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SOLUTIONS

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JP.016. Find all triplets (m, n, p) where m, n are two natural numbers and p is a prime number, satisfying the equation:

$$m^4 = 4(p^n - 1)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Safal Das Biswas –Kolkata- India

$m^4 = 4(p^n - 1)$. *Since* $m^4 \equiv 0(mod\ 4)$ *then* $m^4 = 2^4k^4$ *some integer* k.

Then, this leads, $4k^4 + 1 = p^n$, *so* $(2k^2 + 2k + 1)(2k^2 - 2k + 1) = p^n$.

Since p *is prime then we can set* $2k^2 + 2k + 1 = p^{n_1}$ *and* $2k^2 - 2k + 1 = p^{n_2}$ *for some integer* n_1 *and* n_2 *where* $n_1 + n_2 = n$.

Now, $p^{n_1} - p^{n_2} = 4k$. *Thus,* $k = \frac{p^{n_1} - p^{n_2}}{4}$, *and* $2k^2 + 1 = \frac{p^{n_1} + p^{n_2}}{2}$, *so we have the final set up,* $2\left(\frac{p^{n_1} - p^{n_2}}{4}\right)^2 + 1 = \left(\frac{p^{n_1} + p^{n_2}}{2}\right)$, *implies* $\left(\frac{p^{n_1} - p^{n_2}}{2}\right)^2 = p^{n_1} + p^{n_2} - 2$ *so we have, or* $(p^{n_1} - p^{n_2})^2 = 4(p^{n_1} + p^{n_2}) - 8$, *or,*

$$p^{2n_1} + p^{2n_2} - 4p^{n_1} - 4p^{n_2} + 8 = 2p^{n_1}p^{n_2}, \text{ or,}$$

$$(p^{n_1} - 2)^2 + (p^{n_2} - 2)^2 + 2(p^{n_1} - 2)(p^{n_2} - 2) = 4(p^{n_1}p^{n_2} - p^{n_1} - p^{n_2} + 2).$$

So we have $(p^{n_1} + p^{n_2} - 4)^2 = 4(p^{n_1}p^{n_2} - p^{n_1} - p^{n_2} + 2) =$

$= 4\big((p^{n_1} - 1)(p^{n_2} - 1) + 1\big)$. *So we have:* $\left(\frac{p^{n_1} + p^{n_2} - 4}{2}\right)^2 = (p^{n_1} - 1)(p^{n_2} + 1) + 1$.

As, p *is prime, then:* $(p^{n_1} - 1)(p^{n_2} - 1) \equiv 0(mod\ 4)$

Thus, $(p^{n_1} - 1)(p^{n_2} - 1) + 1 = 4u + 1$. *So,* $\frac{p^{n_1} + p^{n_2} - 4}{2} \equiv 1(mod\ 2)$, *which clearly suffices* $(2a + 1)^2 = (p^{n_1} - 1)(p^{n_2} - 1) + 1$, *which gives,* $(p^{n_1} - 1)(p^{n_2} - 1) = 4a(a + 1)$.

Thus we get:

$\left(\frac{p^{n_1} - 1}{2}\right)\left(\frac{p^{n_2} - 1}{2}\right) = a(a + 1)$, *set,* $\left(\frac{p^{n_1} - 1}{2}\right) = x$ *and* $\left(\frac{p^{n_2} - 1}{2}\right) = y$ *then we have that,* $(x + y - 1)^2 = 4xy + 1$, *and* $xy = a(a + 1)$ *as* $x > y$ *set,* $x = m(mh + 1)$ *and* $y = h$ *and* $a = mh$. *Thus,* $x + y - 1 = 2a + 1$ *so,*

$x + h = x + y = 2(a + 1) = 2(mh + 1)$, *so,* $x = 2(mh + 1) - h$ *again,* $x = m(mh + 1)$, *so by compairing,* $m(mh + 1) = 2(mh + 1) - h$, *so,* $h = (2 - m)(mh + 1)$.

As $h \geq 0$ hence $2 - m \geq 0$ so, $m \in \{1, 2\}$ if $m = 1$ then $h = h + 1$ which is contradiction, hence $m = 2$ and $h = 0$, which gives $n_2 = 0, n_1 = 1, p = 5$, so $n = 1$ and $m = 2$ is the only solution.

Note 1: Here all the variables that are used $\in \mathbb{Z}$ and also observed that since p is prime we have, $(p^{n_j} - 1) \equiv 0 (mod\ 2) \forall\ j \in \{1, 2\}$

Note 2: $(x + y - 1)^2 = 4xy + 1$ is true as: $\left(\frac{p^{n_1} + p^{n_2} - 4}{2}\right)^2 = (p^{n_1} - 1)(p^{n_2} - 1) + 1$ and we have substituted $\left(\frac{p^{n_1} - 1}{2}\right) = x$ and $\left(\frac{p^{n_2} - 1}{2}\right) = y$.

JP.017. Prove the following inequality holds for all positive real numbers a, b, c

$$a^2 + b^2 + c^2 \geq \frac{1}{2}(ab + bc + ca) + \sqrt{\frac{2(a + b + c)(a^3b^3 + b^3c^3 + c^3a^3)}{(a + b)(b + c)(c + a)}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Para todos los reales no negativos: a, b, c, x, y, z se cumple la siguiente desigualdad:

$$(b + c)x + (a + c)y + (a + b)z \geq 2\sqrt{(ab + bc + ac)(xy + yz + zx)}$$

Aplicando: Cauchy – Schwarz:

$$P = (b + c)x + (a + c)y + (a + b)z = (a + b + c)(x + y + z) - (ax + by + cz)$$

$$P = \sqrt{\big((a^2 + b^2 + c^2) + 2(ab + bc + ac)\big)\big((x^2 + y^2 + z^2) + 2(xy + yz + zx)\big)} -$$

$$-(ax + by + cz)$$

$$\geq \sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)} + 2\sqrt{(ab + bc + ac)(xy + yz + zx)} -$$

$$-(ax + by + cz) \geq 2\sqrt{(ab + bc + ac)(xy + yz + zx)}$$

La igualdad se alcanza cuando: $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$. Por lo tanto:

Sea: $a = a^3, b = b^3, c = c^3, x = \frac{1}{b+c}, y = \frac{1}{a+c}, z = \frac{1}{a+b}$

$$\Rightarrow (b^3+c^3)\left(\frac{1}{b+c}\right)+(a^3+c^3)\left(\frac{1}{a+c}\right)+(a^3+b^3)\left(\frac{1}{a+b}\right)\geq$$

$$\geq 2\sqrt{(a^3b^3+b^3c^3+a^3c^3)\left(\frac{2(a+b+c)}{(a+b)(b+c)(a+c)}\right)}$$

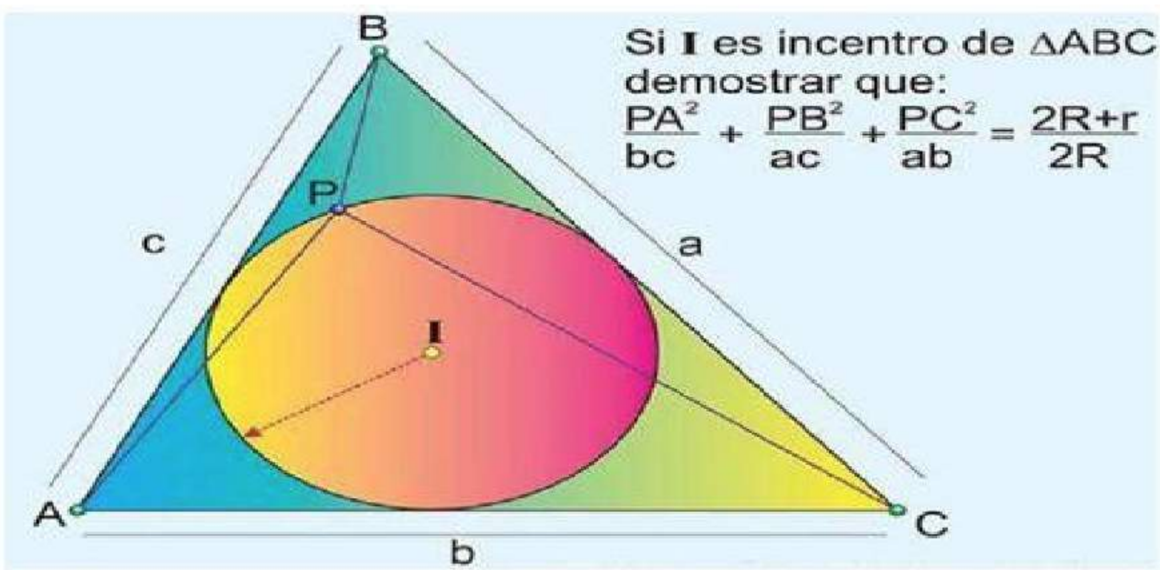
$$\Rightarrow 2a^2+2b^2+2c^2\geq ab+bc+ac+2\sqrt{(a^3b^3+b^3c^3+a^3c^3)\left(\frac{2(a+b+c)}{(a+b)(b+c)(a+c)}\right)}$$

$$\Rightarrow a^2+b^2+c^2\geq \frac{ab+bc+ac}{2}+\sqrt{(a^3b^3+b^3c^3+a^3c^3)\left(\frac{2(a+b+c)}{(a+b)(b+c)(a+c)}\right)}$$

JP.018. Let ABC be a triangle with the known normal notations. Prove that for any point P moving on the incircle,

$$5r\leq \frac{PA^2}{h_a}+\frac{PB^2}{h_b}+\frac{PC^2}{h_c}\leq \frac{5}{2}R$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam



Picture by Kevin Soto Palacios – Huarmey – Peru

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $5r \le \frac{PA^2}{h_a} + \frac{PB^2}{h_b} + \frac{PC^2}{h_c} \le \frac{5R}{2}$. ***De la siguiente identidad:***

$$aMA^2 + bMB^2 + cMC^2 = (a+b+c)MI^2 + abc. \text{ Sea: } M = P$$

$$aPA^2 + bPB^2 + cPC^2 = (2p)PI^2 + abc \quad (A)$$

Del gráfico: $PI = r$ ***(Caso particular). Dividiendo:*** $(\div\, abc)$

$$\frac{PA^2}{bc} + \frac{PB^2}{ac} + \frac{PC^2}{ab} = \frac{2pr^2}{abc} + 1, \quad abc = 4pRr$$

$$\frac{PA^2}{bc} + \frac{PB^2}{ac} + \frac{PC^2}{ab} = \frac{r}{2R} + 1 \to$$

$$\to \frac{2R}{bc}PA^2 + \frac{2R}{ac}PB^2 + \frac{2R}{ab}PC^2 = 2R + r \to \frac{PA^2}{h_a} + \frac{PB^2}{h_b} + \frac{PC^2}{h_c} = 2R + r$$

$$5r \le 2R + r \le \frac{5R}{2} \Rightarrow 2R + r \ge 5r \to R \ge 2r \quad \text{(Desigualdad de Euler)}$$

$$\Rightarrow 2R + r \le \frac{5R}{2} \Rightarrow r \le \frac{R}{2} \to R \ge 2r \text{ (Desigualdad de Euler)}$$

JP.019. If $a, b, c > 0$ and $x, y, z \ge 1$ then:

$$x^{\frac{4a^3}{a^2+bc}} y^{\frac{4b^3}{b^2+ca}} z^{\frac{4c^3}{c^2+ab}} \ge \left(\frac{x^4}{yz}\right)^a \left(\frac{y^4}{zx}\right)^b \left(\frac{z^4}{xy}\right)^c$$

Proposed by Mihály Bencze – Romania

Solution by proposer

We have: $\frac{a^3}{a^2+bc} \ge \frac{4a-b-c}{4} \Leftrightarrow b(a-c)^2 + c(a-b)^2 \ge 0$ ***therefore***

$$\begin{cases} \frac{a^3 \ln x}{a^2+bc} \ge \frac{(4a-b-c)\ln x}{4} \\ \frac{b^3 \ln y}{b^2+ca} \ge \frac{(4b-c-a)\ln y}{4} \\ \frac{c^3 \ln z}{c^2+ab} \ge \frac{(4c-a-b)\ln z}{4} \end{cases}$$ ***. After addition we have:***

$$\sum \ln x^{\frac{a^3}{a^2+bc}} = \sum \frac{a^3 \ln x}{a^2+bc} \ge \sum \frac{(4a-b-c)\ln x}{4} = \sum \frac{a(4\ln x - \ln y - \ln z)}{4} =$$

$$= \sum \ln\left(\frac{x^4}{yz}\right)^{\frac{a}{4}} \textit{ therefore } \prod x^{\frac{4a^3}{a^2+bc}} \ge \prod \left(\frac{x^4}{yz}\right)^a$$

JP.020. Given $x_1, x_2, \dots, x_n$ be positive real numbers such that:

$$\sum_{k=1}^{n} x_k = n.$$

If $\alpha, \beta > 0, 4\alpha(n-1)(2\alpha n\sqrt{n} + \beta) > \beta^2\sqrt{n}$ then:

$$\alpha \sum_{k=1}^{n} \frac{1}{a_k} + \frac{\beta}{\sqrt{\sum_{k=1}^{n} a_k^2}} \ge n\alpha + \frac{\beta}{\sqrt{n}}$$

Proposed by Ngo Minh Ngoc Bao – Vietnam

Solution by proposer

Let $\sum_{k=1}^{n} a_k^2 = n + n(n-1)t^2$, with t be real number, $0 \le t < 1$. We have:

$$\alpha \sum_{k=1}^{n} \frac{1}{a_k} = \frac{\alpha}{1+(n-1)t} \sum_{k=1}^{n} \frac{1+(n-1)t - a_k}{a_k} + \frac{n\alpha}{1+(n-1)t}$$

Using Cauchy-Schwarz inequality:

$$\sum_{k=1}^{n} \frac{1+(n-1)t - a_k}{a_k} \ge \frac{\left(\sum_{k=1}^{n}[1+(n-1)t - a_k]\right)^2}{\sum_{k=1}^{n} a_k + (n-1)t\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k^2} = \frac{n(n-1)t}{1-t}$$

$$\Rightarrow \alpha \sum_{k=1}^{n} \frac{1}{a_k} \ge \frac{n\alpha(n-1)t}{[1+(n-1)t]} + \frac{n\alpha}{1+(n-1)t} = \frac{n\alpha(nt-2t+1)}{[1+(n-1)t](1-t)}$$

$$\Rightarrow \alpha \sum_{k=1}^{n} \frac{1}{a_k} + \frac{\beta}{\sqrt{\sum_{k=1}^{n} a_k^2}} \ge \frac{n\alpha(nt-2t+1)}{[1+(n-1)t](1-t)} + \frac{\beta}{\sqrt{n+n(n-1)t^2}}$$

We need to prove that: $\frac{n\alpha(nt-2t+1)}{[1+(n-1)t](1-t)} + \frac{\beta}{\sqrt{n+n(n-1)t^2}} \ge n\alpha + \frac{\beta}{\sqrt{n}}$, *(*)*

$$(*) \Leftrightarrow \frac{n\alpha(n-1)t^2}{[1+(n-1)t](1-t)} - \frac{\beta(n-1)t^2}{\sqrt{n+n(n-1)t^2}\left(1+\sqrt{1+(n-1)t^2}\right)} \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (n-1)(\alpha n\sqrt{n} + \beta)t^2 - \beta(n-2)t + n\alpha\sqrt{n+n(n-1)t^2} + \alpha n\sqrt{n} - \beta \ge 0$$

Considering the function:

$$f(t)=(n-1)(\alpha n\sqrt{n}+\beta)t^2-\beta(n-2)t+n\alpha\sqrt{n+n(n-1)t^2}+\alpha n\sqrt{n}-\beta$$

We have: $f(t)\geq g(t)=(n-1)(\alpha n\sqrt{n}+\beta)t^2-\beta(n-2)t+2\alpha n\sqrt{n}-\beta$

Considering function: $g(t)=(n-1)(\alpha n\sqrt{n}+\beta)t^2-\beta(n-2)t+2\alpha n\sqrt{n}-\beta,\forall t\in[0,1]$

Three quadractic formula $g(t)$ *reaches the minimum value when* $t=\frac{\beta(n-2)}{2(n-1)(n\sqrt{n}\alpha+\beta)}\Rightarrow$

$$\Rightarrow Ming(t)=2\alpha n\sqrt{n}-\beta-\frac{\beta^2(n-2)^2}{4(n-1)(n\sqrt{n}\alpha+\beta)}=$$

$$\frac{4\alpha(n-1)(2\alpha n\sqrt{n}+\beta)-\beta^2\sqrt{n}}{4(n-1)(n\sqrt{n}a+\beta)}>0$$

$\Rightarrow f(t)\geq g(t)\geq Ming(t)>0$. *Equality occurs when* $a_1=a_2=\cdots=a_n=1$.

