e^{2(n+1)} > (n+1)^{2e}$ ***for*** $n \geq 3$

Combining Lemma 1, 2 and 3 yields: $a_n \leq b_n$ ***for*** $n \geq 3$ ***which completes the proof.***

Solution 3 by Omar Raza-Lahore-Pakistan

When $n = 3$

$3^{2e} \leq e^6; 3^e \leq e^3$ ***(from inequality*** $e^x \geq 1 + x$***, putting*** $x = \frac{k}{e} - 1$ ***we get***

$e^{\frac{k}{e}-1} \geq \frac{k}{e}; e^{\frac{k}{e}} \geq k \quad e^k \geq k^e$ ***hence*** $e^3 \geq 3^e$ ***(when*** $k = 3$***).***

Assuming the inequality is true for $n = k$, i.e. $\left(\frac{k!}{2}\right)^{2e} \leq e^{k^2+k-6}$ ***when*** $n = k+1$

we get $\left(\frac{(k+1)!}{2}\right)^2 \leq e^{(k+1)^2+k+1-6}$, $(k+1)^{2e} * \left(\frac{k!}{2}\right)^{2e} \leq e^{2(k+1)} * e^{k^2+k-6}$ ***since***

$\left(\frac{k!}{2}\right)^{2e} \leq e^{k^2+k-6}$ ***and*** $(k+1)^{2e} \leq e^{2(k+1)}$ ***implies*** $(k+1)^e \leq e^{k+1}$ ***which is true***

from the inequality proved at start. Hence this, $\left(\frac{(k+1)!}{2}\right)^{2e} \leq e^{(k+1)^2+k+1-6}$***, is***

true as well. Thus if inequality is true for $n = k$***, it is true for*** $n = k+1$ ***as well***

and by principal of mathematical induction is true for all n ***where*** $n \geq 3$ ***and*** a ***a natural number)***

UP.010. Find:

$$\int \frac{e^x\ln(1+e^x) - e^{2x}}{(1+e^x)^2\ln^2(1+e^x)}dx; x \in \mathbb{R}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

Put $e^x = t$

$$I = \int \frac{\ln(1+t) - t}{[(1+t)\ln(1+t)]^2} dt = I_1 - I_2$$

where:

$$I_1 = \int \frac{\ln(1+t) + 1}{[(1+t)\ln(1+t)]^2} dt$$

Put $(1+t)\ln(1+t) = u \Rightarrow (1+\ln(1+t))dt = du$

$$I_1 = \int \frac{du}{u^2} = -\frac{1}{u}$$

$$= -\frac{1}{(1+t)\ln(1+t)} = -\frac{1}{(1+e^x)\ln(1+e^x)}$$

$$I_2 = \int \frac{t+1}{(t+1)^2(\ln(t+1))^2} dt$$

Put $\ln(1+t) = v$

$$\frac{1}{1+t} dt = dv$$

$$I_2 = \int \frac{dv}{v^2} = -\frac{1}{v}$$

$$= -\frac{1}{\ln(1+t)} = -\frac{1}{\ln(1+e^x)}$$

$$I = \frac{1}{\ln(1+e^x)} - \frac{1}{(1+e^x)\ln(1+e^x)} + C$$

$$= \frac{e^x}{[\ln(1+e^x)](1+e^x)} + C$$

Solution 2 by Henry Ricardo-New York -USA

Let $u = \frac{e^x}{(1+e^x)\ln(1+e^x)}$. **Then**

$$du = \frac{e^x \ln(1+e^x) - e^{2x}}{(1+e^x)^2 \ln^2(1+e^x)} dx,$$

so that we have

$$\int \frac{e^x \ln(1+e^x) - e^{2x}}{(1+e^x)^2 \ln^2(1+e^x)} dx = \int 1\, du = u + C = \frac{e^x}{(1+e^x)\ln(1+e^x)} + C$$

[Since the denominator of the original integrand is $[(1+e^x)\ln(1+e^x)]^2$, this suggested a possible antiderivative of the form $f(x)/[(1+e^x)\ln(1+e^x)]$. A little calculation indicated that $f(x)=e^x$.]

