ROMANIAN MATHEMATICAL MAGAZINE

Prove the below closed form

$$\int_0^\infty \frac{\tan^{-1}(x^2)}{1+x^2}dx+\frac{1}{2}\int_0^\infty \frac{\tan^{-1}(4x^2)}{1+4x^2}dx+\frac{1}{3}\int_0^\infty \frac{\tan^{-1}(9x^2)}{1+9x^2}dx+\cdots=\frac{\pi^4}{48}$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^\infty \frac{\tan^{-1}(x^2)}{1+x^2}dx+\frac{1}{2}\int_0^\infty \frac{\tan^{-1}(4x^2)}{1+4x^2}dx+\frac{1}{3}\int_0^\infty \frac{\tan^{-1}(9x^2)}{1+9x^2}dx+\cdots$$

$$=\sum_{k=1}^\infty \frac{1}{k}\int_0^\infty \frac{\tan^{-1}(kx)^2}{1+(kx)^2}dx\Bigg]_{kx=y}=\sum_{k=1}^\infty \frac{1}{k^2}\int_0^\infty \frac{\tan^{-1}(y^2)}{1+y^2}dy$$

$$\int_0^\infty \frac{\tan^{-1}(y^2)}{1+y^2}dy=\int_0^1 \frac{\tan^{-1}(y^2)}{1+y^2}dy+\int_1^\infty \frac{\tan^{-1}(y^2)}{1+y^2}dy$$

$$=\int_0^1 \frac{\tan^{-1}(y^2)}{1+y^2}dy+\int_0^1 \frac{\tan^{-1}\left(\frac{1}{y^2}\right)}{1+y^2}dy=\frac{\pi}{2}\int_0^1 \frac{dy}{1+y^2}=\frac{\pi^2}{8}$$

$$I=\sum_{k=1}^\infty \frac{1}{k^2}\int_0^\infty \frac{\tan^{-1}(y^2)}{1+y^2}dy=\frac{\pi^2}{8}\sum_{k=1}^\infty \frac{1}{k^2}=\frac{\pi^2}{8}\zeta(2)=\frac{\pi^4}{48}$$