JP.021.Prove that if $x,y,z>0, xyz=8$ then:

$$x^3+y^3+z^3\geq 2x\sqrt{y+z}+2y\sqrt{z+x}+2z\sqrt{x+y}$$

Proposed by Iuliana Trașcă – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: $x,y,z>0$*, además:* $xyz=8$*. Probar que:*

$x^3+y^3+z^3\geq 2x\sqrt{y+z}+2y\sqrt{x+z}+2z\sqrt{x+y}$. *Tener en cuenta lo siguiente:*

$$x^3+y^3\geq xy(x+y)\to\frac{x^3+y^3}{8}\geq\frac{xy(x+y)}{xyz}=\frac{x+y}{z}$$

$\Rightarrow\frac{x^3+y^3}{8}+\frac{z^3}{4}\geq\frac{x+y}{z}+\frac{z^3}{4}\to$ *Por:* $MA\geq MG$: $\frac{x+y}{z}+\frac{z^3}{4}\geq z\sqrt{(x+y)}$. *Por la trasitividad:*

$$\Rightarrow\frac{x^3+y^3}{8}+\frac{z^3}{4}\geq z\sqrt{(x+y)}\quad(A)$$

$$\Rightarrow\frac{y^3+z^3}{8}+\frac{x^3}{4}\geq x\sqrt{(y+z)}\quad(B)$$

$$\Rightarrow\frac{z^3+x^3}{8}+\frac{y^3}{4}\geq y\sqrt{(x+z)}\quad(C)$$

$\Rightarrow$ *Sumando:* $(A)+(B)+(C)\to x^3+y^3+z^3\geq 2x\sqrt{y+z}+2y\sqrt{x+z}+2z\sqrt{x+y}$

Solution 2 by Soumitra Moukherjee - Chandar Nagore – India

$$\sum_{cyc} 2x\sqrt{y+z} = \sum_{cyc} 2\sqrt{x}\sqrt{xy+2z}$$

$$\le 2\sqrt{(x+y+z)\cdot 2(xy+yz+zx)}$$ ***[applying Cauchy – Schwarz]***

$$= \sqrt{8(x+y+z)(xy+yz+zx)}$$

$$= \sqrt{xyz(x+y+z)(xy+yz+zx)} \le \sqrt{\frac{1}{3}(xy+yz+zx)^3}$$

$[(xy+yz+zx)^2 \ge 3xyz(x+y+z)]$. ***We need to prove,***

$$3(x^3+y^3+z^3)^2 \ge (xy+zy+zx)^3 \quad (1)$$

Applying Holder's Inequality,

$$(1+1+1)(x^3+y^3+z^3)(x^3+y^3+z^3) \ge \left(\sqrt[3]{x^3\cdot x^3\cdot 1} + \sqrt[3]{y^3\cdot y^3\cdot 1} + \sqrt[3]{z^3\cdot z^3\cdot 1}\right)^3 =$$

$$= (x^2+y^2+z^2)^3 \ge (xy+yz+zx)^3$$

$$3(x^3+y^3+z^3) \ge (xy+yz+zx)^3$$

(Statement (1) is proved)

$$x^3+y^3+z^3 \ge 2x\sqrt{y+x} + 2y\sqrt{z+x} + 2z\sqrt{x+y}$$

Solution 3 by Ngô Minh Ngọc Bảo-Gia Lang Province-VietNam

Using $AM-GM$ ***inequality, we have:***

$$2x\sqrt{y+z} + 2y\sqrt{z+x} + 2z\sqrt{x+y} \le x^2+y^2+z^2+2(x+y+z)$$

We need to prove that: $x^3+y^3+z^3 \ge x^2+y^2+z^2+2(x+y+z)(*)$

$$(*) \Leftrightarrow (x^3-x^2-2x)+(y^3-y^2-2y)+(z^2-z^2-2z) \ge 0$$

We have: $x^3-x^2-2x \ge 6x-12 \Leftrightarrow (x-2)^2(x+3) \ge 0$ ***(true)***

Similarly, $y^3-y^2-2y \ge 6y-12 \Leftrightarrow (y-2)^2(y+3) \ge 0$ ***(true)***

and $z^3-z^2-2z \ge 6z-12 \Leftrightarrow (z-2)^2(z+3) \ge 0$ ***(true)***

$$\Rightarrow (x^3-x^2-2x)+(y^3-y^2-2y)+(z^3-z^2-2z) \ge 6(x+y+z)-36 \ge$$

$$\ge 6\cdot 3\sqrt[3]{xyz} - 36 = 0$$

Solution 4 by Imad Zak-Saida-Lebanon

$$AM - GM \Rightarrow \begin{cases} \sqrt{ab} \le \frac{1}{2}(a+b) & (1) \\ a^3 + b^3 \ge ab(a+b) & (2) \end{cases}$$

$$2x\sqrt{y+z} = \frac{2}{2\sqrt{2}}\sqrt{x^2 \cdot 8 \cdot (y+z)} = \frac{1}{\sqrt{2}}\sqrt{x^2(xyz)(y+z)}$$

$$= \frac{1}{\sqrt{2}}\sqrt{x^3yz(y+z)} \le \frac{1}{\sqrt{2}}\sqrt{x^3(y^3+z^3)} \quad \textbf{by (2)}$$

$$\left.\begin{matrix} \le \frac{1}{2}\sqrt{(2x^3)(y^3+z^3)} \le \frac{1}{4}[2x^3+y^3+z^3] \text{ by } (1) \\ Similarly: 2y\sqrt{z+x} \le \frac{1}{4}[2y^3+x^3+z^3] \\ 2z\sqrt{x+y} \le \frac{1}{4}[2z^3+x^3+y^3] \end{matrix}\right\} \Rightarrow \sum 2x\sqrt{y+z} \le \sum x^3$$

We are done! Equality holds for $x = y = z = 2$

JP.022. Let ABC be an acute triangle with the orthocenter H, inradius r, and circumradius R. Prove that:

$$\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ca}} + \frac{HC}{\sqrt{ab}} \le \sqrt{2\left(1 + \frac{r}{R}\right)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Vietnam

Siendo H un Ortocentro: Probar en un triángulo acutángulo ABC:

$\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ac}} + \frac{HC}{\sqrt{ab}} \le \sqrt{2\left(1+\frac{r}{R}\right)}$***. Por la desigualdad de Cauchy:***

$$(HA + HB + HC)\left(\frac{HA}{bc} + \frac{HB}{ac} + \frac{HC}{ab}\right) \ge \left(\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ac}} + \frac{HC}{\sqrt{ab}}\right)^2$$

Para ello desarrollaremos: $\frac{HA}{h_a} + \frac{HB}{h_b} + \frac{HC}{h_c} = \frac{2R\cos A}{\frac{bc}{2R}} + \frac{2R\cos B}{\frac{ac}{2R}} + \frac{2R\cos C}{\frac{ab}{2R}} =$

$$= \frac{2R\cos A}{2R\sin B\sin C} + \frac{2R\cos B}{2R\sin A\sin C} + \frac{2R\cos C}{2R\sin A\sin B}$$

$$\Rightarrow -\frac{\cos(B+C)}{\sin B\sin C}-\frac{\cos(A+C)}{\sin A\sin C}-\frac{\cos(A+B)}{\sin A\sin B}=$$

$$=-(\cot B\cot C-1)-(\cot A\cot C-1)-(\cot A\cot B-1)$$

$$\Rightarrow \frac{HA}{h_a}+\frac{HB}{h_b}+\frac{HC}{h_c}=3-(\cot A\cot B+\cot B\cot C+\cot A\cot C)=2$$

$$\Rightarrow 2R(\cos A+\cos B+\cos C)\left(\frac{HA}{bc}+\frac{HB}{ac}+\frac{HC}{ab}\right)=\left(1+\frac{r}{R}\right)\left(\frac{HA}{h_a}+\frac{HB}{h_b}+\frac{HC}{h_c}\right)=$$

$$=2\left(1+\frac{r}{R}\right)\ge\left(\frac{HA}{\sqrt{bc}}+\frac{HB}{\sqrt{ac}}+\frac{HC}{\sqrt{ab}}\right)^2\Rightarrow\frac{HA}{\sqrt{bc}}+\frac{HB}{\sqrt{ac}}+\frac{HC}{\sqrt{ab}}\le\sqrt{2\left(1+\frac{r}{R}\right)}$$

Solution 2 by Adil Abdullayev-Baku-Azerbaidian

(VON AUBEL) $\sum_{cyc}\frac{AH}{h_a}=2$. ***Lemma:*** $\sum_{cyc}AH=2(R+r)$

$$\left(\sum_{cyc}\frac{HA}{\sqrt{bc}}\right)^2\overset{C-B-S}{\le}\sum_{cyc}AH\cdot\sum_{cyc}\frac{AH}{bc}=2(R+r)\cdot\sum_{cyc}\frac{AH}{bc}\quad (A)$$

$$\sum_{cyc}\frac{AH}{bc}=\frac{1}{2R}\cdot\sum_{cyc}\frac{AH}{\frac{bc}{2R}}=\frac{1}{2R}\cdot\sum_{cyc}\frac{AH}{h_a}\overset{VON\ AUBEL}{=}\frac{1}{R}\quad (B)$$

$$(A)(B)\Rightarrow LHS\le RHS$$

JP.023. Prove that for all positive real numbers a,b,c,d

$$\frac{a}{bc}+\frac{b}{cd}+\frac{c}{da}+\frac{d}{ab}\ge\frac{8}{\sqrt{a^2+b^2+c^2+d^2}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a,b,c,d ***reales positivos. Probar que:*** $\frac{a}{bc}+\frac{b}{cd}+\frac{c}{da}+\frac{d}{ab}\ge\frac{8}{\sqrt{a^2+b^2+c^2+d^2}}$

$\Rightarrow\frac{a^2d+b^2a+c^2b+d^2c}{abcd}\sqrt{a^2+b^2+c^2+d^2}\ge 8\Rightarrow$ ***Siendo:*** $a,b,c,d>0\rightarrow$ ***Por:*** $MA\ge MG$

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$$a^2d+b^2a+c^2b+d^2c\geq 4\sqrt[4]{(abcd)^3}\Leftrightarrow \frac{a^2d+b^2a+c^2b+d^2c}{abcd}\geq 4\sqrt[4]{\frac{1}{abcd}}\quad (A)$$

Además: $\sqrt{a^2+b^2+c^2+d^2}\geq\sqrt{4\sqrt[4]{(abcd)^2}}=2\sqrt[4]{abcd}$ ***(B)***

Desde que: $\frac{a^2d+b^2a+c^2b+d^2c}{abcd}\geq 4\sqrt[4]{\frac{1}{abcd}}\wedge\sqrt{a^2+b^2+c^2+d^2}\geq 2\sqrt[4]{abcd}$

Multiplicando las expresiones se tiene que: $\frac{a^2d+b^2a+c^2b+d^2c}{abcd}\sqrt{a^2+b^2+c^2+d^2}\geq 8$

JP.024. Given a triangle ABC and let P be any point in its plane. Prove that:

$$\frac{PB\cdot PC}{bc}+\frac{PC\cdot PA}{ca}+\frac{PA\cdot PB}{ab}\leq\frac{1}{4}\left(\frac{PA}{h_a}+\frac{PB}{h_b}+\frac{PC}{h_c}\right)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios-Huarmey-Peru

KLAMKIN's INEQUALITY: $\rightarrow x,y,z\in\mathbb{R},n\in\mathbb{Z}$

$$x^2+y^2+z^2\geq(-1)^{n+1}(2yxz\cos(nA)+2zx\cos(nB)+2xy\cos(nC))\Leftrightarrow$$

$\Leftrightarrow$ ***(Demostrado anteriormente). Si:*** $n=2$

$$x^2+y^2+z^2\geq -2yz\cos 2A-2zx\cos 2B-2xy\cos 2C$$

$$x^2+y^2+z^2\geq -2yz(1-2\operatorname{sen}^2A)-2zx(1-2\operatorname{sen}^2B)-2xy(1-2\operatorname{sen}^2C)$$

$$x^2+y^2+z^2+2xy+2yz+2zx\geq 4yz\operatorname{sen}^2A+4zx\operatorname{sen}^2B+4xy\operatorname{sen}^2C$$

$$(x+y+z)^2\geq 4yz\operatorname{sen}^2A+4zx\operatorname{sen}^2B+4xy\operatorname{sen}^2C$$

Sea: $x=\frac{PA}{h_a}>0,y=\frac{PB}{h_b}>0,z=\frac{PC}{h_c}>0$. ***La desigualdad es equivalente:***

$$\frac{1}{4}\left(\frac{PA}{h_a}+\frac{PB}{h_b}+\frac{PC}{h_c}\right)^2\geq\frac{PB\cdot PC}{h_bh_c}\operatorname{sen}^2A+\frac{PA\cdot PC}{h_ah_c}\operatorname{sen}^2B+\frac{PA\cdot PB}{h_ah_b}\operatorname{sen}^2C$$

$$\frac{1}{4}\left(\frac{PA}{h_a}+\frac{PB}{h_b}+\frac{PC}{h_c}\right)^2\geq\frac{PB\cdot PC}{\frac{ac}{2R}\cdot\frac{ab}{2R}}\cdot\frac{a^2}{4R^2}+\frac{PA\cdot PC}{\frac{bc}{2R}\cdot\frac{ab}{2R}}\cdot\frac{b^2}{4R^2}+\frac{PA\cdot PB}{\frac{bc}{2R}\cdot\frac{ac}{2R}}\cdot\frac{c^2}{4R^2}$$

$$\frac{1}{4}\left(\frac{PA}{h_a}+\frac{PB}{h_b}+\frac{PC}{h_c}\right)^2\geq\frac{PB\cdot PC}{bc}+\frac{PA\cdot PC}{ac}+\frac{PA\cdot PB}{ab}$$

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JP.025. Let $n \geq 2$ be an integer and let a, b, c be positive numbers such that $ab + bc + ca \leq 1$. Prove that:

$$\frac{bc}{(2a^2 + bc)^n} + \frac{ca}{(2b^2 + ca)^n} + \frac{ab}{(2c^2 + ab)^n} \geq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sea: $n \geq 2$ ***un entero y sean "***a, b, c***"***$\mathbb{R}^+$***, tal que:*** $ab + bc + ac \leq 1$. ***Probar que:***

$\frac{bc}{(2a^2+bc)^n} + \frac{ac}{(2b^2+ca)^n} + \frac{ab}{(2c^2+ab)^n} \geq \frac{1}{3}$. ***Realizando la desigualdad ponderada "Jensen" para:***

$$f(x) = \frac{1}{x^n}, x > 0, n \geq 2 \Leftrightarrow \textit{(Convexo)}$$

$$\frac{bc}{(2a^2 + bc)^n} + \frac{ac}{(2b^2 + ca)^n} + \frac{ab}{(2c^2 + ab)^n} = bcf(bc + 2a^2) + acf(ac + 2b^2) + abf(ab + 2c^2)$$

$$bcf(bc + 2a^2) + acf(ac + 2b^2) + abf(ab + 2c^2) \geq$$

$$\geq (bc + ac + ab)f\left(\frac{bc(bc + 2a^2) + ac(ac + 2b^2) + ab(ab + 2c^2)}{ab + bc + ac}\right)$$

$$(bc + ac + ab)f\left(\frac{(bc)^2 + (ac)^2 + (ab)^2 + 2abc(a + b + c)}{ab + bc + ac}\right) = (bc + ac + ab)f\left(\frac{(ab + bc + ac)^2}{ab + bc + ac}\right)$$

$$(bc + ac + ab)f\left(\frac{(ab + bc + ac)^2}{ab + bc + ac}\right) = (bc + ac + ab)f(ab + bc + ac) = \frac{1}{(ab + bc + ac)^{n-1}} \geq 1$$