UP.012. If $x>0$ then compute:

$$\int \frac{2e^x+\sin x+1003}{e^x+2\sinh x+\sin x-\cos x+2006}dx$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$f(x)=e^x+2\sinh x+\sin x-\cos x+2006$$

$$f'(x)+f(x)=2e^x+2(\sinh x+\cosh x)+2\sin x+2006=4e^x+2\sin x+2006=$$

$$=2(2e^x+\sin x+1003) \text{ so, } \int\frac{2e^x+\sin x+1003}{e^x+2\sinh x+\sin x}=\frac{1}{2}\int\frac{f'(x)+f(x)}{f(x)}dx=$$

$$=\frac{1}{2}\int dx+\frac{1}{2}\int\frac{f'(x)}{f(x)}dx=\frac{x}{2}+\ln(e^x+2\sinh x+\sin x-\cos x+2006)+C$$

UP.013. Let $(A,+,\cdot)$ be a ring with $1\neq 0$. If $x,y\in A$ such that $x+y=1$ and $x^{2016}=x$ prove that the elements $1-xy$ and $1-yx$ are invertible.

Proposed by Nicolae Papacu – Slobozia – Romania

Solution by Nicolae Papacu – Slobozia – Romania

We have $t=1-xy=1-x(1-x)=1-x+x^2$ *and*

$$1-yx=1-(1-x)x=1-x+x^2=t.$$

Because $x^{2016}=x$, *we have* $x^{2017}=x^2$ *and then*

$$t=1-x+x^2=1-x+x^{2017}=1-x(1-x^{2016})=1-(1-x^{2016})x.$$

Because

$$1-x^{2016}=(1-x^6)\sum_{k=0}^{372}x^{6k}$$

and $1-x^6=(1+x^3)(1-x^3)=(1-x+x^2)(1+x)(1-x^3)$, *we have*

$$1-x^{2016}=(1-x^6)\sum_{k=0}^{372}x^{6k}=(1-x+x^2)(1+x)(1-x^3)\sum_{k=0}^{372}x^{6k}=(1-x+x^2)z$$

and then

$$t=1-x+x^2=1-x+x^{2017}=1-(1-x^{2016})x=1-(1-x+x^2)zx\,,$$

so $t=1-tzx$, ***wherefrom***

$t(1+zx)=1$. ***Analog*** $(1+zx)t=1$, **so** $t=1-xy=1-yx$ ***is invertible.***

UP.014. Prove that

$$\lim_{p\to\infty}\sum_{n=1}^{p}\left(\sum_{m=1}^{p}\frac{1}{mn(m+n)}\right)<\frac{\pi^3}{6}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Cornel Ioan Valean-Romania

$$\frac{1}{mn(m+n)}>0\Rightarrow\sum_{n=1}^{p}\sum_{m=1}^{p}\frac{1}{mn(m+n)}<\left(\sum_{n=1}^{p}\frac{1}{n}\right)\sum_{m=1}^{\infty}\frac{1}{m(m+n)}=\sum_{n=1}^{p}\frac{H(n)}{n^2}$$

$$H(n)=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+n)}=\frac{H(n)}{n}$$

$$\lim_{n\to\infty}\sum_{n=1}^{p}\sum_{m=1}^{p}\frac{1}{mn(m+p)}<\sum_{n=1}^{\infty}\frac{H(n)}{n^2}=1+\sum_{n=2}^{p}\frac{H(n)}{n^2}<1+\lim_{N\to\infty}\sum_{n=2}^{N}\frac{H(n)}{n(n-1)}=$$

$$=1+\lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{H(n)}{n-1}-\frac{H(n+1)}{n}+\frac{1}{n}-\frac{1}{n+1}\right)=3-\lim_{N\to\infty}\left(\frac{1}{N+1}+\frac{H(N+1)}{N}\right)=3<\frac{\pi^3}{6}$$

