

ROMANIAN MATHEMATICAL MAGAZINE

$$X \in M_2(\mathbb{R}), X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X^5 = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$$

$$\text{Find: } \Omega = a - b + c - d$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; X^5 = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \stackrel{\text{say}}{\Rightarrow} A = X^5$$

$$\Rightarrow X_A = (3 - x) \cdot (5 - x) - 8 = x^2 - 8x + 7 = (x - 1) \cdot (x - 7)$$

$$\Rightarrow (x - 1) \cdot (x - 7) \Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$$

$$\Rightarrow U_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow U_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \rightarrow P^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow A = P \cdot D \cdot P^{-1}$$

$$\Rightarrow X = P \cdot D^{\frac{1}{5}} \cdot P^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt[5]{7} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 - \sqrt[5]{7} & -2 - \sqrt[5]{7} \\ 2 + \sqrt[5]{7} & -1 + \sqrt[5]{7} \end{pmatrix}$$

$$\Rightarrow \Omega = 3$$