JP.026. Let a, b, c be non-negative real numbers and let x, y, z be real numbers different from 0, such that $by + cz = x,\ \ cz + ax = y, ax + by = z$. Prove that.

a. $abc \leq \frac{1}{8}$

b. $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+c+a} \leq 1$

c. $a + b + c \geq 2(ab + bc + ca)$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

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Sean: a, b, c ***números reales no negativos y sean:*** x, y, z ***números reales diferentes de cero. Además:*** $by + cz = x, cz + ax = y,\ ax + by = z$***. Probar que:***

a) $abc \leq \frac{1}{8}$

b) $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+a+c} \leq 1$

c) $a + b + c \geq 2(ab + bc + ac)$

Sumando las ecuaciones se tiene:

$$2(ax + by + cz) = x + y + z \rightarrow (ax + by) + cz = \frac{x + y + z}{2} \rightarrow$$

$$\rightarrow cz = \frac{x + y - z}{2} \rightarrow c = \frac{x + y - z}{2z}$$

De forma análoga se tiene que: $a = \frac{y+z-x}{2x};\ b = \frac{z+x-y}{2y}$***. Por la tanto:***

a) $abc \leq \frac{1}{8} \Leftrightarrow (x + y - z)(z + x - y)(y + z - x) \leq xyz$

$$(x + y - z)(z + x - y)(y + z - x) = (x^2 - (y - z)^2)(y + z - x) =$$

$$= (x^2 - y^2 - z^2 + 2yz)(y + z - x)$$

$$\rightarrow (x^2 - y^2 - z^2 + 2yz)(y + z - x) = x^2y + x^2z - x^3 - y^3 - y^2z + y^2x -$$

$$-z^2y - z^3 + z^2x + 2y^2z + 2yz^2 - 2xyz$$

$$\Rightarrow (x + y - z)(z + x - y)(y + z - x) = -x^3 - y^3 - z^3 + xy(x + y) +$$

$$+yz(y + z) + zx(z + x) - 2xyz \leq xyz$$

$$\Rightarrow x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x)$$

$$\Rightarrow x(x - y)(x - z) + y(y - x)(y - z) + z(z - x)(z - y) \geq 0 \rightarrow$$

$\rightarrow$ ***(Válido por desigualdad de Schur)***

b) $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+a+c} \leq 1$

$$\rightarrow (2 + a + c)(2 + b + c) + (2 + a + b)(2 + a + c) + (2 + a + b)(2 + b + c) \leq$$

$$\leq (2 + a + b)(2 + b + c)(2 + a + c)$$

$$\rightarrow 4(3) + \sum(a + c)(b + c) + 2\sum(2a + b + c) \leq 8 + 4(2)(a + b + c) +$$

$$+2\sum(a + c)(b + c) + \prod(a + b)$$

$$\rightarrow 4+8(a+b+c)\le 8(a+b+c)+\sum(a+c)(b+c)+\sum ab(a+b)+2abc$$

$$\Rightarrow 4\le \sum a^2+3\sum ab+\sum ab(a+b)+2abc$$

Tener en cuenta lo siguiente: $a=\frac{y+z-x}{2x}\ge 0;\ b=\frac{z+x-y}{2y}\ge 0;\ c=\frac{x+y-z}{2z}\ge 0$

$$\Rightarrow \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2\rightarrow \sum(1+b)(1+c)=2\prod(1+a)\rightarrow$$

$$\rightarrow 3+2\sum a+\sum ab=2+2\sum a+2\sum ab+2abc$$

$$\Rightarrow 1=ab+bc+ac+2abc\rightarrow ab+bc+ac=1-2abc\ \wedge abc\le\frac{1}{8}\Leftrightarrow$$

$$\Leftrightarrow ab+bc+ac\ge\frac{3}{4}\Rightarrow a+b+c\ge\sqrt{3(ab+bc+ac)}\ge\sqrt{\frac{9}{4}}=\frac{3}{2}\rightarrow a+b+c\ge\frac{3}{2}$$

$$\sum a^2+3\sum ab+\sum ab(a+b)+2abc\ge 4\sum ab+\sum ab\left(\frac{3}{2}-c\right)+2abc=$$

$$=\frac{11}{2}\sum ab-abc\ge\frac{33}{8}-\frac{1}{8}\ge 4$$

c) $a+b+c\ge 2(ab+bc+ca)$

Desde que: $a=\frac{y+z-x}{2x}\ge 0;\ b=\frac{z+x-y}{2y}\ge 0;\ c=\frac{x+y-z}{2z}\ge 0$

$$\Leftrightarrow y=n+m;z=n+p;x=m+p;\ a=\frac{n}{m+p}\ge 0;\ b=\frac{p}{n+m}\ge 0;$$

$$c=\frac{m}{n+p}\ge 0;$$

$$\Rightarrow \frac{n}{m+p}+\frac{p}{m+n}+\frac{m}{n+p}\ge 2\left(\frac{n}{m+p}\cdot\frac{p}{m+n}+\frac{n}{m+p}\cdot\frac{m}{n+p}+\frac{p}{m+n}\cdot\frac{m}{n+p}\right)$$

$$\Rightarrow \frac{n(n+m)(n+p)+p(p+m)(p+n)+m(m+p)(m+n)}{(m+n)(n+p)(m+p)}\ge$$

$$\ge\frac{2np(n+p)+mn(m+n)+mp(m+p)}{(m+n)(n+p)(m+p)}\Rightarrow \sum n^3-\sum mn(m+n)+3mnp\ge 0$$

$$\Rightarrow n(n-m)(n-p)+m(m-n)(m-p)+p(p-m)(p-n)\ge 0\Leftrightarrow$$

$\Leftrightarrow$ ***(Desigualdad de Schur)***

JP.027. Find all real numbers x satisfying the following equation:

$$(x+\{x\})^2-(x+\{x\})=6\lfloor x\rfloor\{x\}-1$$

where $\lfloor x\rfloor$ and $\{x\}$ denote the integer part and fractional part of x, respectively.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Soumava Chakraborty – Kolkata – India

Let $\lfloor x\rfloor = I$ **and** $\{x\}=f\in[0,1)$

$$x=I+f$$

$$(I+f+f)^2-(I+f+f)=6If-1\Rightarrow(I+2f)^2-(I+2f)=6If-1$$

$$\Rightarrow I^2+4f^2-2If-I-2f+1=0$$

$$\Rightarrow I^2-I(2f+1)+4f^2-2f+1=0 \quad (1)$$

$$\Delta=(2f+1)^2-4(4f^2-2f+1)=-3(4f^2-4f+1)=-3(2f-1)^2\le 0$$

But, $I\in\mathbb{Z}\in\mathbb{R},\Delta\ge 0$**. So,** $\Delta\le 0$ **and** $\Delta\ge 0\Rightarrow\Delta=0\Rightarrow 2f-1=0\Rightarrow f=\frac{1}{2}$

$\Delta=0$**, from (1), we get,** $I=\frac{2f+1}{2}=\frac{2\cdot\frac{1}{2}+1}{2}=1;\ x=I+f=1+\frac{1}{2}=\frac{3}{2}$

JP.028. Prove that in any triangle ABC:

$$\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\ge\sqrt{1+\frac{4R}{r}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\ge\sqrt{1+\frac{4R}{r}}$***. Tener en cuenta lo siguiente:***

$$\frac{4R}{r}=\csc\frac{A}{2}\csc\frac{B}{2}\csc\frac{C}{2}=\frac{abc}{(p-a)(p-b)(p-c)}=\frac{8abc}{(b+c-a)(a+c-b)(b+a-c)}$$

$$\frac{r_b}{r_a}=\frac{\frac{S}{p-b}}{\frac{S}{p-a}}=\frac{p-a}{p-b}=\frac{b+c-a}{a+c-b}>0 \quad (IV)$$

$$\frac{r_c}{r_a}=\frac{a+c-b}{b+a-c}>0 \quad (V)$$

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$$\frac{r_a}{r_c}=\frac{a+b-c}{b+c-a}>0 \quad \textit{(VI)}$$

Sean: $b+c-a=2x>0$; $a+c-b=2z>0$; $a+b-c=2y>0$

Por la tanto: $a=y+z, b=x+y, c=x+z$

$$\Rightarrow \frac{x}{z}+\frac{z}{y}+\frac{y}{x}\geq\sqrt{1+\frac{(x+y)(y+z)(x+z)}{xyz}}$$

Desde que: $x,y,z>0\rightarrow$ ***(Elevando al cuadrado la expresión tenemos)***

$$\Rightarrow\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)^2\geq 1+\frac{\sum xy(x+y)+2xyz}{xyz}\Rightarrow\left(\frac{x}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{y}{x}\right)^2+\frac{2x}{y}+\frac{2z}{x}+\frac{2y}{z}\geq$$

$$\geq 3+\frac{x+y}{z}+\frac{z+x}{y}+\frac{y+z}{x}$$

$$\Rightarrow\left(\frac{x}{2}\right)^2+\left(\frac{z}{y}\right)^2+\left(\frac{y}{x}\right)^2+\frac{x}{y}+\frac{z}{x}+\frac{y}{z}\geq 3+\frac{y}{x}+\frac{x}{z}+\frac{z}{y}$$

$$\Rightarrow\left(\left(\frac{x}{z}\right)^2+1\right)+\left(\left(\frac{z}{x}\right)^2+1\right)+\left(\left(\frac{y}{x}\right)^2+1\right)+\frac{x}{y}+\frac{z}{x}+\frac{y}{z}\geq 6+\frac{y}{x}+\frac{x}{z}+\frac{z}{y}$$

Por: $MA\geq MG$

$$\Rightarrow\left(\left(\frac{x}{z}\right)^2+1\right)+\left(\left(\frac{z}{y}\right)^2+1\right)+\left(\left(\frac{y}{x}\right)^2+1\right)+\frac{x}{y}+\frac{z}{x}+\frac{y}{z}\geq\frac{2x}{z}+\frac{2z}{y}+\frac{2y}{x}+\frac{x}{y}+\frac{z}{x}+\frac{y}{z}$$

$\Rightarrow$ ***Lo cual nos falta probar que:***

$$\frac{2x}{z}+\frac{2z}{y}+\frac{2y}{x}+\frac{x}{y}+\frac{z}{x}+\frac{y}{z}\geq 6+\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\Leftrightarrow\frac{y}{x}+\frac{x}{z}+\frac{z}{y}+\frac{x}{y}+\frac{z}{x}+\frac{y}{z}\geq 6\Leftrightarrow(MA\geq MG)$$

Por la tanto; $\frac{r_b}{r_a}+\frac{r_c}{r_b}+\frac{r_a}{r_c}\geq\sqrt{1+\frac{4R}{r}}$

JP.029. In acute angled $\Delta\ ABC$; $L-$ Nagel's point, $M, M'\in(AB)$; $N'\in(AC)$; (M,L,N); (M',L,N') - collinear points. Prove that:

$$(a+c-b)\left(\frac{MB}{MA}+\frac{M'B}{M'A}\right)+(a+b-c)\left(\frac{NC}{NA}+\frac{N'C}{N'A}\right)>b+c-a$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

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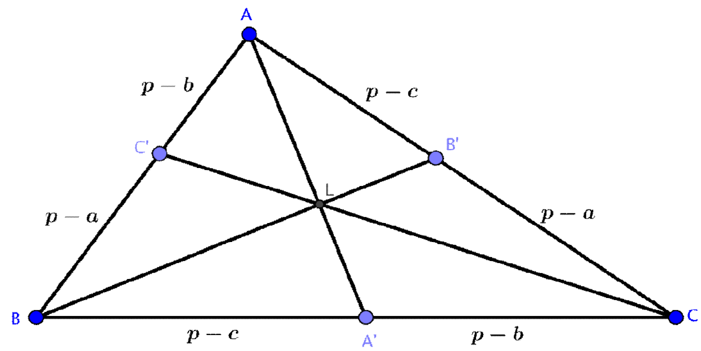

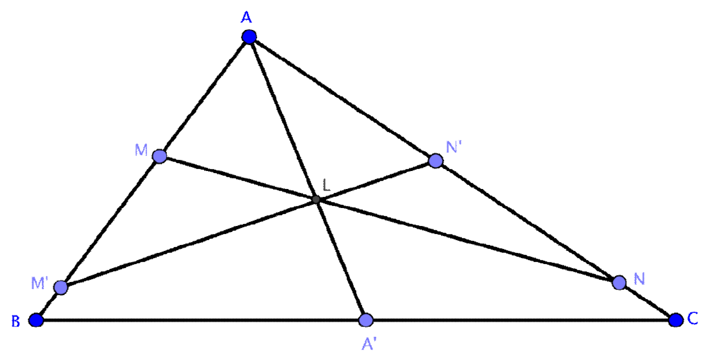

From transversal theorem we have:

$$\frac{MB}{MA}\cdot A'C+\frac{NC}{NA}\cdot BA'=\frac{LA'}{LA}\cdot BC\Rightarrow\frac{MB}{MA}\cdot(p-b)+\frac{NC}{NA}\cdot(p-c)=\frac{LA'}{LA}\cdot a\Rightarrow$$

$$\frac{MB}{MA}(a+c-b)+\frac{NC}{NA}(a+b-c)=\frac{LA'}{LA}2a\quad\text{(1)}$$

Similarly: $\frac{M'B}{MA}(a+c-b)+\frac{N'C}{N'A}(a+b-c)=\frac{LA'}{LA}2a$ ***(2)***

From (1)+(2)$\Rightarrow(a+c-b)\left(\frac{MB}{MA}+\frac{M'B}{M'A}\right)+(a+b-c)\left(\frac{NC}{NA}+\frac{M'C}{M'A}\right)=4a\frac{LA'}{LA}$ ***(3)***

From (3) we must show: $4a\frac{LA'}{LA}>b+c-a\Rightarrow\frac{LA'}{LA}>\frac{b+c-a}{4a}\Leftrightarrow\frac{LA'}{LA}>\frac{p-a}{2a}$ ***(4)***

From Van Aubel theorem we have: $\frac{LA}{LA'}=\frac{AC'}{C'B}+\frac{AB'}{B'C}=\frac{p-b}{p-a}+\frac{p-c}{p-a}=\frac{a}{p-a}\Rightarrow\frac{LA'}{LA}=\frac{p-a}{a}$ ***(5)***

From (4)+(5) we must show: $\frac{p-a}{a}>\frac{p-a}{2a}\Leftrightarrow 1>\frac{1}{2}$ ***true.***

JP.030. If $x_k\in[0,1]$ $(k=1,2,\dots,n)$ then:

$$3\sum_{k=1}^{n}x_k^2\le 2n+x_1x_2x_3+x_2x_3x_4+\dots+x_nx_1x_2$$

Proposed by Mihály Bencze – Romania

Solution by proposer

First we prove that $x^2+y^2+z^2\le xyz+2$ *for all* $x,y,z\in[0,1]$. *Let be*

$f(x)=x^2+y^2+z^2-xyz-2\Rightarrow f(x)\le\max\{f(0),f(1)\}$ *but*

$f(0)=y^2+z^2-2\le 0$ *with equality for* $y=z=1$ *and* $f(1)=y^2+z^2-yz-1$

Let be $g(y)=y^2+z^2-yz-1\Rightarrow g(y)\le\max\{g(0),g(1)\}=\max\{z^2-1,z^2-z\}\le 0$

Equality holds if and only if $x=y=z\Rightarrow$ *or* $x=y=1;z=0$*, etc.*

$$3\sum_{k=1}^{n}x_k^2=\sum_{cyclic}(x_1^2+x_2^2+x_3^2)\le\sum_{cyclic}(x_1x_2x_3+2)=$$
$$=2n+x_1x_2x_3+x_2x_3x_4+\cdots+x_nx_1x_2$$

SP.016. Let a,b,c,s,t,u be positive real numbers such that $a+b+c=1$. Prove that:

$$\frac{sa^2+tb^2+uc^2}{sa+tb+uc}+\frac{sb^2+tc^2+ua^2}{sb+tc+ua}+\frac{sc^2+ta^2+ub^2}{sc+ta+ub}\ge 1$$

