

ROMANIAN MATHEMATICAL MAGAZINE

Solve for natural numbers:

$$\sum_{i=1}^x \sum_{k=1}^{100} |i - k| = 333300$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $1 \leq i \leq x$,

$$\begin{aligned} \sum_{k=1}^{100} |i - k| &= \sum_{k=1}^i |i - k| + \sum_{k=i+1}^{100} |k - i| = \sum_{k=1}^{i-1} k + \sum_{k=1}^{100-i} k = \\ &= \frac{1}{2}(i-1)i + \frac{1}{2}(100-i)(101-i) = \frac{1}{2}[i^2 - i + (100)(101) - 201i + i^2] = \end{aligned}$$

$$= i^2 - 101i + 5050 \Rightarrow \sum_{i=1}^x \sum_{k=1}^{100} |i - k|$$

$$= \frac{1}{6}x(x+1)(2x+1) - \frac{101x(x+1)}{2} + 5050x$$

$$\frac{1}{6}x(x+1)(2x+1) - \frac{1}{2}(101)x(x+1) + 5050x = 333300$$

$$\Rightarrow 2x^3 + 3x^2 + x - 303x^2 - 303x + 30300x = 1999800$$

$$\Rightarrow 2x^3 - 300x^2 + 29998x - 1999800 = 0$$

$$\Rightarrow x^2(x - 100) - 50x(x - 100) + 9999(x - 100) = 0$$

$$\Rightarrow (x - 100)(x^2 - 50x + 9999) = 0 \Rightarrow (x - 100)[(x - 25)^2 + 9374] = 0 \Rightarrow x = 100$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\begin{aligned} \sum_{k=1}^{100} |i - k| &= \sum_{k=1}^i |i - k| + \sum_{k=i+1}^{100} |i - k| = \sum_{k=1}^i |i - k| + \sum_{k=i+1}^{100} |k - i| \\ &= \frac{(i-1) \cdot i}{2} + \frac{(100-i) \cdot (101-i)}{2} = \frac{2i^2 + 10100 - 202i}{2} = i^2 - 101i + 5050 \end{aligned}$$

$$\sum_{i=1}^x \sum_{k=1}^{100} |i - k| = \sum_{i=1}^x (i^2 - 101i + 5050) = \frac{x \cdot (x+1) \cdot (2x+1)}{6} - \frac{101x \cdot (x+1)}{2} + 5050x =$$

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$$= \frac{x \cdot (x^2 - 150x + 14999)}{3} = 333300 \Rightarrow x = 100$$

$$\sum_{i=1}^x \sum_{k=1}^{100} |i - k| = 333300 \Rightarrow x = 100$$