ROMANIAN MATHEMATICAL MAGAZINE

For $a,b,c,x>0$ and $\log_a(bx)=m,\log_c(ax)=n,\log_b(cx)=p$

Find $S=\log_{\frac{a^4b^5}{c^6x^3}}\left(\frac{x^8a^7b^5}{c^{20}}\right)$ in terms of m,n,p

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$$\log_a(bx)=m\ ;\ \log_c(ax)=n\ ;\ \log_c(cx)=p$$

Let's express a,b,c *by* x

$\log_c(ax)=n\ \Rightarrow a=\frac{c^n}{x}$; *Let's write the expression instead of* $\log_a(bx)=m$:

$$bx=\left(\frac{c^n}{x}\right)^m\Rightarrow bx^{m+1}=c^{mn}\ \ (1)$$

$$\log_a(bx)=m\Rightarrow b=\frac{a^m}{x}\ \ (2)$$

From $\log_c(cx)=p\Rightarrow b=(cx)^{\frac{1}{p}}\ \ (3)$

Let's write expression (3) *instead of* (1)*:*

$$bx^{m+1}=c^{mn}\Rightarrow (cx)^{\frac{1}{p}}\cdot x^{m+1}=c^{m+n}\Rightarrow c=x^{\frac{pm+p+1}{pmn-1}}\ (*)$$

Let's write expression $(*)$ *instead of* (3)*:*

$$b^p=x^{\frac{pm+p+1}{pmn-1}}\cdot x=x^{\frac{p(mn+m+1)}{pmn-1}}\Rightarrow b=x^{\frac{mn+m+1}{pmn-1}}\ (**)$$

Let's write expression $(**)$ *instead of* (1)*:*

$$x^{\frac{mn+m+1}{pmn-1}}\cdot x=a^n\Rightarrow x^{\frac{m(pn+n+1)}{pmn-1}}=a^m$$

$$a=x^{\frac{pn+n+1}{pmn-1}}\ (***)$$

Let's use the expressions $(*),(**)$ *and* $(***)$ *in* $\left(\frac{a^4b^5}{c^6x^3}\right)$

$$A=\frac{a^4b^5}{c^6x^3}=\frac{x^{\frac{4(pn+n+1)}{pmn-1}}\cdot x^{\frac{5(mn+m+1)}{pmn-1}}}{x^{\frac{6(pm+p+1)}{pmn-1}}\cdot x^3}=x^{\frac{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3}{mpn-1}}$$

$$B=\frac{x^8a^7b^5}{c^{20}}=\frac{x^8\cdot x^{\frac{7(pn+n+1)}{pmn-1}}\cdot x^{\frac{5(mn+m+1)}{pmn-1}}}{x^{\frac{20(pm+p+1)}{pmn-1}}}==x^{\frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{mpn-1}}$$

$$S=\log_A B=\frac{\frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{mpn-1}}{\frac{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3(mpn-1)}{mpn-1}}$$

$$=\frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3(mpn-1)}$$

$$S=\frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3(mpn-1)}$$