Proposed by Kunihiko Chikaya – Tokyo – Japan

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean: a,b,c,s,t,u *números reales positivos, tal que:* $a+b+c=1$. *Probar que:*

$$\frac{sa^2+tb^2+uc^2}{sa+tb+uc}+\frac{sb^2+tc^2+ua^2}{sb+tc+ua}+\frac{sc^2+ta^2+ub^2}{sc+ta+ub}\ge 1$$

Por desigualdad de Cauchy:

1. $(sa^2+tb^2+uc^2)(s+t+u)\ge(sa+tb+uc)^2\rightarrow\frac{sa^2+tb^2+uc^2}{sa+tb+uc}\ge\frac{sa+tb+uc}{s+t+u}$ *(A)*

2. $\frac{sb^2+tc^2+ua^2}{sb+tc+ua}\ge\frac{sb+tc+ua}{s+t+u}$ *(B)*

3. $\frac{sc^2+ta^2+ub^2}{sc+ta+ub}\ge\frac{sc+ta+ub}{s+t+u}$ *(C)*

Sumando: (A) + (B) + (C): $\frac{sa^2+tb^2+uc^2}{sa+tb+uc}+\frac{sb^2+tc^2+ua}{sb+tc+ua}+\frac{sc^2+ta^2+ub^2}{sc+ta+ub}\ge$

$$\geq \frac{s(a+b+c)+t(a+b+c)+u(a+b+c)}{s+t+u}=1$$

SP.017. Let $a_k(k=1,2,\dots,n)$ be a positive real numbers such that

$$\sum_{k=1}^{n} a_k=\frac{n(n+1)}{2}$$

Prove that:

$$\sum_{k=1}^{n}\frac{(k^2-1)a_k+k^2+2k}{a_k^2+a_k+1}\geq\frac{n(n+1)}{2}$$

Proposed by Kunihiko Chikaya – Tokyo – Japan

Solution by Ngô Minh Ngọc Bảo-Gia Lang Province-VietNam

We known: $2+4+6+\cdots+2n=n(n+1)$, ***with*** $n\in N$. ***We have:***

$$\sum_{k=1}^{n}\frac{(k^2-1)a_k+k^2+2k}{a_k^2+a_k+1}=\frac{3}{a_1^2+a_1+1}+\frac{3a_2+8}{a_2^2+a_2+1}+\cdots+\frac{(n^2-1)a_n+n^2+2n}{a_n^2+a_n+1}$$

We prove that: $\frac{(n^2-1)a_n+n^2+2n}{a_n^2+a_n+1}\geq -a_n+2n$, ***(*). Indeed,***

$$(*)\Leftrightarrow (n^2-1)a_n+n^2+2n\geq(2n-a_n)(a_n^2+a_n+1)$$

$$\Leftrightarrow (n^2-1)a_n+n^2+2n\geq -a_n^3+(2n-1)a_n^2+(2n-1)a_n+2n$$

$$\Leftrightarrow a_n^3-(2n-1)a_n^2+(n^2-2n)a_n+n^2\geq 0\Leftrightarrow (a_n-n)^2(a_n+1)\geq 0 \text{ (True)}$$

$$\Rightarrow\sum_{k=1}^{n}\frac{(k^2-1)a_k+k^2+2k}{a_k^2+a_k+1}\geq(2+4+\cdots+2n)-\sum_{k=1}^{n}a_k=$$

$$=(2+4+\cdots+2n)-\frac{n(n+1)}{2}=\frac{n(n+1)}{2}$$

Equality occurs when $a_1=1, a_2=2, a_3=3,\dots,a_n=n$.

SP.018. If $a,b,c>0$ and $x,y,z\geq 1$ then:

$$x^{\frac{8a^3}{a+b}}y^{\frac{8b^3}{b+c}}z^{\frac{8c^3}{c+a}}\geq\left(\frac{x^5}{z}\right)^{a^2}\left(\frac{y^5}{x}\right)^{b^2}\left(\frac{z^5}{y}\right)^{c^2}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

We have: $\frac{a^3}{a+b} \ge \frac{5a^2-b^2}{8} \Leftrightarrow (a-b)^2(3a+b) \ge 0$ *therefore*

$$\begin{cases} \frac{a^3 \ln x}{a+b} \ge \frac{(5a^2-b^2)\ln x}{8} \\ \frac{b^3 \ln y}{b+c} \ge \frac{(5b^2-c^2)\ln y}{8} \\ \frac{c^3 \ln z}{c+a} \ge \frac{(5c^2-a^2)\ln z}{8} \end{cases}$$ *. After addition we obtain:*

$$\sum \frac{a^3 \ln x}{a+b} = \sum \ln x^{\frac{a^3}{a+b}} \ge \sum \frac{(5a^2-b^2)\ln x}{8} = \sum \frac{a^2(5\ln x - \ln z)}{8} =$$

$$= \sum \ln\left(\frac{x^5}{z}\right)^{\frac{a^2}{8}} \Rightarrow \prod x^{\frac{8a^3}{a+b}} \ge \prod \left(\frac{x^5}{z}\right)^{a^2}$$

SP.019. Prove that:

1) $\sum_{k=1}^{n}(2k+1)(2k^2+2k+5)(k^2+k)^4 = \frac{1}{3}(n^3+3n^2+2n)^4$

2) $\sum_{k=1}^{n}(2k+1)(k^2+k+1)(k^2+k+7)(k^2+k)^6 = \frac{1}{9}(n^3+3n^2+2n)^6$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\sum_{k=1}^{n}(k^2+k)^\alpha((k+2)^\alpha-(k-1)^\alpha) = \sum_{k=1}^{n}(k+1)^\alpha k^\alpha((k+2)^\alpha-(k-1)^\alpha) =$$

$$= \sum_{k=1}^{n}((k+2)^\alpha(k+1)^\alpha k^\alpha - (k+1)^\alpha k^\alpha (k-1)^\alpha) = (n+2)^\alpha(n+1)^\alpha n^\alpha$$

1) If $\alpha = 4 \Rightarrow (k+2)^4-(k-1)^4 = 3(2k+1)(2k^2+2k+5)$

2) If $\alpha = 6 \Rightarrow (k+2)^6-(k-1)^6 = 9(2k+1)(k^2+k+1)(k^2+k+7)$

SP.020. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$**, then prove that:**

$$\frac{\tan^2 x}{(y+x)^2} + \frac{\tan^2 y}{(z+x)^2} + \frac{\tan^2 x}{(x+y)^2} > \frac{3}{4}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $x, y, z \in < 0, \frac{\pi}{2} >$***. Probar que:*** $\frac{\tan^2 x}{(y+z)^2} + \frac{\tan^2 y}{(z+x)^2} + \frac{\tan^2 z}{(x+y)^2} > \frac{3}{4}$

Es bien conocido lo siguiente, ya se ha demostrado anteriormente:

$\tan x > x \ \forall x \in < 0, \frac{\pi}{2} >$***. Por la tanto:*** $\frac{\tan^2 x}{(y+z)^2} + \frac{\tan^2 y}{(z+x)^2} + \frac{\tan^2 z}{(x+y)^2} > \frac{x^2}{(y+z)^2} + \frac{y^2}{(z+x)^2} + \frac{z^2}{(x+y)^2}$

$$\Rightarrow \frac{x^2}{(y+z)^2} + \frac{y^2}{(z+x)^2} + \frac{z^2}{(x+y)^2} \geq \frac{xy}{(y+z)(z+x)} + \frac{yz}{(z+x)(x+y)} + \frac{xz}{(y+z)(x+y)}$$

Por desigualdad de Cauchy:

$$\frac{(xy)^2}{x(y+z)y(z+x)} + \frac{(yz)^2}{y(z+x)z(x+y)} + \frac{(xz)^2}{x(y+z)z(x+y)} \geq$$

$$\geq \frac{(xy+yz+xz)^2}{\sum xy(z+x)(z+y)} \geq \frac{\sum (xy)^2 + 2xyz(x+y+z)}{\sum z^2xy + \sum x^2yz + \sum y^2xz + \sum (xy)^2}$$

$$\Rightarrow \frac{\sum (xy)^2 + 2xyz(x+y+z)}{3xyz(x+y+z) + \sum (xy)^2} \geq \frac{3}{4} \Leftrightarrow \sum (xy)^2 \geq xyz(x+y+z)$$

Por transitividad: $\frac{\tan^2 x}{(y+z)^2} + \frac{\tan^2 y}{(z+x)^2} + \frac{\tan^2 z}{(x+y)^2} > \frac{3}{4}$

Solution 2 by Soumitra Moukherjee-Chandar Nagore-India

Let $f(x) = \tan x - x$ ***for all*** $x \in \left(0, \frac{\pi}{2}\right)$***. Now*** $f'(x) = \tan^2 x$ ***for all*** $x \in \left(0, \frac{\pi}{2}\right)$

Now, $f(x)$ ***is continuous on*** $\left(0, \frac{\pi}{2}\right)$, $f'(x) > 0$ ***for all*** $x \in \left(0, \frac{\pi}{2}\right)$

Hence, $f(x)$ ***is increasing on*** $\left(0, \frac{\pi}{2}\right)$, $f(x) > f(0) = 0$ ***for all*** $x \in \left(0, \frac{\pi}{2}\right)$

So, $\tan x > x$ ***for all*** $x \in \left(0, \frac{\pi}{2}\right)$,

$$\sum_{cyc} \frac{\tan^2 x}{(y+z)^2} > \sum_{cyc} \left(\frac{x}{y+z}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc} \frac{x}{y+z}\right)^2 \geq \frac{1}{3}\left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

$$\left[since, \sum_{cyc} \frac{x}{y+z}\right]. Hence, \sum_{cyc} \frac{\tan^2 x}{(y+z)^2} > \frac{3}{4}$$

SP.021. If $x, y, z > 0$, then prove that:

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$$(x^3y^3+y^3z^3+z^3x^3)\left(\frac{1}{(x+y)^5z}+\frac{1}{(y+z)^5x}+\frac{1}{(z+x)^5y}\right)\geq\frac{9}{32}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Moukherjee – Kolkata – India

$$2\left(\sum_{cyc}x^3y^3\right)\left\{\sum_{cyc}\frac{1}{(x+y)^5z}\right\}\geq\left\{\sum_{cyc}z^3(x^3+y^3)\right\}\left\{\sum_{cyc}\frac{1}{(x+y)^5z}\right\}\geq$$

$$\geq\left\{\sum_{cyc}\frac{z^3}{4}(x+y)^3\right\}\left\{\sum_{cyc}\frac{1}{(x+y)^5z}\right\}\geq$$

$$\geq\frac{1}{4}\left(\frac{x}{y+x}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2$$ ***[Applying Cauchy – Schwarz]***

$$\geq\frac{1}{4}\left(\frac{3}{2}\right)^2=\frac{9}{16}$$ ***[Applying Nesbitt Inequality]***

$$\Rightarrow\left(\sum_{cyc}x^3y^3\right)\left\{\sum_{cyc}\frac{1}{(x+y)^5z}\right\}\geq\frac{9}{32}$$

SP.022. Prove that if $n\in\mathbb{N};n\geq 2;0<a\leq b$ then:

$$\frac{b^{n+1}-a^{n+1}}{n+1}+\frac{ab(b^{n-1}-a^{n-1})}{n-1}\leq(b-a)\sqrt{2(a^{2n}+b^{2n})}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hamza Mahmood – Lahore – Pakistan

Since $0<a\leq b$, we consider the following two cases:

Case 1: $0<a=b;n\in\mathbb{N};n\geq 2,$

To prove for this case, we first prove the following lemma:

Lemma: $\forall\, x\in(1,\infty);\ n\in\mathbb{N};n\geq 2,\frac{(1+x^n)(x-1)}{2x}>\frac{x^{n-1}-1}{n-1}$

Proof: Consider

$$f(x)=\frac{(1+x^n)(x-1)}{2x}-\int_1^x t^{n-2}\,dt,\ x\ge 1,n\in\mathbb{N},n\ge 2,$$

It is easy to show that $f(x)$ ***is continuous on*** $[1,\infty)$ ***and differentiable on*** $(1,\infty)$ ***(1)***

Differentiating $f(x)$ ***gives*** $f'(x)=\frac{nx^{n+1}+1-(n+1)x^n}{2x^2}$

Now if $x>1\Rightarrow x^n>x^i$ ***for*** $i=n-1,n-2,n-3,\dots,3,2,1,0$

$$\Rightarrow nx^n>x^{n-1}+x^{n-2}+x^{n-3}+\dots+x^2+x+1=\frac{x^n-1}{x-1}\Rightarrow$$

$$\Rightarrow x^n-1<nx^n(x-1)$$

$$\Rightarrow 0<nx^{n+1}-nx^n+1-x^n=nx^{n+1}+1-(n+1)x^n\Rightarrow$$

$$\Rightarrow\frac{nx^{n+1}+1-(n+1)x^n}{2x^2}>0$$

$$\Rightarrow f'(x)>0 \text{ for } x>1;n\in\mathbb{N};n\ge 2\quad (2)$$

(1) & (2) $\Rightarrow f(x)$ ***is increasing on*** $[1,\infty)\Rightarrow\forall\, x>1\Rightarrow f(x)>f(1)$

$$\Rightarrow\frac{(1+x^n)(x-1)}{2x}-\int_1^x t^{n-2}\,dt>0\Rightarrow\frac{(1+x^n)(x-1)}{2x}>\frac{x^{n-1}-1}{n-1}$$

$\forall x\in(1,\infty);\ n\in\mathbb{N};n\ge 2$ ***(proved)***

Now $\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}=\frac{b^n+ab^{n-1}+a^2b^{n-2}+\dots+a^{n-1}b+a^n}{n+1}<\sqrt{\frac{b^{2n}+a^2b^{2n-2}+\dots+a^{2n}}{n+1}}$

Because Arithmetic Mean (A. M.) $\le$ ***Quadratic Mean (Q.M.)***

Similarly, $\frac{ab(b^{n-1}-a^{n-1})}{(b-a)(n-1)}=\frac{ab^{n-1}+a^2b^{n-2}+\dots+a^{n-1}b}{n-1}<\sqrt{\frac{a^2b^{2n-2}+a^4b^{2n-4}+\dots+a^{2n-2}b^2}{n-1}}$

$$\Rightarrow\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}+\frac{ab(b^{n-1}-a^{n-1})}{(b-a)(n-1)}<\sqrt{\frac{b^{2n}+a^2b^{2n-2}+\dots+a^{2n}}{n+1}}+$$

$$+\sqrt{\frac{a^2b^{2n-2}+a^4b^{2n-4}+\dots+a^{2n-2}b^2}{n-1}}\quad (A)$$

Applying A.M – Q.M. Inequality again:

$$\sqrt{\frac{b^{2n}+a^2b^{2n-2}+\dots+a^{2n}}{n+1}}+\sqrt{\frac{a^2b^{2n-2}+a^4b^{2n-4}+\dots+a^{2n-2}b^2}{n-1}}<$$

$$< \sqrt{2}\cdot\sqrt{\frac{b^{2n}+a^2b^{2n-2}+\cdots+a^{2n}}{n+1}+\frac{a^2b^{2n-2}+a^4b^{2n-4}+\cdots+a^{2n-2}b^2}{n-1}}$$

Simplifying the Right Hand Side:

$$\frac{b^{2n}+a^2b^{2n-2}+\cdots+a^{2n}}{n+1}+\frac{a^2b^{2n-2}+a^4b^{2n-4}+\cdots+a^{2n-2}b^2}{n-1}=$$

$$=\frac{(n-1)(a^{2n}+b^{2n})+2n(a^2b^{2n-2}+a^4b^{2n-4}+\cdots+a^{2n-2}b^2)}{n^2-1}$$

$$=\frac{a^{2n}+b^{2n}}{n+1}+\frac{2n}{n^2-1}\cdot a^2b^{2n-2}\cdot\frac{1-\left(\frac{a^2}{b^2}\right)^{n-1}}{1-\frac{a^2}{b^2}}$$

(because $a^2b^{2n-2}+a^4b^{2n-4}+\cdots+a^{2n-2}b^2=a^2b^{2n-2}\cdot\frac{1-\left(\frac{a^2}{b^2}\right)^{n-1}}{1-\frac{a^2}{b^2}}$ ***A Geometric Series)***