The precise value of the limit is $2zeta(3)\approx 2,40411$

Solution 2 by Ravi Prakash-New Delhi-India

$$S=\lim_{p\to\infty}\sum_{n=1}^{p}\sum_{m=1}^{p}\frac{1}{mn(m+n)}$$

$$=\frac{1}{1\cdot1\cdot2}+\frac{1}{1\cdot2\cdot3}+\frac{1}{1\cdot2\cdot4}+\cdots+$$
$$+\frac{1}{2\cdot1\cdot3}+\frac{1}{2\cdot2\cdot4}+\frac{1}{2\cdot3\cdot5}+\cdots+$$
$$+\frac{1}{3\cdot1\cdot4}+\frac{1}{3\cdot2\cdot5}+\frac{1}{3\cdot3\cdot6}+\cdots$$

Let's sum up this double series diagonally:

$$S=\sum_{k=2}^{\infty}\cdot\sum_{m=1}^{k-1}\frac{1}{m(k-m)k}=\sum_{k=2}^{\infty}\frac{1}{k^2}\sum_{m=1}^{k-1}\left(\frac{1}{m}+\frac{1}{k-m}\right)=$$
$$=2\sum_{k=2}^{\infty}\frac{1}{k^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}\right)<2\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)$$
$$=2\left[\left(1-\frac{1}{2}\right)(1)+\left(\frac{1}{2}-\frac{1}{3}\right)\left(1+\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\left(1+\frac{1}{2}+\frac{1}{3}\right)+\cdots\right]$$
$$=2\left[\frac{1}{1^2}+\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{2}\right)+\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-1-\frac{1}{2}\right)+\cdots\right]$$
$$=2\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]<2\left(\frac{\pi^2}{6}\right)<\frac{\pi^3}{6}$$

UP.015. Let be $(A,+,\cdot)$. If it does exists $k\in\mathbb{N}^*$ such that for any $a,b\in A$ we have $(a+b)^{2k+1}=a^{2k}+b^{2k}$ and $(a+b)^{2k+3}=a^{2k+2}+b^{2k+2}$, then prove that the ring is commutative.

Proposed by Dana Heuberger – Romania

Solution by Dana Heuberger – Romania

We denote with (1) and (2) the equalities from the hypothesis.

For $a=b=1$, we obtain that $2=0$, so the ring that the characteristic 2.

So $\forall\,\alpha\in A,\alpha=-\alpha.$ (3)

For $a=1,b=x\in A$, from (2) we obtain:

$$(1+x)^{2k+1}\cdot(1+x)^2=1+x^{2k+2}\overset{(1),(3)}{\Leftrightarrow}\left(1+x^{2k}\right)\cdot\left(1+x^2\right)=1+x^{2k+2}\Leftrightarrow$$
$$\Leftrightarrow x^{2k}+x^2=0\overset{(3)}{\Leftrightarrow}x^{2k}=x^2,\text{ so }\forall x\in A,x^{2k+1}=x^3.$$

Replacing x with $x+1$ in the preceding equality and using (1), we deduce:

$$\forall x \in A, (1+x)^3 = (1+x)^{2k+1} = 1 + x^{2k} = 1 \overset{(3)}{\Leftrightarrow} 1 + x + x^2 + x^3 =$$

$$= 1 + x^2 \overset{(3)}{\Leftrightarrow} x^3 = x.$$

Replacing x with $x+1$ in the preceding equality and using (3), it follows:

$$\forall x \in A, (1+x)^3 = 1 + x \Leftrightarrow \forall x \in A, 1 + x + x^2 + x^3 = 1 + x \Leftrightarrow$$

$$\Leftrightarrow \forall x \in A, x^2 = x.$$

So the ring is boolean, hence is commutative.