So, $\sqrt{\frac{b^{2n}+a^2b^{2n-2}+\cdots+a^{2n}}{n+1}}+\sqrt{\frac{a^2b^{2n-2}+a^4b^{2n-4}+\cdots+a^{2n-2}b^2}{n-1}}<$

$$<\sqrt{2}\cdot\sqrt{\frac{a^{2n}+b^{2n}}{n+1}+\frac{2n}{n^2-1}\cdot a^2b^{2n-2}\cdot\frac{1-\left(\frac{a^2}{b^2}\right)^{n-1}}{1-\frac{a^2}{b^2}}}\quad (B)$$

Now since $0<a<b\Rightarrow 1<\frac{b}{a}\Rightarrow 1<\left(\frac{b}{a}\right)^2$ & $n=\mathbb{N}; n\geq 2$***, we use the above lemma for***

$$x=\left(\frac{b}{a}\right)^2>1:$$

$$\frac{(1+x^n)(x-1)}{2x}>\frac{x^{n-1}-1}{n-1}\Rightarrow\frac{\left\{1+\left(\frac{b}{a}\right)^{2n}\right\}\left\{\left(\frac{b}{a}\right)^2-1\right\}}{2\left(\frac{b}{a}\right)^2}>\frac{\left(\frac{b}{a}\right)^{2n-2}-1}{n-1}\Rightarrow$$

$$\Rightarrow\left\{1+\left(\frac{b}{a}\right)^{2n}\right\}\left\{1-\left(\frac{a}{b}\right)^2\right\}>\frac{2}{n-1}\cdot\left\{\left(\frac{b}{a}\right)^{2n-2}-1\right\},$$

Multiplying both sides by $\frac{a^{2n}\cdot n}{n+1}$ ***gives:***

$$\frac{n}{n+1}\cdot(a^{2n}+b^{2n})\cdot\left\{1-\left(\frac{a}{b}\right)^2\right\}>\frac{2n}{n^2-1}(a^2b^{2n-2}-a^{2n})\Rightarrow$$

$$\Rightarrow\left(1-\frac{1}{n+1}\right)\cdot\left(a^{2n}+b^{2n}\right)\geq\frac{2n}{n^2-1}\cdot a^2b^{2n-2}\cdot\frac{1-\left(\frac{a^2}{b^2}\right)^{n-1}}{1-\frac{a^2}{b^2}}$$

$$\Rightarrow a^{2n}+b^{2n}-\frac{a^{2n}+b^{2n}}{n+1}>\frac{2n}{n^2-1}\cdot a^2b^{2n-2}\cdot\frac{1-\left(\frac{a^2}{b^2}\right)^{n-1}}{1-\frac{a^2}{b^2}}\Rightarrow$$

$$\Rightarrow a^{2n}+b^{2n}>\frac{a^{2n}+b^{2n}}{n+1}+\frac{2n}{n^2-1}\cdot a^2b^{2n-2}\cdot\frac{1-\left(\frac{a^2}{b^2}\right)^{n-1}}{1-\frac{a^2}{b^2}}$$

$$\Rightarrow\sqrt{2}\sqrt{\frac{a^{2n}+b^{2n}}{n+1}+\frac{2n}{n^2-1}\cdot a^2b^{2n-2}\cdot\frac{1-\left(\frac{a^2}{b^2}\right)^{n-1}}{1-\frac{a^2}{b^2}}}<\sqrt{2\left(a^{2n}+b^{2n}\right)}\quad (C)$$

From (A), (B) & (C) We finally get:

$\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}+\frac{ab\left(b^{n-1}-a^{n-1}\right)}{(b-a)(n-1)}<\sqrt{2\left(a^{2n}+b^{2n}\right)}$ ***Multiplying both sides by*** $b-a>0$,

$$\Rightarrow\frac{b^{n+1}-a^{n+1}}{n+1}+\frac{ab\left(b^{n-1}-a^{n-1}\right)}{n-1}<(b-a)\sqrt{2\left(a^{2n}+b^{2n}\right)}\text{ for}$$

$0<a<b; n\in\mathbb{N}; n\geq 2$ ***(Hence Case 2 proved)***

Solution 2 by Kunihiko Chikaya – Tokyo – Japan

$$\frac{b^{n+1}-a^{n+1}}{n+1}+\frac{ab\left(b^{n-1}-a^{n-1}\right)}{n-1}\leq(b-a)\sqrt{2\left(a^{2n}+b^{2n}\right)}\quad (*)$$

$$0<a\leq b, n\geq 2\ \ (n=2,3,\dots)\ \ 0<a\leq b$$

LHS of (*)

$$=\int_a^b\left(x^n+abx^{n-2}\right)dx$$

$$f(x)=x^n+abx^{n-2}=x^{n-2}\left(x^2+ab\right)$$

$$f'(x)=x^{n-3}\{nx^2+ab(n-2)\}>0$$

M.V.T of Integral for $a\leq x\leq b$

$$\leq(b-a)f\left(\frac{a+b}{2}\right)\left(0<a<\frac{a+b}{2}<b\right)=(b-a)\left(\frac{a+b}{2}\right)^{n-2}\left\{\left(\frac{a+b}{2}\right)^2+ab\right\}$$

$$\le (b-a)\left(\frac{a+b}{2}\right)^{n-2}\cdot 2\left(\frac{a+b}{2}\right)^{2} = 2(b-a)\left(\frac{a+b}{2}\right)^{n} \overset{Jensen}{\lesseqgtr} 2(b-a)\frac{a^n+b^n}{2}$$

$$\le 2(b-a)\sqrt{\frac{(a^2)^n+(b^2)^n}{2}}$$

SP.023. Let $A, B \in M_n(C)$ such that $\det A = \det B \neq 0$. Prove that:

$$\det(AB + xy(AB)^{-1} + (x+y)I_n) = \det(BA + xy(BA)^{-1} + (x+y)I_n)$$

for all $x, y \in C$.

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\left.\begin{matrix}(A+XB^{-1})B = AB + XI_n \\ A(B+XA^{-1}) = AB + xI_n\end{matrix}\right\} \Rightarrow (A+XB^{-1})B = A(B+XA^{-1}) \Rightarrow$$

$$\Rightarrow \det(A+XB^{-1})\det B = \det A \det(B+XA^{-1}) \Rightarrow \det(A+XB^{-1}) = \det(B+XA^{-1})$$

$$\begin{cases}\det(A+XB^{-1}) = \det(B+XA^{-1}) \\ \det(B+yA^{-1}) = \det(A+yB^{-1})\end{cases}.$$ ***After multiplication:***

$$\det(AB + yI_n + xI_n + xy(AB)^{-1}) = \det(BA + yI_n + xI_n + xy(BA)^{-1}),$$ ***finally***

$$\det(AB + xy(AB)^{-1} + (x+y)I_n) = \det(BA + xy(BA)^{-1} + (x+y)I_n)$$

SP.024. Let ABC be a triangle with the centroid G and denote by S_{ABC} its area. Prove that for any point P in the plane

$$\frac{PA\cdot GA^2}{BC} + \frac{PB\cdot GB^2}{CA} + \frac{PC\cdot GC^2}{AB} \ge \frac{4}{3}S_{ABC}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

First, we will recall without proof two known results below

Lemma 1

For any triangle ABC and all positive real numbers x, y, z then

$$xa^2 + yb^2 + zc^2 \ge 4S_{ABC}\sqrt{xy+yz+zx}$$

***Remark 1.** We have known that there exists a triangle whose side-lenghts are* m_a, m_b, m_c *and its area is* $S' = \frac{3}{4}S_{ABC}$*. Applying lemma 1 for this triangle yields*

$$x \cdot m_a^2 + y \cdot m_b^2 + z \cdot m_c^2 \geq 3S_{ABC}\sqrt{xy + yz + zx} \qquad (1)$$

***Lemma 2.** If* ABC *is a triangle and* P *is any point in its plane, then*

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \geq 1$$

(Hayashi's inequality)

Back to the main problem

***Solution.** Applying inequality (1) for* $(x, y, z) = \left(\frac{PA}{a}, \frac{PB}{b}, \frac{PC}{c}\right)$ *and using lemma 2, we obtain*

$$\frac{PA}{a}m_a^2 + \frac{PB}{b}m_b^2 + \frac{PC}{c}m_c^2 \geq 3S_{ABC}$$

Note that: $m_a = \frac{3}{2}GA, m_b = \frac{3}{2}GB, m_c = \frac{3}{2}GC$*. The inequality above may be rewritten as*

$\frac{PA \cdot GA^2}{BC} + \frac{PB \cdot GB^2}{CA} + \frac{PC \cdot GC^2}{AB} \geq \frac{4}{3}S_{ABC}$*. The proof is complete.*

SP.025. If $a_i > 0$ $(i = 1, 2, \dots, n)$ and $k \geq 1$ then:

$$\left(\sum_{i=1}^{n} a_i\right)^k \leq \sum_{i_1=1,\dots,i_k=1}^{n} \frac{i_1 \dots i_k}{i_i + \dots + i_k - k + 1} a_{i_1} \dots a_{i_k}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\sum_{i_1=1,\dots,i_k=1}^{n} \frac{i_1 \dots i_k a_{i_1} \dots a_{i_k}}{i_1 + \dots + i_k - k + 1} = \sum_{i_1=1,\dots,i_k=1}^{n} i_1 a_{i_1} \dots i_k a_{i_k} \int_0^1 t^{i_1+\dots+i_k-k}\, dt =$$

$$= \int_0^1 \left(\sum_{i_1=1,\dots,i_k=1}^{n} i_1 a_{i_1} \dots i_k a_{i_k} t^{i_1-1+\dots+i_k-1}\right) dt = \int_0^1 \left(\sum_{i=1}^{n} i a_i t^{i-1}\right)^k dt \geq$$

$$\geq \left(\int_0^1 \left(\sum_{i=1}^{n} i a_i t^{i-1}\right) dt\right)^k = \left(\sum_{i=1}^{n} a_i\right)^k$$

SP.026. Let be $n \in \mathbb{N}^*$. Compute:

$$I = \int_n^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x} + \sqrt{x-n}e^{2x-2n-1}} dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hamza Mahmood – Lahore – Pakistan

Let $u = x - n$

$$I = \int_n^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x} + \sqrt{x-n}e^{2x-2n-1}} dx = \int_0^1 \frac{\sqrt{1-u}}{\sqrt{1-u} + \sqrt{u}e^{2u-1}} du \quad (A)$$

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we have:

$$I = \int_0^1 \frac{\sqrt{1-(1-u)}}{\sqrt{1-(1-u)} + \sqrt{1-u}e^{2(1-u)-1}} du = \int_0^1 \frac{\sqrt{u}}{\sqrt{u} + \sqrt{1-u}e^{1-2u}} du \quad (B)$$

Adding (A) & (B): $2I = \int_0^1 \left(\frac{\sqrt{1-u}}{\sqrt{1-u}+\sqrt{u}e^{2u-1}} + \frac{\sqrt{u}}{\sqrt{u}+\sqrt{1-u}e^{1-2u}}\right) du$

Since $\frac{\sqrt{1-u}}{\sqrt{1-u}+\sqrt{u}e^{2u-1}} = \frac{e\sqrt{1-u}}{e\sqrt{1-u}+u^{2u}\sqrt{u}}$. And $\frac{\sqrt{u}}{\sqrt{u}+\sqrt{1-u}e^{1-2u}} = \frac{e^{2u}\sqrt{u}}{e^{2u}\sqrt{u}+e\sqrt{1-u}}$. So

$$2I = \int_0^1 \left(\frac{e\sqrt{1-u}}{e\sqrt{1-u} + e^{2u}\sqrt{u}} + \frac{e^{2u}\sqrt{u}}{e^{2u}\sqrt{u} + e\sqrt{1-u}}\right) du =$$

$$= \int_0^1 \frac{e\sqrt{1-u} + e^{2u}\sqrt{u}}{e\sqrt{1-u} + e^{2u}\sqrt{u}} du = \int_0^1 (1)\, du = 1$$

$I = \int_n^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x}+\sqrt{x-n}e^{2x-2n-1}} dx = \frac{1}{2}$, where $n \in \mathbb{N}^*$

Solution 2 by Soumitra Moukherjee - Chandar Nagore – India

$$I = \int_n^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x} + \sqrt{x-n}e^{2x-2n-1}} dx = \int_n^{n+1} \frac{\sqrt{x-n}}{\sqrt{x-n} + \sqrt{n+1-x}e^{2n+1-2x}} dx$$

[applying $f(x) = f(a+b-x)$] $= \int_n^{n+1} \frac{\sqrt{x-n}e^{2x-2n-1}}{\sqrt{n+1-x}+\sqrt{x-n}e^{2n-2n-1}} dx$

$$2I = \int_{n}^{n+1} dx = 1 \Rightarrow I = \frac{1}{2}$$

SP.027. Solve the following equation in set of real numbers:

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Ravi Prakash-New Delhi-India

For $x \neq 0$, *let* $f(x) = 8^x + 27^{\frac{1}{x}} + 2^{x+1}3^{\frac{x+1}{x}} + 2^x 3^{\frac{2x+1}{x}} = 2^{3x} + 3^{\frac{3}{x}} + (15)\left(2^x 3^{\frac{1}{x}}\right)$

For $x < 0, f(x) < 1 + 1 + 15(1) < 125$. *For* $0 < x < 1$

$$f'(x) = (2^{3x})(3\ln 2) + 3^{\frac{3}{x}}\left(-\frac{3}{x^2}\ln 3\right) + 15\left[2^x 3^{\frac{1}{x}}\ln 2 + 2^x 3^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln 3\right)\right]$$

$$= (3)2^x\left[2^{2x}\ln 2 - \frac{5}{x^2}\left(3^{\frac{1}{x}}\right)\ln 3\right] + \left(3^{\frac{1}{x}}\right)(3)\left[5(2^x)\ln 2 - \frac{3^{\frac{2}{x}}}{x^2}\ln 3\right]$$

For $0 < x < 1, 0 < 2^{2x} < 4, \ln 2 < 0.7 \Rightarrow 0 < 2^{2x}\ln 2 < 2\cdot 8$

For $0 < x < 1, \frac{1}{x^2} > 1,\ 3^{\frac{1}{x}} > 3, \ln 3 > 1$

$2^{2x}\ln 2 - \frac{5}{x^2}\left(3^{\frac{1}{x}}\right)\ln 3 < 0$. *Also, for* $0 < x < 1, 5(2^x)\ln 2 < (10)(0.7) = 7$

and $\frac{3^{\frac{2}{x}}\ln 3}{x^2} > 9 \Rightarrow 5(2^x)\ln 2 - \frac{3^{\frac{2}{x}}}{x^2}\ln 3 < 0$

Thus, $f'(x) < 0$ *for* $0 < x \leq 1$, $f(x)$ *is strictly decreasing* $(0,1]$

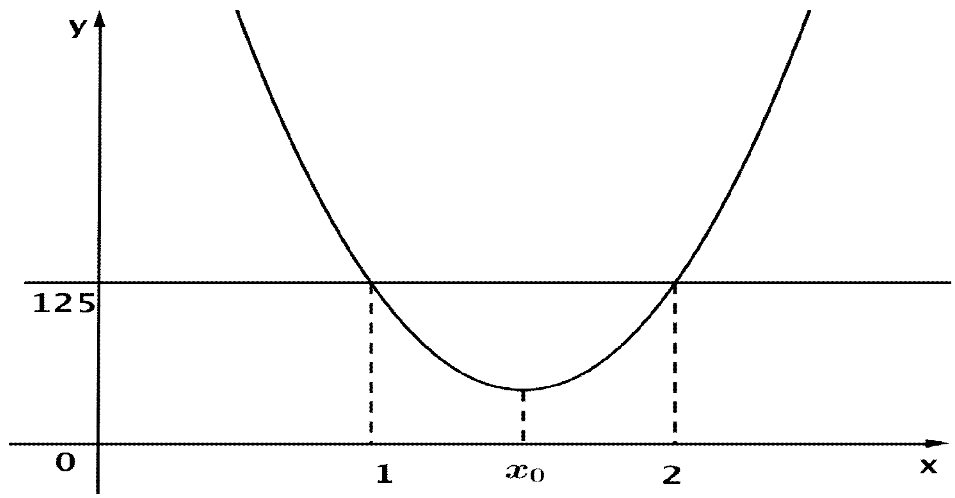
Also, note that $f'(x)$ *is continuous for* $x \geq 1$.

$$f'(1) = 24\ln 2 - 81\ln 3 + 90\ln 2 - 90\ln 3 < 0$$

$$f'(2) = 192\ln 2 - 15\sqrt{3}\ln 3 + 60\sqrt{3}\ln 2 - \frac{9}{4}\sqrt{3}\ln 3 > 0$$

Thus, $\exists$ *some* $x_0 \in (1,2)$ *such that* $f'(x_0) = 0$. *For* $x \geq 2, 2^{3x} \geq 64, 27^{\frac{1}{x}} > 1, 2^x 3^{\frac{1}{x}} > 4$

$f(x) > 64 + 1 + 60 = 125\ \forall x \geq 2$. *Graph of* $y = f(x), x > 0$ *is as follow.*



Thus, $f(x) = 125$ ***has two solutions,*** $\alpha = 1$ ***and*** β ***where*** $1 < \beta < 2$.

SP.028. Compute:

$$L = \lim_{n\to\infty}\frac{1}{n}\int_1^n \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$I = \int_1^n \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243}dx = \int_1^n \frac{x(x^3 + 4x^2 + 12x + 9)}{15x^4 + 90x^3 + 270x^2 + 405x}dx$$

$$= \frac{1}{15}\int_1^n \frac{(x^3 + 4x^2 + 12x + 9)x}{x^4 + 6x^3 + 18x^2 + 27x}dx = \frac{1}{15}\int_1^n \frac{x(x^3 + 4x^2 + 12x + 9)}{x(x+3)(x^2 + 3x^2 + 9)}dx$$

$$= \frac{1}{15}\int_1^n \frac{x(x+1)(x^2 + 3x^2 + 9)}{x(x+3)(x^2 + 3x + 9)}dx = \frac{1}{15}\int_1^n \frac{x + 3 - 1}{x + 3}dx$$

$$= \frac{1}{15}\int_1^n \left[1 - \frac{1}{x+3}\right]dx = \frac{1}{15}[x - \ln|x+3|]_1^n = \frac{1}{15}[n - \ln(n+3) - 1 + \ln 4]$$

$$\lim_{n\to\infty}\frac{1}{n}(I) = \frac{1}{15}\lim_{n\to\infty}\left[1 - \frac{1}{n}\ln(n+3) - \frac{1}{n} + \frac{\ln 4}{n}\right] = \frac{1}{15}$$

Solution 2 by Washma Nayer-Rawalpindi-Pakistan

Using the Binomial Theorem, we have:

$$(x+3)^5 = x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$$

$$\Rightarrow (x+3)^5 - x^5 - 243 = 15x^4 + 90x^3 + 270x^2 + 405x$$

$$\Rightarrow \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243} = \frac{x^4 + 4x^3 + 12x^2 + 9x}{15x^4 + 90x^3 + 270x^2 + 405x} = \frac{x(x^3 + 4x^2 + 12x + 9)}{15x(x^3 + 6x^2 + 18x + 27)}$$

$$\Rightarrow \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243} = \frac{1}{15} \cdot \frac{P(x)}{Q(x)}$$

From the Rational Root Theorem, we find $x = -1$ to be a rational root of $P(x)$ and $x = -3$ to be the rational root of $Q(x) \Rightarrow P(-1) = 0 \,\&\, Q(-3) = 0$

$\Rightarrow$ By the Factor Theorem, $(x+1)$ is a factor of $P(x)$ and $(x+3)$ is a factor of $Q(x)$. By long division, we get:

$$\frac{x^3 + 4x^2 + 12x + 9}{x+1} = x^2 + 3x + 9 \Rightarrow P(x) = x^3 + 4x^2 + 12x + 9 = (x+1)(x^2 + 3x + 9)$$

$$\frac{x^3 + 6x^2 + 18x + 27}{x+3} = x^2 + 3x + 9 \Rightarrow Q(x) = x^3 + 6x^2 + 18x + 27 =$$

$$= (x+3)(x^2 + 3x + 9)$$

$$\Rightarrow \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243} = \frac{1}{15} \cdot \frac{P(x)}{Q(x)} = \frac{1}{15} \cdot \frac{(x+1)(x^2 + 3x + 9)}{(x+3)(x^2 + 3x + 9)} = \frac{1}{15} \cdot \frac{x+1}{x+3}$$

Now we decompose $\frac{x+1}{x+3}$ into Partial Fractions:

Let $\frac{x+1}{x+3} = A + \frac{B}{x+3} = \frac{A(x+3)+B}{x+3}$

$A(x+3) + B = x + 1$ & comparing both sides:

$$A = 1 \,\&\, B = -2. \Rightarrow \frac{x+1}{x+3} = 1 - \frac{2}{x+3}$$

$$\Rightarrow \int_1^n \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243} dx = \int_1^n \frac{1}{15} \cdot \frac{x+1}{x+3} dx = \frac{1}{15} \int_1^n \left(1 - \frac{2}{x+3}\right) dx$$

$$= \frac{1}{15}\{x - 2\ln(x+3)\}\Big|_1^n = \frac{1}{15}\{n - 1 - 2\ln(n+3) + 2\ln 4\}$$

$$\Rightarrow \frac{1}{n} \int_1^n \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243} dx = \frac{1}{15} - \frac{1}{15n} - \frac{2\ln(n+3)}{15n} + \frac{2\ln 4}{15n}$$

Now,

$$L = \lim_{n\to\infty}\frac{1}{n}\int_1^n \frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx$$

$$L = \lim_{n\to\infty}\left\{\frac{1}{15}-\frac{1}{15n}-\frac{2\ln(n+3)}{15n}+\frac{2\ln 4}{15n}\right\}$$

Since $\lim_{n\to\infty}\frac{1}{15}=\frac{1}{15}$, $\lim_{n\to\infty}\frac{1}{15n}=0$, $\lim_{n\to\infty}\frac{2\ln 4}{15n}=0$,

By L'Hospital's Rules,

$$\lim_{n\to\infty}\frac{\ln(n+3)}{n}=\lim_{n\to\infty}\frac{\frac{1}{n+3}}{1}=\lim_{n\to\infty}\frac{1}{n+3}=0$$

$$\Rightarrow L=\frac{1}{15}-0-0+0=\frac{1}{15}\ \therefore\ L=\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx=\frac{1}{15}$$

SP. 029. Compute:

$$L=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1\frac{x\sin\pi x}{x+(1-x)k^{1-2x}}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hamza Mahmood-Lahore-Pakistan

We shall use the following theorem: If $(c_n)_{n\ge1}$ *is a convergent sequence with*

$\lim_{n\to\infty}c_n=L$ *then* $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}c_k=L$

Consider a sequence $(a_n)_{n\ge1}$ *defined as:* $a_n=\int_0^1\frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx$,

we now find $\lim_{n\to\infty}a_n$

$$a_n=\int_0^{\frac12}\frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx+\int_{\frac12}^{1}\frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx$$

For $0<x<\frac12\Rightarrow 0>-2x>-1\Rightarrow 1>1-2x>0\Rightarrow$ as $n\to\infty, n^{1-2x>0}\to\infty$

$$\Rightarrow \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}} \to 0;\ \lim_{n\to\infty}\int_0^{\frac{1}{2}} \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx = 0$$

For $\frac{1}{2} < x < 1 \Rightarrow -1 > -2x > -2 \Rightarrow 0 > 1-2x > -1 \Rightarrow$ ***as*** $n\to\infty, n^{1-2x<0} \to$

$$\to 0 \Rightarrow \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}} \to \frac{x\sin(\pi x)}{x} = \sin(\pi x)$$

$$\lim_{n\to\infty}\int_{\frac{1}{2}}^{1} \frac{x\sin(\pi x)}{x+(1-x)n^{1-2x}}dx = \int_{\frac{1}{2}}^{1}\sin(\pi x) = -\frac{1}{\pi}\left(\cos\pi - \cos\frac{\pi}{2}\right) = \frac{1}{\pi}$$

$\lim_{n\to\infty} a_n = \frac{1}{\pi} \Rightarrow (a_n)_{n\ge 1}$ ***is a convergent sequence with*** $\lim_{n\to\infty} a_n = \frac{1}{\pi}$

so from the above theorem: $\Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{\pi}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1 \frac{x\sin(\pi x)}{x+(1-x)k^{1-2x}}dx = \frac{1}{\pi}$$

Solution 2 by Abdelmalek Metidji-Bouira-Algerie

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1 \frac{x\sin\pi x}{x+(1-x)k^{1-2x}}dx$$

Let $I = \int_0^1 \frac{x\sin\pi x}{x+(1-x)k^{1-2x}}dx$. ***So,*** $I = \int_0^1 \frac{(1-x)\sin\pi(1-x)}{(1-x)+(1-(1-x))k^{1-2(1-x)}}dx$

$= \int_0^1 \frac{(1-x)k^{1-2x}\sin\pi x}{x+(1-x)k^{1-2x}}dx$ as $\sin(\pi-\theta) = \sin\theta$. ***So,***

$$2I = \int_0^1 \frac{\sin\pi x\,(x+(1-x)k^{1-2x})}{x+(1-x)k^{1-2x}}dx = \int_0^1 \sin\pi x\,dx = \frac{2}{\pi}$$

So, $I = \frac{1}{\pi}$

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1 \frac{x\sin\pi x}{x+(1-x)k^{1-2x}}dx =$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\pi} = \frac{1}{\pi}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}1 = \frac{1}{\pi}\lim_{n\to\infty}\frac{1}{n}(n) = \frac{1}{\pi}\lim_{n\to\infty}1 = \frac{1}{\pi}$$

Solution 3 by Soumitra Moukherjee – Chandar Nagore – India

Let $I = \int_0^1 \frac{x\sin\pi x}{x+(1-x)k^{1-2x}}dx = \int_0^1 \frac{(1-x)\sin\pi(1-x)}{xk^{2x-1}+1-x}dx[\text{applying } f(x) = f(a+b-x)]$

$$= \int_0^1 \frac{k^{1-2x}(1-x)\sin\pi x}{x+(1-x)k^{1-2x}}dx = \int_0^1 \frac{k^{1-2x}\sin\pi x}{x+(1-x)k^{1-2x}}dx - k^{1-2x}I$$

$$\Rightarrow I(1+k^{1-2x}) = \int_0^1 \frac{k^{2x-1}\sin\pi(1-x)}{1-x+xk^{2x-1}}dx = \int_0^1 \frac{\sin\pi x}{x+(1-x)k^{1-2x}}dx$$

$$0 \le x \le -1 \Rightarrow 0 \ge -2x \ge -2 \Rightarrow 1 \ge 1-2x \ge -1 \Rightarrow k \ge k^{1-2x} \ge \frac{1}{k}$$

$$x+(1-x)k \ge x+(1-x)k^{1-2x} \ge x - \frac{x-1}{k}$$

$$\Rightarrow \frac{1}{x+\frac{1-x}{k}} \ge \frac{1}{x+(1-x)k^{1-2x}} \ge \frac{1}{x+(1-x)k}$$

Again, $0 \le x \le 1 \Rightarrow \sin 0 \le \sin\pi x \le \sin 1$

$$\frac{\sin\pi}{x+\frac{1-x}{k}} \ge \frac{\sin\pi x}{x+(1-x)k^{1-2x}} \ge 0$$

$$\int_0^1 \frac{\sin\pi}{x+\frac{1-x}{k}}dx \ge \int_0^1 \frac{\sin\pi x}{x+(1-x)k^{1-2x}}dx \ge 0$$

From Sandwich theorems, we have, $\int_0^1 \frac{\sin\pi x}{x+(1-x)k^{1-2x}}dx = 0$

$$L = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_0^1 \frac{x\sin\pi x}{x+(1-x)k^{1-2x}}dx = 0$$

SP.030. Prove that:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2(2015x) - \cos^2(2016x)}{\sin x}dx > 0.0001$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sagar Kumar-Kolkata-India

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos(4030x) - \cos(4032x)}{2\sin x} dx$$

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2\sin(4031x)\sin x}{2\sin x} dx$$

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(4031x)\,dx = \left(\frac{\cos\left(\frac{4031\pi}{3}\right) - \cos\left(\frac{4031\pi}{2}\right)}{4031}\right) =$$

$$= \frac{1}{4031}\cos\left(1353\pi + \frac{2\pi}{3}\right) = -\frac{\cos\left(\frac{2\pi}{3}\right)}{4031} = \frac{1}{8002} = 0.000124 > 0.0001$$

(proved)

Solution 2 by Naren Bhandari-New Delhi-India

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2 2015\,x - \cos^2 2016\,x}{\sin x} dx > 0.0001$$

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\left[2\cos\left[\frac{4031x}{2}\right]\cdot\cos\left(\frac{x}{2}\right)\right]\left[2\omega\sin\left(\frac{40131x}{2}\right)\cdot\sin\left(\frac{x}{2}\right)\right]}{\sin x} dx$$

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\left[d\sin\left(\frac{4031x}{2}\right)\cdot\cos\left(\frac{4031x}{2}\right)\right]\left[2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)\right]}{\sin x} dx$$

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(4031x)\,dx;\ I = -\frac{-\cos(4031x)}{4031}\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}} > 0{,}001$$

$$I = +\frac{1}{4031}\left[\cos\left(4031\frac{\pi}{3}\right) - \cos\left(4031\frac{\pi}{2}\right)\right] > 0{,}0001$$

$$I = \frac{1}{4031}\left[\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{2}\right)\right] > 0.0001$$

$$I = \frac{1}{4031\cdot 2} > \frac{1}{10000} \Rightarrow 10000 > 4031\cdot 2 \therefore I = \frac{1}{4031\cdot 2} > \frac{1}{10000} \quad \textit{(proved)}$$

Solution 3 by Tran Hong-Vietnam

Using: $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$; $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$. *We have:*

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\left[\frac{\cos(4030x) - \cos(4032x)}{\sin x}\right] dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2\sin x \cdot \sin 4031}{2\cdot \sin x}$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(4031x)\, dx = -\frac{1}{4031}\cos(4031x)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{\cos\left(\frac{4031\pi}{3}\right) - \cos\left(\frac{4031\pi}{2}\right)}{4031} =$$

$$= -\frac{-\cos\left(\frac{2\pi}{3}\right)}{4031} = \frac{1}{8002} > \frac{1}{10000}$$

UP.016. Compute the limit:

$$\lim_{n\to\infty}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta} d\theta$$

Proposed by Kunihiko Chikaya - Tokyo – Japan

Solution by Quang Minh Tran - Ho Chi Minh City – VietNam

For all $\theta \in \left[\frac{\pi}{4},\frac{\pi}{3}\right], n \in \mathbb{N}$ *we have:*

$$\left|\left\{\ln\left[\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\right]\right\}\right| \le |\cos\theta|\cdot\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right) \le$$

$$\le \max_{x\in\left[\frac{\pi}{3},\frac{\pi}{4}\right]} \ln(1+\sin\theta\sec^2\theta) = M_1$$

$$\left|\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\right| \le \left(1+\frac{\cos\theta}{n}\right)^{\cot\theta} \le (1+\cos\theta)^{\cot\theta} \le$$

$$\le \max_{x\in\left[\frac{\pi}{3},\frac{\pi}{4}\right]}(1+\cos\theta)^{\cot\theta} = M_2$$

$$\left|\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\cdot\sec^2\theta}\right| \le \left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\cdot\sec^2\theta} \le (1+\cot\theta)^{\sin\theta\cdot\sec^2\theta} \le$$

$$\le \max_{x\in\left[\frac{\pi}{3},\frac{\pi}{4}\right]}(1+\cot\theta)^{\sin\theta\cdot\sec^2\theta} = M_3$$

So for all $\theta \in \left[\frac{\pi}{4},\frac{\pi}{3}\right]$, $n\in\mathbb{N}$ ***exists*** $M>0$ ***such that:***

$$\left|\left\{\ln\left[\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\right]\right\}\cdot\left[\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\right]\cdot\left[\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\cdot\sec^2\theta}\right]\right| \le M$$

For all $\theta \in \left[\frac{\pi}{4},\frac{\pi}{3}\right]$ ***we have:***

$$\lim_{n\to\infty}\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta} = 0$$

Use Lebesgue dominated convergence theorem we have:

$$\lim_{n\to\infty}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\ln\left(1+\frac{\sin\theta\sec^2\theta}{n}\right)^{\cos\theta}\left(1+\frac{\cos\theta}{n}\right)^{\cot\theta}\left(1+\frac{\cot\theta}{n}\right)^{\sin\theta\sec^2\theta}d\theta =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 0\cdot d\theta = 0$$

UP.017. Let $\{a_n\}$ be a sequence defined inductively by

$$a_1 = 1, a_{n+1} = \frac{1}{2}a_n + \frac{n^2-2n-1}{n^2(n+1)^2}\ \ (n = 1, 2, 3, \dots).$$

Find the greatest value of n such that $a_1+a_2+\dots+a_n$ is minimized.

Proposed by Kunihiko Chikaya – Tokyo – Japan

Solution 1 by Ravi Prakash-New Delhi-India

$$a_{n+1}=\frac{1}{2}a_n+\frac{n^2-2n-1}{n^2(n+1)^2},\forall n\geq 1$$

$$=\frac{1}{2}a_n+\frac{2}{(n+1)^2}-\frac{1}{n^2}\Rightarrow a_{n+1}-\frac{2}{(n+1)^2}=\frac{1}{2}\left(a_n-\frac{2}{n^2}\right)\ \forall n\geq 1$$

Let $b_n=a_n-\frac{2}{n^2}$

$b_1=1-2=-1$ and $b_{n+1}=\frac{1}{2}b_n\ \forall n\geq 1\Rightarrow b_n=\left(\frac{1}{2}\right)^{n-1}b_1=-\left(\frac{1}{2}\right)^{n-1}$

$$\therefore a_n=\frac{1}{2n^2}-\left(\frac{1}{2}\right)^{n-1}\quad \forall n\geq 1$$

$$a_1=1,a_2=0,a_3=\frac{2}{a}-\frac{1}{4}=-\frac{1}{36};a_4=\frac{2}{16}-\frac{1}{8}=0;a_5=\frac{2}{25}-\frac{1}{32}>0$$

As $2^n>n^2\ \forall n\geq 5$, we get $a_n>0\ \forall n\geq 5$. Let $s_n=a_1+a_2+\cdots+a_n$, then

$s_1=a_1=1,s_2=1,s_3=\frac{35}{36},s_4=\frac{35}{36},s_5=s_4+a_5>s_4$. In fact $s_{n+1}>s_n>s_5>s_4$ $\forall n\geq 5$. Thus, s_n is minimum when $n=3$ or 4.

Solution 2 by Shafiqur Rahman-Bangladesh

$$a_{n+1}=\frac{1}{2}a_n+\frac{n^2-2n-1}{n^2(n+1)^2}\Rightarrow 2^n a_{n+1}-2^{n-1}a_n=\frac{2^{n+1}}{(n+1)^2}-\frac{2^n}{n^2}\Rightarrow 2^{n-1}a_n-a_1=$$

$$=\frac{2^n}{n^2}-2\Rightarrow a_n=\frac{2}{n^2}-\frac{1}{2^{n-1}}$$

$$S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\left(\frac{2}{n^2}-\frac{1}{2^{n-1}}\right)=2H_n^{(2)}-2+\frac{1}{2^{n-1}}=2\left(\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)+\frac{1}{2^{n-1}}=$$

$=S_{n-1}+\frac{2}{n^2}-\frac{1}{2^{n-1}}$. Here $S_1=1,S_2=1,S_3=\frac{35}{36},S_4=\frac{35}{36},S_5=\frac{3563}{3600}$ & $S_n>S_{n-1}$ when $n\geq 5$. So minimum value of S_n is $\frac{35}{36}$ when $n=3,4$. Thus the greatest value of $n=4$.

UP.018.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{\sqrt{\cos x}}{x}dx\leq\frac{\pi}{24}\left(e^{\frac{\pi}{12}}-1\right)^2+\frac{\sqrt{2}-1}{4}+\frac{152}{\pi^3}$$

Proposed by Soumitra Mandal – Chandar Nagore – India

Solution by proposer

$$\text{Let } A = \begin{pmatrix} a & bc & 1 \\ 1 & cb & a \end{pmatrix}, A^T = \begin{pmatrix} a & 1 \\ b & c \\ c & b \\ 1 & a \end{pmatrix}, \text{ then}$$

$$A \cdot A^T = \begin{pmatrix} 1 + \sum_{cyc} a^2 & 2(a + bc) \\ 2(a + bc) & 1 + \sum_{cyc} a^2 \end{pmatrix}$$

Applying Cauchy – Binet $\det(A \cdot A^T) \geq 0 \Rightarrow 1 + \sum_{cyc} a^2 \geq 2(a + bc)$

putting $b = \sqrt{\cos x}$, $c = \frac{1}{x}$ *and* $a = e^{\frac{\pi}{12}}$, *then* $\left(e^{\frac{\pi}{12}} - 1\right)^2 + \cos x + \frac{1}{x^2} \geq 2\frac{\sqrt{\cos x}}{x}$, *then*

integrating both sides

$$\frac{\pi}{12}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{x^2} \geq 2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx$$

$$\Rightarrow \frac{\pi}{24}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \frac{\sqrt{2} - 1}{4} + \frac{152}{\pi^3} \geq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx$$

UP.019. Let $E = (n, k, p)$ be the total number of $(x_1, x_2, \dots, x_p)$ for which $x_1 + x_2 + \cdots x_p$ is a perfect k power when the integers $x_1, x_2, \dots, x_p$ are selected independently at random from the set $\{1, 2, \dots, n\}$. Compute: $\lim_{n\to\infty} \frac{E(n,k,p)}{n\sqrt[k]{n}}$ for $p = 2$; $^*p \geq 3$.

Proposed by Mihály Bencze – Romania

Solution by proposer

$$p = 2, E(n, k, 2) = 2\sum_{i=1}^{[\sqrt[k]{n}]} i^k - \sum_{i=1}^{[\sqrt[n]{2n}]} i^k - [\sqrt[k]{n}] + (2n + 1)\left(\left[\sqrt[k]{2n} - [\sqrt[k]{n}]\right]\right)$$

and $\lim_{n\to\infty}\frac{[\sqrt[k]{n}]}{\sqrt[k]{n}}\Rightarrow\lim_{n\to\infty}\frac{[\sqrt[k]{2n}]}{\sqrt[k]{2n}}=\sqrt[k]{2}$

$\lim_{n\to\infty}\frac{1}{n\sqrt[k]{n}}\sum_{i=1}^{[\sqrt[n]{2n}]}i^k=\frac{2\sqrt[k]{2}}{k+1}$ *and finally* $\lim_{n\to\infty}\frac{E(n,k,2)}{n\sqrt[k]{n}}=\frac{2k}{k+1}(\sqrt[k]{2}-1)$

UP.020. If $x_k\in\mathbb{R}$ $(k=1,2,\dots,n)$ then:

$$\left(\sum_{k=1}^{n}\frac{x_k}{k(k+1)}\right)^2\le\frac{n}{n+1}\sum_{k=1}^{n}\frac{x_k^2}{k(k+1)}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\left(\sum_{k=1}^{n}\frac{1}{k(k+1)}x_k\right)^2=\left(\sum_{k=1}^{n}\sqrt{\frac{1}{k(k+1)}}\sqrt{\frac{1}{k(k+1)}}x_k\right)^2\le$$

$$\le\sum_{k=1}^{n}\left(\sqrt{\frac{1}{k(k+1)}}\right)^2\sum_{k=1}^{n}\left(\sqrt{\frac{1}{k(k+1)}}x_k\right)^2=$$

$$=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\sum_{k=1}^{n}\frac{x_k^2}{k(k+1)}=\frac{n}{n+1}\sum_{k=1}^{n}\frac{x_k^2}{k(k+1)}$$

UP.021. Prove that:

$$1\le\int_0^1\frac{dx}{\sqrt{1-x^2+x^{2015}-x^{2016}}}\le\frac{\pi}{2}$$

Proposed by Soumitra Mandal – Chandar Nagore – India

Solution by Ravi Prakash - New Delhi – India

For $0<x<1,\ 1-x^2+x^{2015}-x^{2016}=(1-x)[1+x+x^{2015}]>0$

Also, $x^2-x^{2015}+x^{2016}=x^2+x^{2016}-x^{2015}\ge2(x^{1009})-x^{2015}$

$=x^{1009}+x^{1009}-x^{2015}>0\Rightarrow1-x^2+x^{2015}-x^{2016}<1$

$0<1-x^2+x^{2015}-x^{2016}<1$ ***(1)***

Also, $x^{2015}-x^{2016}=x^{2015}(1-x)>0$ *for* $0<x<1$

$\Rightarrow 1 - x^2 + x^{2015} - x^{2016} > 1 - x^2$, ***(2)*** $0 < x < 1$. ***From (1) and (2) for*** $0 < x < 1$

$$1 - x^2 < 1 - x^2 + x^{2015} - x^{2016} < 1 \Rightarrow 1 < \frac{1}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} < \frac{1}{\sqrt{1 - x^2}}$$

$$1 < \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} < \frac{\pi}{2}$$

UP.022. Let ABC be a triangle with the area S and denote by r, r_a, r_b, r_c inradius, exradii respectively. Prove that:

$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) \geq \left(\frac{10}{3}\right)^3 (rS)^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietam

Solution by Soumava Chakraborty – Kolkata – India

$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) = r^6 + r^4\left(\sum r_a r_b\right) + r^2 r_a r_b r_c\left(\sum r_a\right) + (r_a r_b r_c)^2$$

$$= r^6 + r^4 s^2 + r^2\left(\frac{S^2}{r}\right)(4R + r) + \left(\frac{S^2}{r}\right)^2$$

$$\left(\sum r_a r_b = s^2; r_a r_b r_c = \frac{S^2}{r}; \sum r_a = 4R + r\right) = r^6 + r^4 s^2 + r^3 s^2(4R + r) + r^2 s^4$$

$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) \geq \frac{1000}{27} r^2 S^2$$

$$\Leftrightarrow r^6 + r^4 s^2 + r^4 s^2 + 4Rr^3 s^2 + r^2 s^4 \geq \frac{1000}{27} r^4 s^2$$

$$\Leftrightarrow 27r^6 - 946r^4 s^2 + 27r^2 s^4 + 108Rr^3 s^2 \geq 0$$

$$\Leftrightarrow 27r^4 - 946r^2 s^2 + 27s^4 + 108Rrs^2 \geq 0 \quad (1)$$

Now, $27r^4 - 946r^2 s^2 + 27s^4 + 108rs^2 \geq 27r^4 - 946r^2 s^2 + 27s^4 + 216r^2 s^2$

$$(R \geq 2r)$$

$$= 27r^4 - 730r^2 s^2 + 27s^4 = (27r^2 - s^2)(r^2 - 27s^2) = (s^2 - 27r^2)(27s^2 - r^2) \geq 0$$

$\left(s^2 \overset{Gerretsen}{\geq} 16Rr - 5r^2 \overset{Euler}{\geq} 32r^2 - 5r^2 = 27r^2\right)$. (1) ***is proved (Done).***

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UP.023. If $x, y, z, z, b, c > 0$ then:

$$\left(\frac{x+y}{2x}\right)^{b+c}\left(\frac{y+z}{2y}\right)^{c+a}\left(\frac{z+x}{2z}\right)^{a+b} \geq (x+y)^{b-c}(y+z)^{c-a}(z+x)^{a-b}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

We have: $\frac{(y+z)(z+x)}{4yz} \geq \frac{x+z}{y+z} \Leftrightarrow (y-z)^2 \geq 0 \Rightarrow$

$$\begin{cases} \left(\frac{(y+z)(z+x)}{4yz}\right)^a \geq \left(\frac{x+z}{y+z}\right)^a \\ \left(\frac{(z+x)(x+y)}{4zx}\right)^b \geq \left(\frac{y+x}{z+x}\right)^b \\ \left(\frac{(x+y)(y+z)}{4xy}\right)^c \geq \left(\frac{z+y}{x+y}\right)^c \end{cases}$$

After multiplication we obtain: $\prod\left(\frac{x+y}{2x}\right)^{b+c} \geq \prod(x+y)^{b-c}$

UP.024. Calculate:

$$\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right),$$

where F_n is the n th Fibonacci number.

Proposed by Cornel Ioan Vălean – Romania

Solution by Hamza Mahmood-Lahore-Pakistan

Compute $\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right)$, *where F_n is the* nth *Fibonacci number with* $F_1 = 1$

The n^th^ Fibonacci number can be expressed as:

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \Rightarrow F_{2n} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right\}$$

$$\Rightarrow F_{2n} = \frac{1}{\sqrt{5}}\left\{\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right\}$$

$$\textit{Let } a=\frac{3+\sqrt{5}}{2}\Rightarrow\frac{1}{a}=\frac{2}{3+\sqrt{5}}=\frac{3-\sqrt{5}}{2}$$

$$\Rightarrow F_{2n}=\frac{1}{\sqrt{5}}\left\{a^n-\left(\frac{1}{a}\right)^n\right\}=\frac{a^{2n}-1}{\sqrt{5}a^n}\Rightarrow\frac{F_{2n}+1}{F_{2n}-1}=\frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1}$$

$$\textit{Now } \operatorname{arctanh}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), x<1$$

$$\textit{Since } F_{2n}>1 \textit{ for } n\geq 2\Rightarrow\frac{1}{F_{2n}}<1\Rightarrow\operatorname{arctanh}\left(\frac{1}{F_{2n}}\right)=\frac{1}{2}\ln\left(\frac{F_{2n}+1}{F_{2n}-1}\right)=\frac{1}{2}\ln\left(\frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1}\right)$$

$$\textit{Now } a+\left(-\frac{1}{a}\right)=a-\frac{1}{a}=\frac{3+\sqrt{5}}{2}-\frac{3-\sqrt{5}}{2}=\sqrt{5}\ \&\ a\left(-\frac{1}{a}\right)=-1$$

$$\textit{So, } \frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1}=\frac{a^{2n}+\left(a+\left(-\frac{1}{a}\right)\right)a^n+(a)\left(-\frac{1}{a}\right)}{a^{2n}-\left(a+\left(-\frac{1}{a}\right)\right)a^n+(a)\left(-\frac{1}{a}\right)}=\frac{a^{2n}-\left(\frac{1}{a}+(-a)\right)a^n+\left(\frac{1}{a}\right)(-a)}{a^{2n}-\left(a+\left(-\frac{1}{a}\right)\right)a^n+(a)\left(-\frac{1}{a}\right)}$$

$$\Rightarrow\frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1}=\frac{\left(a^n-\frac{1}{a}\right)(a^n+a)}{(a^n-a)\left(a^n+\frac{1}{a}\right)}$$

$$x^2-(p+q)x+pq=(x-p)(x-q)$$

$$\Rightarrow\frac{a^{2n}+\sqrt{5}a^n-1}{a^{2n}-\sqrt{5}a^n-1}=\frac{\left(a^n-\frac{1}{a}\right)a(a^{n-1}+1)}{a(a^{n-1}-1)\left(a^n+\frac{1}{a}\right)}=\frac{(a^{n+1}-1)(a^{n-1}+1)}{(a^{n-1}-1)(a^{n+1}+1)}$$

$$\ln\left\{\frac{(a^{n+1}-1)(a^{n-1}+1)}{(a^{n-1}-1)(a^{n+1}+1)}\right\}=\ln(a^{n+1}-1)-\ln(a^{n-1}-1)+\ln(a^{n-1}+1)-\ln(a^{n+1}+1)$$

Now

$$\sum_{n=2}^{\infty}\operatorname{arctanh}\left(\frac{1}{F_{2n}}\right)=\frac{1}{2}\sum_{n=2}^{\infty}\ln\left\{\frac{(a^{n+1}-1)(a^{n-1}+1)}{(a^{n-1}-1)(a^{n+1}+1)}\right\}$$

$$\sum_{n=2}^{\infty}\operatorname{arctanh}\left(\frac{1}{F_{2n}}\right)=\frac{1}{2}[\{\ln(a^3-1)+\ln(a^4-1)+\ln(a^5-1)+\ln(a^6-1)+\cdots\}$$

$$-\{\ln(a-1)+\ln(a^2-1)+\ln(a^3-1)+\ln(a^4-1)+\cdots\}$$

$$+\{\ln(a+1)+\ln(a^2+1)+\ln(a^3+1)+\ln(a^4+1)+\ln(a^5+1)+\cdots\}$$

$$-\{\ln(a^3+1)+\ln(a^4+1)+\ln(a^5+1)+\cdots\}]$$

$$\sum_{n=2}^{\infty}\operatorname{arctanh}\left(\frac{1}{F_{2n}}\right)=\frac{1}{2}\{-\ln(a-1)-\ln(a^2-1)+\ln(a+1)+\ln(a^2+1)\}=$$

$$= \frac{1}{2}\ln\left\{\frac{(a+1)(a^2+1)}{(a-1)(a^2-1)}\right\}$$

Since $a = \frac{3+\sqrt{5}}{2} \Rightarrow a+1 = \frac{5+\sqrt{5}}{2}, a-1 = \frac{1+\sqrt{5}}{2}, a^2 = \frac{7+3\sqrt{5}}{2}, a^2+1 = \frac{9+3\sqrt{5}}{2},$

$$a^2 - 1 = \frac{5+3\sqrt{5}}{2}$$

$$\frac{(a+1)(a^2+1)}{(a-1)(a^2-1)} = \frac{(5+\sqrt{5})(9+3\sqrt{5})}{(1+\sqrt{5})(5+3\sqrt{5})} = \frac{45+15\sqrt{5}+9\sqrt{5}+15}{5+3\sqrt{5}+5\sqrt{5}+15} = \frac{60+24\sqrt{5}}{20+8\sqrt{5}} = 3$$

$$\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right) = \frac{1}{2}\ln 3 = \ln\sqrt{3}$$

UP.025. Compute:

$$\lim_{n\to\infty}\left(\sqrt[3n+3]{(n+1)!} - \sqrt[3n]{n!}\right)\cdot\sqrt[3]{n^2}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Moukherjee - Chandar Nagore – India

$x_n = \sqrt[3n]{n!} \Rightarrow \ln x_n = \frac{1}{3n}\ln n!$ *. Using Stirling's formula,*

$$\ln n! = \ln\sqrt{2\pi} + \left(n+\frac{1}{2}\right)\ln n - n + O\left(\frac{1}{n}\right)$$

$$\ln x_n = \frac{1}{3n}\ln n! = \frac{1}{3n}\ln\sqrt{2\pi} + \frac{1}{3}\left(1+\frac{1}{2n}\right)\ln n - \frac{1}{3} + \frac{1}{3}O\left(\frac{1}{n^2}\right)$$

$$\ln x_{n+1} = \frac{1}{3(n+1)}\ln\sqrt{2\pi} + \frac{1}{3}\left(1+\frac{1}{2(n+1)}\right)\ln(n+1) - \frac{1}{3} + \frac{1}{3}O\left(\frac{1}{(n+1)^2}\right)$$

$$\ln x_{n+1} - \ln x_n = -\frac{\ln 2\pi}{6n(n+1)} + \frac{1}{3}\ln\left(1+\frac{1}{n}\right) + \frac{1}{6(n+1)}\ln(n+1) - \frac{1}{6n}\ln n + O\left(\frac{1}{n^2}\right)$$

Using Lagrange's Mean Value Theorem:

$$\ln x_{n+1} - \ln x_n = (x_{n+1} - x_n)\frac{1}{c_n} \quad \textit{where } c_n \in (x_n, x_{n+1})$$

Also, $\lim_{n\to\infty} c_n = \frac{1}{e}$; $\sqrt[3]{n^2}(\ln x_{n+1} - \ln x_n)$

$$= -\frac{\sqrt[3]{n^2}\ln 2\pi}{6n(n+1)} + \frac{\sqrt[3]{n^2}}{3}\ln\left(1+\frac{1}{n}\right) + \frac{\sqrt[3]{n^2}}{6(n+1)}\ln(n+1) - \frac{\sqrt[3]{n^2}}{6n}\ln + \sqrt[3]{n^2}\,O\left(\frac{1}{n^2}\right).$$ *Now,*

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(\ln x_{n+1} - \ln x_n\right) = 0$$

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(x_{n+1} - x_n\right) = \frac{1}{e}\lim_{n\to\infty} \sqrt[3]{n^2}\left(\ln a_{n+1} - \ln x_n\right) = 0$$

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3n+3]{(n+1)!} - \sqrt[3n]{n!}\right) = 0$$

UP.026. Compute:

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}$$

Proposed by D. M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by George – Florin Șerban – Romania

Proposition: Let be the sequence:

$$(a_n)_{n\ge 1}, a_n > 0, \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1, \lim_{n\to\infty}\frac{a_n}{n} = a \in (0,\infty), \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = b,$$

then $\lim_{n\to\infty}(a_{n+1} - a_n) = a\ln b$

Solution:

$$\left(\frac{a_{n+1}}{a_n}\right)^n = \left[\left(1 + \left(\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\frac{n}{a_n}(a_{n+1}-a_n)}$$

$$\ln\left(\frac{a_{n+1}}{n}\right)^n = \ln\left[1 + \left(\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\frac{n}{a_n}(a_{n+1}-a_n)},$$

$$\frac{a_n}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)^n = (a_{n+1} - a_n)\ln\left[1 + \left(\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right],$$

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} - \lim_{n\to\infty}\frac{a_n}{a_n} = 1 - 1 = 0,$$

Then $\lim_{n\to\infty}\ln\left[\left(1 + \left(\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right] = \ln e = 1$, ***then***

$$\lim_{n\to\infty}\frac{a_n}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}(a_{n+1} - a_n)\ln\left[\left(1 + \left(\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right],$$

$$\lim_{n\to\infty}\frac{a_n}{n}\ln\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}(a_{n+1}-a_n) = a(\ln b).$$

$$l = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}} =$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[2n+2]{(2n+1)!!}}\cdot\frac{\sqrt{n}}{\sqrt{n+1}} - \frac{n\sqrt{n}}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}},$$

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 1,$$

$$l = \lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n\sqrt{n}}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} =$$

$$= \lim_{n\to\infty}(a_{n+1}-a_n)\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}},$$

$$a_n = \frac{n\sqrt{n}}{\sqrt[2n]{(2n-1)!!}}, \lim_{n\to\infty}\frac{a_n}{n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{(2n-1)!!}} = \sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}} =$$

$$= \sqrt{\lim_{n\to\infty}\frac{(n+1)^n(n+1)}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}},$$

$$\lim_{n\to\infty}\frac{a_n}{n} = \sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{n+1}{2n+1}} = \sqrt{\frac{e}{2}},$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{n+1}\cdot\frac{n}{a_n}\cdot\frac{n+1}{n} = \sqrt{\frac{e}{2}}\cdot\sqrt{\frac{2}{e}}\cdot 1 = 1,$$

$$b = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[2n+2]{(2n+1)!!}}\cdot\frac{\sqrt[2n]{(2n-1)!!}}{n\sqrt{n}}\right)^n =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[2n+2]{(2n+1)!!}}\right)^n, \lim_{n\to\infty}\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}} = 1,$$

$$b=\lim_{n\to\infty}\left(\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt{2n+1}}\right)^{\frac{n}{n+1}}=\sqrt{\lim_{n\to\infty}\left(\sqrt[n]{\frac{(2n-1)!!}{(2n+1)^n}}\right)^{\frac{n}{n+1}}}=$$

$$=\sqrt{\lim_{n\to\infty}\frac{(2n+1)!!}{(2n+3)^n(2n+3)}\cdot\frac{(2n+1)^n}{(2n-1)!!}},$$

$$b=\sqrt{\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)^n\cdot\frac{2n+1}{2n+3}}=\sqrt{\lim_{n\to\infty}\left(1+\frac{-2}{2n+3}\right)^{\frac{2n+3}{-2}\cdot\frac{-2n}{2n+3}}}=\sqrt{e^{-1}}=e^{-\frac{1}{2}},$$

$$\lim_{n\to\infty}\frac{2n+1}{2n+3}=1,$$

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a\cdot\ln b=\sqrt{\frac{e}{2}}\cdot\ln e^{-\frac{1}{2}}=\frac{-1}{2}\sqrt{\frac{e}{2}},$$

$$l=\lim_{n\to\infty}(a_{n+1}-a_n)\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}=\frac{-1}{2}\sqrt{\frac{e}{2}}\cdot 0=0.$$

UP.027. If $\Gamma:(0,\infty)\to(0,\infty)$ is Euler's function, compute:

$$\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{2x+2}}}-\frac{x}{(\Gamma(x+1))^{\frac{1}{2x}}}\right)^{\sqrt{x}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Feti Sinani-Kosovo

$$\Gamma(x)\sim\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}\Rightarrow\Gamma(x)=\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}+$$

$$+o\left(\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}\right),x\to\infty$$

$$(x+1)\Gamma(x+2)^{-\frac{1}{2x+2}}=(x+1)e^{-\frac{\ln\Gamma(x+2)}{2x+2}}=(x+1)e^{-\frac{\ln\left(\left(\frac{x+1}{e}\right)^{x+1}\sqrt{2\pi(x+1)}(1+o(1))\right)}{2x+2}}=$$

$$=(x+1)\frac{\sqrt{e}}{\sqrt{x+1}}\left(1-\frac{\ln\sqrt{2\pi}}{x+1}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)\left(1-\frac{\ln(x+1)}{4(x+1)}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)=$$

$$=\sqrt{e}\sqrt{x}\left(1+\frac{1}{2x}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)\left(1+o\left(\frac{1}{x^{\frac{1}{2}}}\right)\right)=\sqrt{e}\sqrt{x}+o(1)\quad (x\to\infty)$$

$$x\Gamma(x+1)^{-\frac{1}{2x}}=x\frac{\sqrt{e}}{\sqrt{x}}\left(1-\frac{\ln\sqrt{2\pi}}{x}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)\left(1-\frac{\ln x}{4x}+o\left(\frac{1}{x^{\frac{3}{2}}}\right)\right)=\sqrt{e}\sqrt{x}+o(1)$$

$$\therefore \lim_{x\to\infty}\left(\frac{x+1}{\Gamma(x+2)^{\frac{1}{2x+2}}}-\frac{x}{\Gamma(x+1)^{\frac{1}{2x}}}\right)^{\sqrt{x}}=\lim_{x\to\infty}e^{\sqrt{x}\ln(o(1)_+)}=e^{-\infty}=0$$

UP.028. Let be $f:(0,\infty)\to(0,\infty)$,

$$f(x)=(x+1)^{\frac{(m+1)(x+2)}{x+1}}-x^{\frac{(m+1)(x+1)}{x}};\ m\in[0,\infty)\text{. Compute:}$$

$$\lim_{x\to\infty}\frac{f(x)}{x^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

Let $g:[x,x+1]\to\mathbb{R}$ $g(t)=t^{\frac{(m+1)(t+1)}{t}}$. From Lagrange theorem we have: $\exists x\in(x,x+1)$.

Such that $\frac{g(x+1)-g(x)}{x+1-x}=g'(c)\Rightarrow(x+1)^{\frac{(m+1)(x+2)}{x+1}}-x^{\frac{(m+1)(x+1)}{x}}=$

$$=c^{\frac{(m+1)(c+1)}{c}}\left[(m+1)\left(-\frac{1}{t^2}\right)\ln c+(m+1)\frac{c+1}{c}\cdot\frac{1}{c}\right]$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{x^m}=\lim_{x\to\infty}\frac{(m+1)c^{\frac{(m+1)(c+1)}{c}\left[-\frac{\ln c}{c^2}+\frac{c+1}{c^2}\right]}}{x^m}\quad \textit{(1)}$$

Because $c\in(x,x+1)$ and $x\to\infty$ we calculate this.

$$\lim_{x\to\infty}\frac{(m+1)x^{(m+1)\left(1+\frac{1}{x}\right)}\left[-\frac{\ln x}{x^2}+\frac{x+1}{x^2}\right]}{x^m}=(m+1)\lim_{x\to\infty}x^{\frac{m+1}{x}}\cdot x\left[-\frac{\ln x}{x^2}+\frac{x+1}{x^2}\right]=$$

$$=(m+1)\lim_{x\to\infty}x^{\frac{m+1}{x}}\left[-\frac{\ln x}{x}+\frac{x+1}{x}\right]\quad \textit{(2)}$$

$$\lim_{x\to\infty}x^{\frac{m+1}{x}}=e^{\lim_{x\to\infty}\frac{m+1}{x}\cdot\ln x}\overset{L'H}{=}e^{(m+1)\lim_{x\to\infty}\frac{1}{x}}=e^0=1\quad \textit{(3)}$$

$$\lim_{x\to\infty} -\frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} -\frac{1}{x} = 0 \quad (4)$$

$\lim_{x\to\infty}\frac{x+1}{x} = 1$ *(5). From (1)+(2)+(3)+(4)+(5)* $\Rightarrow \lim_{x\to\infty}\frac{f(x)}{x^m} = m+1.$

UP.029. Let $x, y \in (0,\infty)$. Prove that $\frac{2}{\pi}\arctan(x+y)\arctan\left(\frac{1}{x+y}\right) < \frac{x+y}{4xy+1}$

Proposed by Daniel Sitaru – Romania

Solution by Anas Adlany - Khemis Des Zemamra – Morroco

Put $x+y=s$ *and* $p=xy$ *then* $s^2 \ge 4p \Leftrightarrow \frac{s}{1+s^2} < \frac{s}{1+4p}$;

So it suffices to show that $\frac{2}{\pi}\arctan(x+y)\arctan\left(\frac{1}{x+y}\right) < \frac{s}{1+s^2}$

or $\arctan(s) - \frac{2}{\pi}(\arctan(s))^2 < \frac{s}{1+s^2}$

But we know that $\begin{cases} \arctan(s) < s \\ \arctan(s) > \frac{s}{1+s^2} \end{cases}$

$\Rightarrow \arctan(s) - \frac{2}{\pi}(\arctan(s))^2 < s - \frac{2}{\pi}\left(\frac{s}{1+s^2}\right)^2 < \frac{s}{1+s^2} < \frac{s}{1+4p}$ *as desired.*

UP.030. Prove that:

$$\sum_{k=1}^{n}\frac{k^{m-1}+(k+1)^{m-1}+(k+2)^{m-1}}{(2k+3)k^{m+1}+2(k+1)^{m+2}+(2k+1)(k+2)^{m+1}}$$
$$\le \frac{1}{8} - \frac{1}{4(n+1)(n+2)}$$

for all $n, m \in \mathbb{N}^*$.

Proposed by Mihály Bencze – Romania

Solution by proposer

First we show that for all $x, y, z > 0$ *and* $m \in \mathbb{N}^*$ *we have*

$$x^{m+1}(y+z) + y^{m+1}(z+x) + z^{m+1}(x+y) \ge 2xyz(x^{m-1}+y^{m-1}+z^{m-1})$$

Proof.

$\frac{x^m}{y} + \frac{y^m}{x} \ge x^{m-1} + y^{n-1} (x, y > 0)$*. If* $t = \frac{y}{x} > 0 \Rightarrow t^{m+1} + 1 \ge t^m + t \Leftrightarrow$

$$\Leftrightarrow (t-1)^2(t^{m-1}+t^{m-2}+\cdots+t+1)\ge 0$$

$$\sum\left(\frac{x^m}{y}+\frac{y^m}{x}\right)\ge\sum(x^{m-1}+y^{m-1})\Rightarrow\sum x^{m+1}(y+z)\ge 2xyz\sum x^{m-1}$$

If $k=k; y=k+1, z=k+2$ ***then***

$$\sum_{k=1}^{n}\frac{k^{m-1}+(k+1)^{m-1}+(k+2)^{m-1}}{(2k+3)k^{m+1}+2(k+1)^{m+2}+(2k+1)(k+2)^{n+1}}=$$

$$=\sum_{k=1}^{n}\frac{k^{m-1}+(k+1)^{m-1}+(k+2)^{m-1}}{(k+1+k+2)k^{n+1}+(k+k+2)(k+1)^{n+1}+(k+k+1)(k+2)^{n+1}}\le$$

$$\le\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k}-\frac{1}{k+1}+\frac{1}{2(k+2)}\right)=\frac{1}{8}-\frac{1}{4(n+1)(n+2)}$$